PRODUCTS OF H-SEPARABLE SPACES IN THE LAVER MODEL

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Abstract. We prove that in the Laver model for the consistency of the Borel's conjecture, the product of any two H-separable spaces is M-separable.

1. Introduction

This paper is devoted to products of H-separable spaces. A topological space X is said [3] to be *H-separable*, if for every sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, one can pick finite subsets $F_n \subset D_n$ so that every nonempty open set $O \subset X$ meets all but finitely many F_n's. If we only demand that $\bigcup_{n\in\omega} F_n$ is dense we get the definition of *M-separable* spaces introduced in [14]. It is obvious that second-countable spaces (even spaces with a countable π-base) are H-separable, and each H-separable space is M-separable. The main result of our paper is the following

Theorem 1.1. *In the Laver model for the consistency of the Borel's conjecture, the product of any two countable H-separable spaces is M-separable.*

Consequently, the product of any two H-separable spaces is M-separable provided that it is hereditarily separable.

It worth mentioning here that by [12, Theorem 1.2] the equality $\mathfrak{b} = \mathfrak{c}$ which holds in the Laver model implies that the M-separability is not preserved by finite products of countable spaces in the strong sense.

Let us recall that a topological space X is said to have the *Menger property* (or, alternatively, is a *Menger space*) if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that each $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ and the collection $\{\cup \mathcal{V}_n : n \in \omega\}$ is a cover of X. This property was introduced by Hurewicz, and the current name (the Menger property) is used because Hurewicz proved in [7] that for metrizable spaces his property is equivalent to a certain property of a base considered by Menger in [10]. If in the definition above we additionally require that $\{n \in \omega : x \notin \cup \mathcal{V}_n\}$ is finite for each $x \in X$, then we obtain the definition of the *Hurewicz property* introduced in [8]. The original idea behind the

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Menger's property, as it is explicitly stated in the first paragraph of [10], was an application in dimension theory, one of the areas of interest of Mardešić. However, this paper concentrates on set-theoretic and combinatorial aspects of the property of Menger and its variations.

Theorem 1.1 is closely related to the main result of [13] asserting that in the Laver model the product of any two Hurewicz metrizable spaces has the Menger property. Let us note that our proof in [13] is conceptually different, even though both proofs are based on the same main technical lemma of [9]. Regarding the relation between Theorem 1.1 and the main result of [13], each of them implies a weak form of the other one via the following duality results: For a metrizable space X, $C_p(X)$ is M-separable (resp. H-separable) if and only if all finite powers of X are Menger (resp. Hurewicz), see [14, Theorem 35] and [3, Theorem 40], respectively. Thus Theorem 1.1 (combined with the well-known fact that $C_p(X)$ is hereditarily separable for metrizable separable spaces X) implies that in the Laver model, if all finite powers of metrizable separable spaces X_0, X_1 are Hurewicz, then $X_0 \times X_1$ is Menger. And vice versa: The main result of [13] implies that in the Laver model, the product of two H-separable spaces of the form $C_p(X)$ for a metrizable separable X, is M-separable.

The proof of Theorem 1.1, which is based on the analysis of names for reals in the style of [9], unfortunately seems to be rather tailored for the H-separability and we were not able to prove any analogous results even for small variations thereof. Recall from [6] that a space X is said to be *wH-separable* if for any *decreasing* sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, one can pick finite subsets $F_n \subset D_n$ such that for any non-empty open $U \subset X$ the set $\{n \in \omega : U \cap F_n \neq \emptyset\}$ is co-finite. It is clear that every H-separable space is wH-separable, and it seems to be unknown whether the converse is (at least consistently) true. Combining [6, Lemma 2.7(2) and Corollary 4.2] we obtain that every countable Fréchet-Urysohn space is wH-separable, and to our best knowledge it is open whether countable Fréchet-Urysohn spaces must be H-separable. The statement "finite products of countable Fréchet-Urysohn spaces are M-separable" is known to be independent from ZFC: It follows from the PFA by [2, Theorem 3.3], holds in the Cohen model by [2, Corollary 3.2], and fails under CH by [1, Theorem 2.24]. Moreover[1], CH implies the existence of two countable Fréchet-Urysohn H-separable topological groups whose product is not M-separable, see [11, Corollary 6.2]. These results motivate the following

Question 1.2. (1) Is it consistent that the product of two countable wH-separable spaces is M-separable? Does this statement hold in the Laver model?

