Probabilistic cellular automata with memory two

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ALEA in Europe Workshop

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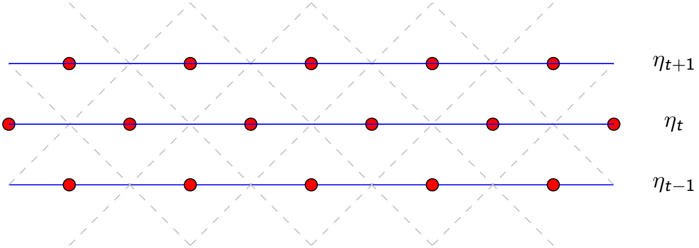
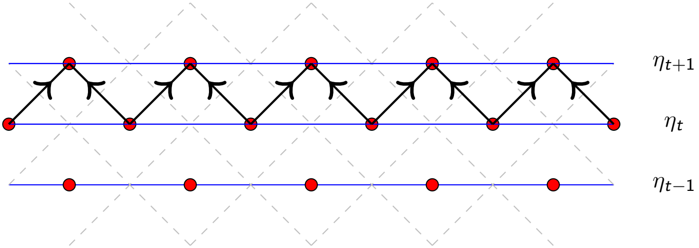
Content

1. Introductory example (the 8-vertex model)



Finite symbol set: S



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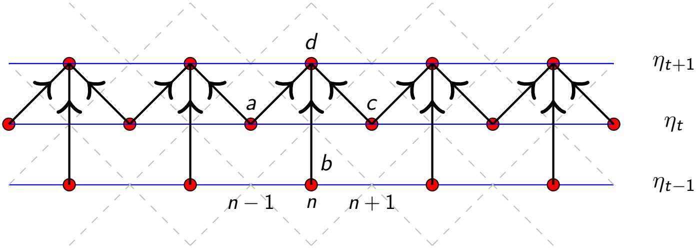

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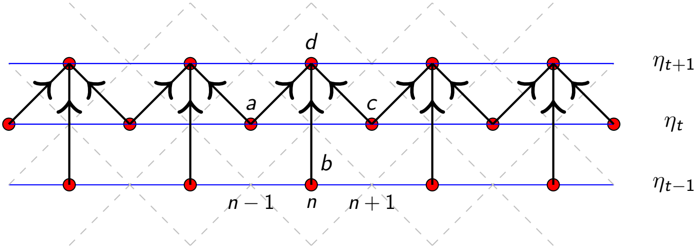

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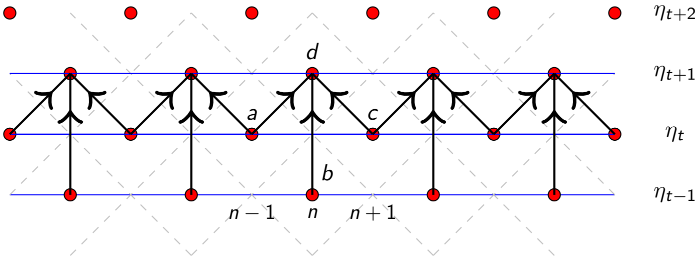
For any $a, b, c \in S$, $T(a, b, c; \cdot)$ is a probability distribution on S.

- The value $\eta_{t+1}(n)$ is equal to d with probability $T(a, b, c; d)$.
- Conditionnally to η_t and η_{t-1}, the values $(\eta_{t+1}(n))_{n \in \mathbb{Z}_{t+1}}$ are independent.



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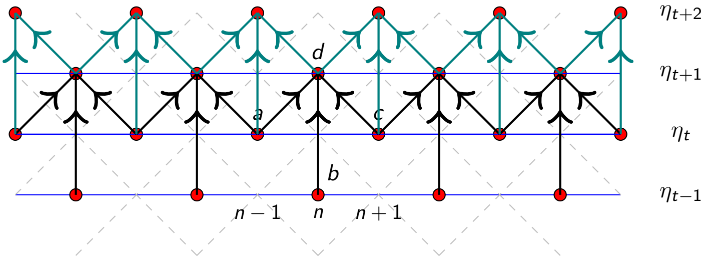

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Example: the 8-vertex PCA

The 8-vertex PCA of parameters $p, r \in (0,1)$:

