Stochastic fixed-points and periodicities in combinatorial structures

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Goethe-University
Frankfurt am Main

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October 9–13, 2017

Bucket Selection

Given: $U_1, \ldots, U_n$ indep, unif$[0,1]$ distributed.

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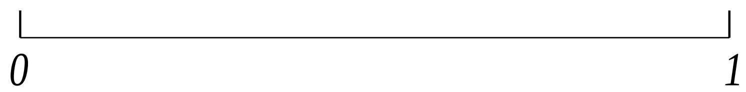

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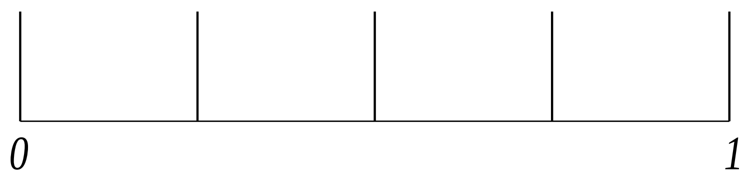

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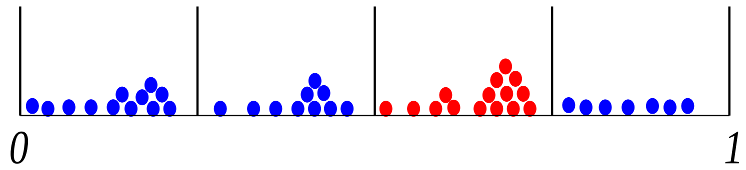

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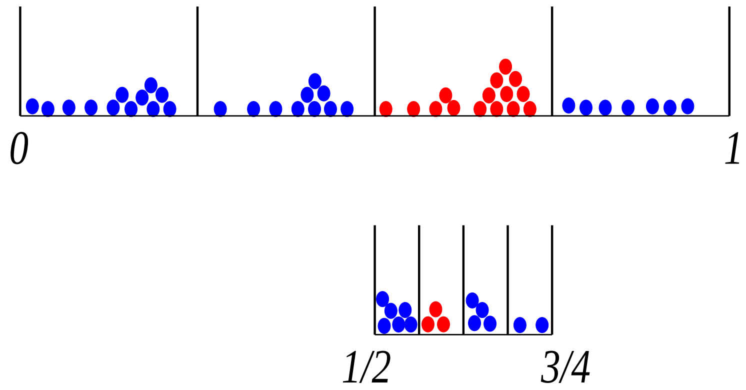

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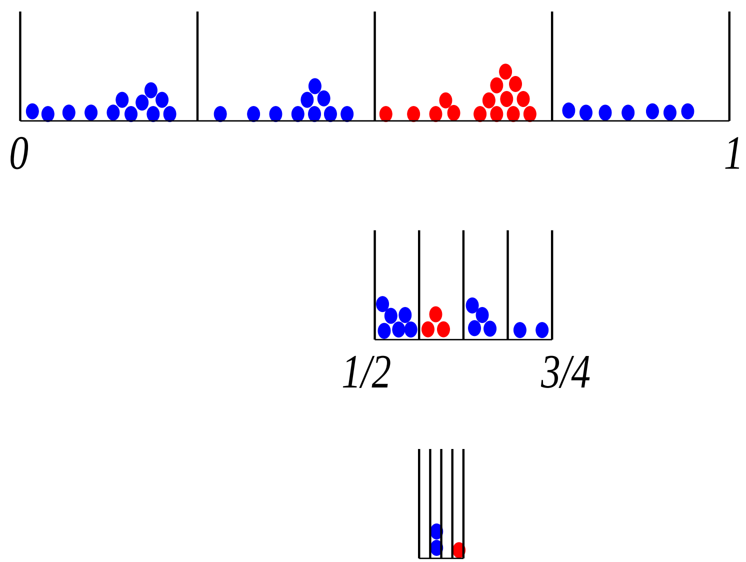

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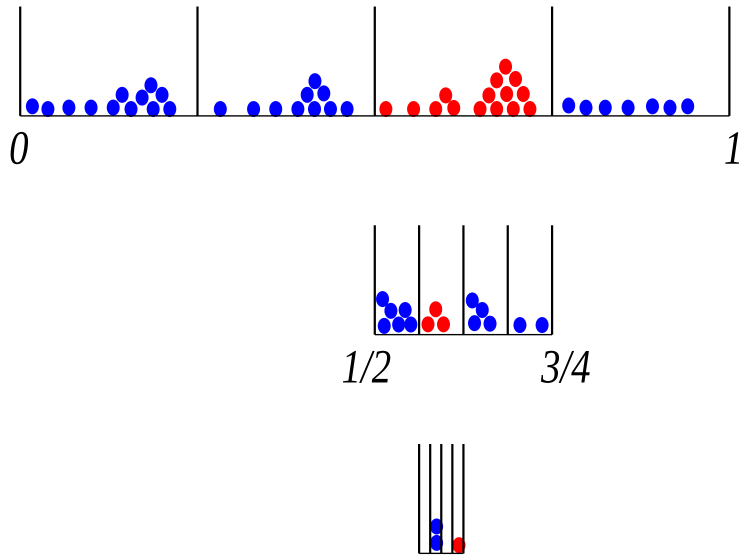

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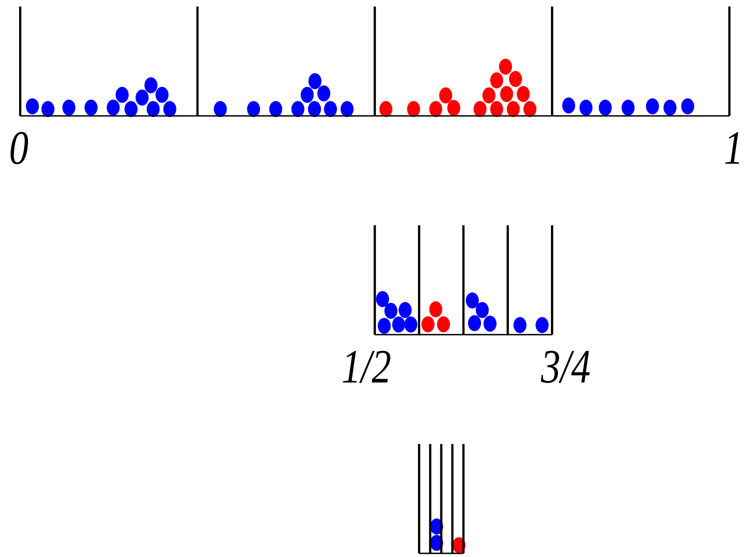

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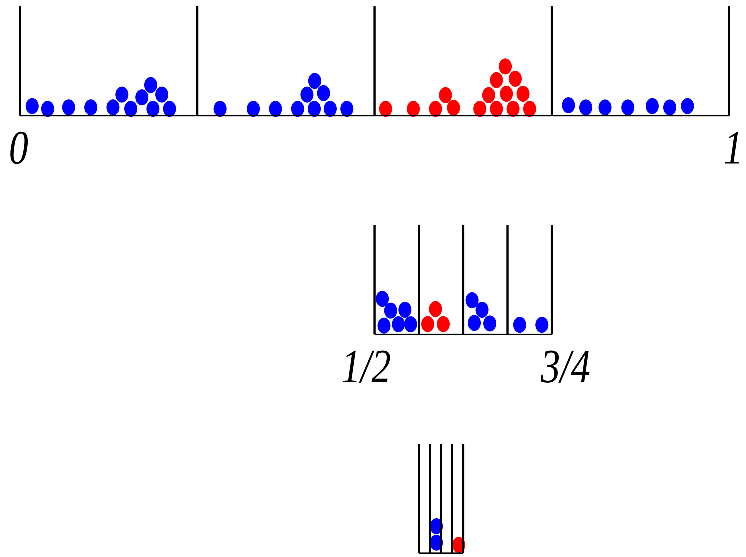

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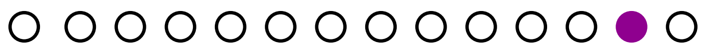

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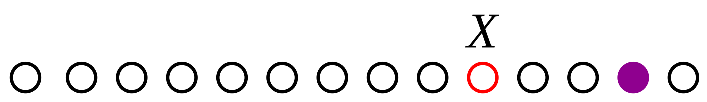

