COVERING PROPERTIES OF ω-MAD FAMILIES

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Abstract. We prove that $\mathfrak{p} = \mathfrak{c}$ implies the existence of a Cohen-indestructible mad family such that the Mathias forcing associated to its filter adds dominating reals, while $\mathfrak{b} = \mathfrak{c}$ is consistent with the negation of this statement as witnessed by the Laver model for the consistency of Borel's conjecture.

1. Introduction

Recall that an infinite $\mathcal{A} \subset [\omega]^\omega$ is called a *mad family*, if $|A_0 \cap A_1| < \omega$ for any distinct $A_0, A_1 \in \mathcal{A}$, and for every $B \in [\omega]^\omega \setminus \mathcal{A}$ there exists $A \in \mathcal{A}$ such that $|B \cap A| = \omega$. In [2, Theorem 2.1] Brendle constructed under CH a mad family $\mathcal{A}$ on ω such that the Mathias forcing[1] $M_{\mathcal{F}(\mathcal{A})}$ associated to the filter

$$\mathcal{F}(\mathcal{A}) = \{F \subset \omega : \exists \mathcal{A}' \in [\mathcal{A}]^{<\omega}(\omega \setminus \cup\mathcal{A}' \subset^* F)\}$$

adds a dominating real. In the same paper Brendle asked whether such a mad family can be constructed outright in ZFC. This question has been answered in the affirmative in [5] and later independently also in [4] using different methods. Since the goal of these studies was to find forcings destroying a given mad family while keeping (certain subsets of) the ground model reals unbounded (and perhaps having other useful properties), this motivates the following version of Brendle's question: Suppose that a mad family $\mathcal{A}$ cannot be destroyed by some very "mild" forcing $\mathbb{P}$, i.e., it remains maximal in $V^{\mathbb{P}}$, must then $M_{\mathcal{F}(\mathcal{A})}$ add dominating reals? This approach seems natural because if $\mathcal{A}$ is already destroyed by $\mathbb{P}$, there is no need to use its Mathias forcing for its destruction in a hypothetic construction of a model where, e.g., $\mathfrak{b}$ should stay small. $\mathfrak{b}$ as well as other notions used in the introduction will be defined in the next section. In this note we consider this question for $\mathbb{P}$ being the Cohen forcing $\mathbb{C}$. Mad families $\mathcal{A}$ which remain maximal in $V^{\mathbb{C}}$ will be called Cohen-indestructible.

Theorem 1.1. $\mathfrak{p} = \mathfrak{c}$ *implies the existence of a Cohen-indestructible mad family $\mathcal{A}$ such that $M_{\mathcal{F}(\mathcal{A})}$ adds a dominating real.*

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[1] Since we shall not analyze this poset directly but rather use certain topological characterizations, we refer the reader to, e.g., [2] for its definition.

Recall from [10] that a mad family $\mathcal{A}$ is called *ω-mad* if for every sequence $\langle X_n : n \in \omega\rangle$ of elements of $\mathcal{F}(\mathcal{A})^+$ there exists $A \in \mathcal{A}$ such that $|A \cap X_n| = \omega$ for all n. Cohen-indestructible mad families are closely related to ω-mad ones, see [12] or [10, Theorem 4]: Every ω-mad family is Cohen-indestructible, and if $\mathcal{A}$ is Cohen-indestructible, then for every $X \in \mathcal{F}(\mathcal{A})^+$ there exists $Y \subset X$, $Y \in \mathcal{F}(\mathcal{A})^+$, such that $\mathcal{A} \restriction Y = \{A \cap Y : A \in \mathcal{A}, A \cap Y$ is infinite$\}$ is ω-mad as a mad family on Y.

In the proof of Theorem 1.1 we actually construct an ω-mad family. The next theorem shows that $\mathfrak{b} = \mathfrak{c}$ would not suffice in Theorem 1.1.