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\begin{aligned}
T(0,0,1;\cdot) &= T(1,0,0;\cdot) = \mathcal{B}(p),\\
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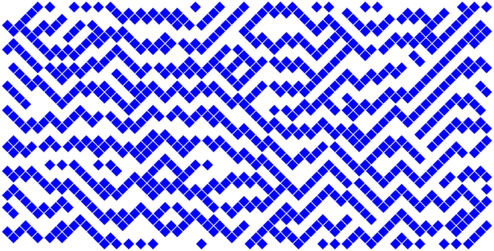

$r = 0.2$ and $p = 0.9$

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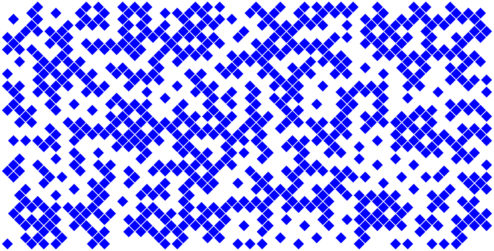
As a special case, for $p = r$, we have:

$$T(a,b,c;\cdot) = p\,\delta_{a+b+c \mod 2} + (1-p)\,\delta_{a+b+c+1 \mod 2}.$$



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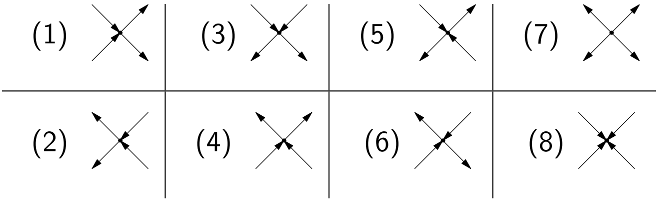

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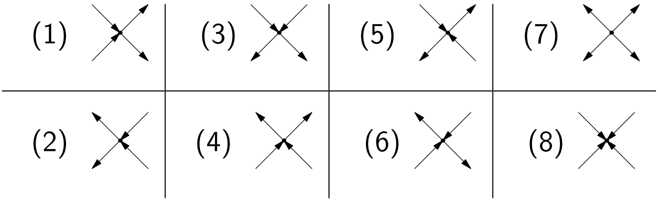
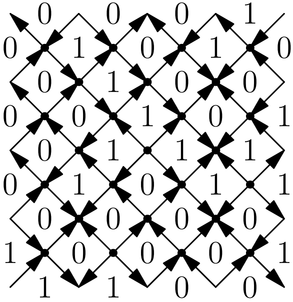
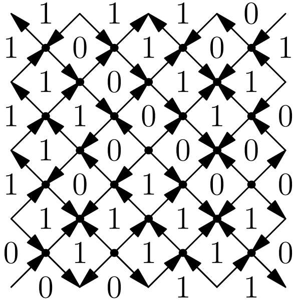


Arrow pointing up iff same colour.





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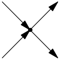
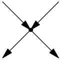
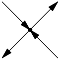
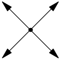
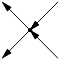
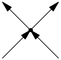
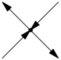
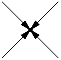
(1) 	(3) 	(5) 	(7) 
(2) 	(4) 	(6) 	(8) 

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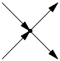
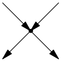
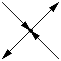
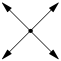
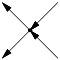
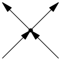
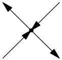
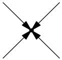
(1) 	(3) 	(5) 	(7) 
(2) 	(4) 	(6) 	(8) 

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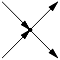
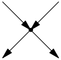
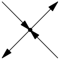
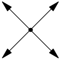
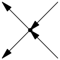
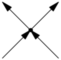
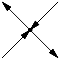
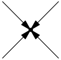
(1) 	(3) 	(5) 	(7) 
(2) 	(4) 	(6) 	(8) 

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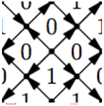
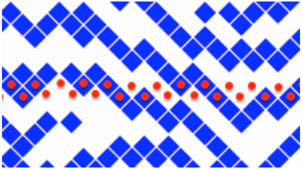


Example: the 8-vertex PCA

The uniform *Horizontal Zig-zag Product Measure* (HZPM) is invariant.