(2) Is the product of two countable Fréchet-Urysohn space M-separable in the Laver model?

(3) Is the product of three (finitely many) countable H-separable spaces M-separable in the Laver model?

[1] We do not know whether the spaces constructed in the proof of [1, Theorem 2.24] are H-separable.

(4) Is the product of finitely many countable H-separable spaces H-separable in the Laver model?

2. Proof of Theorem 1.1

We need the following

Definition 2.1. A topological space $\langle X, \tau\rangle$ is called *box-separable* if for every function R assigning to each countable family $\mathcal{U}$ of non-empty open subsets of X a sequence $R(\mathcal{U}) = \langle F_n : n \in \omega\rangle$ of finite non-empty subsets of X such that $\{n : F_n \subset U\}$ is infinite for every $U \in \mathcal{U}$, there exists $\mathsf{U} \subset [\tau \setminus \{\emptyset\}]^\omega$ of size $|\mathsf{U}| = \omega_1$ such that for all $U \in \tau \setminus \{\emptyset\}$ there exists $\mathcal{U} \in \mathsf{U}$ such that $\{n : R(\mathcal{U})(n) \subset U\}$ is infinite.

Any countable space is obviously box-separable under CH, which makes the latter notion uninteresting when considered in arbitrary ZFC models. However, as we shall see in Lemma 2.3, the box-separability becomes useful under $\mathfrak{b} > \omega_1$. Here $\mathfrak{b}$ denotes the minimal cardinality of a subspace X of ω^ω which is not eventually dominated by a single function, see [4] for more information on $\mathfrak{b}$ and other cardinal characteristics of the reals.

The following lemma is the key part of the proof of Theorem 1.1. We will use the notation from [9] with the only difference being that smaller conditions in a forcing poset are supposed to carry more information about the generic filter, and the ground model is denoted by V.

A subset C of ω_2 is called an ω_1-*club* if it is unbounded and for every $\alpha \in \omega_2$ of cofinality ω_1, if $C \cap \alpha$ is cofinal in α then $\alpha \in C$.

Lemma 2.2. *In the Laver model every countable H-separable space is box-separable.*

Proof. We work in $V[G_{\omega_2}]$, where G_{ω_2} is $\mathbb{P}_{\omega_2}$-generic and $\mathbb{P}_{\omega_2}$ is the iteration of length ω_2 with countable supports of the Laver forcing, see [9] for details. Let us fix an H-separable space of the form $\langle \omega, \tau\rangle$ and a function R such as in the definition of box-separability. By a standard argument (see, e.g., the proof of [5, Lemma 5.10]) there exists an ω_1-club $C \subset \omega_2$ such that for every $\alpha \in C$ the following conditions hold:

(*i*) $\tau \cap V[G_\alpha] \in V[G_\alpha]$ and for every sequence $\langle D_n : n \in \omega\rangle \in V[G_\alpha]$ of dense subsets of $\langle \omega, \tau\rangle$ there exists a sequence $\langle K_n : n \in \omega\rangle \in V[G_\alpha]$ such that $K_n \in [D_n]^{<\omega}$ and for every $U \in \tau \setminus \emptyset$ the intersection $U \cap K_n$ is non-empty for all but finitely many $n \in \omega$;

(*ii*) $R(\mathcal{U}) \in V[G_\alpha]$ for any $\mathcal{U} \in [\tau \setminus \{\emptyset\}]^\omega \cap V[G_\alpha]$; and

(*iii*) For every $A \in \mathcal{P}(\omega) \cap V[G_\alpha]$ the interior $Int(A)$ also belongs to $V[G_\alpha]$.

