PRESERVATION OF γ-SPACES AND COVERING PROPERTIES OF PRODUCTS

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Abstract. We prove that the Hurewicz property is not preserved by finite products in the Miller model. This is a consequence of the fact that the Miller forcing preserves ground model γ-spaces.

1. Introduction

When trying to describe σ-compactness in terms of open covers, Hurewicz [5] introduced the following property, nowadays called *the Menger property*: A topological space X is said to have this property if for every sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega\rangle$ such that each $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ and the collection $\{\cup\mathcal{V}_n : n \in \omega\}$ is a cover of X. The current name (the Menger property) has been adopted because Hurewicz proved in [5] that for metrizable spaces his property is equivalent to a certain basis property considered by Menger in [10]. If in the definition above we additionally require that $\{\cup\mathcal{V}_n : n \in \omega\}$ is a γ-*cover* of X (this means that the set $\{n \in \omega : x \notin \cup\mathcal{V}_n\}$ is finite for each $x \in X$), then we obtain the definition of the Hurewicz covering property introduced in [6]. Contrary to a conjecture of Hurewicz, the class of metrizable spaces having the Hurewicz property turned out to be wider than the class of σ-compact spaces [7, Theorem 5.1].

Like for most of the topological properties, it is interesting to ask whether the Hurewicz property is preserved by finite products. One of the motivations behind this question comes from spaces of continuous functions, see [8, Theorem 21]. In the case of general topological spaces there are ZFC examples of Hurewicz spaces whose product is not even Menger, see [18, §3] and the discussion in the introduction of [14]. That is why *we concentrate in what follows on subspaces of the Cantor space* 2^ω. (Let us note that the preservation of the Hurewicz property by finite products of metrizable spaces reduces to subspaces of 2^ω, see the end of the proof of [14, Theorem 1.1] on p. 331 of that paper.) The covering properties of products of subspaces of 2^ω with the Hurewicz property turned out to be sensitive to the ambient set-theoretic universe: Under CH there exists a Hurewicz space whose square is not Menger, see [7, Theorem 2.12]. Later, a similar

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construction has been carried out under a much weaker assumption, see [17, Theorem 43]. In particular, under the Martin Axiom there are Hurewicz subspaces of the Cantor space whose product is not Menger.

On the other hand, the product of any two Hurewicz subspaces of 2^ω is Menger in the Laver and Miller models, see [14] and [19], respectively. In the Miller model we actually know that the product of finitely many Hurewicz subspaces of 2^ω is Menger (for the Laver model this is unknown even for three Hurewicz subspaces), because in this model the Menger property is preserved by products of subspaces of 2^ω, see [19]. That is why the Miller model seemed to be the best candidate for a model where the Hurewicz property is preserved by finite products of metrizable spaces. The next theorem refutes this expectation, and hence the question whether one can find ZFC examples of Hurewicz subspaces X, Y of 2^ω with non-Hurewicz product remains open.

Standardly, by the Miller model we mean a forcing extension of a ground model of GCH by adding a generic for the forcing obtained by an iteration of length ω_2 with countable supports of the poset defined by Miller in [11]. We recall the definition of this poset in the proof of Lemma 2.5.

Theorem 1.1. *In the Miller model there are two γ-subspaces X, Y of 2^ω such that $X \times Y$ is not Hurewicz. In particular, in this model the Hurewicz property is not preserved by finite products of metrizable spaces.*

A family $\mathcal{U}$ of subsets of a set X is called an *ω-cover* of X if $X \notin \mathcal{U}$ and for every finite subset F of X there exists $U \in \mathcal{U}$ such that $F \subset U$. A space X is called a *γ-set* if every open ω-cover of X contains a γ-subcover. This notion was introduced in [4] where it was proved that X is a γ-space if and only if $C_p(X)$, the space of all continuous functions from X to $\mathbb{R}$ with the topology of the pointwise convergence, has the Frechet-Urysohn property, i.e., for every $f \in C_p(X)$ and $A \subset C_p(X)$ with $f \in \bar{A}$ there exists $\langle f_n : n \in \omega\rangle \in A^\omega$ converging to f.