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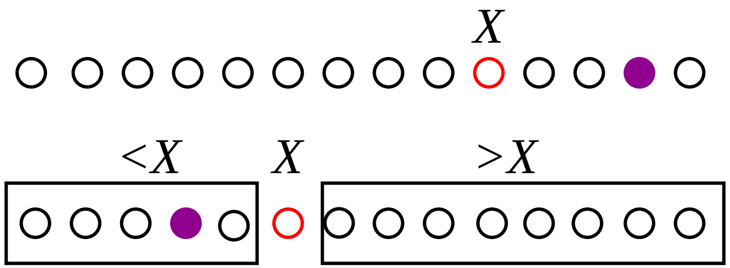

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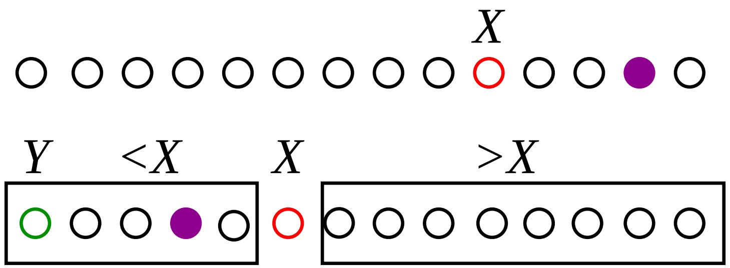

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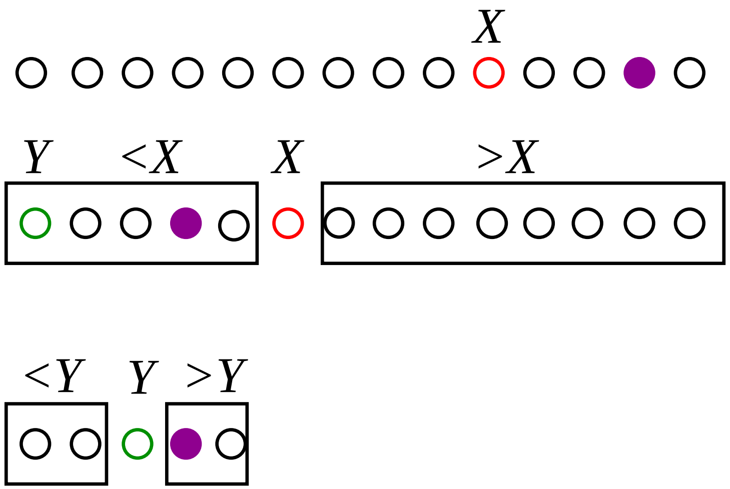

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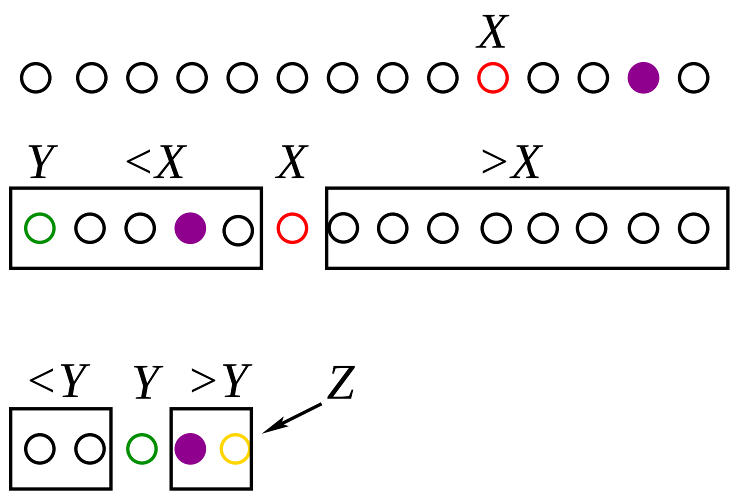

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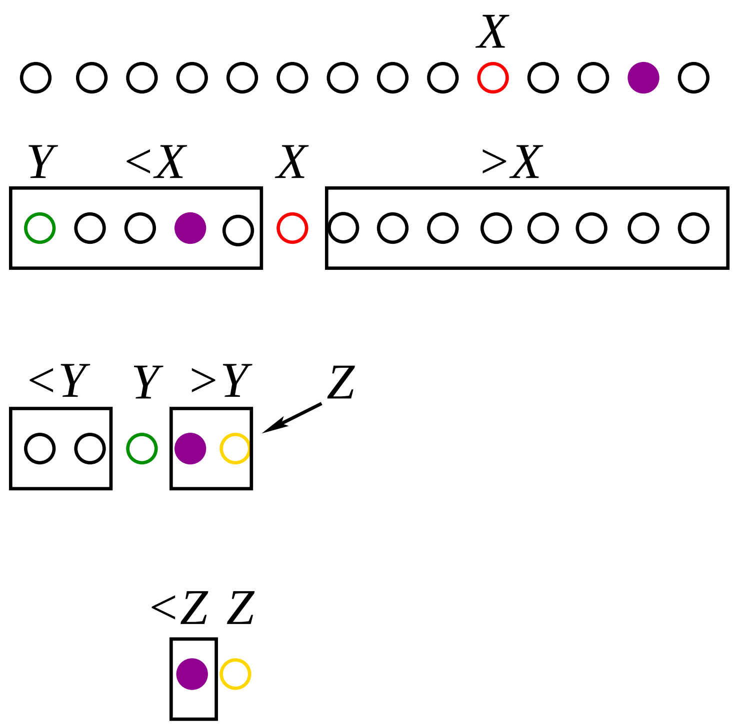

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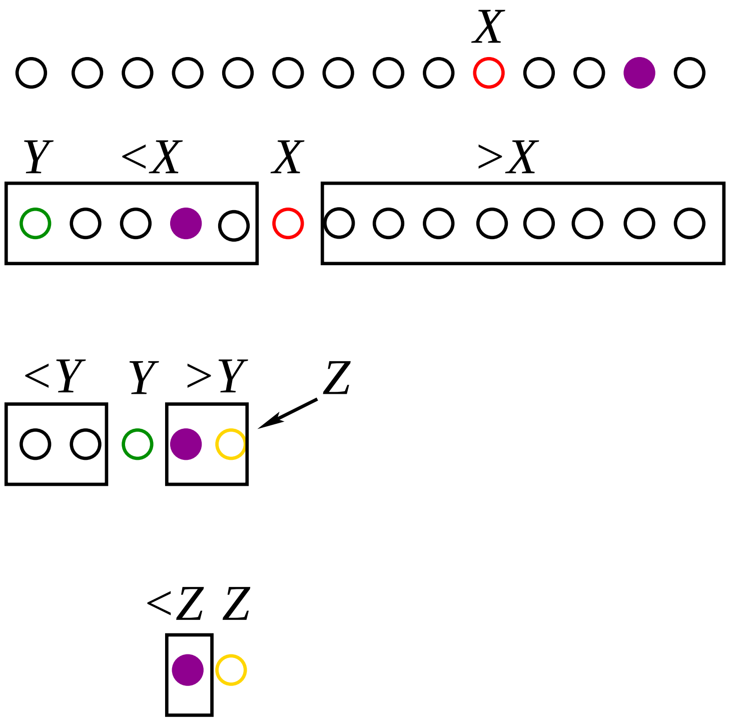

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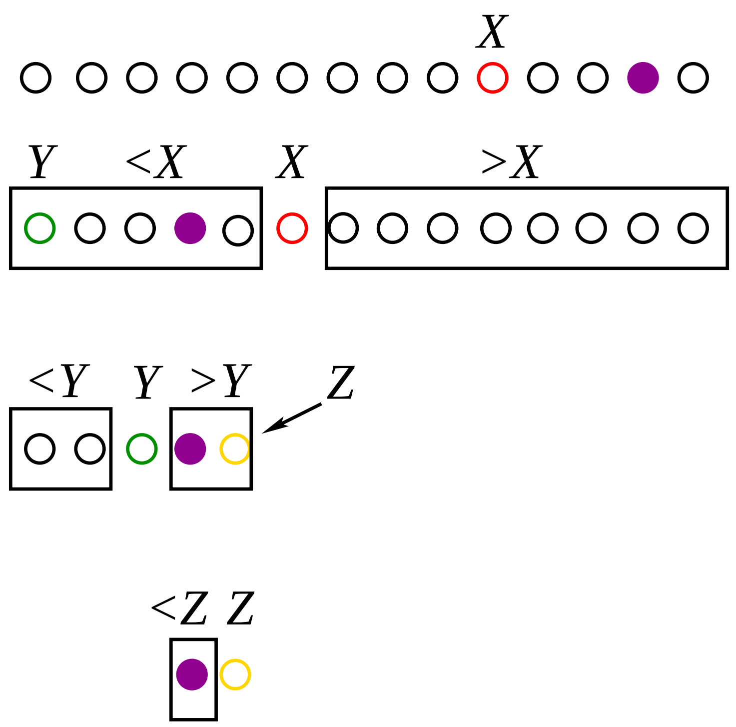

