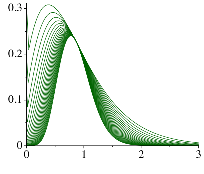
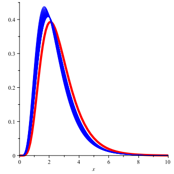
PROBABILISTIC ANALYSIS OF THE (1 + 1)-EVOLUTIONARY ALGORITHM

Hsien-Kuei Hwang (joint with *Alois Panholzer, Nicolas Rolin, Tsung-Hsi Tsai, Wei-Mei Chen*)

October 13, 2017



ALEA Network



Evolutionary Computation Just Accepted MS.
doi:10.1162/EVCO_a_00212

Probabilistic Analysis of the $(1+1)$-Evolutionary Algorithm[1]

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WE ARE ALWAYS *SEARCHING & RESEARCHING*

Life

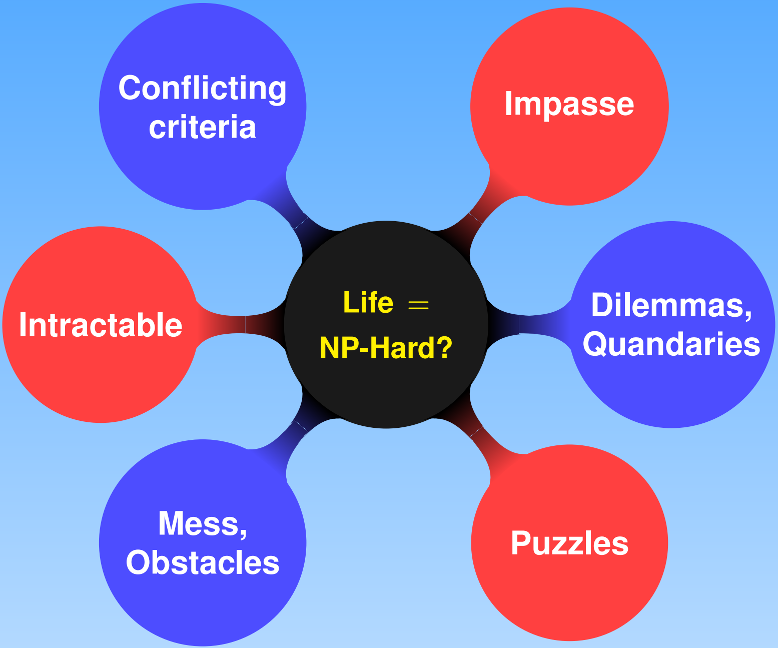
Searching

Algorithms

Searching

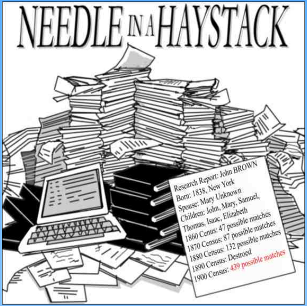
Life $=$ Algorithms ?

OPTIMIZATION PROBLEMS EVERYWHERE



SEARCH ALGORITHMS

- Backtracking
- Branch-and-bound
- Greedy
- Dynamic programming
- *Simulated annealing*
- *Evolutionary algorithms*
- *Ant colony optimization*
- *Particle swarm*
- *Tabu search*
- *GRASP*
- … *Meta-heuristics*



EVOLUTIONARY ALGORITHMS

Evolutionary algorithm

From Wikipedia, the free encyclopedia

In artificial intelligence, an **evolutionary algorithm** (**EA**) is a subset of evolutionary computation, a generic population-based metaheuristic optimization algorithm. An EA uses mechanisms inspired by biological evolution, such as reproduction, mutation, recombination, and selection. Candidate solutions to the optimization problem play the role of individuals in a population, and the fitness function determines the quality of the solutions (see also loss function). Evolution of the population then takes place after the repeated application of the above operators. *Artificial evolution* (AE) describes a process involving individual *evolutionary algorithms*; EAs are individual components that participate in an AE.

EVOLUTIONARY ALGORITHM

The use of Darwinian principles for automated problem solving originated in the 1950s.

Darwin's theory of evolution: survival of the fittest

- **stochastic evolution on computers**

  ☞ **cultivating problem solutions instead of calculating them**

- **randomized search heuristics**

  ☞ ***generate* and *test* (or trial and error)**

- **useful for global optimization, if the problem is**
  - **too complex to be handled by an exact method or**
  - **no exact method is available**

Pioneers: John Holland, Lawrence J. Fogel, Ingo Rechenberg, . . .

EVOLUTIONARY ALGORITHMS (EAs)

Anne-Wil Harzing's s/w ***Publish or Perish***

1	Cites	Authors	Title	Year
2	13687	K Deb	Multi-objective optimization using evolutionary algorithms	2001
3	6223	CAC Coello, GB Lamont, DA Van Veldh	Evolutionary algorithms for solving multi-objective problems	2007
4	6113	E Zitzler, L Thiele	Multiobjective evolutionary algorithms: a comparative case study and the str	1999
5	5977	E Zitzler, M Laumanns, L Thiele	SPEA2: Improving the strength Pareto evolutionary algorithm	2001
6	5849	T Back	Evolutionary algorithms in theory and practice: evolution strategies, evolutio	1996
7	4562	E Zitzler, K Deb, L Thiele	Comparison of multiobjective evolutionary algorithms: Empirical results	2000
8	3529	K Deb, S Agrawal, A Pratap, T Meyariv	A fast elitist non-dominated sorting genetic algorithm for multi-objective opti	2000
9	3394	HPP Schwefel	Evolution and optimum seeking: the sixth generation	1993
10	3091	T Bäck, DB Fogel, Z Michalewicz	Handbook of evolutionary computation	1997
11	2688	CM Fonseca, PJ Fleming	An overview of evolutionary algorithms in multiobjective optimization	1995
12	2328	P Larrañaga, JA Lozano	Estimation of distribution algorithms: A new tool for evolutionary computatic	2001
13	2195	E Zitzler	Evolutionary algorithms for multiobjective optimization: Methods and applica	1999
14	2168	T Bäck, HP Schwefel	An overview of evolutionary algorithms for parameter optimization	1993
15	2105	DA Van Veldhuizen, GB Lamont	Multiobjective evolutionary algorithms: Analyzing the state-of-the-art	2000
16	1848	RC Eberhart, Y Shi	Comparison between genetic algorithms and particle swarm optimization	1998
17	1838	ÁE Eiben, R Hinterding, ...	Parameter control in evolutionary algorithms	1999
18	1711	E Zitzler, L Thiele	Multiobjective optimization using evolutionary algorithms—a comparative ca	1998
19	1685	Z Michalewicz, M Schoenauer	Evolutionary algorithms for constrained parameter optimization problems	1996
20	1623	CAC Coello	Theoretical and numerical constraint-handling techniques used with evolutio	2002

Most popular: Multiobjective optimization problems

ELITISM IN MULTIOBJECTIVE EVOLUTIONARY ALGORITHM

nsga

About 43,500 results (**0.03** sec)

A fast and elitist multiobjective genetic algorithm: **NSGA**-II

K Deb, A Pratap, S Agarwal… - IEEE transactions on …, 2002 - ieeexplore.ieee.org

Abstract: Multi-objective evolutionary algorithms (MOEAs) that use non-dominated sorting and sharing have been criticized mainly for:(1) their O (MN/sup 3/) computational complexity (where M is the number of objectives and N is the population size);(2) their non-elitism

☆ 99 Cited by 22270 Related articles All 45 versions

A fast elitist non-dominated sorting genetic algorithm for multi-objective optimization: **NSGA**-II

