УДК 512, 515.1

L. S. Zdomskyy

# CAN A BOREL GROUP BE GENERATED BY A HUREWICZ SUBSPACE?

L. S. Zdomskyy. *Can a Borel group be generated by a Hurewicz subspace?* Matematychni Studii, **25** (2006) 219–224.

In this paper we formulate three problems concerning topological properties of sets generating Borel non-$\sigma$-compact groups. In the case of a concrete $F_{\sigma\delta}$-subgroup of $\{0,1\}^{\omega\times\omega}$ this gives an equivalent reformulation of the Scheepers diagram problem.

Л. С. Здомский. *Может ли борелевская группа порождаться подпространством Гуревича?* // Математичні Студії. – 2006. – Т.25, №2. – С.219–224.

В данной работе мы формулируем три проблемы о топологических свойствах пространств порождающих борелевские не $\sigma$-компактные группы. В случае конкретной $F_{\sigma\delta}$ -подгруппы $\{0,1\}^{\omega\times\omega}$ мы получаем эквивалентную формулировку одной проблемы М. Шиперза.

**Introduction.** The Hurewicz property was introduced in [5] as a cover counterpart of the $\sigma$-compactness: a topological space $X$ is said to have this property, if for every sequence $(u_n)_{n\in\omega}$ of open covers of $X$ there exists a sequence $(v_n)_{n\in\omega}$, where each $v_n$ is a finite subset of $u_n$ such that each element $x \in X$ belongs to $\bigcup v_n$ for all but finitely many $n \in \omega$. It is easy to see that each $\sigma$-compact space is Hurewicz (= has the Hurewicz property). The converse statement is known to fail in ZFC, see [6]. By a *Borel* space we mean a separable metrizable space which is a Borel subset of its completion. This paper is devoted to problems close to the following one.

**Problem 1.** *Can a Borel non-$\sigma$-compact group be generated by its Hurewicz subspace?*

This problem is especially interesting for the concrete subgroup $\mathsf{G}$ of $\{0,1\}^{\omega\times\omega}$ (standardly endowed with the coordinatewise addition modulo 2) being equivalent to the "Hurewicz" part of the Scheepers diagram problem (see [6, Problems 1,2], [14, Problems 4.1,4.2], [12, Problem 1], and [13, Problem 3.2]), where

$$\mathsf{G} = \left\{x \in \{0,1\}^{\omega^2} : x_{i,j} = 0 \text{ for every } j \in \omega \text{ and all but finitely many } i\right\}.$$

In order to formulate the Scheepers diagram problem we have to recall some definitions. M. Scheepers in his paper [10] introduced a long list of new properties looking similar to the Hurewicz one, and thus gave rise to the branch of set-theoretic topology known as *Selection Principles.* Selection principles may be thought as some combinatorial conditions on the family of open covers of a topological space. Let $\mathcal{A}$ and $\mathcal{B}$ be families of covers of a topological space $X$. Following [10] we say that $X$ has the property

2000 *Mathematics Subject Classification:* 54D20, 22A10, 28A05.

- $\bigcup_{\text{fin}}(\mathcal{A},\mathcal{B})$, if for every sequence $(u_n)_{n\in\omega}\mathcal{A}^\omega$ there exists a sequence $(v_n)_{n\in\omega}$, where each $v_n$ is a finite subset of $u_n$, such that $\{\bigcup v_n : n\in\omega\}\in\mathcal{B}$;
- $\mathrm{S}_{\text{fin}}(\mathcal{A},\mathcal{B})$, if for every sequence $(u_n)_{n\in\omega}\in\mathcal{A}^\omega$ there exists a sequence $(v_n)_{n\in\omega}$ where each $v_n$ is a finite subset of $u_n$ such that $\bigcup\{v_n : n\in\omega\}\in\mathcal{B}$.