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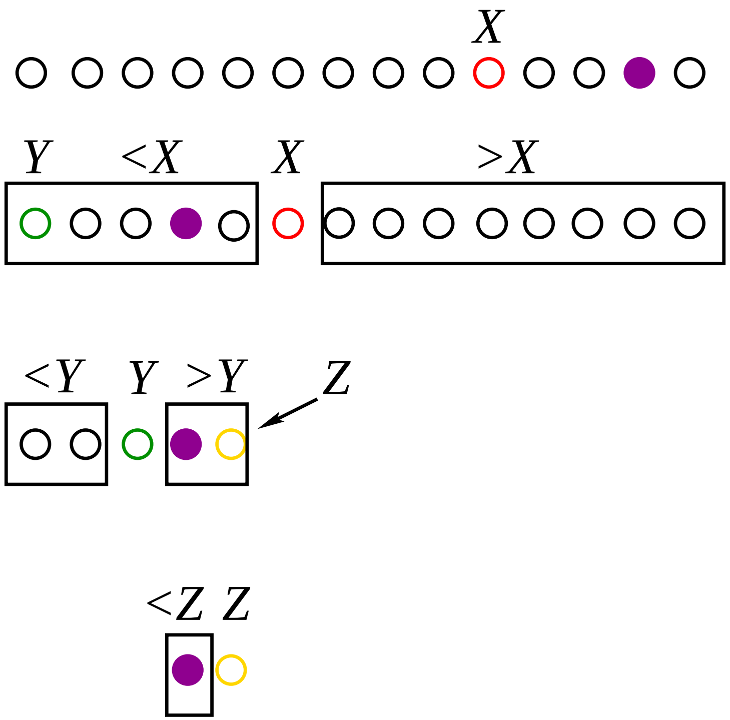

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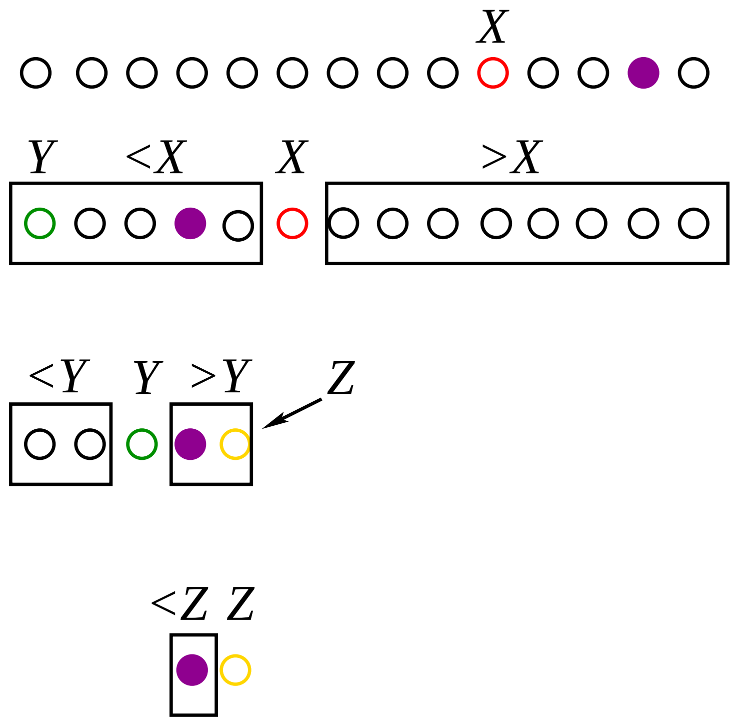

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Binary search tree

Given numbers: $6, 1, 8, 7, 5, 3, 10, 2, 11, 4, 9.$

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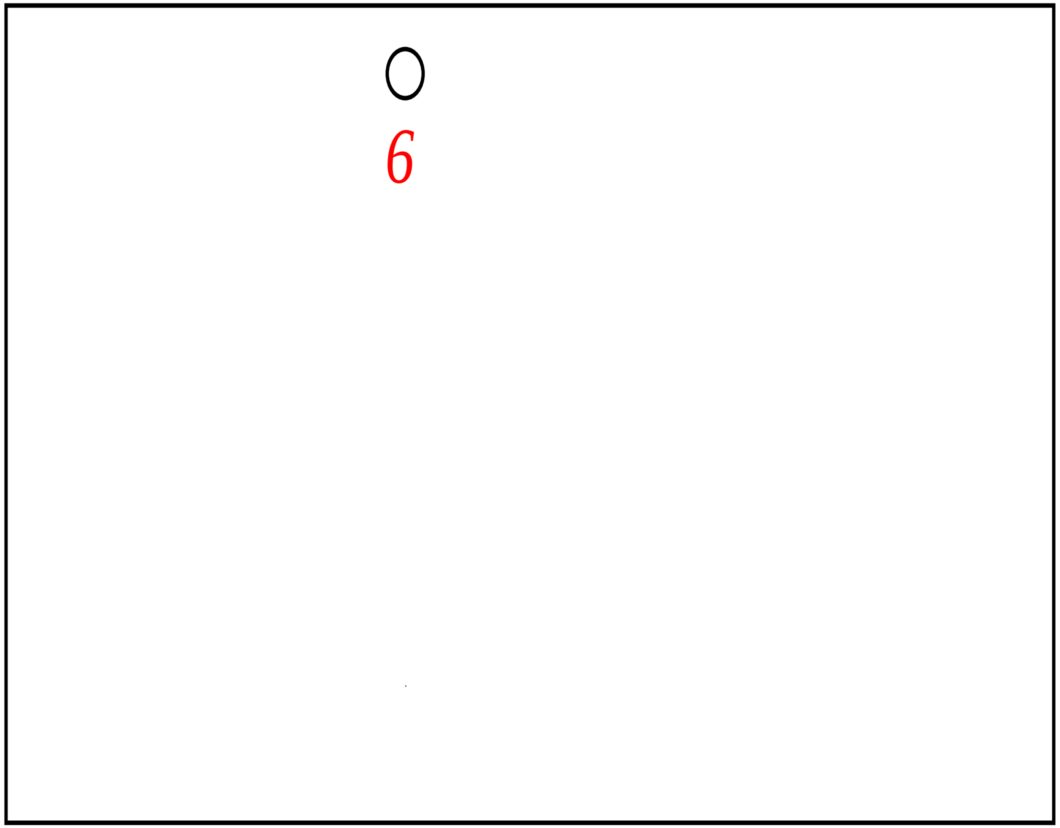

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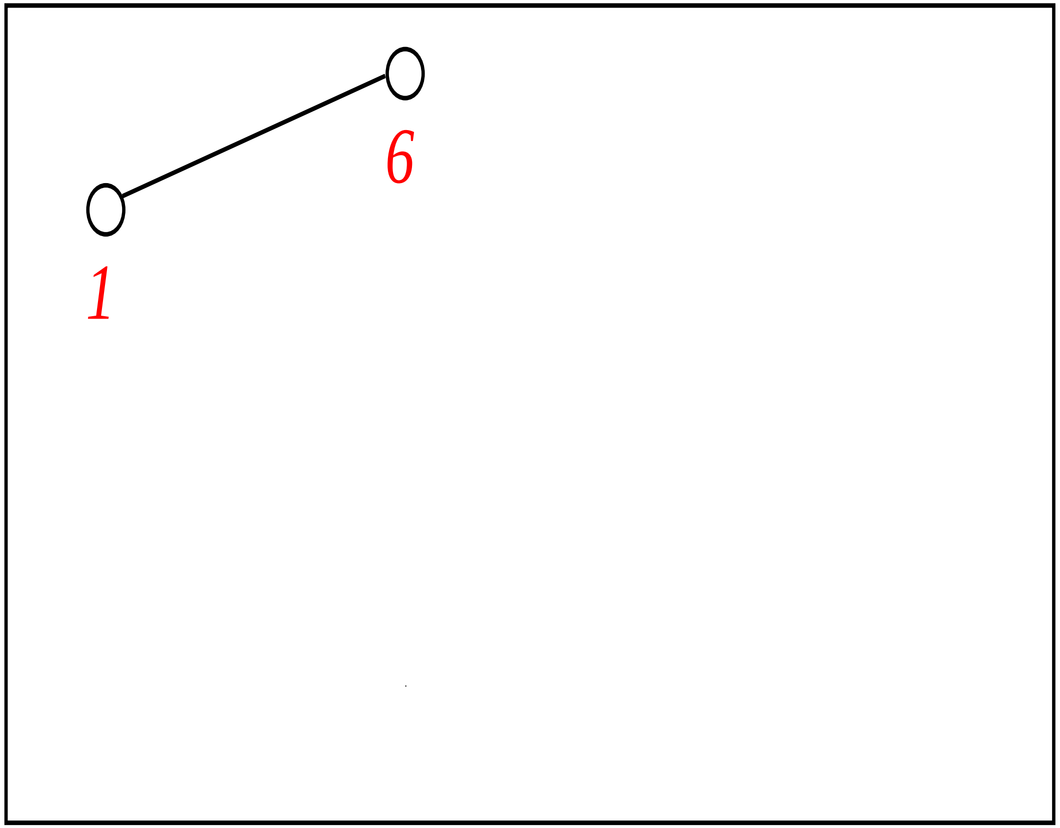

