PRODUCTS OF MENGER SPACES IN THE MILLER MODEL

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Abstract. We prove that in the Miller model the Menger property is preserved by finite products of metrizable spaces. This answers several open questions and gives another instance of the interplay between classical forcing posets with fusion and combinatorial covering properties in topology.

1. Introduction

A topological space X has the *Menger* property (or, alternatively, is a Menger space) if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ for all n and $X = \bigcup\{\cup \mathcal{V}_n : n \in \omega\}$. This property was introduced by Hurewicz, and the current name (the Menger property) is used because Hurewicz proved in [14] that for metrizable spaces his property is equivalent to one considered by Menger in [18]. Each σ-compact space has obviously the Menger property, and the latter implies lindelöfness (that is, every open cover has a countable subcover). The Menger property is the weakest one among the so-called *selection principles* or *combinatorial covering properties*, see, e.g., [4, 16, 24, 25, 31] for detailed introductions to the topic. Menger spaces have recently found applications in such areas as forcing [12], Ramsey theory in algebra [33], combinatorics of discrete subspaces [2], and Tukey relations between hyperspaces of compacts [13].

In this paper we proceed our investigation of the interplay between posets with fusion and selection principles initiated in [23]. More precisely, we concentrate on the question whether the Menger property is preserved by finite products. For general topological spaces the answer negative: In ZFC there are two normal spaces X, Y with a covering property much stronger than the Menger one such that $X \times Y$ does not have the Lindelöf property, see [28, §3]. However, the above situation becomes impossible if we restrict our attention to metrizable spaces. This case, on which we concentrate in the sequel, turned out to be sensitive to the ambient set-theoretic universe. Indeed, by [21, Theorem 3.2] under CH there exist $X, Y \subset \mathbb{R}$ which have the Menger property (in fact, they have the strongest combinatorial covering property considered thus far), whose product $X \times Y$ is not Menger. There

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are many results of this kind where CH is relaxed to an equality between cardinal characteristics, see, e.g., [3, 15, 22, 27]. Surprisingly, there are also inequalities between cardinal characteristics which imply that the Menger property is not productive even for sets of reals, see [26]. The following theorem, which is the main result of our paper, shows that an additional set-theoretic assumption in all these results was indeed necessary.

Theorem 1.1. *In the Miller model, the product of any two Menger spaces is Menger provided that it is Lindelöf. In particular, in this model the product of any two Menger metrizable spaces is Menger.*

Theorem 1.1 answers [26, Problem 7.9(2)], [29, Problem 8.4] (restated as [30, Problem 4.11]), and [32, Problem 6.7] in the affirmative; implies that the affirmative answer to [1, Problem II.2.8], [5, Problem 3.9], and [15, Problem 2] (restated as [31, Problem 3.2] and [32, Problem 2.1]) is consistent; implies that the negative answer to and [1, Problem II.2.7], [6, Problem 8.9], and [35, Problems 1,2,3] is consistent; and answers [24, Problem 7] in the negative.

By the *Miller model* we mean a generic extension of a ground model of GCH with respect to the iteration of length ω_2 with countable support of the Miller forcing, see the next section for its definition. This model has been first considered by Miller in [19] and since then found numerous applications, see [11] and references therein. The Miller forcing is similar to the Laver one introduced in [17], the main difference being that the splitting is allowed to occur less often. The main technical part of the proof of Theorem 1.1 is Lemma 2.3 which is an analog of [17, Lemma 14]. The latter one was the key ingredient in the proof that all strong measure zero sets of reals are countable in the Laver model given in [17], the arguably most quotable combinatorial feature of the Laver model.

As we shall see in Section 2, a big part of the proof of Theorem 1.1 requires only the inequality $\mathfrak{u} < \mathfrak{g}$ which holds in the Miller model. However, we do not know the answer to the following

Question 1.2. Is the Menger property preserved by finite products of metrizable spaces under $\mathfrak{u} < \mathfrak{g}$? If yes, can $\mathfrak{u} < \mathfrak{g}$ be weakened to the Filter Dichotomy, NCF, or $\mathfrak{u} < \mathfrak{d}$?

We refer the reader to [11, § 9] for corresponding definitions.

We assume that the reader is familiar with the basics of forcing. The paper is essentially self-contained in the sense that we give all the definitions needed to understand our proofs.