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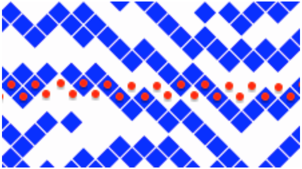

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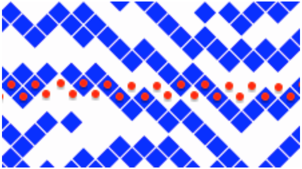

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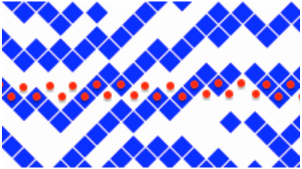

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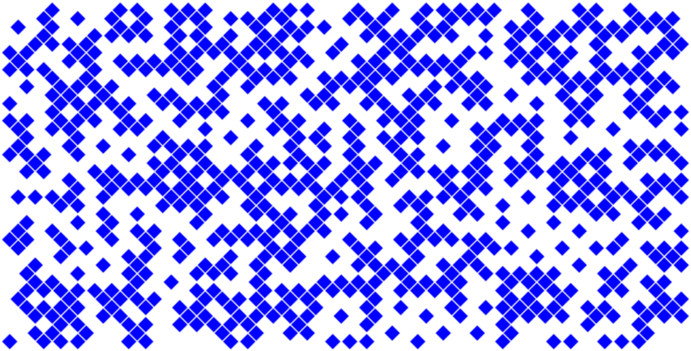


$p = r = 0.2$

Multi-directional reversibility

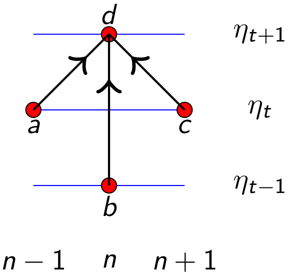
2. Invariant product measure and ergodicity

Condition for having an invariant HZPM

Theorem

Let A be a PCA with transition kernel T and let p be a probability vector on S. The HZPM π_p is invariant for A if and only if

$$\forall a, c, d \in S, \quad p(d) = \sum_{b \in S} p(b) T(a, b, c; d).$$



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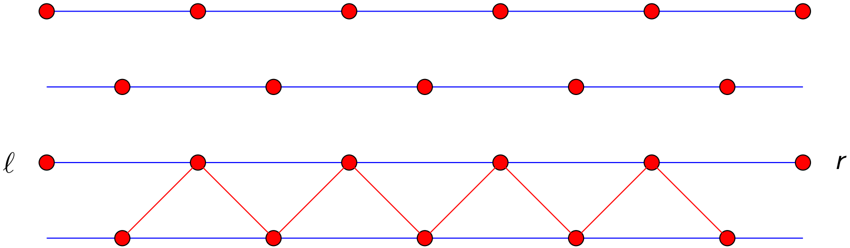

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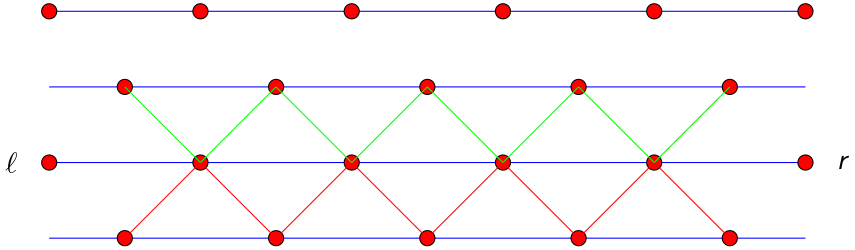

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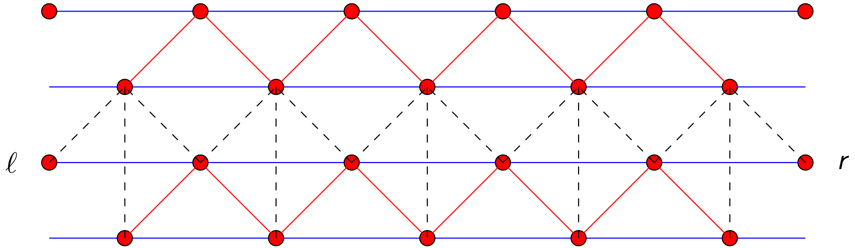