It is well-known that γ-spaces have the Hurewicz property in all finite powers, see, e.g., [15] and references therein. This follows from the following characterization proved in [4]: X is a γ-space iff for every sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of open ω-covers of X there exists a sequence $\langle U_n \in \mathcal{U}_n : n \in \omega\rangle$ such that $\{U_n : n \in \omega\}$ is a γ-cover of X.

Our proof of Theorem 1.1 is based on the fact that if $X \subset 2^\omega$, $X \in V$, and X is a γ-space in V, then X remains a γ-space in the forcing extension by a countable support iteration of posets satisfying property (†) introduced in Definition 2.1 below. This seems to be the first attempt to find iterable properties of forcing posets guaranteeing the preservation of ground model γ-spaces. Previously, only specific posets were treated: By [13] and [16] γ-spaces are preserved by Cohen and random forcing, respectively, whereas the Hechler forcing kills all ground model uncountable γ-spaces, see [12].

Let us note that the Cohen forcing satisfies (†) but fails to preserve Hurewicz spaces, see the discussion in [13] after Problem 4.1 therein. That is why our proof of Theorem 1.1 leaves open the following

Question 1.2. Does the Miller forcing preserve the Hurewicz property of ground model spaces? What about subspaces of 2^ω?

2. Proof of Theorem 1.1

Theorem 1.1 is a direct consequences of Lemmata 2.2, 2.3, 2.4, and 2.5 proved below, combined with one of the main results of [13]. First we need to introduce some auxiliary notions.

Definition 2.1.
- A poset $\mathbb{P}$ has property (†) if for every countable elementary submodel $M \ni \mathbb{P}$ of $H(\theta)$ for θ big enough, $p \in \mathbb{P} \cap M$, and $\phi_i : \mathbb{P} \cap M \to \mathbb{P} \cap M$ for all $i \in \omega$ such that $\phi_i(p) \leq p$ for all $p \in \mathbb{P} \cap M$ and $i \in \omega$, there exists an $(M, \mathbb{P})$-generic $q \leq p$ forcing
$$\dot{G} \cap \{\phi_i(p) : p \in M \cap \mathbb{P}\} \text{ is infinite for all } i \in \omega,$$
where $\dot{G}$ is the canonical $\mathbb{P}$-name for $\mathbb{P}$-generic filter.
- Let $\mathcal{B} = \{B_n : n \in \omega\}$ be a bijective enumeration of the standard clopen base of the topology on 2^ω, i.e., $\mathcal{B}$ consists of finite unions of elements of the family $\big\{[s] = \{x \in 2^\omega : x \restriction |s| = s\} : s \in 2^{<\omega}\big\}$. Let $X \subset 2^\omega$ and M be as above. $\mathcal{W} \subset \mathcal{B}$ is called $\langle X, M, \omega\rangle$-hitting if $\mathcal{W} \cap \mathcal{U}$ is infinite for every ω-cover $\mathcal{U}$ of X such that $\mathcal{U} \in M$ and $\mathcal{U} \subset \mathcal{B}$.
- The poset $\mathbb{P}$ is called $\langle X, \gamma\rangle$*-preserving* if for every countable elementary submodel M such that $X, \mathbb{P} \in M$, $\langle X, M, \omega\rangle$-hitting $\mathcal{W} \subset \mathcal{B}$, and $p \in \mathbb{P} \cap M$ there exists an $(M, \mathbb{P})$-generic condition $q \leq p$ forcing $\mathcal{W}$ to be $\langle X, M[\dot{G}], \omega\rangle$-hitting.

The following lemma justifies our terminology.