2. Proofs

The proof of Theorem 1.1 is based on the fact that in the Miller model spaces with the Menger property enjoy certain concentration properties defined below. Recall that a subset R of a topological space X is called a *G_{ω_1}-set* if R is an intersection of ω_1-many open subsets of X.

Definition 2.1. A topological space X is called *weakly G_{ω_1}-concentrated* (resp. *weakly ωG_{ω_1}-concentrated*) if for every collection $\mathsf{Q} \subset [X]^\omega$ which is cofinal with respect to inclusion, and for every function $R : \mathsf{Q} \to \mathcal{P}(X)$ assigning to each $Q \in \mathsf{Q}$ a G_{ω_1}-set $R(Q)$ containing Q, there exists $\mathsf{Q}_1 \in [\mathsf{Q}]^{\omega_1}$ such that $X \subset \bigcup_{Q\in\mathsf{Q}_1} R(Q)$ (resp. for every $Q \in [X]^\omega$ there exists $Q_1 \in \mathsf{Q}_1$ with the property $Q \subset R(Q_1)$).

Let A be a countable set and $x, y \in \omega^A$. As usually $x \leq^* y$ means that $\{a \in A : x(a) > y(a)\}$ is finite. If $x(a) \leq y(a)$ for all $a \in A$, then we write $x \leq y$. The smallest cardinality of a dominating with respect to $\leq^*$ subset of ω^ω is denoted by $\mathfrak{d}$. The smallset cardinality of a family $\mathcal{B} \subset [\omega]^\omega$ generating an ultrafilter (i.e., such that $\{A : \exists B \in \mathcal{B}\,(B \subset A)\}$ is an ultrafilter) is denoted by $\mathfrak{u}$. By [7, Theorem 2] combined with the results of [10] the inequality $\omega_1 = \mathfrak{u} < \mathfrak{g} = \omega_2$ holds in the Miller model, see [7] or [11] for the definition of $\mathfrak{g}$ as well as systematic treatment of cardinal characteristics of reals.

As the following fact established in [20] shows, the inequality $\mathfrak{u} < \mathfrak{g}$ imposes strong restrictions on the structure of Menger spaces.

Lemma 2.2. *In the Miller model, for every Menger space $X \subset \mathcal{P}(\omega)$ and a G_δ-subset G such that $X \subset G \subset \mathcal{P}(\omega)$, there exists a family $\mathcal{K}$ of compact subsets of G such that $|\mathcal{K}| = \omega_1$ and $X \subset \bigcup\mathcal{K}$.*

Consequently, in this model for every Menger space $X \subset \mathcal{P}(\omega)$ and continuous $f : X \to \omega^\omega$ there exists $F \in [\omega^\omega]^{\omega_1}$ such that for every $x \in X$ there exists $f \in F$ with $f(x) \leq f$.

Proof. The first statement is [20, Theorem 4.4] combined with the fact that $\mathfrak{u} = \omega_1$ in the Miller model.

Regarding the second statement, since the Menger property is preserved by continuous images and ω^ω is homeomorphic to a G_δ-subset of $\mathcal{P}(\omega)$, there exists a family $\mathcal{K}$ of compact subsets of ω^ω such that $|\mathcal{K}| = \omega_1$ and $X \subset \bigcup\mathcal{K}$. For every $K \in \mathcal{K}$ there exists $f_K \in \omega^\omega$ such that $y \leq f_K$ for all $y \in K$. It follows that the family $F = \{f_K : K \in \mathcal{K}\}$ is as required. □

By a *Miller tree* we understand a subtree T of $\omega^{<\omega}$ consisting of increasing finite sequences such that the following conditions are satisfied:

- Every $t \in T$ has an extension $s \in T$ which is splitting in T, i.e., there are more than one immediate successors of s in T;
- If s is splitting in T, then it has infinitely many successors in T.

The *Miller forcing* is the collection $\mathbb{M}$ of all Miller trees ordered by inclusion, i.e., smaller trees carry more information about the generic. This poset has been introduced in [19] and since then found numerous applications see, e.g., [10]. We denote by $\mathbb{P}_\alpha$ an iteration of length α of the Miller forcing with countable support. If G is $\mathbb{P}_\beta$-generic and $\alpha < \beta$, then we denote the intersection $G \cap \mathbb{P}_\alpha$ by G_α.