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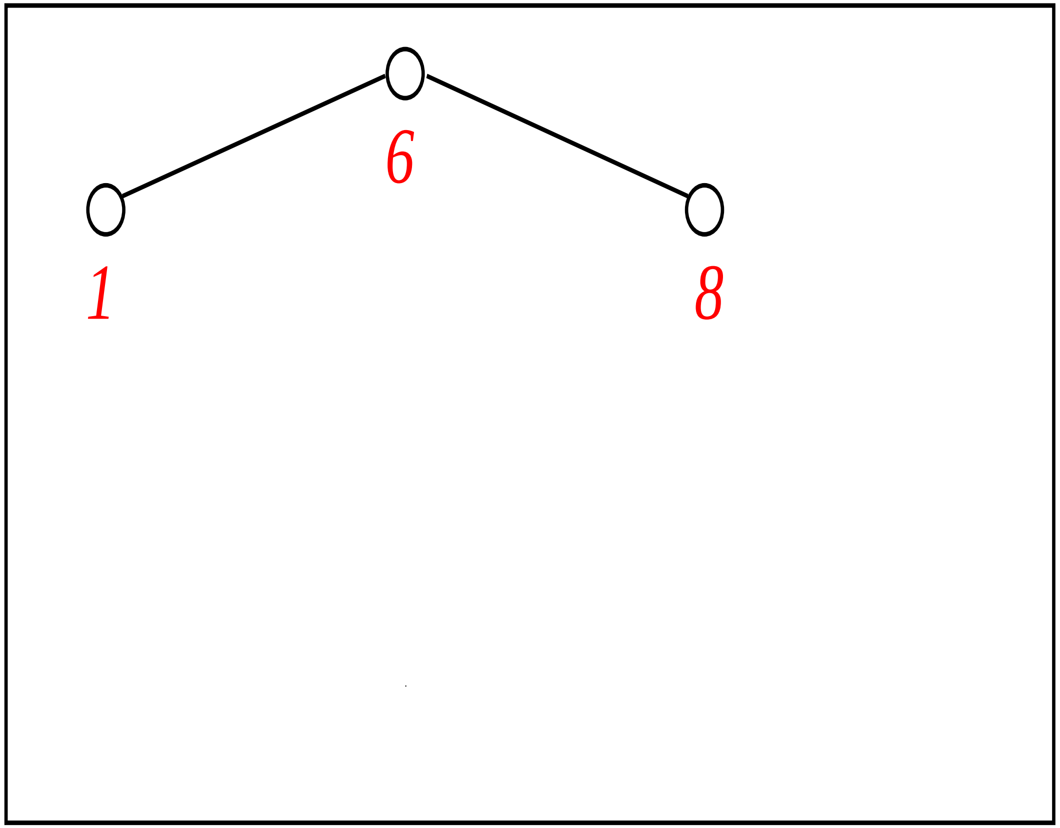

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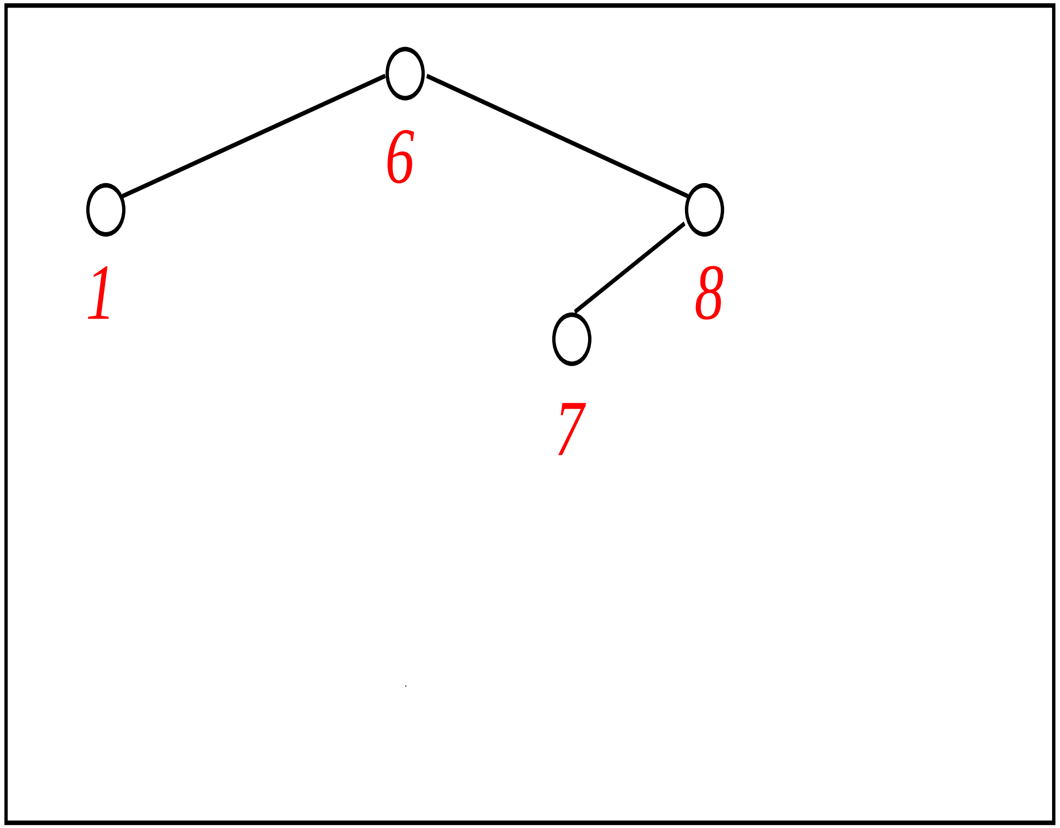

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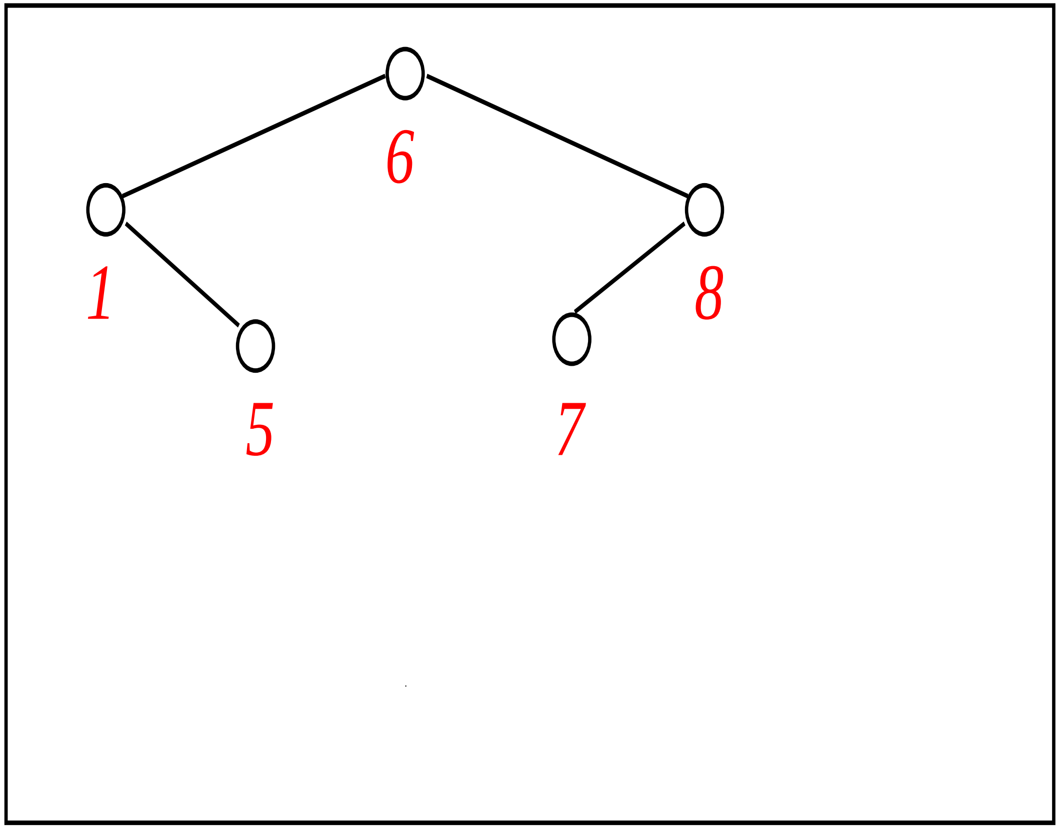