K Deb, S Agrawal, A Pratap, T Meyarivan - International Conference on …, 2000 - Springer

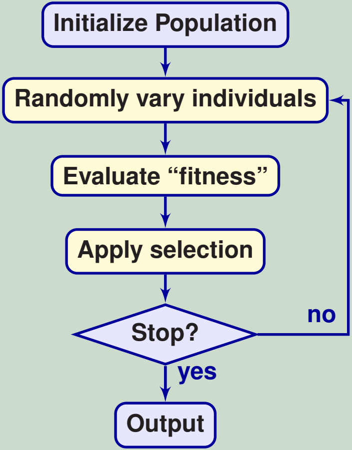
Abstract Multi-objective evolutionary algorithms which use non-dominated sorting and sharing have been mainly criticized for their (i) O (MN 3) computational complexity (where M is the number of objectives and N is the population size),(ii) non-elitism approach, and (iii)

☆ 99 Cited by 3529 Related articles All 22 versions

Our motivation: from maxima (skylines) to elites to EA

COMPONTENTS OF EAs

- Representation
  - ➠ Coding of solutions
- Initialization
- Parent selection
- Evaluation:
  - ➠ Fitness function
- Survivor selection
- Offspring Reproduction
  - ➠ Genetic operators
- Termination condition

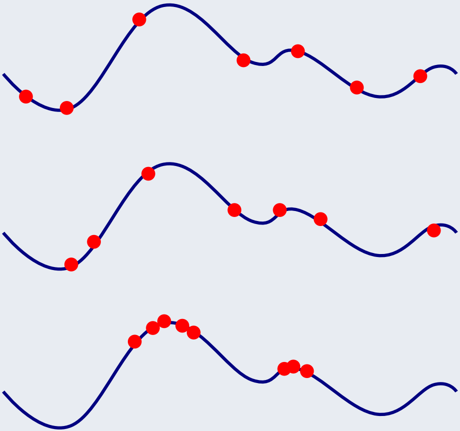


TYPICAL PROGRESS OF AN EA

Initialization

Halfway

Termination



PROS AND CONS OF EAs

Disadvantages

- Large convergence time
- Difficult adjustment of parameters
- Heuristic principle
- No guarantee of global max

Advantages

- Reasonably good solutions quickly
- Suitable for complex search spaces
- Easy to parallelize
- Scalable to higher dimensional problems

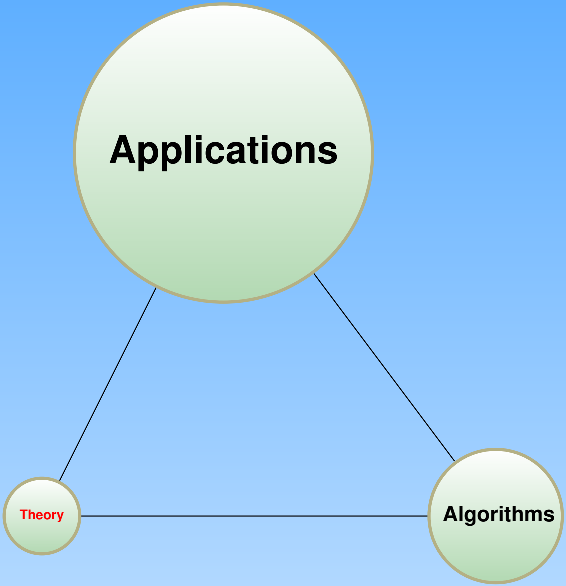
DIFFICULTY OF ANALYSIS OF EA

A typical EA comprises several ingredients

- coding of solution
- population of individuals
- selection for reproduction
- operations for breeding new individuals
- fitness function to evaluate the new individual
- ...

Mathematical description of the dynamics of the algorithms or the asymptotics of the complexity $\Longrightarrow$ ***challenging***

Droste et al. (2002): ***Theory is far behind experimental knowledge . . . rigorous research is hard to find.***



SIMPLEST VERSION: 1 PARENT, 1 CHILD, AND MUTATION ONLY

Algorithm $(1+1)$-EA

1. *Choose an initial string* $\mathbf{x} \in \{0,1\}^n$ *uniformly at random*
2. *Repeat until a terminating condition*
   - *(mutation) Create* $\mathbf{y}$ *by flipping each bit of* x *independently with probability* p
   - *Replace* $\mathbf{x}$ *by* $\mathbf{y}$ *iff* $f(\mathbf{y}) \geqslant f(\mathbf{x})$

f: ***fitness (or objective) function***

ANALYSIS OF $(1+1)$-EA UNDER ONEMAX

Known results for ONEMAX $f(\mathbf{x}) = x_1 + \cdots + x_n$

$X_n :=$ ***# steps used by the*** $(1+1)$***-EA to reach the optimum state*** $f(\mathbf{x}) = n$ ***when the mutation rate is*** $\frac{1}{n}$

- **Bäck (1992): transition probabilities**
- **Mühlenbein (1992):** $\mathbb{E}(X_n) \approx n \log n$
- **Droste et al. (1998, 2002):** $\mathbb{E}(X_n) \asymp n \log n$

• • •

ONEMAX function		Linear functions $\sum w_i x_i$	
Doerr et al. (2010)	**lower bound** $(1-o(1))en\log(n)$	**Jagerskupper (2011)**	**upper bound** $2.02en\log(n)$
Sudholt 2010	**lower bound** $en\log(n) - 2n\log\log(n)$	**Doerr et al. (2010)**	**upper bound** $1.39en\log(n)$
Doerr et al. (2011)	$en\log(n) - \Theta(n)$	**Witt (2013)**	**upper bound** $en\log(n) + O(n)$

Approaches used: Markov chain, martingale, coupon collection, . . .

KNOWN BOUNDS FOR $\mathbb{E}(X_n)$ UNDER ONEMAX

ONEMAX

Droste et al. (2002)	$en(\log n + \gamma)$
Doerr et al. (2011)	$en \log n - 0.1369n$
Our result	$= e(n + \frac{1}{2}) \log n - c_1 n + O(1)$ $c_1 = 1.8254\,17883\ldots$
Lehre & Witt (2014)	$en \log n - 7.81791n - O(\log n)$
Doerr et al. (2011)	$en \log n - \Theta(n)$
Sudholt (2010)	$en \log n - 2n \log \log n$
Doerr et al. (2010)	$(1 - o(1))en \log n$
Droste et al. (2002)	$0.196n \log n$

GARNIER, KALLEL & SCHOENAUER (1999)

"Rigorous hitting times for binary mutations"

Strongest results obtained so far but proof incomplete (probabilistic arguments)

- $\mathbb{E}(X_n) = en \log n + c_1 n + o(n)$, **where** $c_1 \approx -1.9$
- $\frac{X_n}{en} - \log n - c_1 \xrightarrow{d} \log \mathbf{Exp}(1)$ **(double-exponential)**

their results had remained obscure in the EA-literature

OUR RESULTS

$$\mathbb{E}(X_n) = en\log n + c_1 n + \tfrac{1}{2}e\log n + c_2 + O\left(\frac{\log n}{n}\right)$$

$$\begin{aligned} c_1 &= -e\left(\log 2 - \gamma - \phi_1\left(\tfrac{1}{2}\right)\right) \\ &\approx -1.89254\,17883\,44686\,82302\,25714\ldots, \end{aligned}$$

where γ is Euler's constant,

$$\phi_1(z) := \int_0^z \left(\frac{1}{S_1(t)} - \frac{1}{t}\right)\mathrm{d}t,$$