Theorem 1.2. *In the Laver model for the consistency of the Borel conjecture, for every ω-mad family $\mathcal{A}$ the poset $M_{\mathcal{F}(\mathcal{A})}$ does not add dominating reals. In particular, if $\mathcal{A}$ is Cohen-indestructible, then there exists $X \in \mathcal{F}(\mathcal{A})^+$ such that $M_{\mathcal{F}(\mathcal{A})\restriction X}$ does not add dominating reals, where $\mathcal{F}(\mathcal{A}) \restriction X$ denotes the filter on ω generated by the centered family $\{F \cap X : F \in \mathcal{F}(\mathcal{A})\}$.*

In our proofs of Theorems 1.1 and 1.2 we shall not work with the Mathias forcing directly, but rather use the following characterization obtained in [4]: For a filter $\mathcal{F}$ on ω the poset $M_{\mathcal{F}}$ adds no dominating reals iff $\mathcal{F}$ has the Menger covering property when considered with the topology inherited from $\mathcal{P}(\omega)$, which is identified with the Cantor space 2^ω via characteristic functions. Recall from [6] that a topological space X is said to have the Menger property if for every sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega\rangle$ such that each $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ and the collection $\{\cup\mathcal{V}_n : n \in \omega\}$ is a cover of X. The current name (the Menger property) has been adopted because Hurewicz proved in [6] that for metrizable spaces his property is equivalent to a certain basis property considered by Menger in [13]. If in the definition above we additionally require that $\{\cup\mathcal{V}_n : n \in \omega\}$ is a *γ-cover* of X (this means that the set $\{n \in \omega : x \notin \cup\mathcal{V}_n\}$ is finite for each $x \in X$), then we obtain the definition of the Hurewicz covering property introduced in [7]. These properties are related as follows:

$$\sigma\text{-compact} \rightarrow \text{Hurewicz} \rightarrow \text{Menger} \rightarrow \text{Lindelöf'}$$

Contrary to a conjecture of Hurewicz, the class of metrizable spaces having the Hurewicz property turned out to be wider than the class of σ-compact spaces [8, Theorem 5.1]. Also, there are ZFC examples of non-Hurewicz subspaces X of the real line whose all finite powers are Menger, see [3] or [17].

In light of Theorem 1.2 we would like to ask whether it is consistent that $\mathcal{F}(\mathcal{A})$ is Hurewicz for any ω-mad family $\mathcal{A}$. However, since it is unknown whether ω-mad families exist in ZFC, we suggest the following

Question 1.3. Is it consistent that there exist ω-mad families and $\mathcal{F}(\mathcal{A})$ is Hurewicz for any such a family $\mathcal{A}$? Is this the case in the Laver model?

2. Proofs

Let us first recall the definitions of cardinal characteristics appearing in this paper. $\mathfrak{p}$ is the minimal cardinality of a family $\mathcal{X} \subset [\omega]^\omega$ such that

$\cap\mathcal{X}' \in [\omega]^\omega$ for any $\mathcal{X}' \in [\mathcal{X}]^{<\omega}$, but there is no $Y \in [\omega]^\omega$ such that $Y \subset^* X$ for all $X \in \mathcal{X}$. $\mathfrak{b}$ is the minimal cardinality of an unbounded subset B of ω^ω with respect to the following pre-order: $x \leq^* y$ iff $\{n \in \omega : x(n) > y(n)\}$ is finite. Finally, $cov(\mathcal{N})$ is the minimal cardinality of a cover of $\mathbb{R}$ by Lebesgue null sets. It is well-known that $\mathfrak{p} \leq \min\{\mathfrak{b}, cov(\mathcal{N})\}$, see, e.g., [1] and references therein.

We shall first prove Theorem 1.1. Here we shall often use the following easy fact without mentioning it: For any countable collection $\mathcal{A}$ of countable sets, for every $A \in \mathcal{A}$ there exists $B(A) \in [A]^\omega$ such that $B(A) \cap B(A') = \emptyset$ for any distinct $A, A' \in \mathcal{A}$.