Lemma 2.2. *If $\mathbb{P}$ is $\langle X, \gamma\rangle$-preserving and $X \subset 2^\omega$ is a γ-set, then X remains a γ-set in $V^{\mathbb{P}}$.*

Proof. Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for an ω-cover of X by elements of $\mathcal{B}$, $p \in \mathbb{P}$, and $M \ni \dot{\mathcal{U}}, p$ be a countable elementary submodel. Let $\{\mathcal{U}_i : i \in \omega\}$ be an enumeration of $\Omega(X) \cap M \cap \mathcal{P}(\mathcal{B})$ and $U_i \in \mathcal{U}_i$ be such that $\mathcal{W} = \{U_i : i \in \omega\} \in \Gamma(X)$. Then $\mathcal{W}$ is $\langle X, M, \omega\rangle$-hitting, and hence there exists an $(M, \mathbb{P})$-generic $q \leq p$ forcing $\mathcal{W} \cap \dot{\mathcal{U}}$ to be infinite. Thus q forces $\mathcal{W} \cap \dot{\mathcal{U}}$ to be a γ-subcover of $\dot{\mathcal{U}}$. □

Lemma 2.3. *If $\mathbb{P}$ satisfies (†), then it is $\langle X, \gamma\rangle$-preserving for every $X \subset 2^\omega$.*

Proof. Let us enumerate $V^{\mathbb{P}} \cap M$ as $\{\dot{\mathcal{U}}_i : i \in \omega\}$. For every $p \in \mathbb{P} \cap M$ and $i \in \omega$, if p does not force $\dot{\mathcal{U}}_i$ to be an ω-cover of X consisting of elements of $\mathcal{B}$, find $r_{i,p} \leq p$ which forces that $\dot{\mathcal{U}}_i$ is not an ω-cover of X by elements of $\mathcal{B}$. Otherwise set $\mathcal{U}_{i,p} = \{B \in \mathcal{B} : \exists r \leq p (r \Vdash B \in \dot{\mathcal{U}}_i)\}$ and note that $\mathcal{U}_{i,p} \in \Omega(X) \cap M$. Furthermore, by the elementarity we have that for every $B \in \mathcal{U}_{i,p}$ there exists $M \ni r \leq p$ such that $r \Vdash B \in \dot{\mathcal{U}}_i$. Let $\{p_n : n \in \omega\}$ be an enumeration of $M \cap \mathbb{P}$ and for every n, i set $\mathcal{U}'_{i,p_n} = \mathcal{U}_{i,p_n} \setminus \{B_k : k \leq n\}$. Since $\mathcal{W}$ is $\langle X, M, \omega\rangle$-hitting, $|\mathcal{W} \cap \mathcal{U}'_{i,p}| = \omega$ for every $p \in M \cap \mathbb{P}$ and $i \in \omega$.

For every p, i as above pick $U_{i,p} \in \mathcal{W} \cap \mathcal{U}'_{i,p}$ and $r_{i,p} \leq p$ such that $r_{i,p} \in M$ and $r_{i,p} \Vdash U_{i,p} \in \dot{U}_i$.

Now let us fix $p_* \in \mathbb{P} \cap M$ and consider maps $\phi_i : p \mapsto r_{i,p}$, $i \in \omega$. It follows that there exists an $(M, \mathbb{P})$-generic $q \leq p_*$ forcing the set $\dot{G} \cap \{r_{i,p} : p \in \mathbb{P} \cap M\}$ to be infinite for all $i \in \omega$. Let $G \ni q$ be $\mathbb{P}$-generic and $i \in \omega$. If $\dot{\mathcal{U}}_i^G$ is an ω-cover of X by elements of $\mathcal{B}$, then no $r_{i,p} \in G$ can force the negation thereof, and hence for each such $r_{i,p}$ we have $U_{i,p} \in \mathcal{W} \cap \dot{\mathcal{U}}_i^G$. Therefore $|\mathcal{W} \cap \dot{\mathcal{U}}_i^G| = \omega$ since no $B \in \mathcal{B}$ can belong to $\mathcal{U}'_{i,p}$ for infinitely many $p \in M \cap \mathbb{P}$. □