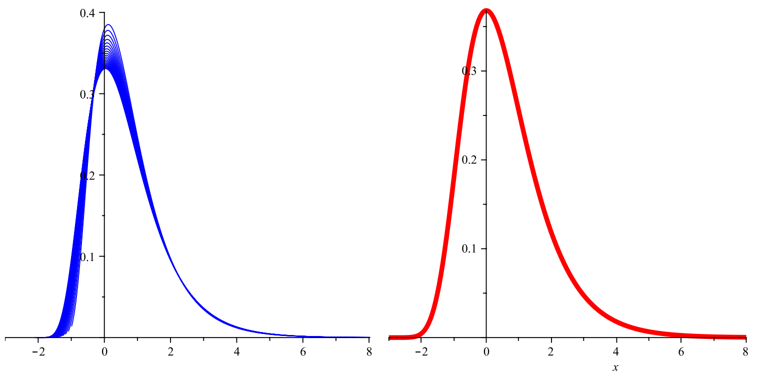
$$S_1(z) := \sum_{\ell\geqslant 1}\frac{z^\ell}{\ell!}\sum_{0\leqslant j<\ell}(\ell-j)\frac{(1-z)^j}{j!}.$$

Indeed $\mathbb{E}(X_n) \sim n\sum_{k\geqslant 0}\frac{c_k'\log n + c_k}{n^k}$

LIMIT GUMBEL DISTRIBUTION

$$\mathbb{P}\left(\frac{X_n}{en} - \log n + \log 2 - \phi_1(\tfrac{1}{2}) \leqslant x\right) \to e^{-e^{-x}}$$

$$\phi_1(\tfrac{1}{2}) \approx -0.58029\,56799\,84283\,81332\,29240\ldots$$



left: $n = 10..30$ **right:** $e^{-x-e^{-x}}$

RECURRENCE

$\Rightarrow$ ASYMPTOTICS

THE RANDOM VARIABLE $X_{n,m}$

$$X_n := \sum_{0\leqslant m\leqslant n} \binom{n}{m} p^m q^{n-m} X_{n,m}$$

$X_{n,m} := \#$ steps used by the $(1+1)$-EA to reach $f(\mathbf{x}) = n$ when starting from $f(\mathbf{x}) = n - m$

Let $Q_{n,m}(t) := \mathbb{E}(t^{X_{n,m}})$**. Then** $Q_{n,0}(t) = 1$ **and**

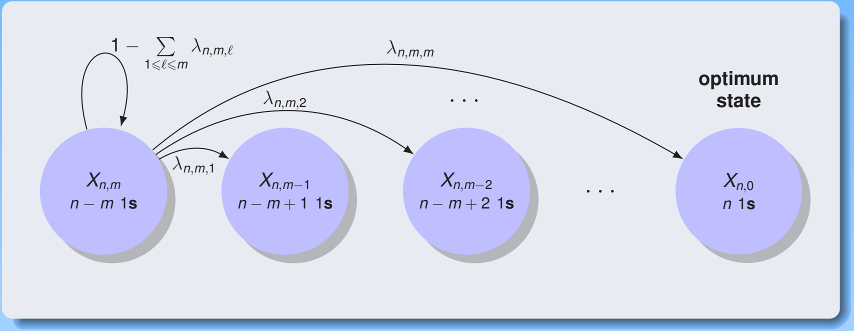
$$Q_{n,m}(t) = \frac{t \displaystyle\sum_{1\leqslant \ell\leqslant m} \lambda_{n,n-m,\ell} Q_{n,m-\ell}(t)}{1 - \left(1 - \displaystyle\sum_{1\leqslant \ell\leqslant m} \lambda_{n,n-m,\ell}\right) t}$$

for $1 \leqslant m \leqslant n$**, where (**$\mathbb{P}(m$ **1's** $\mapsto m + \ell$ **1's))**

$$\lambda_{n,m,\ell} := \left(1 - \frac{1}{n}\right)^n \sum_{0\leqslant j\leqslant \min\{m,n-m-\ell\}} \binom{m}{j}\binom{n-m}{j+\ell}(n-1)^{-\ell-2j}$$

$\lambda_{n,m,\ell} := \mathbb{P}(m \text{ 1's} \mapsto m+\ell \text{ 1's})$

$$\sum_{0\leqslant j\leqslant \min\{m,n-m-\ell\}} \underbrace{\binom{m}{j}\left(\frac{1}{n}\right)^{j}\left(1-\frac{1}{n}\right)^{m-j}}_{1\to 0} \underbrace{\binom{n-m}{j+\ell}\left(\frac{1}{n}\right)^{j+\ell}\left(1-\frac{1}{n}\right)^{n-m-j-\ell}}_{0\to 1}$$



Q: How to solve this recurrence?

$$Q_{n,m}(t) = \frac{t \sum_{1\leqslant \ell \leqslant m} \lambda_{n,n-m,\ell} Q_{n,m-\ell}(t)}{1 - \left(1 - \sum_{1\leqslant \ell \leqslant m} \lambda_{n,n-m,\ell}\right) t}$$

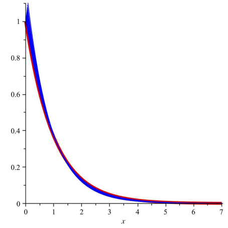
$$m = O(1)$$

SIMPLEST CASE: $m = 1$

$m = 1$ ($n-1$ 1s $\Longrightarrow$ n 1s)

$$\lambda_{n,n-1,1} = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1} \Longrightarrow \textbf{Geometric}\left(\frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}\right)$$

$$Q_{n,1}(t) = \frac{\frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1} t}{1-\left(1-\frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}\right) t}.$$

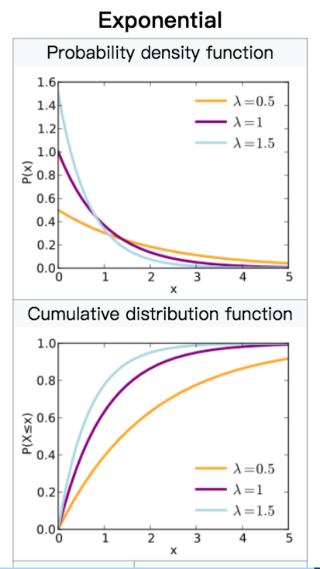


$$\Longrightarrow \quad \frac{X_{n,1}}{en} \xrightarrow{d} \textbf{Exp}(1)$$

$$\mathbb{P}\left(\frac{X_{n,1}}{en} \leqslant x\right) \to 1 - e^{-x}$$

EXPONENTIAL DISTRIBUTION

Parameters	$\lambda > 0$ rate, or inverse scale
Support	$x \in [0, \infty)$
pdf	$\lambda e^{-\lambda x}$
CDF	$1 - e^{-\lambda x}$
Mean	λ^{-1}
Median	$\lambda^{-1} \ln 2$
Mode	0
Variance	λ^{-2}
Skewness	2
Ex. kurtosis	6
Entropy	$1 - \ln(\lambda)$
MGF	$\left(1 - \frac{t}{\lambda}\right)^{-1}$ for $t < \lambda$
CF	$\left(1 - \frac{it}{\lambda}\right)^{-1}$



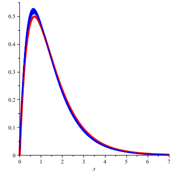
BY INDUCTION $m = O(1)$

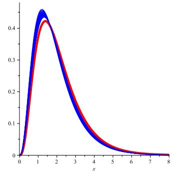
$$\frac{X_{n,m}}{en} \xrightarrow{d} \mathrm{Exp}(1) + \cdots + \mathrm{Exp}(m)$$