By [9, Lemma 11] there is no loss of generality in assuming that $0 \in C$. We claim that $\mathsf{U} := [\tau \setminus \{\emptyset\}]^\omega \cap V$ is a witness for $\langle \omega, \tau\rangle$ being box-separable. Suppose, contrary to our claim, that there exists $A \in \tau \setminus \{\emptyset\}$ such that $R(\mathcal{U})(n) \not\subset A$ for all but finitely many $n \in \omega$ and $\mathcal{U} \in \mathsf{U}$. Let $\dot{A}$ be a $\mathbb{P}_{\omega_2}$-name for A and $p \in \mathbb{P}_{\omega_2}$ a condition forcing the above statement. Applying [9, Lemma 14] to the sequence $\langle \dot{a}_i : i \in \omega\rangle$ such that $\dot{a}_i = \dot{A}$ for all $i \in \omega$, we get a condition $p' \leq p$ such that $p'(0) \leq^0 p(0)$, and a finite set $\mathcal{U}_s \subset \mathcal{P}(\omega)$ for every $s \in p'(0)$ with $p'(0)\langle 0\rangle \leq s$, such that for each $n \in \omega$, $s \in p'(0)$ with $p'(0)\langle 0\rangle \leq s$, and for all but finitely many immediate successors t of s in $p'(0)$ we have

$$p'(0)_t\hat{\ }p' \upharpoonright [1, \omega_2) \Vdash \exists U \in \mathcal{U}_s\ (\dot{A} \cap n = U \cap n).$$

Of course, any $p'' \leq p'$ also has the property above with the same $\mathcal{U}_s$'s. However, the stronger p'' is, the more elements of $\mathcal{U}_s$ might play no role any more. Therefore throughout the rest of the proof we shall call $U \in \mathcal{U}_s$ *void for* $p'' \leq p'$ *and* $s \in p''(0)$, where $p''(0)\langle 0\rangle \leq s$, if there exists $n \in \omega$ such that for all but finitely many immediate successors t of s in $p''(0)$ there is *no* $q \leq p''(0)_t\hat{\ }p'' \upharpoonright [1, \omega_2)$ with the property $q \Vdash \dot{A} \cap n = U \cap n$. Note that for any $p'' \leq p'$ and $s \in p''(0)$, $p''(0)\langle 0\rangle \leq s$, there exists $U \in \mathcal{U}_s$ which is non-void for p'', s. Two cases are possible.

a) For every $p'' \leq p'$ there exists $s \in p''(0)$, $p''(0)\langle 0\rangle \leq s$, and a non-void $U \in \mathcal{U}_s$ for p'', s such that $Int(U) \neq \emptyset$. In this case let $\mathcal{U} \in \mathsf{U}$ be any countable family containing $\{Int(U) : U \in \bigcup_{s\in p'(0), p'(0)\langle 0\rangle \leq s} \mathcal{U}_s\} \setminus \{\emptyset\}$. It follows from the above that p forces $R(\mathcal{U})(k) \not\subset \dot{A}$ for all but finitely many $k \in \omega$. Let $p'' \leq p'$ and $m \in \omega$ be such that p'' forces $R(\mathcal{U})(k) \not\subset \dot{A}$ for all $k \geq m$. Fix a non-void U for p'', s, where $s \in p''(0)$ and $p''(0)\langle 0\rangle \leq s$, such that $Int(U) \neq \emptyset$ (and hence $Int(U) \in \mathcal{U}$). It follows from the above that there exists $k \geq m$ such that $R(\mathcal{U})(k) \subset Int(U) \subset U$. Let $n \in \omega$ be such that $R(\mathcal{U})(k) \subset n$. By the definition of being non-void there are infinitely many immediate successors t of s in $p''(0)$ for which there exists $q_t \leq p''(0)_t\hat{\ }p'' \upharpoonright [1, \omega_2)$ with the property $q_t \Vdash \dot{A} \cap n = U \cap n$. Then for any q_t as above we have that q_t forces $R(\mathcal{U})(k) \subset \dot{A}$ because $R(\mathcal{U})(k) \subset U \cap n$, which contradicts the fact that $q_t \leq p''$ and $p'' \Vdash R(\mathcal{U})(k) \not\subset \dot{A}$.

b) There exists $p'' \leq p'$ such that for all $s \in p''(0)$, $p''(0)\langle 0\rangle \leq s$, every $U \in \mathcal{U}_s$ with $Int(U) \neq \emptyset$ is void for p'', s. Note that this implies that every $U \in \mathcal{U}_s$ with $Int(U) \neq \emptyset$ is void for q, s for all $q \leq p''$ and $s \in q(0)$ such that $q(0)\langle 0\rangle \leq s$.