Remark. It is a simple exercise to check that if in the definition of (†) we restrict ourselves to only one $\phi : M \cap \mathbb{P} \to M \cap \mathbb{P}$ then we get an equivalent statement. The longer formulation we have chosen seems to be easier to apply, though. □

By the definition we have that for every $X \subset 2^\omega$ finite iterations of $\langle X, \gamma\rangle$-preserving posets are again $\langle X, \gamma\rangle$-preserving. The proof of the next fact is modelled after that of [1, Lemma 2.8]. In fact, we just "add an ϵ" to it, using ideas from [3].

Lemma 2.4. *Let $X \subset 2^\omega$. Then countable support iterations of $\langle X, \gamma\rangle$-preserving posets are again $\langle X, \gamma\rangle$-preserving.*

Proof. We shall inductively prove the following formally stronger statement:

> Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta\rangle$ be a countable support iteration of $\langle X, \gamma\rangle$-preserving posets, M a countable elementary submodel of $H(\lambda)$ for a sufficiently large regular cardinal λ such that $\delta, \mathbb{P}_\delta \in M$, and $\mathcal{W} \subset \mathcal{B}$ be $\langle X, M, \omega\rangle$-hitting. For any $\delta_0 \in \delta \cap M$ and $(M, \mathbb{P}_{\delta_0})$-generic condition q_0 forcing $\mathcal{W}$ to be $\langle X, M[\dot{G}_{\delta_0}], \omega\rangle$-hitting, the following holds: If $\dot{p}_0 \in V^{\mathbb{P}_{\delta_0}}$ is such that
>
> $$q_0 \Vdash_{\mathbb{P}_{\delta_0}} \dot{p}_0 \in \mathbb{P}_\delta \cap M \text{ and } \dot{p}_0 \restriction \delta_0 \in \dot{G}_{\delta_0},$$
>
> where $\dot{G}_{\delta_0}$ is the canonical name for $\mathbb{P}_{\delta_0}$-generic, then there is an $(M, \mathbb{P}_\delta)$-generic condition q such that
>
> $$q \restriction \delta_0 = q_0 \text{ and } q \Vdash_{\mathbb{P}_\delta} \text{"}\dot{p}_0 \in \dot{G}_\delta \wedge \mathcal{W} \text{ is } \langle X, M[\dot{G}_\delta], \omega\rangle\text{-hitting."}$$

We are going to prove this statement by induction on δ, the only non-trivial case (modulo [1, Lemma 2.6] and the proof thereof) is when δ is a limit ordinal. Fix a strictly increasing sequence $\langle \delta_n : n \in \omega\rangle$ of ordinals in M cofinal in $M \cap \delta$. For every $\nu < \mu < \delta$ let us denote by $\mathbb{P}_{[\nu,\mu)}$ a $\mathbb{P}_\nu$-name for the iteration of $\dot{\mathbb{Q}}_\beta$, $\beta \in \mu \setminus \nu$, in $V^{\mathbb{P}_\nu}$. As usual, (see, e.g., [9]) we shall identify $\mathbb{P}_{[\nu,\mu)}$ with the set of all functions p with domain $\mu \setminus \nu$ such that $1_{\mathbb{P}_\nu}\hat{\ } p \in \mathbb{P}_\mu$, ordered as follows: Given a $\mathbb{P}_\nu$-generic G and $p_0, p_1 \in \mathbb{P}_{[\nu,\mu)}$, $p_1^G \leq p_0^G$ in $\mathbb{P}^G_{[\nu,\mu)}$ if there exists $s \in G$ such that $s\hat{\ }p_1 \leq s\hat{\ }p_0$ in $\mathbb{P}_\mu$.