Throughout the paper, $\mathcal{A}$ and $\mathcal{B}$ run over the families $\mathcal{O}$, $\Omega$, and $\Gamma$ of all open ($\omega$-, $\gamma$-) covers of $X$. Given a family $u=\{U_i : i\in I\}$ of subsets of a set $X$, we define the map $\mu_u\colon X\to\mathcal{P}(I)$ letting $\mu_u(x)=\{i\in I : x\in U_i\}$ ($\mu_u$ is the Marczewski "dictionary" map introduced in [9]). In what follows, $I\in\{\omega,w^2\}$. Depending on the properties of $\mu_u(X)$ a family $u=\{U_n : n\in\omega\}$ is defined to be

- an *$\omega$-cover* [4], if the family $\mu_u(X)$ is centered, i.e. for every finite subset $K$ of $X$ the intersection $\bigcap_{x\in K}\mu_u(x)$ is infinite;
- a *$\gamma$-cover* of $X$ [4], if for every $x\in X$ the set $\mu_u(x)$ is cofinite in $\omega$, i.e. $\omega\setminus\mu_u(x)$ is finite.

We shall consider here four selection principles: $\bigcup_{\text{fin}}(\mathcal{O},\Gamma)$, $\bigcup_{\text{fin}}(\mathcal{O},\Omega)$, $\bigcup_{\text{fin}}(\mathcal{O},\mathcal{O})$ and $\mathrm{S}_{\text{fin}}(\Gamma,\Omega)$. Let us note that $\bigcup_{\text{fin}}(\mathcal{O},\Gamma)$ is nothing else but the Hurewicz property. Concerning $\bigcup_{\text{fin}}(\mathcal{O},\mathcal{O})$, it is the classical Menger covering property introduced in [8]. We are in a position now to formulate the

**Scheepers diagram problem.**

(1) *Does the property* $\bigcup_{\text{fin}}(\mathcal{O},\Omega)$ *imply* $\mathrm{S}_{\text{fin}}(\Gamma,\Omega)$*?*

(2) *And if not, then does* $\bigcup_{\text{fin}}(\mathcal{O},\Gamma)$ *imply* $\mathrm{S}_{\text{fin}}(\Gamma,\Omega)$*?*

One may ask the same question as in Problem 1 for the properties $\bigcup_{\text{fin}}(\mathcal{O},\Omega)$ and $\bigcup_{\text{fin}}(\mathcal{O},\mathcal{O})$.

**Problem 2.** *Can a Borel non-$\sigma$-compact group be generated by its subspace with the property* $\bigcup_{\text{fin}}(\mathcal{O},\Omega)$*?*

**Problem 3.** *Can a Borel non-$\sigma$-compact group be generated by its subspace with the property* $\bigcup_{\text{fin}}(\mathcal{O},\mathcal{O})$*?*

The following theorem, which is the main result of this paper, is a reformulation of a Scheepers diagram problem in algebraic manner.

**Theorem 4.** *The property* $\bigcup_{\text{fin}}(\mathcal{O},\Gamma)$ *(resp.* $\bigcup_{\text{fin}}(\mathcal{O},\Omega)$, $\bigcup_{\text{fin}}(\mathcal{O},\mathcal{O})$*) implies* $\mathrm{S}_{\text{fin}}(\Gamma,\Omega)$ *if and only if the group* $\mathsf{G}$ *is not generated by its subspace with the property* $\bigcup_{\text{fin}}(\mathcal{O},\Gamma)$ *(resp.* $\bigcup_{\text{fin}}(\mathcal{O},\Omega)$, $\bigcup_{\text{fin}}(\mathcal{O},\mathcal{O})$*).*

*In other words, the affirmative answer to the Scheepers diagram problem (1) (resp. (2)) is equivalent to the negative answer onto Problem 2 (resp. Problem 1) in the case of the group* $\mathsf{G}$.

