COMPLETELY BAIRE SPACES, MENGER SPACES, AND PROJECTIVE SETS

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ABSTRACT. W. Hurewicz proved that analytic Menger sets of reals are σ-compact and that co-analytic completely Baire sets of reals are completely metrizable. It is natural to try to generalize these theorems to projective sets. This has previously been accomplished by $V = L$ for projective counterexamples, and the **Axiom of Projective Determinacy** for positive results. For the first problem, the first author, S. Todorcevic, and S. Tokgöz have produced a finer analysis with much weaker axioms. We produce a similar analysis for the second problem, showing the two problems are essentially equivalent. We also construct in ZFC a separable metrizable space with ω^{th} power completely Baire, yet lacking a dense completely metrizable subspace. This answers a question of Eagle and Tall in *Abstract Model Theory*.

I. INTRODUCTION

It is a common theme in *Descriptive Set Theory* that statements about simply definable sets are true, e.g. "All Borel sets are Lebesgue measurable," but that the Axiom of Choice entails the existence of a non-constructive counterexample: "There is a non-measurable set of reals." For subsets of $\mathbb{R}$ that are definable but not so simply, the situation is more complex; e.g. $V = L$ implies there is a continuous image of the complement of a continuous image of a Borel set that is not measurable, but the **Axiom of Projective Determinacy** (**PD**) implies that the class of subsets of $\mathbb{R}$ obtained by closing the class of Borel sets under complement and continuous image contains only measurable sets. Among other places, these phenomena have been investigated with regard to two classical theorems of W. Hurewicz. We first give the relevant definitions.

Definition 1.1. A topological space is **Menger** if whenever $\{\mathcal{U}_n\}_{n<\omega}$ is a sequence of open covers, there exists $\{\mathcal{V}_n\}_{n<\omega}$, $\mathcal{V}_n \subseteq \mathcal{U}_n$, $\mathcal{V}_n$ finite, such that $\{\bigcup \mathcal{V}_n : n < \omega\}$ is a cover.

Definition 1.2. A subset A of $\mathbb{R}$ is **analytic** if it is a continuous image of a Borel set. $C \subseteq \mathbb{R}$ is **co-analytic** if $\mathbb{R} \setminus A$ is analytic. $P \subseteq \mathbb{R}$ is **projective** if it is in the class of subsets of $\mathbb{R}$ obtained by closing the Borel sets under complementation and continuous real-valued image.

Definition 1.3. A topological space is **Baire** if the intersection of any countable family of dense open sets is dense. A space is **completely Baire** if each closed subspace is Baire.

Definition 1.4. A topological space is **Polish** if it is separable and completely metrizable.

Proposition 1.1 [H1]. Analytic Menger subsets of $\mathbb{R}$ are σ-compact.

Proposition 1.2 [H2]. Co-analytic completely Baire subsets of $\mathbb{R}$ are Polish.

Problem 1. Are "definable" Menger subsets of $\mathbb{R}$ σ-compact?

Problem 2. Are "definable" completely Baire subsets of $\mathbb{R}$ Polish?

II. Results from the literature

We refer to [Ke] for descriptive set theory, and to [K] for $V = L$ and large cardinals. Problem 1 was investigated by Miller and Fremlin [MF] in 1988. They proved that:

Proposition 2.1. $V = L$ implies there is a co-analytic Menger set of reals that is not σ-compact.

Proposition 2.2. PD implies every projective Menger set of reals is σ-compact.

PD is regarded as "true" by many descriptive set theorists but has quite large cardinal consistency strength. [MF] was extended by the first author and S. Tokgöz to consider spaces that were not necessarily metrizable [TT]. They also noted that:

Proposition 2.3. The **Axiom of Co-analytic Determinacy** ($\mathbf{\Pi^1_1}$-Determinacy) implies co-analytic Menger sets of reals are σ-compact.

Corollary 2.4. If there is a measurable cardinal, then co-analytic Menger sets of reals are σ-compact.

Problem 2 is solved from the appropriate determinacy assumptions in [Ke, 28.20]. Medini and Zdomskyy in 2015 proved

Theorem 2.5 [MZ]. $V = L$ implies there is an analytic, completely Baire set of reals which is not Polish.

The hypotheses concerning Problem 1 were considerably sharpened in [TTT]:

Theorem 2.6 [TTT]. $\omega_1^{L[a]} < \omega_1$, for all reals a, if and only if every co-analytic Menger set of reals is σ-compact.

Corollary 2.7 [TTT]. The assertion that every co-analytic Menger set of reals is σ-compact is equiconsistent with the existence of an inaccessible cardinal.

We shall prove analogous results with respect to Problem 2:

Theorem 2.8. $\omega_1^{L[a]} < \omega_1$, for all reals a, if and only if every analytic, completely Baire set of reals is Polish.

Corollary 2.9. The assertion that every analytic completely Baire set of reals is Polish is equiconsistent with the existence of an inaccessible cardinal.

III. The Menger property and the completely Baire property

There is a surprising connection between **Problems 1 and 2**:

Theorem 3.1. Suppose X is a Menger set of reals. Then $\mathbb{R} \setminus X$ is completely Baire.