Let $\{D_i : i \geq 1\}$ be the set of all open dense subsets of $\mathbb{P}_\delta$ which belong to M and $\{\dot{\mathcal{U}}_i : i \in \omega\}$ an enumeration of $V^{\mathbb{P}_\delta} \cap M$ such that each $\tau \in V^{\mathbb{P}_\delta} \cap M$

equals $\dot{\mathcal{U}}_i$ for infinitely many i. Fix $p \in \mathbb{P}_\delta$. We shall define by induction on $n \in \omega$ a condition $q_n \in \mathbb{P}_{\delta_n}$ and a name $\dot{p}_n \in V^{\mathbb{P}_{\delta_n}}$ such that:

(1) q_0 and $\dot{p}_0$ are like in the quoted claim at the beginning of the proof; q_n is $(M, \mathbb{P}_{\delta_n})$-generic; $q_{n+1} \restriction \delta_n = q_n$;

(2) $\dot{p}_n$ is a $\mathbb{P}_{\delta_n}$-name such that

$q_n \Vdash_{\mathbb{P}_{\delta_n}}$ "$\dot{p}_n$ is a condition in $\mathbb{P}_\delta \cap M$ such that

(a) $\dot{p}_n \restriction \delta_n \in \dot{G}_{\delta_n}$;

(b) $\dot{p}_n \le \dot{p}_{n-1}$;

(c) $\dot{p}_n \in D_n$; and

(d) If $\dot{p}_{n-1} \Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathcal{U}}_n$ is an ω-cover of X by elements of $\mathcal{B}$," then $\dot{p}_n \Vdash_{\mathbb{P}_\delta} \exists m \ge n(B_m \in \dot{\mathcal{U}}_n \cap \mathcal{W})$; otherwise $\dot{p}_n \Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathcal{U}}_n$ is **not** an ω-cover of X by elements of $\mathcal{B}$"."

Assume that q_n and $\dot{p}_n$ have already been constructed. For a while we shall work in $V[G]$, where $G \ni q_n$ is $\mathbb{P}_{\delta_n}$-generic. Then $p_n := \dot{p}_n^G \in D_n \cap M$ and $p_n \restriction \delta_n \in G$. Find $p'_n \le p_n$ such that $p'_n \restriction \delta_n \in G$ and $p'_n \in D_{n+1} \cap M$. Wlog we may assume that each condition in D_{n+1} decides whether $\dot{\mathcal{U}}_{n+1}$ is an ω-cover of X by elements of $\mathcal{B}$. If p'_n decides that it is not, then we set $p_{n+1} = p'_n$ and take q_{n+1} to be any $(M, \mathbb{P}_{\delta_{n+1}})$-generic satisfying (1), (2), its existence follows by our inductive assumption. Otherwise fix a $\mathbb{P}_{\delta_n}$-name $\dot{p}'_n$ for a condition in $\mathbb{P}_\delta$ such that q_n forces that $\dot{p}'_n$ has all the properties of p'_n stated above, and an $(M, \mathbb{P}_{\delta_{i+1}})$-generic q_{n+1} such that $q_{n+1} \restriction \delta_n = q_n$, $q_{n+1} \Vdash_{\mathbb{P}_{\delta_{n+1}}} \dot{p}'_n \restriction \delta_{n+1} \in \dot{G}_{\delta_{n+1}}$, and $q_{n+1} \Vdash_{\mathbb{P}_{\delta_{n+1}}}$ "$\mathcal{W}$ is $\langle X, M[\dot{G}_{\delta_{n+1}}], \omega\rangle$-hitting". Consider the $\mathbb{P}_{\delta_{n+1}}$-name $\dot{\mathcal{W}}_{n+1}$ which equals

$$\big\{B \in \mathcal{B} : \exists r \in \mathbb{P}_{[\delta_n, \delta)}(r \le p'_n \restriction [\delta_n, \delta) \wedge r \Vdash_{\mathbb{P}_{[\delta_n,\delta)}} B \in \dot{\mathcal{U}}_{n+1})\big\} =$$
$$= \big\{B \in \mathcal{B} : \exists r \in \mathbb{P}_{[\delta_n, \delta)} \cap M(r \le p'_n \restriction [\delta_n, \delta) \wedge r \Vdash_{\mathbb{P}_{[\delta_n,\delta)}} B \in \dot{\mathcal{U}}_{n+1})\big\}.$$