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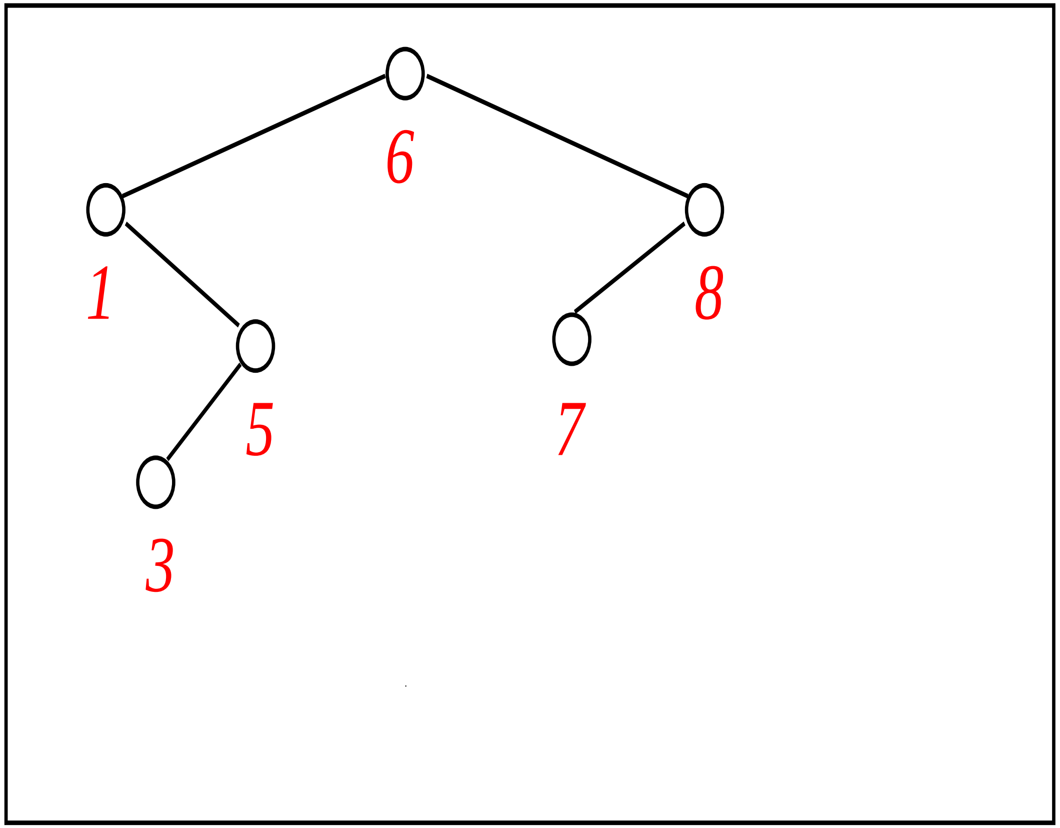

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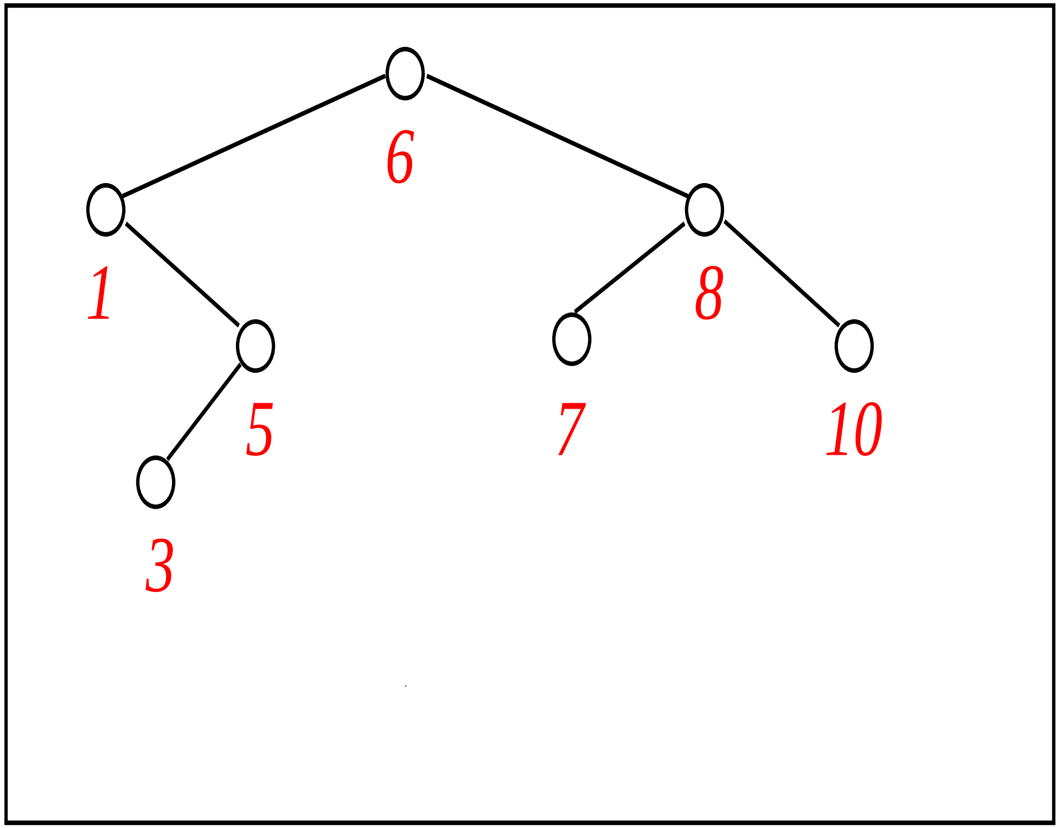

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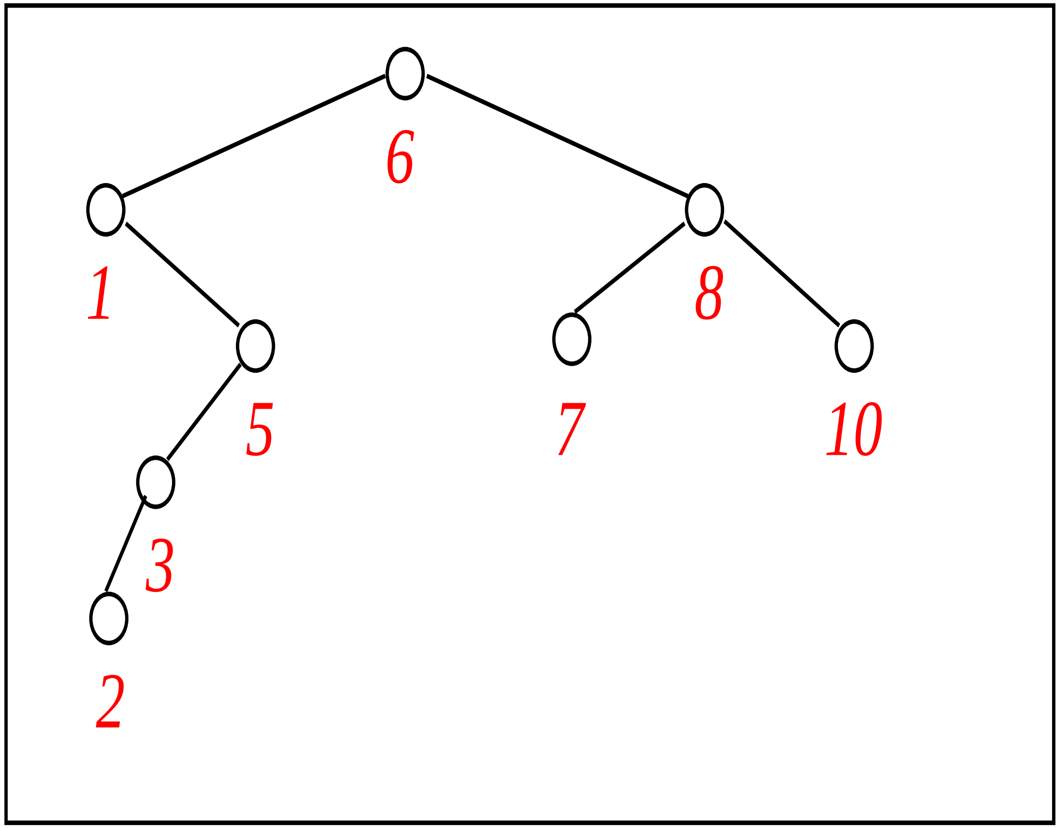