For a Miller tree T we shall denote by *Split*(T) the set of all splitting nodes of T, and for some $t \in$ *Split*(T) we denote the size of $\{s \in$ *Split*$(T) : s \subsetneq t\}$ by *Lev*(t, T). For a node t in a Miller tree T we denote by T_t the set $\{s \in T : s$ is compatible with $t\}$. It is clear that T_t is also a Miller tree. The *stem* of a Miller tree T is the (unique) $t \in$ *Split*(T) such that *Lev*$(t) = 0$.

We denote the stem of T by $T\langle 0\rangle$. If $T_1 \leq T_0$ and $T_1\langle 0\rangle = T_0\langle 0\rangle$, then we write $T_1 \leq^0 T_0$.

The following lemma may be proved by an almost literal repetition of the proof of [17, Lemma 14].

Lemma 2.3. *Let $\langle \dot{x}_i : i \in \omega\rangle$ be a sequence of $\mathbb{P}_{\omega_2}$-names for reals and $p \in \mathbb{P}_{\omega_2}$. Then there exists $p' \leq p$ such that $p'(0) \leq^0 p(0)$, and a finite set of reals U_s for each $s \in Split(p'(0))$, such that for each $\varepsilon > 0$, $s \in Split(p'(0))$ with $Lev(s, p'(0)) = i$, $j \leq i$, and for all but finitely many immediate successors t of s in $p'(0)$ we have*

$$(p'(0))_t \hat{}\, p' \restriction [1, \omega_2) \Vdash \exists u \in U_s \, (|\dot{x}_j - u| < \varepsilon).$$

A subset C of ω_2 is called an ω_1-*club* if it is unbounded and for every $\alpha \in \omega_2$ of cofinality ω_1, if $C \cap \alpha$ is cofinal in α then $\alpha \in C$.

Lemma 2.4. *In the Miller model every Menger subspace of $\mathcal{P}(\omega)$ is weakly ωG_{ω_1}-concentrated (and hence also weakly G_{ω_1}-concentrated[1]).*

Proof. We work in $V[G_{\omega_2}]$, where G_{ω_2} is $\mathbb{P}_{\omega_2}$-generic. Let us fix a Menger space $X \subset \mathcal{P}(\omega)$, consider a cofinal $\mathsf{Q} \subset [X]^\omega$, and let $R(Q) \supset Q$ be a G_{ω_1}-set for all $Q \in \mathsf{Q}$.

In the Miller model the Menger property is preserved by unions of ω_1-many spaces, see [34, Theorem 4] and [7] for the fact that $\mathfrak{g} = \omega_2$ in this model. This implies in particular that a complement $X \setminus R$ of an arbitrary G_{ω_1}-subset $R \subset \mathcal{P}(\omega)$ is Menger. Therefore by Lemma 2.2 and a standard closing off argument (see, e.g., the proof of [9, Lemma 5.10]) there exists an ω_1-club $C \subset \omega_2$ such that for every $\alpha \in C$ the following condition is satisfied:

> $\mathsf{Q} \cap V[G_\alpha]$ is cofinal in $[X]^\omega \cap V[G_\alpha]$, and for every continuous f from a subset of $\mathcal{P}(\omega)$ into ω^ω such that f is coded in $V[G_\alpha]$, and every $Q \in \mathsf{Q} \cap V[G_\alpha]$ such that $X \setminus R(Q) \subset \mathrm{dom}(f)$, for every $x \in X \setminus R(Q)$ there exists $b \in \omega^\omega \cap V[G_\alpha]$ such that $f(x) < b$.

Let us fix $\alpha \in C$. We claim that $\mathsf{Q} \cap V[G_\alpha]$ has the required property. Suppose, contrary to our claim, there exists $p \in G_{\omega_2}$ and a $\mathbb{P}_{\omega_2}$-name $\dot{Q}_*$ such that p forces "$\dot{Q}_* \in [\dot{X}]^\omega$ and $\dot{Q}_* \not\subset \dot{R}(\dot{Q})$ for any $\dot{Q} \in [\dot{X}]^\omega \cap V[\Gamma_\alpha]$", where Γ_α is the standard $\mathbb{P}_\alpha$-name for $\mathbb{P}_\alpha$-generic filter. There is no loss of generality in assuming that $\alpha = 0$. Applying Lemma 2.3 to a sequence $\langle \dot{q}_i : i \in \omega\rangle$ enumerating $\dot{Q}_*$, we get a condition $p' \leq p$ such that $p'(0) \leq^0 p(0)$, and a finite set U_s of reals for every $s \in Split(p'(0))$ such that for each $\varepsilon > 0$ and each $s \in Split(p'(0))$ with $Lev(s, p'(0)) = i$, for all but finitely many immediate successors t of s in $p'(0)$ and all $j \leq i$ we have