It follows that $\dot{\mathcal{W}}_{n+1} \in M$ is a $\mathbb{P}_{\delta_{n+1}}$-name for an ω-cover of X by elements of $\mathcal{B}$, and hence $q_{n+1} \Vdash_{\mathbb{P}_{\delta_{n+1}}} |\mathcal{W} \cap \dot{\mathcal{W}}_{n+1}| = \omega$. Let $H \ni q_{n+1}$ be $\mathbb{P}_{\delta_{n+1}}$-generic over V and p'_n the interpretation $(\dot{p}'_n)^H$.

Now we shall work in $V[H]$ for a while. It follows from the above that there exists $m > n$ such that $B_m \in \mathcal{W} \cap \dot{\mathcal{W}}^H_{n+1}$. Consequently, there exists $\rho \in \mathbb{P}_{[\delta_{n+1}, \delta)}$ such that $\rho \le p'_n \restriction [\delta_{n+1}, \delta)$ and $\rho \Vdash_{\mathbb{P}_{[\delta_{n+1},\delta)}} B_m \in \dot{\mathcal{U}}_{n+1}$. Let $s \in \mathbb{P}_{\delta_{n+1}} \cap M[H] \cap H = \mathbb{P}_{\delta_{n+1}} \cap M \cap H$ be such that $s \le p'_n \restriction \delta_{n+1}$ and $s \Vdash_{\mathbb{P}_{\delta_{n+1}}}$ "ρ is a function with domain $\delta \setminus \delta_{n+1}$ such that $1_{\delta_n}{}^\frown \rho \in \mathbb{P}_\delta$, $\rho \le p'_n \restriction [\delta_{n+1}, \delta)$, and $\rho \Vdash_{\mathbb{P}_{[\delta_{n+1},\delta)}} B_m \in \dot{\mathcal{U}}_{n+1}$". Now set $p_{n+1} = s{}^\frown \rho \in \mathbb{P}_\delta$ and let $\dot{p}_{n+1}$ be a $\mathbb{P}_{\delta_{n+1}}$-name such that q_{n+1} forces that $\dot{p}_{n+1} \restriction \delta_{n+1}$ and $\dot{p}_{n+1} \restriction [\delta_{n+1}, \delta)$ have all the properties of s and ρ stated above, respectively. Its existence follows by the maximality principle. This completes our inductive construction. Exactly as in the proof of [1, Lemma 2.8] one can verify that $q = \bigcup_{n \in \omega} q_n$ is $(M, \mathbb{P}_\delta)$-generic, and (2)(d) clearly ensures that it forces $\mathcal{W}$ to be $\langle X, M[\dot{G}_\delta], \omega\rangle$-hitting. This completes our proof. $\square$

Lemma 2.5. *The Miller, Sacks, and Cohen posets satisfy* (†).

Proof. We shall present the proof only for the Miller forcing because it is exactly what is needed for the proof of Theorem 1.1, and because the Sacks case is completely analogous, whereas the Cohen is trivial.

Before we pass to the proof, let us recall the definition of the Miller forcing and fix our notation. By a Miller tree we understand a subtree T of $\omega^{<\omega}$ consisting of increasing finite sequences such that the following conditions are satisfied:

- Every $t \in T$ has an extension $s \in T$ which is splitting in T, i.e., there are more than one immediate successors of s in T;
- If s is splitting in T, then it has infinitely many immediate successors in T.

The Miller forcing is the collection $\mathbb{M}$ of all Miller trees ordered by inclusion, i.e., smaller trees carry more information about the generic. This poset was introduced in [11].