Proof of Theorem 1.1. We shall first present the proof under CH, and then indicate what should be changed to make the proof work under $\mathfrak{p} = \mathfrak{c}$.

Let $\langle I_n : n \in \omega \rangle$ be a sequence of infinite mutually disjoint subsets of ω. For every $k \in \omega$ set $P_k = 2^{k+1} \setminus 2^k$ and note that elements of $\{P_k : k \in \omega\}$ are mutually disjoint. Let $\{\langle X^\alpha_n : n \in \omega \rangle : \alpha < \omega_1\}$ be the family of all sequences of infinite subsets of ω. Let us also fix an enumeration $\{f_\alpha : \alpha < \omega_1\}$ of all increasing sequences in ω^ω. By transfinite induction on α we shall construct a sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ of infinite subsets of ω satisfying the following properties:

- (i) $|A_\beta \cap A_\gamma| < \omega$ for all $\beta \neq \gamma$;
- (ii) $|A_\beta \cap P_k| \leq 2$ for every $\beta \in \omega_1$ and $k \in \omega$;
- (iii) For every $a \in [\omega_1]^{<\omega}$ and $k \in \omega$ the set $\{n \in I_k : \bigcup_{\beta \in a} A_\beta \cap P_n = \emptyset\}$ is infinite;
- (iv) For every $\beta \in \omega_1$, if $|X^\beta_n \setminus \bigcup_{\gamma \in a} A_\gamma| = \omega$ for all $n \in \omega$ and finite $a \subset \beta$, then $|A_\beta \cap X^\beta_n| = \omega$ for all $n \in \omega$; and
- (v) $A_\beta \cap P_k \neq \emptyset$ provided that $k \in I_n$ and $P_k \subset f_\beta(n)$.

Assuming that conditions (i)-(v) are satisfied for all $\beta, \gamma < \alpha$ and $a \subset \alpha$, let us consider the sequence $\langle X^\alpha_n : n \in \omega \rangle$. Two cases are possible.

1. $|X^\alpha_n \setminus \bigcup_{\gamma \in a} A_\gamma| = \omega$ for all $n \in \omega$ and finite $a \subset \alpha$, i.e., the premises of (iv) hold for α. Let us note that if we shrink the sets X^α_n's, the property (iv) becomes harder to fulfill. Thus passing to an infinite pseudointersection of the countable family

$$\{X^\alpha_n \setminus \bigcup_{\gamma \in a} A_\gamma : a \in [\alpha]^{<\omega}\}$$

of infinite subsets of X^α_n, we may assume that $|X^\alpha_n \cap A_\beta| < \omega$ for all $n \in \omega$ and $\beta < \alpha$. Let $g \in \omega^\omega$ be such that for all $\beta < \alpha$ there exists $n \in \omega$ with the property $X^\alpha_m \cap A_\beta \subset g(m)$ for all $m \geq n$. Letting $Y_n = X^\alpha_n \setminus g(n)$, we get that

(vi) $\bigcup_{n \in \omega} Y_n$ is almost disjoint from A_β for all $\beta < \alpha$.

Claim 2.1. *For every $m \in \omega$ there exists $B_m \in [Y_m]^\omega$ such that $B = \bigcup_{m \in \omega} B_n$ has the following properties:*

$$\forall k \in \omega\, \forall a \in [\alpha]^{<\omega} \big(\{n \in I_k : P_n \cap (B \cup \bigcup_{\beta \in a} A_\beta) = \emptyset\} \textit{ is infinite}\big)$$

and $|B \cap P_n| \leq 1$ *for all* $n \in \omega$.