Proof. We need:

Proposition 3.2 [H2] (For a proof in English, see [vM]). A metrizable space is completely Baire if and only if it does not include a closed copy of the space $\mathbb{Q}$ of rationals.

Now suppose $\mathbb{R} \setminus X$ is not completely Baire. Then there is a copy Q of $\mathbb{Q}$ closed in $\mathbb{R} \setminus X$. Then $\overline{Q} \setminus Q$ is Polish, nowhere locally compact and zero-dimensional (since it does not include any interval). But then it is homeomorphic to the space $\mathbb{P}$ of irrationals, which is not Menger, despite being a closed subspace of X. This is a contradiction. Thus $\mathbb{R} \setminus X$ is completely Baire, which was to be proved. $\square$

IV. An Open Graph Axiom

In [To], Todorcevic introduced what he called the **Open Coloring Axiom**. Unfortunately that name had earlier been used by Abraham–Rubin–Shelah [ARS]. This has caused some confusion, so Todorcevic has renamed his axiom the **Open Graph Axiom**. A variation of this axiom was introduced in [F]. Slightly modifying Feng's notation, we have:

Definition 4.1. Let Γ be a collection of subsets of $\mathbb{R}$. OGA*(Γ): Let X be any member of Γ. Let $[X]^2 = K_1 \cup K_2$ be a partition with K_1 open in the topology on $[X]^2$ inherited from X^2. Either there is a perfect $A \subseteq X$ with $[A]^2 \subseteq K_1$ or $X = \bigcup_{n<\omega} A_n$ with $[A_n]^2 \subseteq K_2$ for all $n < \omega$.

Replacing "perfect" with "uncountable" one gets the **Open Graph Axiom**. The **Open Graph Axiom** implies the continuum hypothesis fails

[To], but **OGA*(projective)** holds in the model obtained by collapsing an inaccessible to ω_1 by finite conditions [F]. CH can be arranged to hold in such a model, so **OGA*(projective)** surprisingly does not imply **OGA**.

Theorem 4.1. The following are equivalent.

a) **OGA*(co-analytic)**;
b) co-analytic Menger subsets of $\mathbb{R}$ are σ-compact;
c) analytic completely Baire subsets of $\mathbb{R}$ are Polish;
d) for every $a \in \mathbb{R}$, $\omega_1^{L[a]} < \omega_1$;
e) every uncountable co-analytic set includes a perfect set.

Proof. The equivalence of a) and b) is in [TTT]; of a), d), and e) in [F]. The implication from c) to b) follows from Theorem 3.1. To obtain c) from a), we recall that in [TTT] it is shown that:

Proposition 4.2. If Γ is closed under continuous pre-images, then $\text{OGA*}(\Gamma)$ implies that if $A \in \Gamma$ is not σ-compact, then there is a compact $K \subseteq \mathbb{R}$ such that $K \cap A$ is homeomorphic to $\mathbb{P}$, the space of irrationals, while $K \cap (\mathbb{R} \setminus A)$ is homeomorphic to $\mathbb{Q}$.

Now suppose B is completely Baire, analytic, and not Polish. Then $\mathbb{R} \setminus B$ is co-analytic and not σ-compact, so there is a compact K with $K \cap B$ homeomorphic to $\mathbb{Q}$. But $K \cap B$ is closed in B, contradicting B being completely Baire. $\square$

The next result extends Theorems 2.7 and 4.1 to projective sets.

Theorem 4.3. The following are equiconsistent:

a) **OGA*(projective)**;
b) projective Menger subsets of $\mathbb{R}$ are σ-compact;
c) projective completely Baire subsets of $\mathbb{R}$ are Polish;
d) there is an inaccessible cardinal;
e) every uncountable projective set includes a perfect set.

Proof. Recall that the complement of a projective set is projective and that the continuous pre-image of a projective set is projective [Ke]. Also recall that if $\omega_1^{L[a]} < \omega_1$, then ω_1 is inaccessible in L, so it is consistent there is an inaccessible cardinal. The implication from a) to b) is in [TTT] and is clear from 4.2. That c) implies b) is by 3.1. The consistency of b) implies the consistency of d) by 4.1. Feng [F] proved the equiconsistency of a) and d), and that a) is equivalent to e). $\square$

V. An Application to Model Theory

Definition 5.1. A *theory* is a set of sentences closed under logical consequence. A *type* is a collection of formulas. An *n-type* is a collection of formulas, each with exactly n free variables. An *n*-type is called *isolated* for a theory T if there is a formula φ with exactly n free variables such that in every model $\mathcal{M}$ of T, any n-tuple satisfying φ must satisfy all the

elements of the type. The *Omitting Types Theorem* (**OTT**) (for first-order logic) asserts:

Let $\{\Sigma_k\}_{k<\omega}$ be a countable collection of non-isolated types. Then there is a model $\mathcal{M}$ of T such that for each k and each $\sigma \in \Sigma_k$, $\mathcal{M} \models \neg\sigma$.