For a Miller tree T we shall denote by $\mathrm{Split}(T)$ the set of all splitting nodes of T, and for some $t \in \mathrm{Split}(T)$ we denote the size of $\{s \in \mathrm{Split}(T) : s \subsetneq t\}$ by $Lev(t,T)$. For a node t in a Miller tree T we denote by T_t the set $\{s \in T : s \text{ is compatible with } t\}$. It is clear that T_t is also a Miller tree. If $T_1 \leq T_0$ and each $t \in T_0$ with $Lev(t,T_0) \leq k$ belongs to T_1, where $k \in \omega$, then we write $T_1 \leq_k T_0$. It is easy to check (and is well-known) that if $T_{n+1} \leq_n T_n$ for all $n \in \omega$, then $\bigcap_{n\in\omega} T_n \in \mathbb{M}$.

We are now in a position to start the proof. Let M and $\{\phi_i : i \in \omega\}$ be such as in the formulation of (†). We can additionally assume that for each $\phi \in \{\phi_i : i \in \omega\}$ there are infinitely many i such that $\phi = \phi_i$. Let $\{D_n : n \in \omega\}$ be the set of all open dense subsets of $\mathbb{M}$ which belong to M. Given $T_0 \in M \cap \mathbb{M}$, construct a sequence $\langle T_n : n \in \omega\rangle \in \mathbb{M}^\omega$ as follows: Assume that T_n has been constructed such that $(T_n)_t \in M$ for every $t \in T_n$ with $Lev(t,T_n) = n$. Given such a $t \in T_n$ and $k \in \omega$ such that $t\hat{\ }k \in T_n$, find $R_{t,k} \leq \phi_n((T_n)_{t\hat{\ }k})$ such that $R_{t,k} \in D_n \cap M$. Now set $T_{n+1} = \bigcup\{R_{t,k} : t \in T_k, Lev(t,T_n) = n, t\hat{\ }k \in T_n\}$ and note that $T_{n+1} \leq_n T_n$ and $(T_{n+1})_r \in M$ for all $r \in T_{n+1}$ with $Lev(r,T_{n+1}) = n+1$. This completes our construction. It is straightforward to check that $T = \bigcap_{n\in\omega} T_n$ is a $(M,\mathbb{M})$-generic condition forcing $\dot{G} \cap \phi_n[M \cap \mathbb{M}]$ to be infinite for all n. □

Finally we have all necessary ingredients to complete the proof of Theorem 1.1. Let V be a model of GCH. By [13, Theorem 3.2] there exist γ-subspaces X, Y of 2^ω and a continuous map $\phi : X \times Y \to \omega^\omega$ such that $\phi[X \times Y]$ is dominating, i.e., for every $f \in \omega^\omega$ there exists $\langle x, y\rangle \in X \times Y$ such that $f \leq^* \phi\langle x, y\rangle$. (As usual, $f \leq^* g$ for $f, g \in \omega^\omega$ means that the set $\{n \in \omega : f(n) > g(n)\}$ is finite. Whenever we speak about unbounded or dominating subsets of ω^ω, we always mean with respect to $\leq^*$.) Let $\mathbb{P}$ be the iteration of $\mathbb{M}$ of length ω_2 with countable supports, and G be $\mathbb{P}$-generic. It is well known that $V \cap \omega^\omega$ is unbounded[1] in $V[G]$, and hence so

[1]Even more is true: there exists an ultrafilter $\mathcal{U} \in V$ which remains a base for an ultrafilter in $V[G]$, see [2]. It is easy to see that the set of anumerating functions of a base of an ultrafilter cannot be bounded.

is $\phi[X \times Y]$. By a result of Hurewicz [6] (see also [7, Theorem 4.3]) this implies that $X \times Y$ is not Hurewicz in $V[G]$. On the other hand, X and Y remain γ-spaces in $V[G]$ by a combination of Lemmata 2.2, 2.3, 2.4, and 2.5. This completes our proof.

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