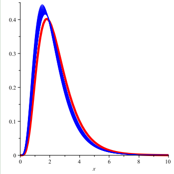
$$\mathbb{P}\left(\frac{X_{n,m}}{en} \leqslant x\right) \to \left(1 - e^{-x}\right)^m$$

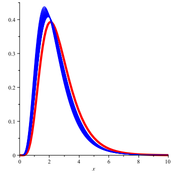
$$\mathbb{E}(X_{n,m}) \sim eH_m n \quad \textbf{and} \quad \mathbb{V}(X_{n,m}) \sim e^2 H_m^{(2)} n^2$$

$m = 2, 4, 6, 8$ & $n = 5, \dots, 50$

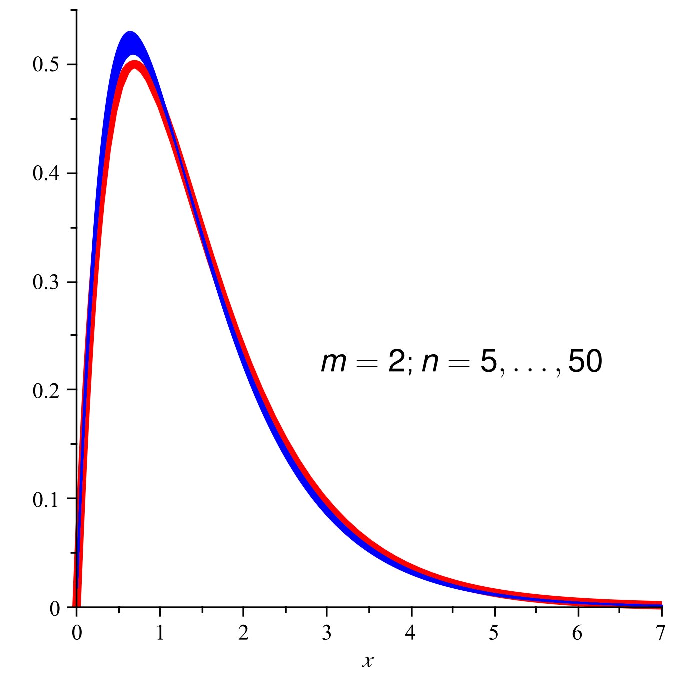








An LLT also holds


0.5
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7
x
m = 2; n = 5, . . . , 50

SKETCH OF PROOF

$$\lambda_{n,n-m,\ell} = \left(1-\frac{1}{n}\right)^n \sum_{0\leqslant j\leqslant \min\{n-m,m-\ell\}} \binom{m}{j+\ell}\binom{n-m}{j}(n-1)^{-\ell-2j}$$
$$= \binom{m}{\ell} e^{-1} n^{-\ell}\left(1+O\left(\frac{m-\ell}{n(\ell+1)}+\frac{\ell}{n}\right)\right)$$

$m = O(1) \Longrightarrow j = 1$ ***is dominant***

SKETCH OF PROOF

$$\lambda_{n,n-m,\ell} = \left(1-\frac{1}{n}\right)^n \sum_{0\leqslant j\leqslant \min\{n-m,m-\ell\}} \binom{m}{j+\ell}\binom{n-m}{j}(n-1)^{-\ell-2j}$$
$$= \binom{m}{\ell} e^{-1} n^{-\ell}\left(1+O\left(\frac{m-\ell}{n(\ell+1)}+\frac{\ell}{n}\right)\right)$$

$m = O(1) \implies j = 1$ ***is dominant***

$$Q_{n,m}(t) = \frac{t \sum_{1\leqslant \ell\leqslant m} \lambda_{n,n-m,\ell} Q_{n,m-\ell}(t)}{1-\left(1-\sum_{1\leqslant \ell\leqslant m}\lambda_{n,n-m,\ell}\right)t}$$

$$\implies Q_{n,m}(t) \sim \frac{\frac{m}{en}t}{1-\left(1-\frac{m}{en}\right)t} Q_{n,m-1}(t)$$

SKETCH OF PROOF: BY INDUCTION

$$Q_{n,m}(t) \sim \prod_{1 \leqslant r \leqslant m} \frac{\frac{r}{en} t}{1 - \left(1 - \frac{r}{en}\right) t}$$

SKETCH OF PROOF: BY INDUCTION

$$Q_{n,m}(t) \sim \prod_{1 \leqslant r \leqslant m} \frac{\frac{r}{en} t}{1 - \left(1 - \frac{r}{en}\right) t}$$

$$\Longrightarrow Q_{n,m}\left(e^{s/(en)}\right) \sim \prod_{1 \leqslant r \leqslant m} \frac{1}{1 - \frac{s}{r}} \Longrightarrow \sum_{1 \leqslant r \leqslant m} \mathbf{Exp}(r)$$

Fails when $m \to \infty$

SKETCH OF PROOF: BY INDUCTION

$$Q_{n,m}(t) \sim \prod_{1\leqslant r\leqslant m} \frac{\frac{r}{en}t}{1-\left(1-\frac{r}{en}\right)t}$$

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Fails when $m \to \infty$

Let $Y_m := \sum_{1\leqslant r\leqslant m} \mathrm{Exp}(r)$. Then as $m \to \infty$

$$\mathbb{E}\left(e^{(Y_m-H_m)i\theta}\right) = \prod_{1\leqslant r\leqslant m} \frac{e^{-\frac{i\theta}{r}}}{1-\frac{i\theta}{r}} \to \prod_{r\geqslant 1} \frac{e^{-\frac{i\theta}{r}}}{1-\frac{i\theta}{r}} = e^{-\gamma i\theta}\Gamma(1-i\theta)$$

SKETCH OF PROOF: BY INDUCTION

$$Q_{n,m}(t) \sim \prod_{1\leqslant r\leqslant m} \frac{\frac{r}{en}t}{1-\left(1-\frac{r}{en}\right)t}$$

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$$\Longrightarrow \mathbb{P}\left(Y_m - \log m \leqslant x\right) \to e^{-e^{-x}} \qquad (x\in\mathbb{R})$$

$$m \to \infty$$

EXPECTED VALUES $\mathbb{E}(X_{n,m})$

$\mu_{n,m} := \mathbb{E}(X_{n,m}) = Q'_{n,m}(1) \quad (\mu_{n,0} = 0)$

$$\mu_{n,m} = \frac{1 + \sum_{1\leqslant \ell \leqslant m} \lambda_{n,n-m,\ell}\mu_{n,m-\ell}}{\sum_{1\leqslant \ell \leqslant m} \lambda_{n,n-m,\ell}\mu_{n,m-\ell}} \qquad (1 \leqslant m \leqslant n)$$

$$\Longrightarrow \mathbb{E}(X_n) = 2^{-n} \sum_{0\leqslant m \leqslant n} \binom{n}{m} \mu_{n,m}$$

Let $e_n := \left(1 - \frac{1}{n+1}\right)^{n+1}$ and $\mu^*_{n,m} := \frac{e_n}{n}\mu_{n+1,m}$

$$\sum_{1\leqslant \ell \leqslant m} \lambda^*_{n,m,\ell}\left(\mu^*_{n,m} - \mu^*_{n,m-\ell}\right) = \frac{1}{n}$$

$$\lambda^*_{n,m,\ell} := \frac{\lambda_{n+1,n+1-m,\ell}}{e_n} = \sum_{0\leqslant j \leqslant \min\{n+1-m,m-\ell\}} \binom{n+1-m}{j}\binom{m}{j+\ell} n^{-\ell-2j}$$