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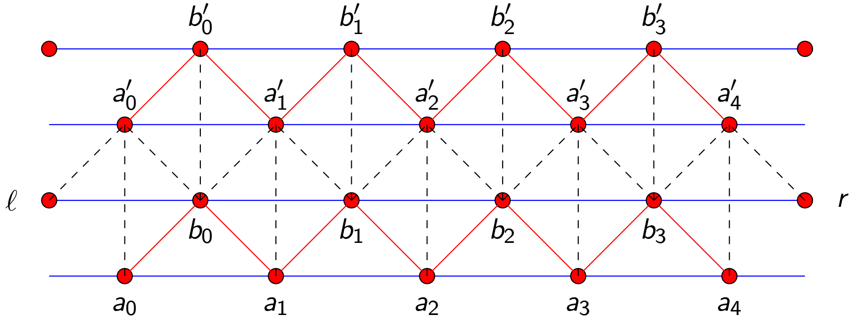

For given boundary conditions ℓ, r, probability transition:
$P^{(\ell,r)}((a_0, b_0, a_1, b_1, \ldots, b_{k-1}, a_k), (a'_0, b'_0, a'_1, b'_1, \ldots, b'_{k-1}, a'_k))$
For any ℓ, r, the product measure with parameter p is invariant.

Ergodicity

There exists $\theta_{(\ell,r)} < 1$ such that for any probability distributions ν, ν' on S^{2k+1}, we have: $||P^{(\ell,r)}\nu - P^{(\ell,r)}\nu'||_1 \leq \theta^{(\ell,r)}||\nu - \nu'||_1$.

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Define: $\theta = \max\{\theta^{(\ell,r)} \ : \ (\ell, r) \in S^2\}$

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For any sequence $(\ell_t, r_t)_{t\geq 0}$, we have:
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This is true in particular for $\nu' =$ product measure of parameter p.

There exists $\theta_{(\ell,r)} < 1$ such that for any probability distributions ν, ν' on S^{2k+1}, we have: $||P^{(\ell,r)}\nu - P^{(\ell,r)}\nu'||_1 \leq \theta^{(\ell,r)}||\nu - \nu'||_1$.

Define: $\theta = \max\{\theta^{(\ell,r)} \ : \ (\ell, r) \in S^2\}$

For any sequence $(\ell_t, r_t)_{t\geq 0}$, we have:
$||P^{(\ell_{t-1},r_{t-1})} \dots P^{(\ell_1,r_1)} P^{(\ell_0,r_0)}\nu - P^{(\ell_{t-1},r_{t-1})} \dots P^{(\ell_1,r_1)} P^{(\ell_0,r_0)}\nu'||_1 \leq \theta^t||\nu - \nu'||_1$.

This is true in particular for $\nu' =$ product measure of parameter p.

Theorem

Let A be a PCA with positive rates, having an invariant HZPM. Then, A is ergodic. Precisely, whatever the distribution of (η_0, η_1) is, the distribution of (η_t, η_{t+1}) converges (weakly) to π_p.

3. Directional reversibility

Directional reversibility

If μ is an invariant measure of a PCA A, we can extend the space-time diagram to the whole lattice $\mathbb{Z}_e^2$ (Kolmogorov theorem).

The extension $(\eta_t(i) : t \in \mathbb{Z}, i \in \mathbb{Z}_t)$ is called the **stationary space-time diagram** of A under μ, and denoted by $G(A, \mu)$.

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Let $D_4 =$ symmetry group of the square.

Definition

For $g \in D_4$, (A, μ) is g-**quasi-reversible**, if there exists a PCA A_g and a measure μ_g such that $G(A, \mu) \overset{(d)}{=} g^{-1} \circ G(A_g, \mu_g)$.

In that case, the pair (A_g, μ_g) is the g-**reverse** of (A, μ).

If, moreover, $(A_g, \mu_g) = (A, \mu)$, then (A, μ) is g-**reversible**.