$$p'(0)_t \hat{}\, p' \restriction [1, \omega_2) \Vdash \exists u \in U_s \, (|\dot{q}_j - u| < \varepsilon). \tag{1}$$

Note that any condition stronger than p' will also satisfy condition (1), because for $q_0, q_1 \in \mathbb{M}$ the inequality $q_1 \leq q_0$ implies $Lev(s, q_1) \leq Lev(s, q_0)$

[1] The proof of Theorem 1.1 requires only that Menger spaces are weakly G_{ω_1}-concentrated in the Miller model.

for all $s \in \mathit{Split}(q_1)$. Fix $Q \in \mathsf{Q} \cap V$ containing $X \cap \bigcup\{U_s : s \in \mathit{Split}(p'(0))\}$ and set $F = X \setminus R(Q)$. It follows from the above that $p' \Vdash \dot{Q}_* \not\subset \dot{R}(Q)$. By passing to a stronger condition, if necessary, we may additionally assume that $p' \Vdash \dot{q}_j \notin \dot{R}(Q)$ for a given $j \in \omega$.

Consider the map $f : F \to \omega^{\mathit{Split}(p'(0))}$ defined as follows:

$$f(y)(s) = [1/\min\{|y-u| : u \in U_s\}] + 1$$

for[2] all $s \in \mathit{Split}(p'(0))$ and $y \in F$. Since F is disjoint from Q, f is well-defined. Since f is coded in V, there exist $p'' \leq p'$ and $b \in \omega^{\mathit{Split}(p'(0))} \cap V$ such that p'' forces $\dot{f}(\dot{q}_j) \leq b$. Without loss of generality we may additionally assume that $\mathit{Lev}(p''(0)\langle 0\rangle, p'(0)) \geq j$. Letting $s'' = p''(0)\langle 0\rangle$, we conclude that $p'' \Vdash \dot{f}(\dot{q}_j)(s'') \leq b(s'')$, which means that

$$p'' \Vdash \min\{|\dot{q}_j - u| : u \in U_{s''}\} \geq 1/b(s'').$$

On the other hand, by our choice of p', $p'' \leq p'$, condition (1), and $\mathit{Lev}(s'', p'(0)) \geq j$, for all but finitely many immediate successors t of s'' in $p''(0)$ we have

$$p''(0)_t \hat{\ } p'' \restriction [1, \omega_2) \Vdash \exists u \in U_{s''}\ |\dot{q}_j - u| < 1/b(s'')$$

which means $p''(0)_t \hat{\ } p'' \restriction [1, \omega_2) \Vdash \min\{|\dot{q}_j - u| : u \in U_{s''}\} < 1/b(s'')$ and thus leads to a contradiction. $\square$

The next lemma relates the weak G_{ω_1}-concentration to products with Menger spaces.

Lemma 2.5. *In the Miller model, let $Y \subset \mathcal{P}(\omega)$ be Menger and $Q \subset \mathcal{P}(\omega)$ be countable. Then for every G_{ω_1}-subset O of $\mathcal{P}(\omega)^2$ containing $Q \times Y$ there exist a G_{ω_1}-subsets $R \supset Q$ such that $R \times Y \subset O$.*

Proof. Without loss of generality we shall assume that O is open. Let us write Q in the form $\{q_n : n \in \omega\}$ and set $O_n = \{z \in \mathcal{P}(\omega) : \langle q_n, z\rangle \in O\} \supset Y$. By Lemma 2.2 there exists a collection $\mathcal{Z} = \{Z_\alpha : \alpha \in \omega_1\}$ of compact subsets of $\bigcap_{n\in\omega} O_n$ covering Y. It follows from the above that $Q \times Z_\alpha \subset O$ for all α, and hence there exists an open $R_\alpha \subset \mathcal{P}(\omega)$ containing Q such that $R_\alpha \times Z_\alpha \subset O$. Letting $R = \bigcap_{\alpha<\omega_1} R_\alpha$ we get that $R \times Y \subset R \times \bigcup_{\alpha<\omega_1} Z_\alpha \subset O$. $\square$