$\mu^*_{n,1} = 1$ & $\sum_{1\leqslant\ell\leqslant m} \lambda^*_{n,m,\ell}\left(\mu^*_{n,m} - \mu^*_{n,m-\ell}\right) = \frac{1}{n}$

$$\mu^*_{n,2} = \frac{3\,n^2+n-1}{2\,n^2+2\,n-1}$$

$$\mu^*_{n,3} = \frac{22\,n^6+40\,n^5-19\,n^4-42\,n^3+14\,n^2+15\,n-6}{\left(2\,n^2+2\,n-1\right)\left(6\,n^4+12\,n^3-7\,n^2-9\,n+6\right)}$$

$\mu^*_{n,1} = 1$ & $\sum_{1\leqslant\ell\leqslant m} \lambda^*_{n,m,\ell}\left(\mu^*_{n,m} - \mu^*_{n,m-\ell}\right) = \frac{1}{n}$

$$\mu^*_{n,2} = \frac{3n^2+n-1}{2n^2+2n-1}$$

$$\mu^*_{n,3} = \frac{22n^6+40n^5-19n^4-42n^3+14n^2+15n-6}{\left(2n^2+2n-1\right)\left(6n^4+12n^3-7n^2-9n+6\right)}$$

$$\mu^*_{n,4} = \frac{\begin{pmatrix} 600n^{12}+2616n^{11}+1128n^{10}-7460n^9 \\ -4958n^8+11506n^7+6167n^6-10887n^5 \\ -2862n^4+5917n^3-153n^2-1398n+360 \end{pmatrix}}{\begin{array}{l}(2n^2+2n-1)(6n^4+12n^3-7n^2-9n+6) \\ \times(24n^6+72n^5-48n^4-140n^3+93n^2+83n-60)\end{array}}$$

$$\mu^*_{n,5} = \frac{\begin{pmatrix} 78912n^{20}+626112n^{19}+1150848n^{18}-2455104n^{17} \\ -8313432n^{16}+4491096n^{15}+27182504n^{14}-5263508n^{13} \\ -55021022n^{12}+7628986n^{11}+74466297n^{10}-15193087n^9 \\ -67391443n^8+21902962n^7+38443857n^6-18491957n^5 \\ -11698973n^4+8358804n^3+827844n^2-1576800n+302400 \end{pmatrix}}{\begin{array}{l}(2n^2+2n-1)(6n^4+12n^3-7n^2-9n+6) \\ \times(24n^6+72n^5-48n^4-140n^3+93n^2+83n-60) \\ \times(120n^8+480n^7-360n^6-1720n^5+1145n^4 \\ \quad +2394n^3-1685n^2-1118n+840)\end{array}}$$

ASYMPTOTICS OF $\mu^*_{n,m}$

$$\sum_{1\leqslant\ell\leqslant m} \lambda^*_{n,m,\ell}\left(\mu^*_{n,m} - \mu^*_{n,m-\ell}\right) = \frac{1}{n}$$

$$\mu^*_{n,1} = 1$$

$$\mu^*_{n,2} = \frac{3}{2} - n^{-1} + \frac{5}{4}n^{-2} - \frac{7}{4}n^{-3} + \frac{19}{8}n^{-4} - \frac{13}{4}n^{-5} + \cdots$$

$$\mu^*_{n,3} = \frac{11}{6} - \frac{13}{6}n^{-1} + \frac{155}{36}n^{-2} - \frac{323}{36}n^{-3} + \frac{4007}{216}n^{-4} + \cdots$$

$$\mu^*_{n,4} = \frac{25}{12} - \frac{41}{12}n^{-1} + \frac{329}{36}n^{-2} - \frac{917}{36}n^{-3} + \frac{61841}{864}n^{-4} + \cdots$$

$$\mu^*_{n,5} = \frac{137}{60} - \frac{283}{60}n^{-1} + \frac{2839}{180}n^{-2} - \frac{19859}{360}n^{-3} + \frac{848761}{4320}n^{-4} + \cdots$$

$$\mu^*_{n,6} = \frac{49}{20} - \frac{121}{20}n^{-1} + \frac{1453}{60}n^{-2} - \frac{36709}{360}n^{-3} + \frac{70451}{160}n^{-4} + \cdots$$

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$$\left\{H_m = \sum_{1\leqslant j\leqslant m}\frac{1}{j}\right\} = \left\{1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \ldots\right\}$$

HEURISTICS

An Ansatz approximation: $\mu^*_{n,m} \approx \sum_{k\geqslant 0} \frac{d_k(m)}{n^k}$

$$d_0(m) = H_m \ (m \geqslant 0)$$

$$d_1(m) = H_m + \tfrac{1}{2} - \tfrac{3}{2}\, m \ (m \geqslant 1)$$

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$$d_3(m) = \tfrac{1}{2}\, H_m + \tfrac{7}{24} - \tfrac{575}{432}\, m + \tfrac{23}{18}\, m^2 - \tfrac{283}{432}\, m^3 \ (m \geqslant 2)$$

$$d_4(m) = \tfrac{5}{18}\, H_m - \tfrac{59}{720} - \tfrac{3439}{3456}\, m + \tfrac{15101}{11520}\, m^2 - \tfrac{19951}{17280}\, m^3 + \tfrac{5759}{11520}\, m^4 \ (m \geqslant 4)$$

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Complication: $d_k(m)$ ***holds for*** $m \geqslant 2\left\lfloor \frac{k}{2} \right\rfloor$

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General pattern: $\mu^*_{n,m} \approx \sum_{k\geqslant 0} n^{-k}\left(b_k H_m + \sum_{0\leqslant j\leqslant k} \varpi_{k,j} m^j\right)$

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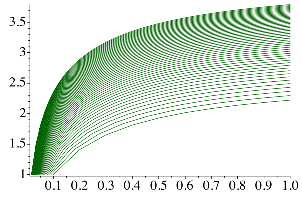
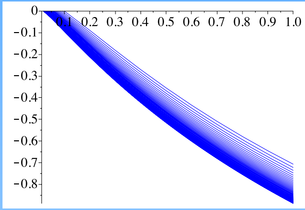
$$\alpha := \frac{m}{n} \Longrightarrow \boxed{\mu^*_{n,m} \approx H_m + \phi(\alpha) \quad \textbf{for } 1 \leqslant m \leqslant n}$$

A MORE GENERAL ANSATZ

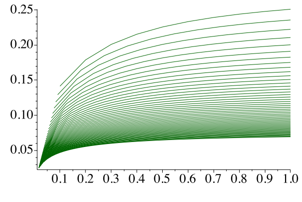
$$\mu^*_{n,m} \approx H_m + \phi_1(\alpha) + \frac{b_1 H_m + \phi_2(\alpha)}{n} + \frac{b_2 H_m + \phi_3(\alpha)}{n^2} + \cdots$$



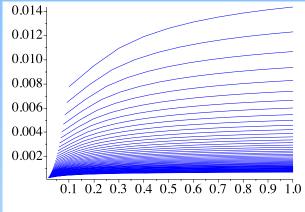
$\mu^*_{n,m}$



$\mu^*_{n,m} - H_m$



$\mu^*_{n,m} - (H_m + \phi_1(\alpha))$



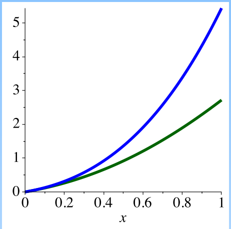
$\mu^*_{n,m} - \left(H_m + \phi_1(\alpha) + \frac{H_m + \phi_2(\alpha)}{n}\right)$

$$S_r(z) := \sum_{\ell \geqslant 1} \frac{z^\ell}{\ell!} \sum_{0 \leqslant j < \ell} (\ell - j)^r \frac{(1-z)^j}{j!}$$