Directional reversibility

1. (A, μ) is *id*-reversible.
2. (A, μ) is v-quasi-reversible and the transition matrix of its v-reverse is $T_v(c, b, a; d) = T(a, b, c; d)$.
3. If (A, μ) is g-quasi-reversible, then its g-reverse (A_g, μ_g) is g^{-1}-quasi-reversible and (A, μ) is the g^{-1}-reverse of (A_g, μ_g).
4. If (A, μ) is g-quasi-reversible and if (A_g, μ_g) is g'-quasi-reversible, then (A, μ) is $g'g$-quasi-reversible and $(A_{g'g}, \mu_{g'g})$ is its $g'g$-reverse.
5. For any subset E of D_4, if (A, μ) is E-reversible, then (A, μ) is $< E >$-reversible.

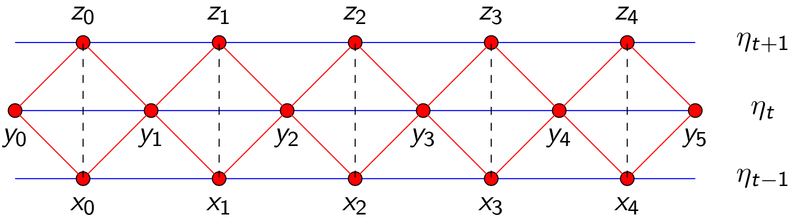
Directional reversibility

We denote by $\mathcal{T}_S(p)$ the set of PCA having a p-HZPM.

Proposition

Any PCA $A \in \mathcal{T}_S(p)$ is r^2-quasi-reversible, and the transition matrix T_{r^2} of its r^2-reverse A_{r^2} is given by:

$$\forall a, b, c, d \in S, \quad T_{r^2}(c, d, a; b) = \frac{p(b)}{p(d)} T(a, b, c; d).$$



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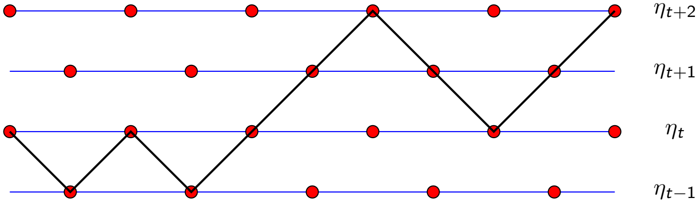
Corollary

Any PCA $A \in \mathcal{T}_S$ is $\{h, r^2, v\}$-quasi-reversible.

Zig-zag polylines

Proposition

If $A \in \mathcal{T}_S(p)$, then any zig-zag polyline of the stationary space-time diagram is made of i.i.d. random variables.



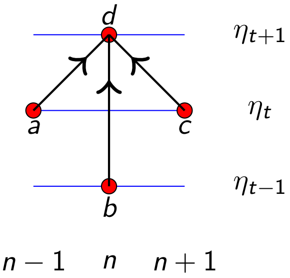
Proposition

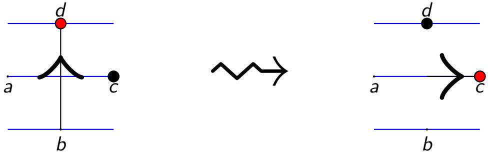
Let $A \in \mathcal{T}_S(p)$. A is r-quasi-reversible iff:

$$\forall a, b, d \in S, \sum_{c \in S} p(c) T(a, b, c; d) = p(d).$$

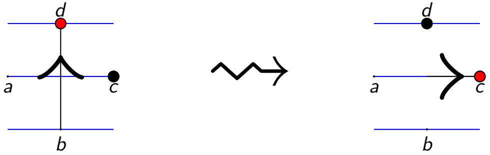
In that case, the transition matrix T_r of its r-reverse A_r is given by:

$$\forall a, b, c, d \in S, \quad T_r(d, a, b; c) = \frac{p(c)}{p(d)} T(a, b, c; d).$$





$$p(c)\,T(a,b,c;d) = p(d)\,T_r(d,a,b;c)$$



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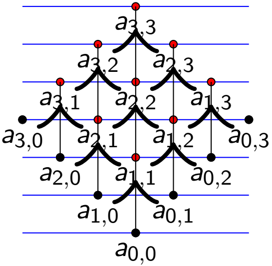