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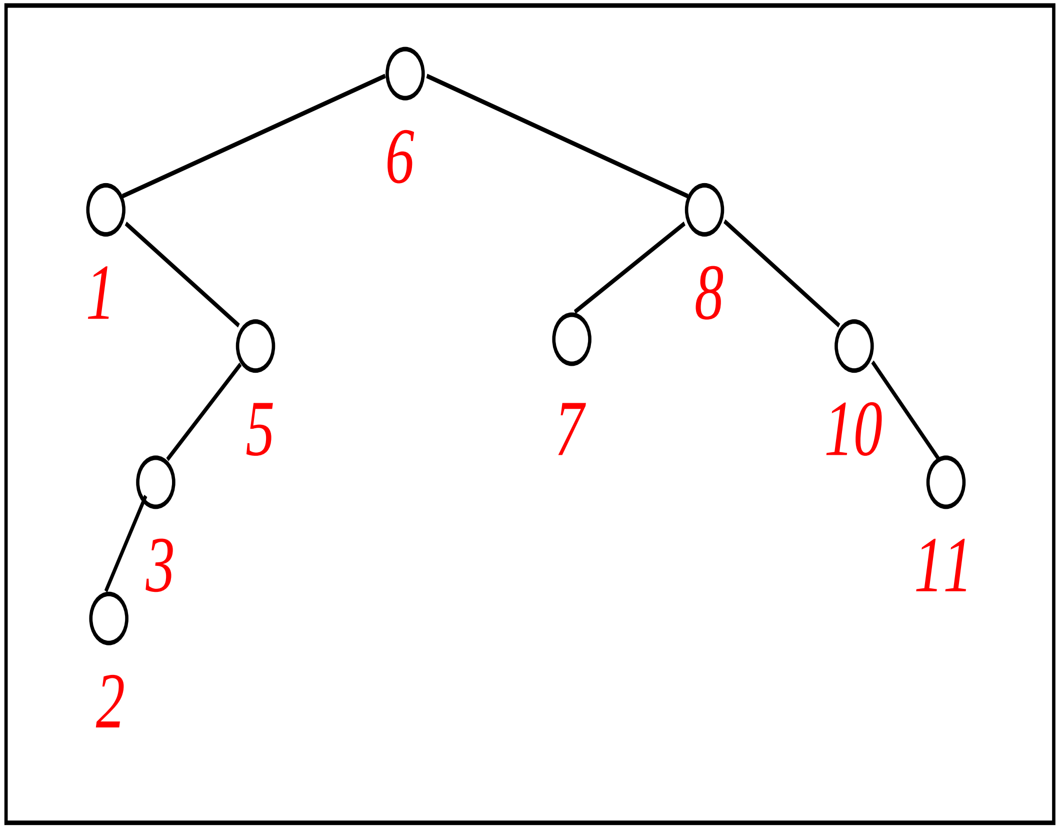

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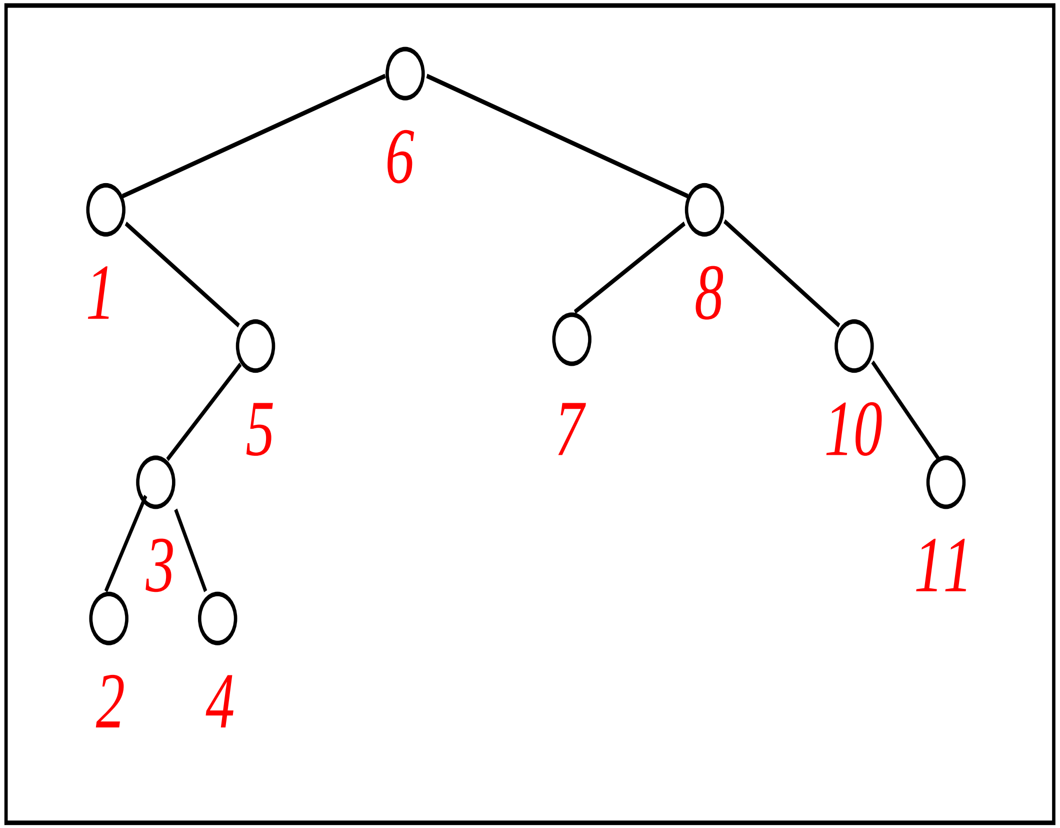