Proof. For every $k \in \omega$ and $a \in [\alpha]^{<\omega}$ set $N^k_a = \{n \in I_k : P_n \cap \bigcup_{\beta \in a} A_\beta = \emptyset\}$ and note that by our assumptions $\{N^k_a : a \in [\alpha]^{<\omega}\}$ is a countable centered family of infinite subsets of I_k, and hence there exists $N^k \in [I_k]^\omega$ such that $N^k \subset^* N^k_a$ for all a as above. Let

$$M_\infty = \{m \in \omega : \exists^\infty k \exists n \in N^k (Y_m \cap P_n) \neq \emptyset\}$$

and for every $m \in M_\infty$ set $J_m = \{k \in \omega : \exists n \in N^k (Y_m \cap P_n) \neq \emptyset\} \in [\omega]^\omega$. Pick $J'_m \in [J_m]^\omega$ for all $m \in M_\infty$ such that $J'_{m_0} \cap J'_{m_1} = \emptyset$ for arbitrary $m_0 \neq m_1$ in M_∞. Given $m \in M_\infty$, for every $k \in J'_m$ pick $n_{m,k} \in N^k$ such that $Y_m \cap P_{n_{m,k}} \neq \emptyset$, and fix $l_{m,k} \in P_{n_{m,k}} \cap Y_m$. For every $m \in M_\infty$ set $B_m = \{l_{m,k} : k \in J'_m\}$.

Suppose now that $m \in \omega \setminus M_\infty$. Two cases are possible.

$a)$ There exists $k_m \in \omega$ such that $L_m := \{n \in N^{k_m} : Y_m \cap P_n \neq \emptyset\}$ is infinite. Given $k \in \omega$, for every m such that $k = k_m$ find $Q_m \in [L_m]^\omega$, and $R_k \in [N^k]^\omega$ such that $Q_{m_0} \cap Q_{m_1} = \emptyset$ for any distinct m_0, m_1 such that $k = k_{m_0} = k_{m_1}$, and $R_k \cap Q_m = \emptyset$ for all m with $k = k_m$. Now for every $n \in Q_m$ pick $q_{m,n} \in Y_m \cap P_n$ and set $B_m = \{q_{m,n} : n \in Q_m\}$.

$b)$ The set

$$S_m := \{\langle k, n\rangle : k \in \omega, n \in N^k, Y_m \cap P_n \neq \emptyset\}$$

is finite. Then let

$$B_m \in \big[Y_m \setminus \bigcup \{P_n : \exists k(\langle k, n\rangle \in S_m)\}\big]^\omega$$

be such that for each n we have $|B_m \cap P_n| \leq 1$.

Thus we have already constructed the sequence $\langle B_m : m \in \omega\rangle$. We claim that $B = \bigcup_{m \in \omega} B_m$ is as required. By the choice of N^k it suffices to prove that

$$\forall k \in \omega \big(\{n \in N^k : P_n \cap B = \emptyset\} \text{ is infinite}\big).$$

We shall show that if $k = k_m$ for some m, then $P_n \cap B = \emptyset$ for all but maybe one $n \in R_k$. Otherwise $P_n \cap B = \emptyset$ for all but maybe one $n \in N^k$. Indeed, by the construction (more precisely, since all J'_m, $m \in M_\infty$ are mutually disjoint), the union $B_\infty := \bigcup_{m \in M_\infty} B_m$ has the property that for every $k \in \omega$ there exists at most one $n \in N^k$ such that $B_\infty \cap P_n \neq \emptyset$. Now if $m \in \omega \setminus M_\infty$ and case $b)$ takes place, then B_m intersects no P_n for $n \in \bigcup_{k \in \omega} N^k$. And finally, if $m \in \omega \setminus M_\infty$ and $a)$ takes place with $k = k_m$, then $B_m \subset \bigcup_{n \in N^k} P_n$ and $B_m \cap \bigcup_{n \in R_k} P_n = \emptyset$. Since the I_k's (and hence also the N^k's) are mutually disjoint, this completes our proof. □

Claim 2.2. *Let* $\langle n_i : i \in \omega\rangle$ *be the increasing enumeration of the set* $\{n \in \omega : \exists k(n \in I_k \wedge P_n \subset f_\alpha(k))\}$. *Then there exists* $C \subset \omega$ *such that* $|C \cap A_\beta| < \omega$ *for all* $\beta < \alpha$, $|C \cap P_{n_i}| = 1$ *for all* i, *and* $C \cap P_n = \emptyset$ *if* $n \notin \{n_i : i \in \omega\}$.