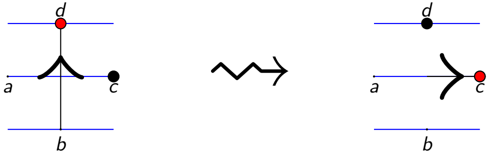

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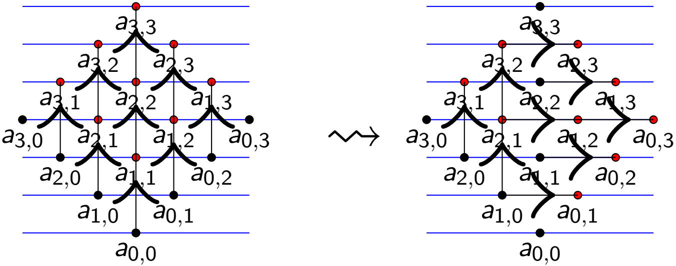

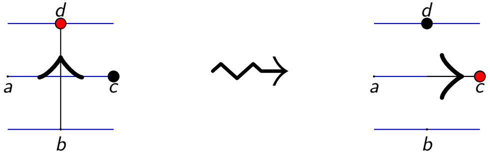

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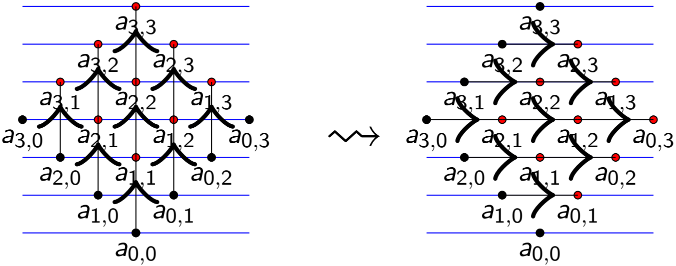

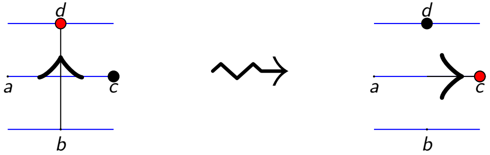

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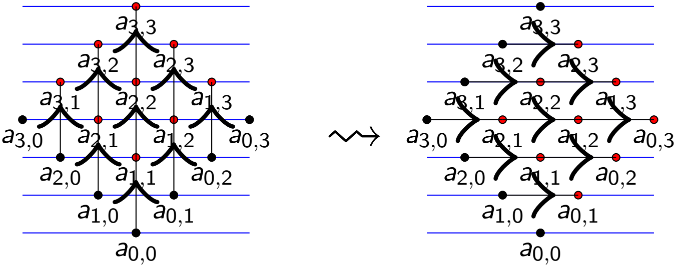


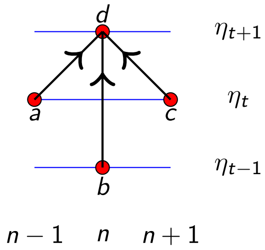
Remark

The PCA A_r does not always have an invariant product measure!

Proposition

Let $A \in \mathcal{T}_S(p)$. The following properties are equivalent:

1. A is $\{r, r^{-1}\}$-quasi-reversible.
2. A is r-quasi-reversible and $A_r \in \mathcal{T}_S(p)$,
3. A is r^{-1}-quasi-reversible and $A_{r^{-1}} \in \mathcal{T}_S(p)$,
4. $\forall a, b, d \in S, \sum_{c \in S} p(c) T(a, b, c; d) = p(d)$ and $\forall b, c, d \in S, \sum_{a \in S} p(a) T(a, b, c; d) = p(d)$.
5. A is D_4-quasi-reversible.
6. A is 3-to-3 i.i.d.



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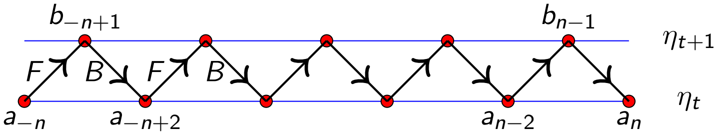
Proposition

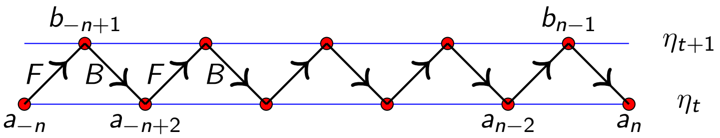
- A is $< r >$-reversible iff $p(a) T(a, b, c; d) = p(d) T(b, c, d; a)$ for any $a, b, c, d \in S$.
- A is D_4-reversible iff $T(a, b, c; d) = T(c, b, a; d)$ and $p(a) T(a, b, c; d) = p(d) T(b, c, d; a)$ for any $a, b, c, d \in S$.