The group $\mathsf{G}$ is a rather simple object from the point of view of the Descriptive Set Theory. For every $j\in\omega$ its projection onto $\{0,1\}^{\omega\times\{j\}}$ is homeomorphic to $\mathbb{Q}$ being a countable metrizable space without isolated points. From the above it follows that $\mathsf{G}$ is a countable intersection of $F_\sigma$ – subsets of $\{0,1\}^{\omega^2}$ (i.e. it is an $F_{\sigma\delta}$- or, equivalently, $\Pi^0_3$-subset) homeomorphic to $\mathbb{Q}^\omega$. Therefore, it is a nowhere locally-compact, and it fails to have the property $\bigcup_{\text{fin}}(\mathcal{O},\mathcal{O})$. For more simple groups from the point of view of Borel hierarchy Problem 1 can be answered in the negative.

**Proposition 1.** *No Borel non-$\sigma$-compact group $B$ can be generated by its subspace $X$ with the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$ provided $B$ is an $F_\sigma$- or $G_\delta$-subspace of a complete metric space.*

Recall that a map $f$ from a topological space $X$ to a topological space $Y$ is *Borel,* if for every Borel subset $B$ of $Y$ its preimage $f^{-1}(B)$ is a Borel subset of $X$. The following statement answers Problem 3 in the affirmative under the Continuum Hypothesis. On the other hand, it is known that the properties $\bigcup_{\mathrm{fin}}(\mathcal{O},\Omega)$ and $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ coincide in some models of ZFC, see [17]. Therefore the negative answer to Problem 2 would imply that the negative answer onto Problem 3 is consistent as well.

**Proposition 2.** *Under the Continuum Hypothesis, a metrizable separable group $B$ can be generated by its subspace $X$ with the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ provided it is a Borel homomorphic image of a nonmeager metrizable separable group. In particular, $\mathsf{G}$ is generated by its subspace with the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ under CH.*

**Remark.** None of the known methods of contruction of spaces with the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$ can give a subspace of a Borel non-$\sigma$-compact group generating it. All finite powers of spaces with the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$ constructed in [6, Theorem 5.1], [15, Theorem 5.1], and [2, Theorem 10(1)] have the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ or even $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$, and hence so is any group they generate. But every Borel (even analytic) space with the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ is $\sigma$-compact, see [1]. While the Sierpinski sets $S$ considered in [6] and [11] have the following property: for every Borel subset $B$ containing $S$ there exists a $\sigma$-compact $L$ such that $S \subset L \subset B$, see [3].

Concerning the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\Omega)$, all known examples (excepting the Sierpinski sets) have the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ in all finite powers, and hence cannot generate non-$\sigma$-compact Borel group. □

**Proofs.** In what follows, $A \subset^* B$ standardly means that $A \setminus B$ is finite. In our proofs we shall exploit set-valued maps. By a *set-valued map* $\Phi$ from a set $X$ into a set $Y$ we understand a map from $X$ into $\mathcal{P}(Y)$ and write $\Phi : X \Rightarrow Y$ (here $\mathcal{P}(Y)$ denotes the set of all subsets of $Y$). For a subset $A$ of $X$ we put $\Phi(A) = \bigcup_{x\in A}\Phi(x) \subset Y$. The set-valued map $\Phi$ between topological spaces $X$ and $Y$ is said to be

- *compact-valued*, if $\Phi(x)$ is compact for every $x \in X$;
- *upper semicontinuous*, if for every open subset $V$ of $Y$ the set $\Phi^{-1}_{\subset}(V) = \{x \in X : \Phi(x) \subset V\}$ is open in $X$.

For a set $X$ we can identify $\mathcal{P}(X)$ with the compact space $\{0,1\}^X$ via the map $X \supset A \mapsto \chi_A \in \{0,1\}^X$ assigning to a subset of $X$ its characteristic function. A family $\mathcal{A}$ of subsets of a set $X$ is called *upward closed*, for every $A \in \mathcal{A}$ and $B \supset A$ we have $B \in \mathcal{A}$. For a set $A \subset X$ we let $\uparrow A = \{B \subset X : A \subset B\}$. The following lemma is a more convenient reformulation of Theorem 4.