Let $\langle D_k : k \in \omega\rangle \in V$ be a sequence of dense subsets of $\langle \omega, \tau\rangle$ such that for every $U \in \bigcup_{s\in p''(0), p''(0)\langle 0\rangle \leq s} \mathcal{U}_s$, if $Int(U) = \emptyset$, then $\omega \setminus U = D_k$ for infinitely many $k \in \omega$. Let $\langle K_k : k \in \omega\rangle \in V$ be such as in item (i) above. Then p'' forces that $K_k \cap \dot{A} \neq \emptyset$ for all but finitely many $k \in \omega$. Passing to a stronger condition, we may additionally assume if necessary, that there exists $m \in \omega$ such that $p'' \Vdash \forall k \geq m\ (K_k \cap \dot{A} \neq \emptyset)$.

Fix $U \in \mathcal{U}_{p''(0)\langle 0\rangle}$ non-void for $p'', p''(0)\langle 0\rangle$. Then $Int(U) = \emptyset$ by the choice of p'' and hence there exists $k \geq m$ such that $\omega \setminus U = D_k$. It follows that $K_k \cap U = \emptyset$ because $K_k \subset D_k$. On the other hand, since U is non-void for $p'', p''(0)\langle 0\rangle$, for $n = \max K_k + 1$ we can find infinitely many immediate successors t of $p''(0)\langle 0\rangle$ in $p''(0)$ for which there exists $q_t \leq p''(0)_t\hat{\ }p'' \upharpoonright [1, \omega_2)$ forcing $\dot{A} \cap n = U \cap n$. Then any such q_t forces $K_k \cap \dot{A} = \emptyset$ (because $K_k \subset n$ and $K_k \cap U = \emptyset$), contradicting the fact that $p'' \geq q_t$ and $p'' \Vdash K_k \cap \dot{A} \neq \emptyset$.

Contradictions obtained in cases $a)$ and $b)$ above imply that $\mathsf{U} := [\tau \setminus \{\emptyset\}]^\omega \cap V$ is a witness for $\langle\omega, \tau\rangle$ being box-separable, which completes our proof. □

Theorem 1.1 is a direct consequence of Lemma 2.2 combined with the following

Lemma 2.3. *Suppose that $\mathfrak{b} > \omega_1$, X is box-separable, and Y is H-separable. Then $X \times Y$ is M-separable provided that it is separable.*

Proof. Let $\langle D_n : n \in \omega\rangle$ be a sequence of countable dense subsets of $X \times Y$. Let us fix a countable family $\mathcal{U}$ of open non-empty subsets of X and a partition $\omega = \sqcup_{U\in\mathcal{U}}\Omega_U$ into infinite pieces. For every $n \in \Omega_U$ set $D^{\mathcal{U}}_n = \{y \in Y : \exists x \in U(\langle x, y\rangle \in D_n)\}$ and note that $D^{\mathcal{U}}_n$ is dense in Y for all $n \in \omega$. Therefore there exists a sequence $\langle L^{\mathcal{U}}_n : n \in \omega\rangle$ such that $L^{\mathcal{U}}_n \in [D^{\mathcal{U}}_n]^{<\omega}$ and for every open non-empty $V \subset Y$ we have $L^{\mathcal{U}}_n \cap V \neq \emptyset$ for all but finitely many n. For every $n \in \Omega_U$ find $K^{\mathcal{U}}_n \in [U]^{<\omega}$ such that for every $y \in L^{\mathcal{U}}_n$ there exists $x \in K^{\mathcal{U}}_n$ such that $\langle x, y\rangle \in D_n$, and set $R(\mathcal{U}) = \langle K^{\mathcal{U}}_n : n \in \omega\rangle$. Note that R is such as in the definition of box-separability because $K^{\mathcal{U}}_n \subset U$ for all $n \in \Omega_U$ and the latter set is infinite. Since X is box-separable there exists a family U of countable collections of open non-empty subsets of X of size $|\mathsf{U}| = \omega_1$, and such that for every open non-empty $U \subset X$ there exists $\mathcal{U} \in \mathsf{U}$ with the property $R(\mathcal{U})(n) \subset U$ for infinitely many n. Since each D_n is countable and $|\mathcal{U}| < \mathfrak{b}$, there exists a sequence $\langle F_n : n \in \omega\rangle$ such that $F_n \in [D_n]^{<\omega}$ and for every $\mathcal{U} \in \mathsf{U}$ we have $F_n \supset (K^{\mathcal{U}}_n \times L^{\mathcal{U}}_n) \cap D_n$ for all but finitely many $n \in \omega$.