Proof. By (ii) we can find a countable family G of functions in $\prod_{i \in \omega} P_{n_i}$ such that $A_\beta \cap (\bigcup_{i \in \omega} P_{n_i})$ is covered by graphs of at most 2 elements of G, for all $\beta < \alpha$. Now it is easy to construct $h \in \prod_{i \in \omega} P_{n_i}$ eventually different from each element of G. It follows that $C := \mathrm{range}(h)$ is as required. □

Set $A_\alpha = B \cup C$, where B, C are such as in Claims 2.1 and 2.2, respectively. Since $\{n_i : i \in \omega\} \cap I_k$ is finite for all $k \in \omega$, it is easy to see that all conditions (i)-(v) are also satisfied for $\beta, \gamma \leq \alpha$ and $a \in [\alpha+1]^{<\omega}$.

2. There exists $n \in \omega$ and a finite $a \subset \alpha$ such that $X_n^\alpha \subset^* \bigcup_{\gamma \in a} A_\gamma$. Set $A_\alpha = C$, where C is such as in Claim 2.2. Again, all conditions (i)-(v) are satisfied for $\beta, \gamma \leq \alpha$ and $a \in [\alpha+1]^{<\omega}$.

This completes our construction of a sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ satisfying (i)-(v). By (i) and (iv), $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ is an ω-mad family. By (iii) the family $\mathcal{U}_k = \{O_n : n \in I_k\}$ is an open cover of $\mathcal{F}(\mathcal{A})$ for all $k \in \omega$, where $O_n = \{X \subset \omega : P_n \subset X\}$. We claim that the sequence $\langle \mathcal{U}_k : k \in \omega \rangle$ witnesses that $\mathcal{F}(\mathcal{A})$ is not Menger. Indeed, otherwise there exists α such that

$$\mathcal{U} := \{O_n : \exists k \in \omega (n \in I_k \wedge P_n \subset f_\alpha(k))\}$$

covers $\mathcal{F}(\mathcal{A})$. However, $P_n \cap A_\alpha \neq \emptyset$ for all $n \in I_k$ such that $P_n \subset f_\alpha(k)$ for some $k \in \omega$, which means that $\mathcal{F}(\mathcal{A}) \ni \omega \setminus A_\alpha \notin \cup \mathcal{U}$. This leads to a contradiction and thus finishes our proof under CH.

Except for the proof of Claim 2.2, we have used CH to produce at stage α a pseudointersection of a centered family of infinite subsets of ω of size $|\alpha|$, and $\mathfrak{p} = \mathfrak{c}$ suffices for finding such pseudointersections by the definition of $\mathfrak{p}$.

Regarding Claim 2.2, we shall show[2] that for any family $G \subset \prod_{i\in\omega} P_{n_i}$ of size $< \mathfrak{p}$ there exists $h \in \prod_{i\in\omega} P_{n_i}$ eventually different from all elements of H (here we use the same notation as in the formulation of Claim 2.2). Indeed, let μ be the Borel measure on $\prod_{i\in\omega} P_{n_i}$ such that for every $i \in \omega$ and $s \in \prod_{j\leq i} P_{n_j}$ we have $\mu([s]) = \prod_{j \leq i} 2^{-n_j}$, where

$$[s] = \{x \in \prod_{i\in\omega} P_{n_i} : x \upharpoonright (i+1) = s\}.$$

By [9, Theorem 17.41] the measurable space $\langle \prod_{i\in\omega} P_{n_i}, \mu \rangle$ is isomorphic to $\mathbb{R}$ equipped with the standard Lebesgue measure λ. A simple calculation shows that

$$\mu\{x \in \prod_{i\in\omega} P_{n_i} : \exists^\infty i \in \omega (x(i) = g(i))\} = 0$$

for every $g \in \prod_{i\in\omega} P_{n_i}$. Since $\mathbb{R}$ cannot be covered by fewer than $\mathfrak{p}$ many null subsets, neither $\langle \prod_{i\in\omega} P_{n_i}, \mu \rangle$ can, and hence Claim 2.2 holds for families G of size $< \mathfrak{p}$. This completes our proof. $\square$