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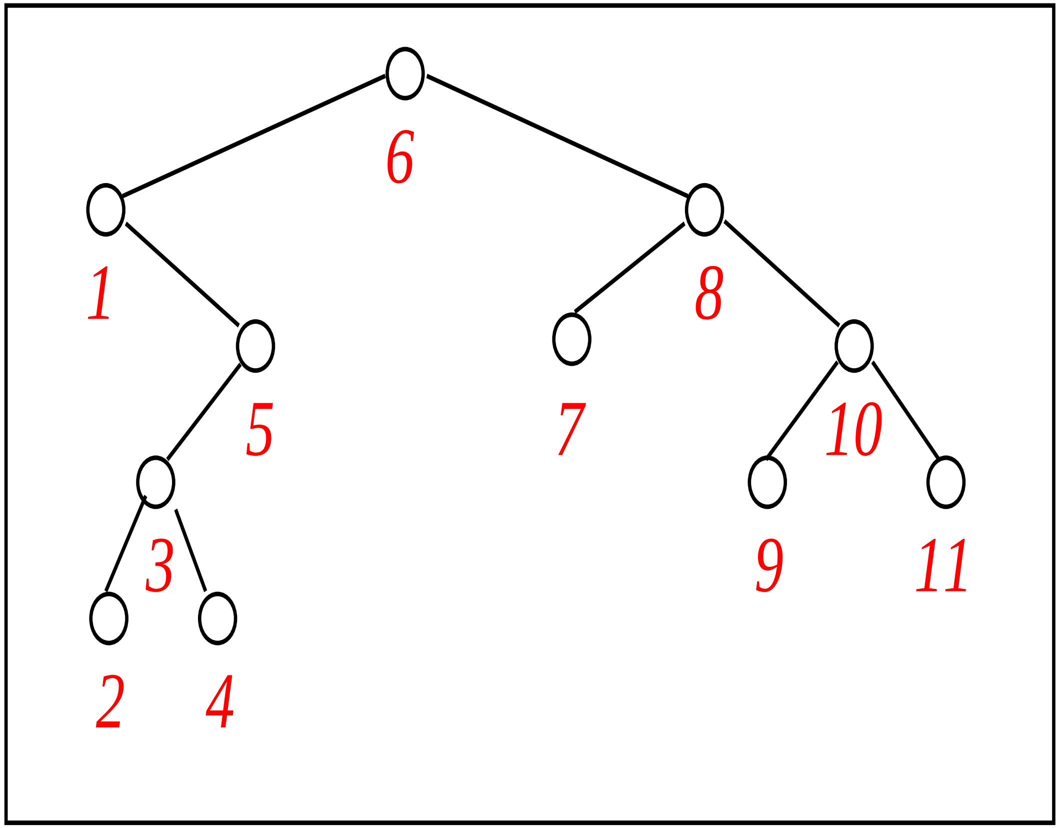
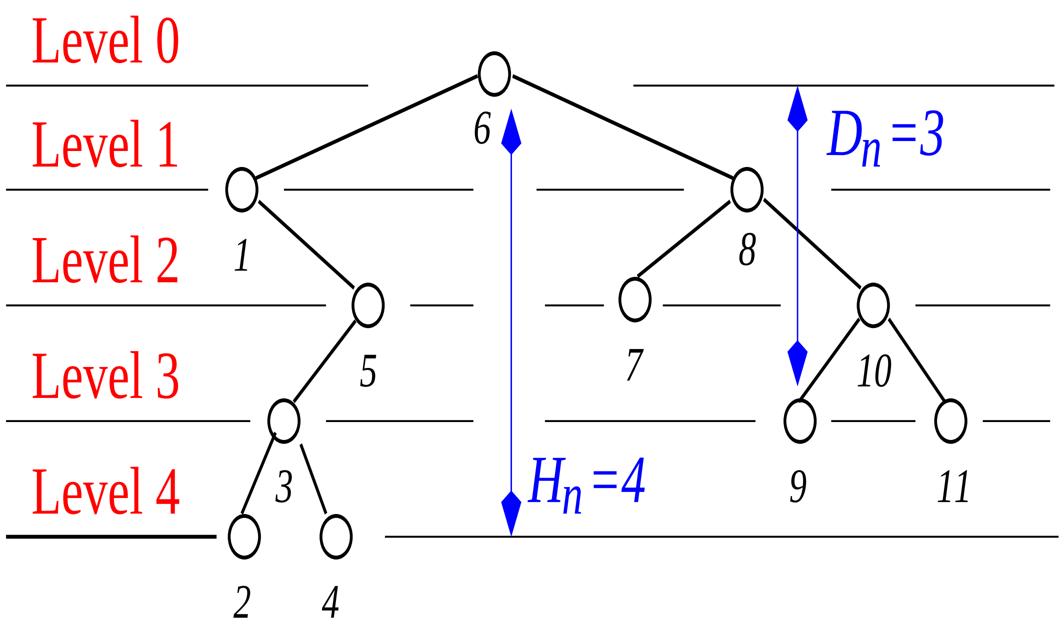


Quantities in BST



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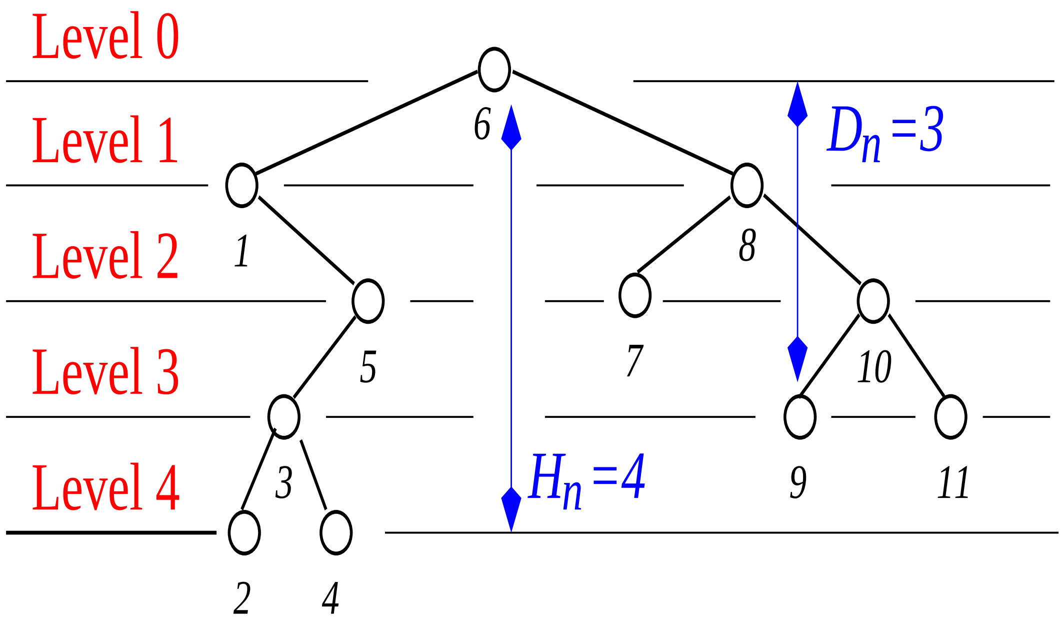
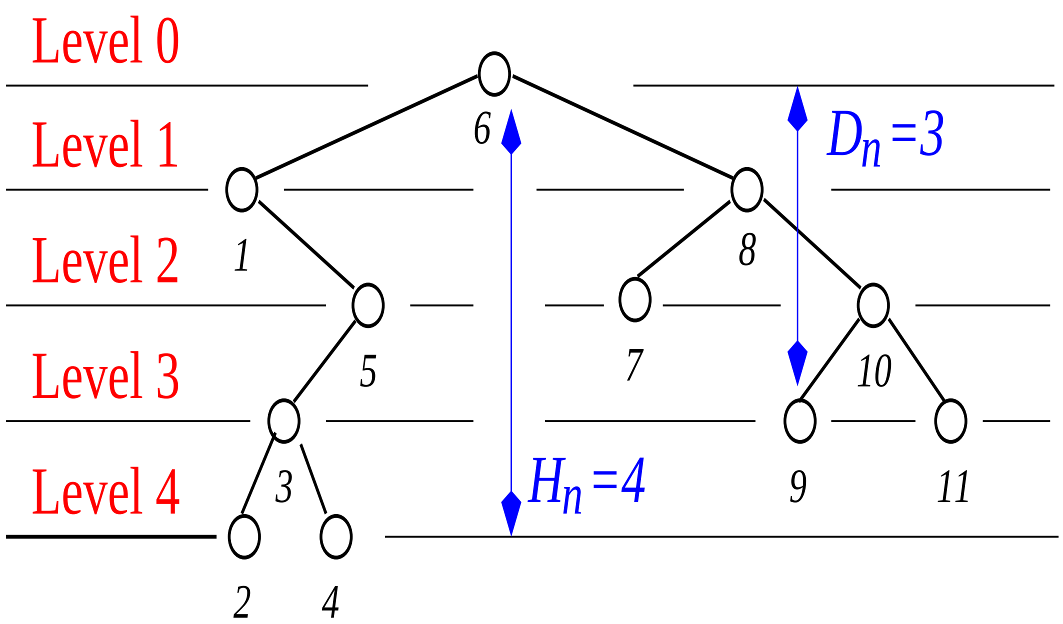