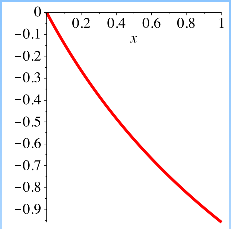
$$\phi_1(z) := \int_0^z \left(\frac{1}{S_1(t)} - \frac{1}{t} \right) \mathrm{d}t$$

$$\phi_2(z) = \frac{1}{2} - \int_0^z \left(\frac{S_2(t)S_1'(t)}{2S_1(t)^3} - \frac{S_0(t)}{S_1(t)^2} - \frac{1}{2S_1(t)} - \frac{1}{2t^2} + \frac{1}{t} \right) \mathrm{d}t$$

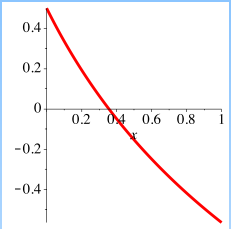
(analytic in $|z| \leqslant 1$)



$S_1(x)$ & $S_2(x)$



$\phi_1(x)$



$\phi_2(x)$

A GENERATING FUNCTION APPROACH?

$$\lambda^*_{n,m,\ell} = \sum_{0\leqslant j\leqslant \min\{n+1-m,m-\ell\}} \binom{n+1-m}{j}\binom{m}{j+\ell} n^{-\ell-2j}$$

$$f_n(z) := \sum_{m\geqslant 1} \mu^*_{n,m} z^m$$

$$\sum_{0\leqslant \ell<m} \lambda^*_{n,m,m-\ell}\left(\mu^*_{n,m} - \mu^*_{n,\ell}\right) = \frac{1}{n}$$

$$\implies \frac{1}{2\pi i}\oint \left(\frac{1}{1-t} - \frac{z\left(t+\frac{1}{n}\right)}{t\left(1+\frac{t}{n} - z\left(t+\frac{1}{n}\right)\right)}\right) \times \left(1+\frac{t}{n}\right)^{n+1} f_n\left(\frac{z\left(t+\frac{1}{n}\right)}{t\left(1+\frac{t}{n}\right)}\right) \mathrm{d}t = \frac{z}{n(1-z)}$$

A HEURISTIC

$$\frac{1}{2\pi i}\oint f_n(w)\Phi_n(z,w)\,\mathrm{d}w = \frac{z}{n(1-z)}$$

$$\Phi_n(z,w) := \left(1+\frac{\tau}{n}\right)^{n+1}\left(\frac{1}{1-\tau} - \frac{z\left(\tau+\frac{1}{n}\right)}{\tau\left(1+\frac{\tau}{n}-z\left(\tau+\frac{1}{n}\right)\right)}\right)\frac{\mathrm{d}\tau}{\mathrm{d}w}$$

$$\sim \frac{z(w-1)}{n(w-z)^2} + \cdots$$

Assume $f_n(w) \sim \phi(w)$.

$$\frac{z}{2\pi i n}\oint \frac{\phi(w)(w-1)}{(w-z)^2}\,\mathrm{d}w = \frac{z}{n}\left(\phi(z) - (1-z)\phi'(z)\right) = \frac{z}{n}\frac{1}{1-z} = \textbf{RHS}$$

Then ($\phi(0) = 0$)

$$\phi(z) - (1-z)\phi'(z) = \frac{1}{1-z} \implies \phi(z) = \frac{1}{1-z}\log\frac{1}{1-z}.$$

$$\implies \mu^*_{n,m} \sim H_m.$$

HOW TO GUESS $\phi_1(\alpha)$?

Assume $\mu^*_{n,m} \sim H_m + \phi(\alpha)$ $(\alpha := \frac{m}{n})$

$$H_m - H_{m-\ell} = \frac{\ell}{m} + \frac{\ell(\ell-1)}{2m^2} + \cdots$$

$$\phi\left(\frac{m}{n}\right) - \phi\left(\frac{m-\ell}{n}\right) = \phi'(\alpha)\frac{\ell}{m} + O\left(\frac{\ell^2}{m^2}\right)$$

Matched asymptotics

$$\begin{aligned}
\frac{1}{n} &= \sum_{1 \leqslant \ell \leqslant m} \lambda^*_{n,m,\ell}\left(\mu^*_{n,m} - \mu^*_{n,m-\ell}\right)\\
&\sim \sum_{1 \leqslant \ell \leqslant m} \lambda^*_{n,m,\ell}\left(\frac{\ell}{m} + \phi'(\alpha)\frac{\ell}{n}\right)\\
&\sim \frac{1}{n}\left(\frac{1}{\alpha} + \phi'(\alpha)\right) \sum_{1 \leqslant \ell \leqslant m} \ell\lambda^*_{n,m,\ell},
\end{aligned}$$

$$\frac{1}{n} \sim \frac{1}{n}\left(\frac{1}{\alpha} + \phi'(\alpha)\right) \sum_{1\leqslant \ell \leqslant m} \ell \lambda^*_{n,m,\ell}$$

$$\sum_{1\leqslant \ell \leqslant m} \ell \lambda^*_{n,m,\ell} = \sum_{j\geqslant 1} \binom{n+1-m}{j} n^{-j} \sum_{j<\ell\leqslant m} \ell \binom{m}{\ell} n^{-\ell}$$

$$\sim \sum_{j\geqslant 1} \frac{(1-\alpha)^j}{j!} \sum_{\ell > j} \ell \frac{\alpha^\ell}{\ell!}$$

$$= S_1(\alpha)$$

Then we see that ϕ must satisfy

$$\phi'(x) = \frac{1}{S_1(x)} - \frac{1}{x} = -\frac{3}{2} + \frac{11}{6}x - \cdots \Longrightarrow \boxed{\boxed{\phi = \phi_1}}$$

The justification relies on a careful error analysis

TOOLS NEEDED

Lemma 1. Asymptotics of $A^*_{n,m} := \sum_{1\leqslant\ell\leqslant m} a_\ell \lambda^*_{n,m,\ell}$

Assume that $A(z) = \sum_{\ell\geqslant 1} a_\ell z^{\ell-1}$ has a nonzero radius of convergence in the z-plane. Then

$$A^*_{n,m} = \tilde{A}_0(\alpha) - \frac{\tilde{A}_1(\alpha)}{2n} + O\left(n^{-2}\right),$$

where

$$\tilde{A}_0(\alpha) := \sum_{\ell\geqslant 1} \frac{\alpha^\ell}{\ell!} \sum_{0\leqslant j<\ell} a_{\ell-j} \frac{(1-\alpha)^j}{j!}$$

$$\tilde{A}_1(\alpha) := \sum_{\ell\geqslant 1} \frac{\alpha^\ell}{\ell!} \sum_{0\leqslant j<\ell} a_{\ell-j} \left(\alpha\frac{(1-\alpha)^{j+2}}{(j+2)!} - 2\frac{(1-\alpha)^{j-1}}{(j-1)!} + (1-\alpha)\frac{(1-\alpha)^{j-2}}{(j-2)!}\right)$$

$$A^*_{n,m} = \frac{1}{2\pi i}\oint_{|z|=c} A(z)\left(1+\frac{1}{nz}\right)^m\left(1+\frac{z}{n}\right)^{n+1-m} \mathrm{d}z$$

Lemma 2. (Asymptotic tranfer)

$$\sum_{1\leqslant\ell\leqslant m} \lambda^*_{n,m,\ell}(a_{n,m} - a_{n,m-\ell}) = b_{n,m}$$