We claim that $\bigcup_{n\in\omega} F_n$ is dense in $X \times Y$. Indeed, let us fix open non-empty subset of $X \times Y$ of the form $U \times V$ and find $\mathcal{U} \in \mathsf{U}$ with the property $R(\mathcal{U})(n) = K^{\mathcal{U}}_n \subset U$ for infinitely many n, say for all $n \in I \in [\omega]^\omega$. Passing to a co-finite subset of I, we may assume if necessary, that $F_n \supset (K^{\mathcal{U}}_n \times L^{\mathcal{U}}_n) \cap D_n$ for all $n \in I$. Finally, fix $n \in I$ such that $L^{\mathcal{U}}_n \cap V \neq \emptyset$ and pick $y \in L^{\mathcal{U}}_n \cap V$. By the definition of $D^{\mathcal{U}}_n$ and $L^{\mathcal{U}}_n \subset D^{\mathcal{U}}_n$ we can find $x \in K^{\mathcal{U}}_n$ such that $\langle x, y\rangle \in D_n$. Then $\langle x, y\rangle \in U \times V$ and $\langle x, y\rangle \in F_n$ because $\langle x, y\rangle \in K^{\mathcal{U}}_n \times L^{\mathcal{U}}_n$ and $\langle x, y\rangle \in D_n$. This completes our proof. □

References

[1] Barman, D.; Dow, A., *Selective separability and SS+,* Topology Proc. **37** (2011), 181–204.

[2] Barman, D.; Dow, A., *Proper forcing axiom and selective separability,* Topology Appl. **159** (2012), 806–813.

[3] Bella, A., Bonanzinga, M., Matveev, M., *Variations of selective separability,* Topology Appl. **156** (2009), 1241–1252.

[4] Blass, A., *Combinatorial cardinal characteristics of the continuum*, in: *Handbook of Set Theory* (M. Foreman, A. Kanamori, and M. Magidor, eds.), Springer, 2010, pp. 395–491.

[5] Blass, A.; Shelah, S., *There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed,* Ann. Pure Appl. Logic **33** (1987), 213–243.

[6] Gruenhage, G.; Sakai, M., *Selective separability and its variations,* Topology Appl. **158** (2011) 1352–1359.

[7] Hurewicz, W., *Über die Verallgemeinerung des Borellschen Theorems*, Math. Z. **24** (1925), 401–421.

[8] Hurewicz, W., *Über Folgen stetiger Funktionen,* Fund. Math. **9** (1927), 193–204.

[9] Laver, R., *On the consistency of Borel's conjecture*, Acta Math. **137** (1976), 151–169.

[10] Menger, K., *Einige Überdeckungssätze der Punktmengenlehre, Sitzungsberichte*. Abt. 2a, Mathematik, Astronomie, Physik, Meteorologie und Mechanik (Wiener Akademie) **133** (1924), 421–444.

[11] Miller, A.W.; Tsaban, B.; Zdomskyy, L., *Selective covering properties of product spaces, II: gamma spaces,* Trans. Amer. Math. Soc. **368** (2016), 2865–2889.

[12] Repovš, D.; Zdomskyy, L., *On M-separability of countable spaces and function spaces,* Topology Appl. **157** (2010), 2538–2541.

[13] Repovš, D.; Zdomskyy, L., *Products of Hurewicz spaces in the Laver model,* Bull. Symb. Log. **23**:3 (2017), 324–333.

[14] Scheepers, M., *Combinatorics of open covers. VI. Selectors for sequences of dense sets,* Quaest. Math. **22** (1999), 109–130.

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