Every filter $\mathcal{F}$ on ω gives rise to the filter $\mathcal{F}^{(<\omega)}$ on $Fin := [\omega]^{<\omega} \setminus \{\emptyset\}$ generated by sets $[F]^{<\omega} \setminus \{\emptyset\}$, where $F \in \mathcal{F}$. For a family $\mathcal{B}$ of infinite subsets of a countable set X we denote by $\mathcal{B}^+$ the family $\{Z \subset X : \forall B \in \mathcal{B}(|Z \cap B| = \omega)\}$. For every $E \subset Fin$ let us denote by $\mathcal{K}(E)$ the family $\{K \subset \omega : \forall e \in E (e \cap K \neq \emptyset)\}$. It is easy to see that $\mathcal{K}(E)$ is always compact and $\mathcal{K}(E) \subset [\omega]^\omega$ if for every $n \in \omega$ there exists $e \in E$ such that

[2]We believe that this straightforward argument is well-known, but we were unable to locate it in the literature.

$\min e > n$. It is a straightforward exercise to check that $E \in (\mathcal{F}^{(<\omega)})^+$ iff $\mathcal{K}(E) \subset \mathcal{F}^+$.

In the next proof, we will use the notation $\omega^{\uparrow\omega}$ for the set of the increasing functions from ω to ω. Also, we will use the fact that $\mathfrak{b} = \omega_2$ holds in the Laver model.

Proof of Theorem 1.2. Let $\mathcal{F} = \mathcal{F}(\mathcal{A})$. By [4, Corollary 2.2] it suffices to prove that for every decreasing sequence $\langle S_n : n \in \omega\rangle$ of elements of $(\mathcal{F}^{(<\omega)})^+$ there exists $f \in \omega^\omega$ such that $S_f := \bigcup_{n\in\omega}(S_n \cap \mathcal{P}(f(n)))$ belongs to $(\mathcal{F}^{(<\omega)})^+$, i.e., $\mathcal{K}(S_f) \subset \mathcal{F}^+$. Without loss of generality we may assume that $\min s > n$ for all $s \in S_n$.

Since $\mathcal{A}$ is ω-mad, for every countable family

$$\{\langle X^i_n : n \in \omega\rangle : i \in \omega\} \subset \prod_{n\in\omega} \mathcal{K}(S_n)$$

there exists $A \in \mathcal{A}$ such that $|A \cap X^i_n| = \omega$ for all $i, n \in \omega$. We claim that there are actually ω_2-many $A \in \mathcal{A}$ as above. Indeed, suppose that for some $\mathcal{A}' \in [\mathcal{A}]^{\omega_1}$ there is no $A \in \mathcal{A} \setminus \mathcal{A}'$ such that $|A \cap X^i_n| = \omega$ for all $i, n \in \omega$. Fix a sequence $\langle A_n : n \in \omega\rangle$ of mutually different elements of $\mathcal{A} \setminus \mathcal{A}'$ and find $h \in \omega^{\uparrow\omega}$ such that

$$\langle \max(A \cap A_n) + 1 : n \in \omega\rangle \leq^* h$$

for all $A \in \mathcal{A}'$. Such an h exists because $|\mathcal{A}'| < \mathfrak{b} = \omega_2$. Set $X = \bigcup_{n\in\omega}(A_n \setminus h(n))$ and note that $X \in \mathcal{F}^+$ and $|X \cap A| < \omega$ for all $A \in \mathcal{A}'$. It follows that there is no $A \in \mathcal{A}$ which intersects infinitely often all elements of the family $\{X^i_n : i, n \in \omega\} \cup \{X\}$, a contradiction.