If $|b_{n,m}| \leqslant c/n$**, uniformly for** $1 \leqslant m \leqslant n$ **and** $n \geqslant 1$**, where** $c > 0$**, then**

$$|a_{n,m}| \leqslant cH_m \qquad (1 \leqslant m \leqslant n).$$

In particular, $\mu^*_{n,m} \leqslant H_m$

$$\Lambda^*_{n,m} := \textstyle\sum_{1\leqslant\ell\leqslant m} \lambda^*_{n,m,\ell} \geqslant \frac{m}{n} \quad (1 \leqslant m \leqslant n)$$

$$|a_{n,m}| \leqslant \frac{|b_{n,m}|}{\Lambda^*_{n,m}} + |a_{n,m-1}| \leqslant \frac{c}{n} \cdot \frac{n}{m} + cH_{m-1} = cH_m,$$

Useful for error analysis

TOOLS NEEDED

Lemma 3

If $\phi \in C^2[0,1]$ **and** $\phi'(x) \neq 0$ **for** $x \in [0,1]$**, then**

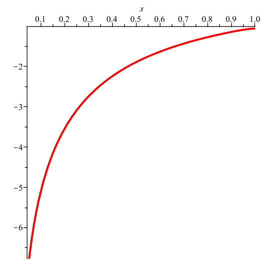
$$\sum_{1\leqslant \ell \leqslant m} \lambda^*_{n,m,\ell}\left(\phi\left(\frac{m}{n}\right) - \phi\left(\frac{m-\ell}{n}\right)\right)$$
$$= \frac{\phi'(\alpha)}{n} \sum_{1\leqslant \ell \leqslant m} \ell\lambda^*_{n,m,\ell} + O\left(n^{-2}\right)$$
$$= \frac{\phi'(\alpha)\, S_1(\alpha)}{n} + O\left(n^{-2}\right)$$

uniformly for $1 \leqslant m \leqslant n$**.**

Bootstrapping & induction $\Longrightarrow \mu^*_{n,m} \sim \sum_{k\geqslant 0} \frac{b_k H_{m+\phi_{k+1}(\alpha)}}{n^k}$

INITIAL BITS ARE BERNOULLI(p): $\sum_m \binom{n}{m} p^m q^{n-m} \mu^*_{n,m}$

$$\frac{\mathbb{E}(X_n)}{en} = \log qn + \gamma + \phi_1(q) + \frac{1}{2n}\Big(\log qn + \gamma + 3 - \phi_1(q) + 2q\phi'(q) + pq\phi_1''(q) + 2\phi_2(q)\Big) + O\left(n^{-2}\log n\right)$$



$p = \frac{1}{2}$

$$\frac{\mathbb{E}(X_n)}{en} = \log n - \log 2 + \gamma + \phi_1(\tfrac{1}{2}) + \frac{1}{2n}\Big(\log n - \log 2 + \gamma + 3 - \phi_1(\tfrac{1}{2}) + \phi'(\tfrac{1}{2}) + \tfrac{1}{4}\phi_1''(\tfrac{1}{2}) + 2\phi_2(\tfrac{1}{2})\Big) + \cdots$$

VARIANCE OF $X_{n,m}$

Uniformly for $1 \leqslant m \leqslant n$

$$\mathbb{V}(X_{n,m}) = e^2 H_m^{(2)} n^2 - e(2e+1)\left(n+\tfrac{1}{2}\right) H_m + \psi_1(\alpha) n + \psi_2(\alpha) + O\left(n^{-1} H_m\right)$$

The two dominating terms independent of p

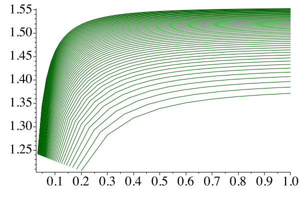
$$\mathbb{V}(X_n) = \frac{e^2\pi^2}{6} n^2 - e(2e+1)\left(n+\tfrac{1}{2}\right)\log n + c_1' n + c_2' + O\left(n^{-1}\log n\right)$$

$$\psi_1(\alpha) = \int_0^\alpha \left(\frac{S_2(x)}{S_1(x)^3} - \frac{1}{x^2} + \frac{2}{x}\right) \mathrm{d}x$$

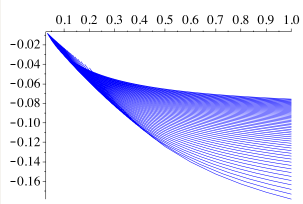
$$\psi_2(\alpha) = \frac{7}{12} - \int_0^\alpha \left(\frac{5S_1'(x)S_2(x)^2}{2S_1(x)^5} - \frac{2S_1'(x)S_3(x)+S_2(x)S_2'(x)+6S_0(x)S_2(x)}{2S_1(x)^4} - \frac{S_0(x)}{S_1(x)^3} + \frac{2}{S_1(x)^2} - \frac{1}{x^3} + \frac{3}{x^2} - \frac{11}{2x}\right) \mathrm{d}x$$

$$V_{n,m}^* := \frac{\left(1-\frac{1}{n+1}\right)^{n+1}}{n^2}\left(\mathbb{V}(X_{n+1,m}) + \mathbb{E}(X_{n+1,m})\right)$$

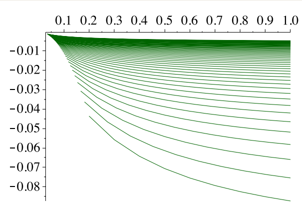
$$= H_m^{(2)} + \sum_{1\leqslant k<K} \frac{r_k H_m + s_k H_m^{(2)} + t_k}{n^k} + O\left(H_m n^{-K}\right)$$



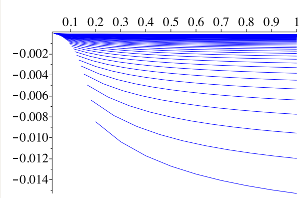
$V_{n,m}^*$



$V_{n,m}^* - H_m^{(2)}$



$K = 2$



$K = 3$

LIMIT GUMBEL DISTRIBUTION OF $X_{n,m}$

$$\mathbb{P}\left(\sum_{1\leqslant r\leqslant m} \mathbf{Exp}(r) - \log m \leqslant x\right) \to e^{-e^{-x}}$$

If $m \to \infty$ with n and $m \leqslant n$, then

$$\mathbb{P}\left(\frac{X_{n,m}}{en} - \log m - \phi_1\left(\tfrac{m}{n}\right) \leqslant x\right) \to e^{-e^{-x}}$$

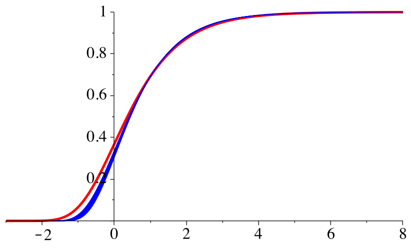
By induction

$$\mathbb{E}\left(e^{X_{n,m}s/(en)-(H_m+\phi_1(\frac{m}{n}))s}\right) = \left(1 + O\left(\frac{H_m}{n}\right)\right) \prod_{1\leqslant r\leqslant m} \frac{e^{-s/r}}{1-\frac{s}{r}},$$

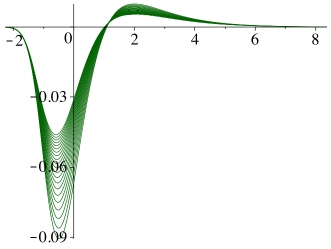
uniformly for $1 \leqslant m \leqslant n$ (proof long and messy).

$$X_n = \sum_{0\leqslant m\leqslant n} \binom{n}{m} p^m q^{n-m} X_{n,m}$$

$$\mathbb{P}\left(\frac{X_n}{en} - \log pn - \phi_1(\rho) \leqslant x\right) \to e^{-e^{-x}} \qquad (x \in \mathbb{R})$$