As it was proved in [10], in the Miller model there exists an ultrafilter $\mathcal{F}$ generated by ω_1-many sets, say $\{F_\alpha : \alpha \in \omega_1\}$. There is a natural linear pre-order $\leq_{\mathcal{F}}$ on ω^ω associated to $\mathcal{F}$ defined as follows: $x \leq_{\mathcal{F}} y$ if and only if $\{n \in \omega : x(n) \leq y(n)\} \in \mathcal{F}$. By [8, Theorem 3.1], in this model for every $X \subset \omega^\omega$ of size ω_1 there exists $b \in \omega^\omega$ such that $x \leq_{\mathcal{F}} b$ for all $x \in X$.

Lemma 2.6. *In the Miller model, suppose that $\mathcal{U}_n = \{U^n_k : k \in \omega\}$ is an open cover of a Menger space $X \subset \mathcal{P}(\omega)$, for every $n \in \omega$. Then there exists $b \in \omega^\omega$ such that $X \subset \bigcup_{n\in F_\alpha} \bigcup_{k\leq b(n)} U^n_k$ for all α. (Equivalently, $\{n \in \omega : x \in \bigcup_{k\leq b(n)} U^n_k\} \in \mathcal{F}$ for all $x \in X$.)*

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[2] Here $[a]$ is the largest integer not exceeding a.

Proof. The equivalence of two statements follows from the equality $\mathcal{F} = \mathcal{F}^+$, where for $\mathcal{X} \subset [\omega]^\omega$ we standardly denote by $\mathcal{X}^+$ the set $\{Y \subset \omega : Y \cap X \neq \emptyset$ for all $X \in \mathcal{X}\}$.

To prove the second statement set $G = \bigcap_{n\in\omega} \bigcup \mathcal{U}_n$ and find a collection $\mathcal{K}$ of compact subsets of G such that $|\mathcal{K}| = \omega_1$ and $X \subset \bigcup \mathcal{K} \subset G$. This is possible by Lemma 2.2. For every $K \in \mathcal{K}$ find $b_K \in \omega^\omega$ such that $K \subset \bigcup_{k\le b_K(n)} U^n_k$ for all $n \in \omega$. Then any $b \in \omega^\omega$ such that $b_K \le_{\mathcal{F}} b$ for all $K \in \mathcal{K}$ is easily seen to be as required. □

The second part of Theorem 1.1 is a direct consequence of Lemma 2.4 and the following

Proposition 2.7. *In the Miller model, let $Y \subset \mathcal{P}(\omega)$ be a Menger space and $X \subset \mathcal{P}(\omega)$ be weakly G_{ω_1}-concentrated. Then $X \times Y$ is Menger.*

Proof. Fix a sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of covers of $X \times Y$ by clopen subsets of $\mathcal{P}(\omega)^2$. For every $Q \in [X]^\omega$ fix a sequence $\langle \mathcal{W}^Q_n : n \in \omega\rangle$ such that $\mathcal{W}^Q_n \in [\mathcal{U}_n]^{<\omega}$ and $Q \times Y \subset O_{Q,\alpha}$ for all $\alpha \in \omega_1$, where $O_{Q,\alpha} = \bigcup_{n\in F_\alpha} \cup \mathcal{W}^Q_n$. (The latter is possible by Lemma 2.6.) Letting $O_Q = \bigcap_{\alpha\in\omega_1} O_{Q,\alpha}$ and using Lemma 2.5, we can find a G_{ω_1}-subset $R(Q) \supset Q$ of $\mathcal{P}(\omega)$ such that $R(Q) \times Y \subset O_Q$. Since X is weakly G_{ω_1}-concentrated, there exists $\mathsf{Q}_1 \in [[X]^\omega]^{\omega_1}$ such that $X \subset \bigcup_{Q\in\mathsf{Q}_1} R(Q)$. For every $Q \in \mathsf{Q}_1$ let us find $b_Q \in \omega^\omega$ such that $\mathcal{W}^Q_n \subset \{U^n_k : k \le b_Q(n)\}$ for all $n \in \omega$, where $\mathcal{U}_n = \{U^n_k : k \in \omega\}$ is an enumeration. Let $b \in \omega^\omega$ be an upper bound of $\{b_Q : Q \in \mathsf{Q}_1\}$ with respect to $\le_{\mathcal{F}}$. We claim that $X \times Y \subset \bigcup_{n\in\omega} \bigcup_{k\le b(n)} U^n_k$. Indeed, fix $y \in Y$, $x \in X$, and find $Q \in \mathsf{Q}_1$ such that $x \in R(Q)$. It follows that $\langle x, y\rangle \in O_{Q,\alpha}$ for all $\alpha \in \omega_1$, therefore for every α there exists $n \in F_\alpha$ with $\langle x, y\rangle \in \cup\mathcal{W}^Q_n \subset \bigcup_{k\le b_Q(n)} U^n_k$, and hence $F := \{n \in \omega : \langle x, y\rangle \in \bigcup_{k\le b_Q(n)} U^n_k\} \in \mathcal{F}^+ = \mathcal{F}$. Then $\langle x, y\rangle \in \bigcup_{k\le b(n)} U^n_k$ for all $n \in F \cap \{k : b_Q(k) \le b(k)\} \in \mathcal{F}$, which completes our proof. □