Conditions on the parameters	Property of the PCA	Dimension of the submanifold (number of degrees of freedom)
Cond. 1: $\forall a, c, d \in S$, $p(d) = \sum_{b \in S} p(b) T(a, b, c; d)$	HZPM invariant $\{r^2, h\}$-quasi-reversible	$n^2(n-1)^2$
Cond. 1 + Cond. 2: $\forall a, b, d \in S$, $p(d) = \sum_{c \in S} p(c) T(a, b, c; d)$.	r-quasi-reversible	$n(n-1)^3$
Cond. 1 + Cond. 3: $\forall b, c, d \in S$, $p(d) = \sum_{a \in S} p(a) T(a, b, c; d)$.	r^{-1}-quasi-reversible	$n(n-1)^3$
Cond. 1 + Cond. 2 + Cond. 3	D_4-quasi-reversible	$(n-1)^4$
Cond. 1 + $\forall a, b, c, d \in S$, $T(a, b, c; d) = T(c, b, a; d)$	v-reversible	$\frac{(n-1)^2 n(n+1)}{2}$
Cond. 1 + $\forall a, b, c, d \in S$, $p(b) T(a, b, c; d) = p(d) T(c, d, a; b)$	r^2-reversible	$\frac{(n-1)^2 n(n+1)}{2}$
Cond. 1 + $\forall a, b, c, d \in S$, $p(b) T(a, b, c; d) = p(d) T(a, d, c; b)$	h-reversible	$\frac{n^3(n-1)}{2}$
Cond. 1 + $\forall a, b, c, d \in S$, $T(a, b, c; d) = T(c, b, a; d)$ and $p(b) T(a, b, c; d) = p(d) T(c, d, a; b)$	$< r^2, v >$-reversible	$\frac{(n-1)n^2(n+1)}{4}$
Cond. 1 + $\forall a, b, c, d \in S$, $p(a) T(a, b, c; d) = p(d) T(b, c, d; a)$	$< r >$-reversible	$\frac{n(n-1)(n^2-3n+4)}{4}$
Cond. 1 + $\forall a, b, c, d \in S$, $p(a) T(a, b, c; d) = p(d) T(d, c, b; a)$	$< r \circ v >$-reversible	$\frac{(n-1)^2(n^2-2n+2)}{2}$
Cond. 1 + $\forall a, b, c, d \in S$, $p(a) T(a, b, c; d) = p(d) T(b, c, d; a)$ and $T(a, b, c; d) = T(c, b, a; d)$	D_4-reversible	$\frac{n(n-1)(n^2-n+2)}{8}$

4. Horizontal Zig-zag Markov Chains

Horizontal Zig-zag Markov Chains

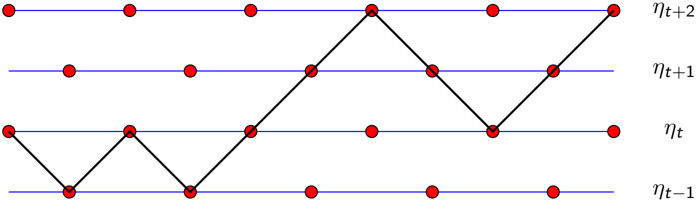




$$FB = BF \qquad \rho \text{ such that } \rho B = B \text{ and } \rho F = F$$

$$\mathbb{P}\left((\zeta_{F,B}(i,t) = a_i, \zeta_{F,B}(i,t+1) = b_i : -n \leq i \leq n)\right)$$
$$= \rho(a_{-n}) \prod_{i=-n+1}^{n-1} F(a_{i-1}; b_i) B(b_i; a_{i+1}).$$

Same kind of result for zig-zag polylines: computation of the distribution using F and B.