Let $f \in \omega^\omega$ be increasing and such that $A \cap X^i_n \cap f(n) \neq \emptyset$ for every i and all but finitely many $n \in \omega$. Set

$$G_{A,f} = \big\{\langle X_n : n \in \omega\rangle \in \prod_{n\in\omega} \mathcal{K}(S_n) : \exists^\infty n(A \cap X_n \cap f(n) \neq \emptyset)\big\}$$

and note that $G_{A,f}$ is a G_δ-subset of $\prod_{n\in\omega}\mathcal{K}(S_n)$ containing $\langle X^i_n : n \in \omega\rangle$ for all $i \in \omega$. Thus we have proven that for every countable $Q \subset \prod_{n\in\omega}\mathcal{K}(S_n)$ there exists $A \in \mathcal{A}$ and $f \in \omega^{\uparrow\omega}$ such that $Q \subset G_{A,f}$. Moreover, there are ω_2-many such pairs $\langle A, f\rangle$ with mutually different first coordinates. Let us fix $\mathcal{A}' \in [\mathcal{A}]^{\omega_1}$. Applying [14, Lemma 2.2] we conclude that there exists a family $\{\langle A_\alpha, f_\alpha\rangle : \alpha < \omega_1\} \subset \mathcal{A} \times \omega^{\uparrow\omega}$ such that $\prod_{n\in\omega}\mathcal{K}(S_n) \subset \bigcup_{\alpha<\omega_1} G_{A_\alpha, f_\alpha}$ and $\mathcal{A}' \cap \{A_\alpha : \alpha < \omega_1\} = \emptyset$. Since $\mathcal{A}'$ was chosen arbitrarily, it follows from the above that we can additionally assume that each $\langle X_n : n \in \omega\rangle \in \prod_{n\in\omega}\mathcal{K}(S_n)$ is contained in G_{A_α, f_α} for infinitely many α. Pick $f \in \omega^{\uparrow\omega}$ such that $f_\alpha \leq^* f$ for all α. We claim that $\mathcal{K}(S_f) \subset \mathcal{F}^+$. Indeed, for every $n \in \omega$ and $s \in S_n \cap \mathcal{P}(f(n))$ select $k_{s,n} \in s$. We are left with the task to prove that $X = \{k_{s,n} : s \in S_n \cap \mathcal{P}(f(n))\} \in \mathcal{F}^+$. In order to do this for every n and $s \in S_n \setminus \mathcal{P}(f(n))$ select $l_{s,n} \in s \setminus f(n)$ and consider the sequence $\langle X_n : n \in \omega\rangle \in \prod_{n\in\omega}\mathcal{K}(S_n)$, where

$$X_n = \big\{k_{s,n} : s \in S_n \cap \mathcal{P}(f(n))\big\} \cup \big\{l_{s,n} : s \in S_n \setminus \mathcal{P}(f(n))\big\}.$$

Our proof will be completed as soon as we show that $X \cap A_\alpha$ is infinite for all α such that $\langle X_n : n \in \omega \rangle \in G_{A_\alpha, f_\alpha}$. So let us fix such an α and $m_0 \in \omega$. Let $m \geq m_0$ be such that $f_\alpha(n) \leq f(n)$ for all $n \geq m$. By the definition of G_{A_α, f_α} there exists $n \geq m$ such that $\emptyset \neq X_n \cap A_\alpha \cap f_\alpha(n)$, and hence $\emptyset \neq X_n \cap A_\alpha \cap f(n)$. Fix j in the latter intersection. It follows that j cannot be of the form $l_{s,n}$ for $s \in S_n \setminus \mathcal{P}(f(n))$ because $j \in f(n)$, an hence $j = k_{s,n}$ for some $s \in S_n \cap \mathcal{P}(f(n))$, which yields $j \in X$. This completes our proof. $\square$

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