The **OTT** follows from the **Compactness Theorem** for first-order logic, but also holds for certain other logics for which compactness fails, and is a useful substitute for compactness. The **OTT** is often proved by a Baire category argument; the exact relationship between the **OTT** and the **Baire Category Theorem** is investigated in Eagle and Tall [ET]. Just as the existence of a winning strategy for Non-empty in the Banach-Mazur game (*weak α-favorability*) is strictly stronger than just being a Baire space, one can formulate a game version of the **OTT** and ask whether it too is strictly stronger. Eagle and Tall show how to define abstract logics from arbitrary topological spaces and prove:

Theorem 5.1. There is an abstract logic satisfying the **OTT** but not its game version if there exists a separable metrizable Y such that Y^ω is completely Baire but does not include a dense completely metrizable subspace.

They then note that a *non-meager P-filter* is such a Y, but such a filter is not known to exist in ZFC, although its existence follows from a variety of consistent set-theoretic hypotheses.

Theorem 5.2. Such a space and hence such a logic exist in ZFC.

We shall exploit the work of Repovš, Zdomskyy, and Zhang, 2014.

Proposition 5.3 [RZZ]. There is a subspace X of 2^ω such that X is Menger, non-meager, and is a filter extending the Fréchet filter.

$Y = 2^\omega \setminus X$ **is the desired space!**

First we prove:

Lemma 5.4. If $X \subseteq 2^\omega$ is Menger, then $(2^\omega \setminus X)^\omega$ is completely Baire.

Proof. It suffices to prove that $(2^\omega)^\omega \setminus (2^\omega \setminus X)^\omega$ is Menger. $(2^\omega)^\omega \setminus (2^\omega \setminus X)^\omega = \bigcup_{n<\omega} Y_n$, where

$$Y_n = \prod_{k<\omega} Y_{n,k},$$

$$Y_{n,k} = \begin{cases} X & \text{for } n = k \\ 2^\omega & \text{for } n \neq k \end{cases}$$

Each Y_n is Menger, so $(2^\omega)^\omega \setminus (2^\omega \setminus X)^\omega$ is Menger, because a countable union of Menger spaces is Menger. $\qquad\square$

To prove $Y^\omega = (2^\omega \setminus X)^\omega$ has no completely metrizable dense subspace, suppose on the contrary that it has such a subspace P. Let $Z = \pi_1(P)$ be the projection onto the first coordinate of the product. Then Z is dense in

Y. Since P is G_δ in the separable metrizable space Y^ω, Z is analytic and therefore has the **Baire Property**, i.e. $Z = U \triangle N$, where U is open and N is meager. We divide into cases, depending on whether or not Z is meager. Either case leads to a contradiction.

Case 1: Z is meager in Y.

Then, considering Y^ω as $Y \times Y^\omega$, we have $Z \times Y^\omega$ is dense in Y^ω. But since Z is meager in Y, $Z \times Y^\omega$ is meager in Y^ω. But $P \subseteq Z \times Y^\omega$, and a meager set can't include a dense Polish subspace.

Case 2: Then since Z has the Baire Property there is a non-empty open $U \subseteq Y$ and a dense G_δ T in U such that $T \subseteq Z$ [Ke, 8.26]. Without loss of generality, let U be $Y \cap [s]$, where $[s]$ is basic open in 2^ω, $[s] = \{f \in 2^\omega : f|n = s\}$, where $s \in 2^n$. Let us enumerate $2^n = \{s_i : i \leq 2^n\}$. Without loss of generality, $s = s_0$. For $i \neq 0$, let $\varphi_i : [s_0] \to [s_i]$ be defined by

$$\varphi_i(y) = s_i^n (y \restriction [n, +\infty)).$$

Then observe that $\varphi_i[[s_0 \cap X]] = [s_i] \cap X$, because X is a filter extending the Fréchet filter, and so is closed under finite modifications. Let $T_0 = T$. For $0 < i \leq 2^n$, let $T_i = \varphi_i[T]$. Then T_i is a dense G_δ in $[s_i]$, since φ_i is a homeomorphism. Moreover, $T_i \cap X = \varnothing$. Then $R = \bigcup_{i \leq 2^n} T_i$ is a G_δ in 2^ω and is dense there. But $R \cap X = \varnothing$, contradicting X being non-meager. $\square$

Remark. The complement of a completely Baire subset of $\mathbb{R}$ need not be Menger. A Bernstein set B and its complement are both completely Baire. We shall show that B has a non-Menger closed subspace and so is not Menger. Let $\mathbb{K}$ be the Cantor set. Since both B and $\mathbb{K} \setminus B$ are Bernstein in $\mathbb{K}$, $\mathbb{K} \setminus B$ is dense in $\mathbb{K}$. Fix a countable dense set $Q \subseteq \mathbb{K} \setminus B$. Then $\mathbb{K} \setminus Q$ is a copy of the space of irrationals. $B \cap \mathbb{K} = B \cap (\mathbb{K} \setminus Q)$ is closed in B and is a Bernstein subset of $\mathbb{K} \setminus Q$. But $\mathbb{K} \setminus Q$ is homeomorphic to $^\omega\omega$. It remains to quote:

Lemma 5.5 [BTZ]. Bernstein subsets of $^\omega\omega$ are not Menger. $\square$

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