Proposition

Let A be a PCA having a (F, B)-HZMC invariant distribution. Then, the stationary space-time diagram $(A, \zeta_{F,B})$ is $\{h, r^2, v\}$-quasi-reversible.

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Proposition

Let A be a PCA having an (F, B)-HZMC invariant distribution. $(A, \zeta_{F,B})$ is r-quasi-reversible iff for any $a, c, d \in S$,

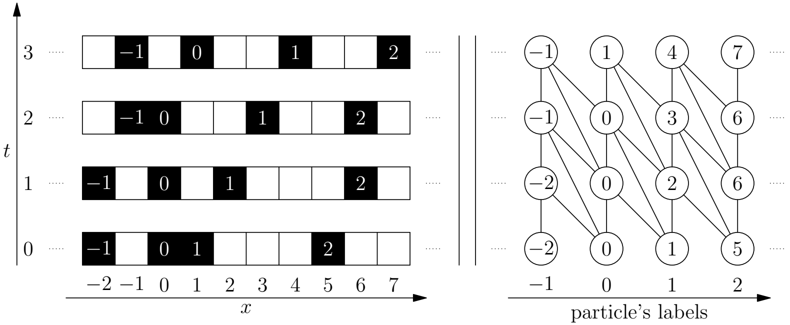
$$F(a; d) = \sum_{c \in S} F(b; c) T(a, b, c; d).$$

In that case, the transition matrix of A_r is given by: for any $a, b, c, d \in S$,

$$T_r(d, a, b; c) = \frac{F(b; c)}{F(a; d)} T(a, b, c; d).$$

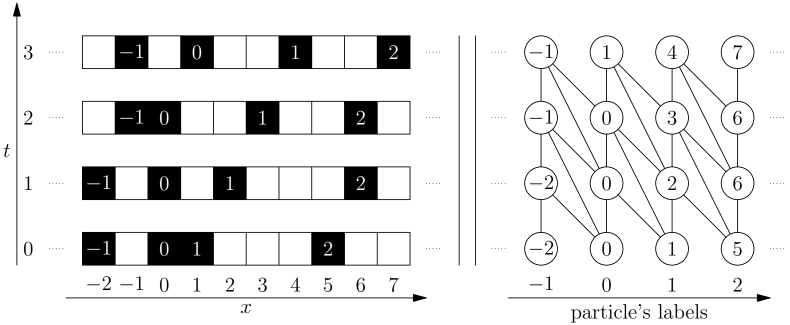
5. A TASEP model

A TASEP model



Parameters: $T(0,k,k;1)$ for $k \geq 2$ and $T(0,k,k+1;1)$ for $k \geq 1$.

A TASEP model



Parameters: $T(0,k,k;1)$ for $k \geq 2$ and $T(0,k,k+1;1)$ for $k \geq 1$.

Classical TASEP: $T(0,k,k;1) = T(0,k,k+1;1) = p$.

Lemma

Let q be a probability distribution on $\{0,1\}$ and let p be a distribution on $\mathbb{N} \setminus \{0\}$. If:

$$p(k)q(1)T(0,k,k+1;0) = p(k+1)q(0)T(0,k+1,k+1;1), \qquad (\star)$$

then there is a stable family of HZMC, given by:

$$F(a;a+k) = q(k) \text{ and } B(a;a+k) = p(k),$$

with starting point

$$\mathbb{P}\left(\eta_t(0) = k\right) = \binom{t}{k} q(1)^k q(0)^{t-k}.$$

Remark: q represents the speed law and p the distance law between two successive particles.

Theorem

For any T, for any distribution q on $\{0,1\}$ such that

$$Z = \sum_{k=0}^{\infty} \left(\frac{q(1)}{q(0)}\right)^{k} \prod_{m=1}^{k} \frac{T(0,m,m+1;0)}{T(0,m+1,m+1;1)} < \infty,$$

there exists a unique distribution p on $\mathbb{N}^*$ such that $(\star)$ hold. Moreover, this distribution p is, for any $k \geq 1$,

$$p(k) = \frac{\left(\frac{q(1)}{q(0)}\right)^{k-1} \prod_{m=1}^{k-1} \frac{T(0,m+1,m+1;0)}{T(0,m,m+1;1)}}{Z}.$$

Thank you for your attention...