Note that Lemma 2.4 together with Proposition 2.7 imply that in the Miller model, a subspace X of $\mathcal{P}(\omega)$ is Menger iff it is weakly G_{ω_1}-concentrated iff it is weakly ωG_{ω_1}-concentrated.

As we have already notices above, Lemma 2.4 and Proposition 2.7 imply Theorem 1.1 for subspaces of $\mathcal{P}(\omega)$. The general case of arbitrary Menger spaces can be reduced to subspaces of $\mathcal{P}(\omega)$ in the same way as in the proof of [23, Theorem 1.1], the only difference being that in some places "Hurewicz" should be replaced with "Menger". However, we present this proof for the sake of completeness. We will need characterizations of the Menger property obtained in [34]. Let $u = \langle U_n : n \in \omega\rangle$ be a sequence of subsets of a set X. For every $x \in X$ let $I_s(x, u, X) = \{n \in \omega : x \in U_n\}$. If every $I_s(x, u, X)$ is infinite (the collection of all such sequences u will be denoted by $\Lambda_s(X)$), then we shall denote by $\mathcal{U}_s(u, X)$ the smallest semifilter on ω containing all $I_s(x, u, X)$. (Recall that a family $\mathcal{F} \subset [\omega]^\omega$ is called a *semifilter* if for every $F \in \mathcal{F}$ and $X^* \supset F$ we have $X \in \mathcal{F}$, where $F \subset^* X$ means $|F \setminus X| < \omega$.) By [34, Theorem 3], a Lindelöf topological space X is Menger if and only if for every $u \in \Lambda_s(X)$ consisting of open sets, the semifilter $\mathcal{U}_s(u, X)$ is Menger. The proof given there also works if we consider only those $\langle U_n : n \in \omega\rangle \in \Lambda_s(X)$ such that all U_n's belong to a given base of X.

Proof of Theorem 1.1. Suppose that X, Y are arbitrary Menger spaces such that $X \times Y$ is Lindelöf and fix $w = \langle U_n \times V_n : n \in \omega\rangle \in \Lambda_s(X \times Y)$ consisting of open sets. Set $u = \langle U_n : n \in \omega\rangle$, $v = \langle V_n : n \in \omega\rangle$, and note that $u \in \Lambda_s(X)$ and $v \in \Lambda_s(Y)$. It is easy to see that

$$\mathcal{U}_s(w, X \times Y) = \{A \cap B : A \in \mathcal{U}_s(u, X), B \in \mathcal{U}_s(v, Y)\},$$

and hence $\mathcal{U}_s(w, X \times Y)$ is a continuous image of $\mathcal{U}_s(u, X) \times \mathcal{U}_s(v, Y)$. By [34, Theorem 3] both of latter ones are Menger, considered as subspaces of $\mathcal{P}(\omega)$, and hence by Lemma 2.4 and Proposition 2.7 their product is Menger as well. Thus $\mathcal{U}_s(w, X \times Y)$ is Menger, being a continuous image of a Menger space. It suffices to use [34, Theorem 3] again. □

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