**Lemma 1.** *Let $\mathsf{P}$ be a topological property preserved by images under upper semicontinuous compact-valued maps. Then the following conditions are equivalent:*

(1) *The property $\mathsf{P}$ implies $\mathrm{S}_{\mathrm{fin}}(\Gamma,\Omega)$;*

(2) *for every (upward-closed) $\mathcal{F} \subset \mathcal{P}(\omega^2)$ with the property $\mathsf{P}$ such that $\omega \times \{j\} \subset^* F$ for every $F \in \mathcal{F}$ and $j \in \omega$, there exists a sequence $(K_j)_{j\in\omega}$ of finite subsets of $\omega$ such that each element of the smallest filter containing $\mathcal{F}$ meets $\bigcup_{n\in\omega} K_j \times \{j\}$.*

*Proof.* (1) $\Rightarrow$ (2). It simply follows from the definition of the property $S_{\mathrm{fin}}(\Gamma,\Omega)$ and the observation that $\{\{F\in\mathcal{F} : F\ni(i,j)\} : i\in\omega\}$ is an open $\gamma$-cover of $\mathcal{F}$ for every $j\in\omega$.

(2) $\Rightarrow$ (1). Let $X$ be a topological space with the property $\mathsf{P}$ and $(u_j)_{j\in\omega}$ be a sequence of open $\gamma$-covers of $X$. Let us write $u_j$ in the form $u_j=\{U_{i,j} : i\in\omega\}$. Set $u=\{U_{i,j} : i,j\in\omega\}$. Consider the set-valued map $\Phi : X\Rightarrow\mathcal{P}(\omega^2)$, $\Phi : x\mapsto\uparrow\mu_u(x)$. Applying Lemma 2 of [17], we conclude that $\Phi$ is compact-valued and upper semicontinuous, and hence $\mathcal{F}:=\Phi(X)$ has the property $\mathsf{P}$. The definition of $\Phi$ implies that $\mathcal{F}$ is upward closed. Since $u_j$ is a $\gamma$-cover of $X$ for every $j\in\omega$, $\omega\times\{j\}\subset^* F$ for each $F\in\mathcal{F}$. From the above it follows that there exists a sequence $(K_j)_{j\in\omega}$ of finite subsets of $\omega$ such that each element of the smallest filter $\mathcal{U}$ containing $\mathcal{F}$ meets some $K_j\times\{j\}$. Then the family $\{U_{i,j} : i\in K_j\}$ is easily seen to be an $\omega$-cover of $X$, which finishes our proof. $\square$

The properties $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$, $\bigcup_{\mathrm{fin}}(\mathcal{O},\Omega)$, and $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$ satisfy the conditions of the above lemma by [17, Lemma 1].

*Proof of Theorem 4.* Let $\mathsf{P}$ be any of the properties $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$, $\bigcup_{\mathrm{fin}}(\mathcal{O},\Omega)$, and $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$. Assuming that $\mathsf{P}$ implies $\mathrm{S}_{\mathrm{fin}}(\Gamma,\Omega)$, fix a subspace $X$ of $\mathsf{G}$ with the property $\mathsf{P}$. Let us denote by $\varphi$ the map assigning to any subset $A$ of $\omega^2$ its characteristic function $\chi_A\in\{0,1\}^{\omega^2}$. Then the space $\mathcal{F}=\{\omega^2\setminus A : A\in\varphi^{-1}(X)\}$ has the property $\mathsf{P}$ being homeomorphic to $X$, and $\omega\times\{j\}\subset^* F$ for every $F\in\mathcal{F}$ by our choice of $\mathsf{G}\supset X$. Applying Lemma 1, we conclude that there exists a sequence $(K_j)_{j\in\omega}$ of finite subsets of $\omega$ such that $\bigcup_{j\in\omega}K_j\times\{j\}$ meets all elements of the smallest filter containing $\mathcal{F}$. Now, a direct verification shows that the charateristic function $\chi_{\cup_{j\in\omega}K_j\times\{j\}}$ cannot be represented as a sum of elements of $X$, which means that $X$ does not generate $\mathsf{G}$.