$\mathbb{P}\left(\frac{X_n}{en} - \log \frac{n}{2} - \phi_1(\frac{1}{2}) \leqslant x\right)$



convergence rates

MAIN STEPS

$$F_{n,m}(s) := \frac{\mathbb{E}\left(e^{X_{n,m}s/(en)}\right) e^{-\phi\left(\frac{m}{n}\right)s}}{\prod_{1\leqslant r\leqslant m} \frac{1}{1-\frac{s}{r}}}$$

$$= \frac{Q_{n,m}\left(e^{s/(en)}\right) e^{-H_m s-\phi\left(\frac{m}{n}\right)s}}{\prod_{1\leqslant r\leqslant m} \frac{e^{-s/r}}{1-\frac{s}{r}}}.$$

$$F_{n,m}(s) = \frac{\sum_{1\leqslant \ell\leqslant m} \lambda_{n,n-m,\ell} F_{n,m-\ell}(s) e^{-\left(\phi\left(\frac{m}{n}\right)-\phi\left(\frac{m-\ell}{n}\right)\right)s} \prod_{m-\ell+1\leqslant r\leqslant m} \left(1-\frac{s}{r}\right)}{e^{-s/(en)} - \left(1 - \sum_{1\leqslant \ell\leqslant m} \lambda_{n,n-m,\ell}\right)}$$

Prove $|F_{n,m}(s) - 1| \leqslant C n^{-1} H_m$ **when** $\phi = \phi_1$

AN AUXILIARY FUNCTION

$$G_{n,m}(s) := \frac{\sum_{1\leqslant \ell \leqslant m} \lambda_{n,n-m,\ell}\, e^{-\left(\phi\left(\frac{m}{n}\right)-\phi\left(\frac{m-\ell}{n}\right)\right)s} \prod_{m-\ell+1\leqslant r\leqslant m}\left(1-\frac{s}{r}\right)}{e^{-s/(en)} - \left(1-\sum_{1\leqslant \ell\leqslant m}\lambda_{n,n-m,\ell}\right)e^{s/(en)}}$$

If $\phi \in C^2[0,1]$, then

$$G_{n,m}(s) = \frac{1-\frac{s}{m}\left(1+\alpha\phi'(\alpha)\right)\frac{S_1(\alpha)}{S(\alpha)} + O\left(\frac{1}{mn}\right)}{1-\frac{s}{m}\cdot\frac{\alpha}{S(\alpha)} + O\left(\frac{1}{mn}\right)},$$

If $\phi = \phi_1$ then $G_{n,m}(s) = 1 + O((mn)^{-1})$

Recurrence

$\Downarrow$

Ansatz

$\Downarrow$

Error analysis

(1 + 1)-EA FOR LEADINGONES

$$f(\mathbf{x}) = \sum_{1 \leqslant k \leqslant n} \prod_{1 \leqslant j \leqslant k} x_j; \; Y_n := \text{optimization time}$$

- **Rudolph (1997): introduced LEADINGONES**
- **Droste et al. (2002):** $\mathbb{E}(Y_n) \asymp n^2$
- **Ladret (2005): CLT**$(c_1 n^2, c_2 n^3)$
- **Böttcher et al. (2010): re-derived mean**
- **many other papers**

Prove Ladret's results by a direct analytic approach

TIME TO OPTIMUM STATE UNDER LEADINGONES

Y_n (starting with n random bits (each being 1 with probability $\frac{1}{2}$)

$$\mathbb{E}\left(e^{Y_n s}\right) := 2^{-n} + \sum_{1 \leqslant m \leqslant n} 2^{m-n-1} Q_{n,m}(s)$$

where the conditional moment generating function $Q_{n,m}(s)$ satisfies the recurrence relation

$$\left(1 - (1 - pq^{n-m})e^s\right) Q_{n,m}(s) = pq^{n-m} e^s \left(2^{1-m} + \sum_{1 \leqslant \ell < m} \frac{Q_{n,\ell}(s)}{2^{m-\ell}}\right),$$

for $1 \leqslant m \leqslant n$, where $q = 1 - p$.

$$p \asymp n^{-1}$$

CLOSED-FORM SOLUTION FOR $Q_{n,m}$

$$Q_{n,m}(s) = \frac{1}{1 - \frac{1-e^{-s}}{pq^{n-m}}} \prod_{1\leqslant j<m} \frac{1 - \frac{1-e^{-s}}{2pq^{n-j}}}{1 - \frac{1-e^{-s}}{pq^{n-j}}}$$

$$Y_{n,m} \stackrel{d}{=} \underbrace{Z_{n,m}^{[0]}}_{\textbf{geom}} + \underbrace{Z_{n,m}^{[m-1]} + \cdots + Z_{n,m}^{[m-1]}}_{\frac{1}{2}+\frac{1}{2}\textbf{geom}}$$

$$R_m(t) := \mathbb{E}\left(t^{Z_{n,m}^{[0]}}\right) = \frac{pq^{n-m}t}{1-(1-pq^{n-m})t}$$

$$\mathbb{E}\left(t^{Z_{n,m}^{[j]}}\right) = \frac{1}{2}\cdot\frac{1-(1-2pq^{n-j})t}{1-(1-pq^{n-j})t} = \frac{1}{2} + \frac{R_j(t)}{2}$$

$$(j = 1, \ldots, m-1)$$

$$\frac{Y_n - \mathbb{E}(Y_n)}{\sqrt{\mathbb{V}(Y_n)}} \longrightarrow \mathscr{N}(0,1)$$

$$\mathbb{E}(Y_{n,m}) = \frac{1}{pq^{n-1}}\left(\frac{1-q^{m-1}}{2p} + q^{m-1}\right)$$
$$\overset{p=\frac{c}{n}}{=} \frac{n^2}{2c^2}\left(e^c - e^{c(1-\alpha)} + O\left(\frac{c(c+1)}{n}\right)\right).$$

$$\mathbb{E}(Y_n) = \sum_{1\leqslant m\leqslant n} 2^{-n+m-1}\mathbb{E}(Y_{n,m}) = \frac{q}{2p^2}\left(q^{-n}-1\right)$$
$$= \frac{e^c-1}{2c^2}n^2 + \frac{(c-2)e^c+2}{4c}n + \frac{ce^c(3c-4)}{48} + \cdots$$

$$\mathbb{V}(Y_n) = \frac{3q^2}{4p^3(1+q)}\left(q^{-2n}-1\right) - \mu_n$$
$$\overset{p=\frac{c}{n}}{=} \frac{e^{2c}-1}{8c^3}n^3 + \frac{3e^{2c}(2c-3)-8e^c+17}{16c^2}n^2$$
$$+ \frac{(6c^2-10c+3)e^{2c}-8(c-2)e^c-19}{32c}n + O(1)$$

Properties	OneMax (X_n)	LeadingOnes (Y_n)
Mean $\sim$	$en\log n + c_1 n$	$\frac{e-1}{2} n^2$
Variance $\sim$	$\frac{\pi^2}{6}(en)^2 - (2e+1)en\log n$	$\frac{e^2-1}{8} n^3$
Limit law	**Gumbel distribution** $\mathbb{P}\left(\frac{X_n}{en} - \log\frac{n}{2} - \phi_1(\tfrac{1}{2}) \leqslant x\right) \to e^{-e^{-x}}$	**Gaussian distribution** $\mathbb{P}\left(\frac{Y_n - \frac{e-1}{2} n^2}{\sqrt{\frac{e^2-1}{8} n^3}} \leqslant x\right) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,\mathrm{d}t$
Approach	Ansatz & error analysis	Analytic combinatorics

THANK YOU

1+1=2

$$(1+1)\text{-}\boldsymbol{EA} = 232$$