Next, let us assume that $\mathsf{P}$ does not imply $\mathrm{S}_{\mathrm{fin}}(\Gamma,\Omega)$ and apply Lemma 1 to find an upward closed family $\mathcal{F}$ of subsets of $\omega^2$ such that for every sequence $(K_j)_{j\in\omega}$ of finite subsets of $\omega$ there exists a finite subset $\mathcal{A}$ of $\mathcal{F}$ such that

$$\left(\bigcup_{j\in\omega}K_j\times\{j\}\right)\cap\bigcap\mathcal{A}=\varnothing.$$

Set $X=\{\chi_{\omega^2\setminus F} : F\in\mathcal{F}\}$. Then $X$ has the property $\mathsf{P}$ being homeomorphic to $\mathcal{F}$. We claim that $X$ is a set of generators of $\mathsf{G}$. Indeed, let us fix any $g\in\mathsf{G}$ and set $K_j=\{i\in\omega : g_{i,j}=1\}$. Then each $K_j$ is finite by the definition of $\mathsf{G}$. For the sequence $(K_j)_{j\in\omega}$ find a finite subset $\mathcal{A}=\{A_i : i\leq n\}$ of $\mathcal{F}$ as above. Using the upward closedness of $\mathcal{F}$, define inductively a finite subset $\mathcal{B}=\{B_i : i\leq n\}$ of $\mathcal{F}$ letting $B_0=A_0$ and $B_k=A_k\cup\bigcup_{l<k}(\omega^2\setminus B_l)$ for all $0<k\leq n$. It is easy to prove by induction over $k\leq n$ that $(\omega^2\setminus B_l)\cap(\omega^2\setminus B_k)=\varnothing$ for all $l<k$ and $\bigcap_{l\leq k}B_k=\bigcap_{l\leq k}A_k$, consequently $\bigcap\mathcal{B}=\bigcap\mathcal{A}\subset\left(\omega^2\setminus\bigcup_{j\in\omega}K_j\times\{j\}\right)$. Let $C_k=B_k\cup\left(\omega^2\setminus\bigcup_{j\in\omega}K_j\times\{j\}\right)$, $k\leq n$. Then $\mathcal{C}=\{C_k : k\leq n\}$ has the following properties:

*(i)* $\bigcup\mathcal{C}=\omega^2\setminus\bigcup_{j\in\omega}K_j\times\{j\}$;

*(ii)* $(\omega^2\setminus C)\cap(\omega^2\setminus D)=\varnothing$ for all $C,D\in\mathcal{C}$;

*(iii)* $\mathcal{C}\subset\mathcal{F}$.

It suffices to note that $\{\chi_{\omega^2\setminus C_k} : k\leq n\}\subset X$ by *(iii)* and $\chi_{\omega^2\setminus C_0}+\dots+\chi_{\omega^2\setminus C_n}=\chi_{\cup_{j\in\omega}K_j\times\{j\}}=g$, which finishes our proof. $\square$

*Proof of Proposition 1.* First assume that $B$ is a non-$\sigma$-compact $G_\delta$-subspace of a complete metric space and fix a subspace $X$ of $B$ with the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$. The same argument as in [6, Theorem 5.7] gives a $\sigma$-compact subset $L$ of $B$ such that $X \subset L$. Since $B$ is not $\sigma$-compact, it is not generated by $L$, and hence by $X$ as well.

Now consider a non-$\sigma$-compact Borel group $B$ which is an $F_\sigma$-subset of a complete metric space $Y$ and write $B$ in the form $\bigcup_{n\in\omega} B_n$, where each $B_n$ is closed in $Y$. Let $X$ be a subspace of $B$ with the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$. Since the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$ is preserved by closed subspaces, $X \cap B_n$ has the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$ for all $n \in \omega$. In addition, each $B_n$ is a $G_\delta$-subspace of $Y$ being closed. From the above it follows that there exists a $\sigma$-compact $L_n$ such that $X \cap B_n \subset L_n \subset B_n$, and consequently $X \subset \bigcup_{n\in\omega} L_n \subset B$. It suffices to apply the same argument as in the first part of the proof. □

*Proof of Proposition 2.* Let $C$ be a nonmeager metrizable separable topological group and $f\colon C \to B$ be a surjective Borel homomorphism. Almost literal repetition of the proof of Lemma 29 from [11] gives us a subspace $Z$ of $C$ such that $Z$ generates $C$ and each Borel image of $Z$ has the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$, see [11, Corollary 30]. It suffices to note that $B$ is generated by $f(Z)$.

Next, let us show that under CH the group $\mathsf{G}$ is generated by its subspace with the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. Indeed, let us denote by $\tau$ the Tychonoff product topology on $\{0,1\}^{\omega^2} = \prod_{j\in\omega}\{0,1\}^{\omega\times\{j\}}$, where $\{0,1\}^{\omega\times\{j\}}$ is considered with the discrete topology for each $j \in \omega$. Then $\tau|\mathsf{G}$ is stronger than the natural topology on $\mathsf{G}$, and $(\mathsf{G}, \tau|\mathsf{G})$ is a completely metrizable topological group being a countable product of countable discrete groups. □

**Acknowledgements.** The author wishes to express his thanks to Professor Taras Banakh for supervising the writing of this paper.

## REFERENCES

1. Arhangel'skiĭ A.V. *Hurewicz spaces, analytic sets, and fan tightness of function spaces*, Soviet Math. Doklady **33** (1986), 396–399.
2. Bartoszyński T., Tsaban B. *Hereditary Topological Diagonalizations and the Menger-Hurewicz Conjectures*, Proc. Amer. Math. Soc., to appear.
3. Banakh T., Zdomskyy L. *Separation properties between $\sigma$-compactness and the Hurewicz property*, in progress.
4. Gerlits J., Nagy Zs. *Some properties of $C(X)$, I*, Topology Appl. **14** (2) (1982), 151–163.
5. Hurewicz W. *Über Folgen stetiger Functionen*, Fund. Math. **9** (1927), 193-204.
6. Just W., Miller A., Scheepers M., Szeptycki S. *The combinatorics of open covers II*, Topology Appl. **73** (1996), 241–266.
7. Kechris A. Classical Descriptive Set Theory. – Grad. Texts in Math. **156**, Springer, 1995.
8. Menger K. *Einige Überdeckungssätze der Punktmengenlehre, Sitzungsberichte.* Abt. 2a, Mathematic, Astronomie, Physic, Meteorologie und Mechanic (Wiener Akademie) **133** (1924), 421– 444.
9. Marczewski E. (Szpilrajn) *The characteristic function of a sequence of sets and some of its applications,* Fund. Math. **31** (1938), 207–233.
10. Scheepers M. *Combinatorics of open covers I: Ramsey Theory*, Topology Appl. **69** (1996), 31–62.

11. Scheepers M., Tsaban B. *The combinatorics of Borel covers*, Topology Appl. **121** (2002), 357-382.

12. Tsaban B. *Selection principles in mathematics: A milestone of open problems*, Note Mat. **22** (2003/2004), no 2, 179–208.

13. Tsaban B. *Some new directions in infinite-combinatorial topology*, submitted. http://arxiv.org/abs/math.GN/0409069

14. Tsaban B. (eds.), SPM Bulletin **2** (2003). http://arxiv.org/abs/math.GN/0302062

15. Tsaban B., Zdomskyy L. *Scales, fields, and a problem of Hurewicz*, submitted to J. Amer. Math. Soc.. http://arxiv.org/abs/math.GN/0507043.

16. Vaughan J. *Small uncountable cardinals and topology*, In: Open Problems in Topology (J. van Mill, G.M. Reed, Eds.), Elsevier Sci. Publ., Amsterdam, 1990, 195-218.

17. Zdomskyy L. *A semifilter approach to selection principles,* Comment. Math. Univ. Carolin. **46** (2005), 525–540.

Ivan Franko National University of Lviv
lzdomsky@rambler.ru

*Received 01.09.2005*