D_n — depth = distance root to n-th inserted node

Quantities in BST



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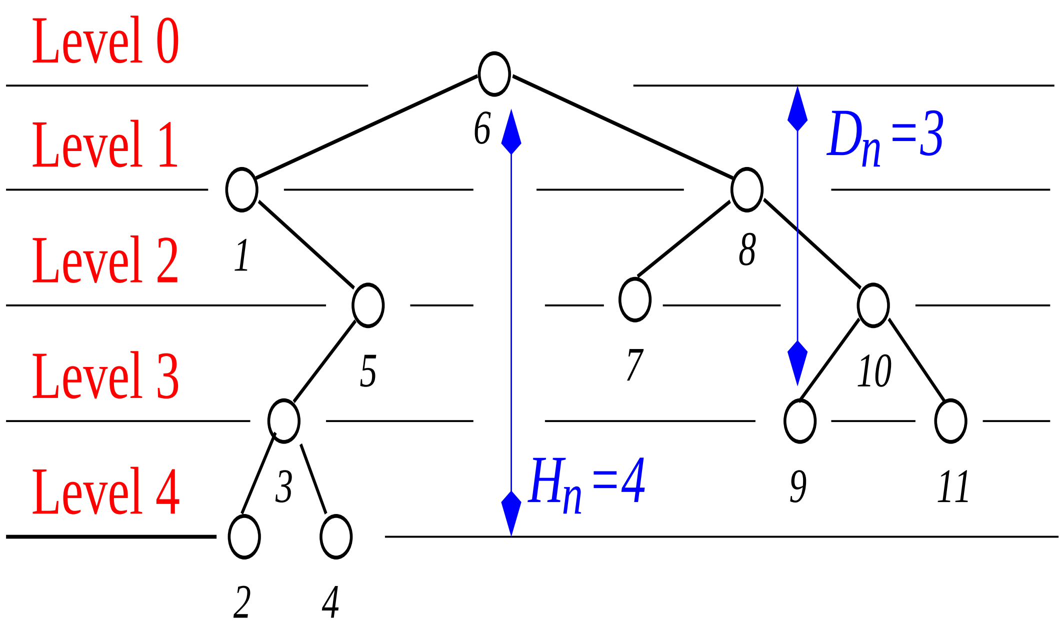
Quantities in BST



D_n — depth = distance root to n-th inserted node

$H_n = \max_{1 \le j \le n} D_j$ — height

$Q_n = \sum_{1 \le j \le n} D_j$ — internal path length

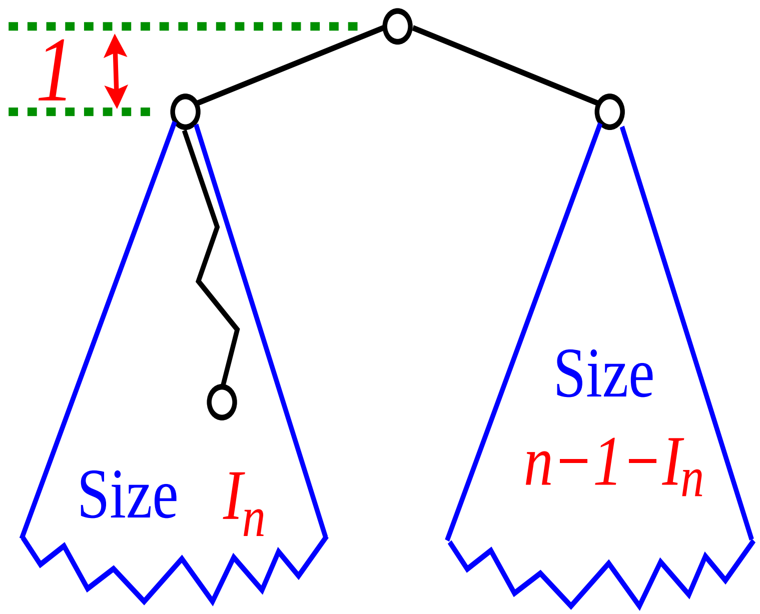
Random binary search tree

Model of randomness:

All permutations of $1, \ldots, n$ equally likely.

Equivalent: Use $U_1, \ldots, U_n$ i.i.d. unif$[0, 1]$.

Internal path length



Internal path length Q_n:

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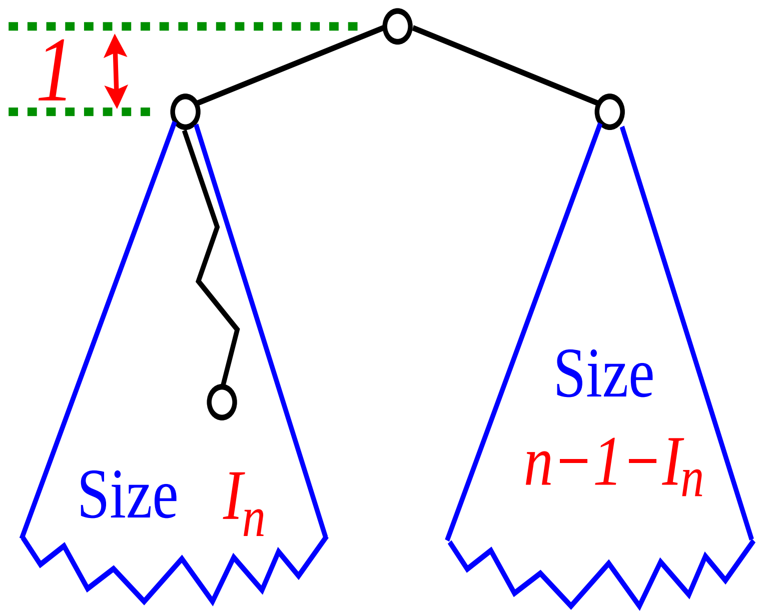
Internal path length

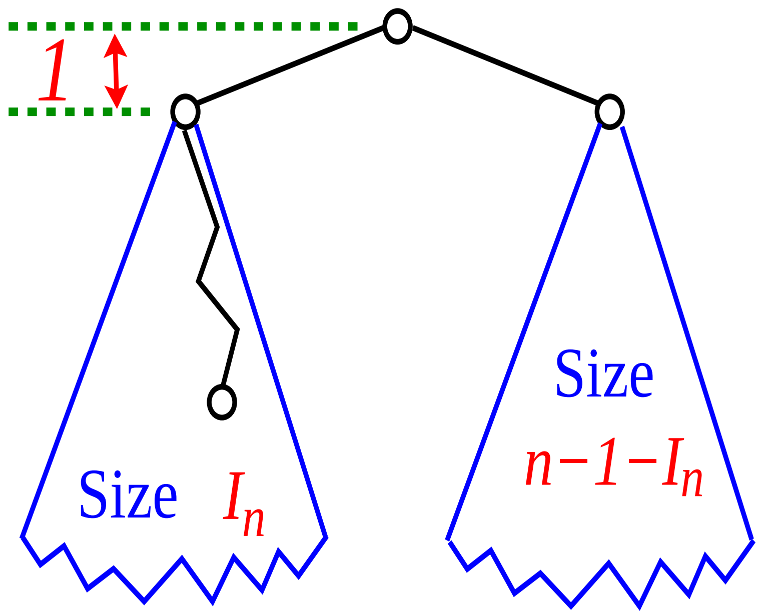


Internal path length Q_n:

$$Q_n \stackrel{d}{=} Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1$$

.

Internal path length



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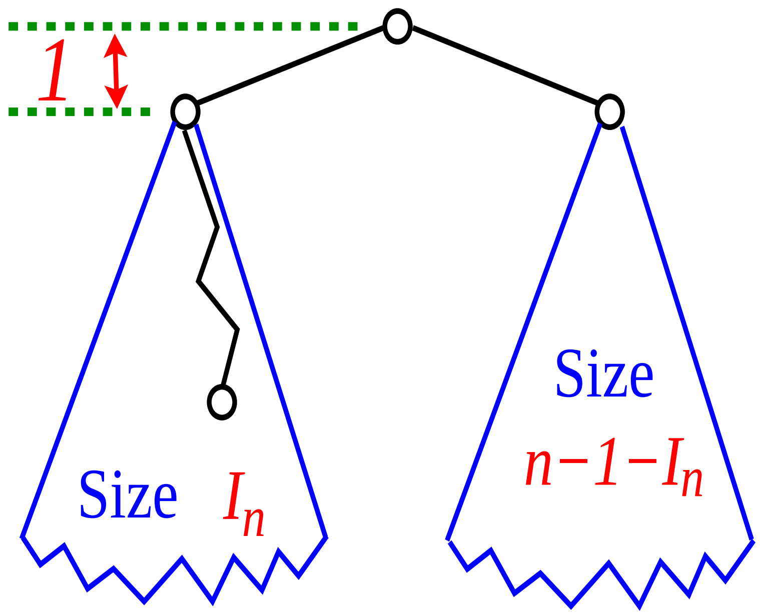
$Q^*_0, \dots, Q^*_{n-1}, Q^{**}_0, \dots, Q^{**}_{n-1}, I_n$ indep.,

.

Internal path length



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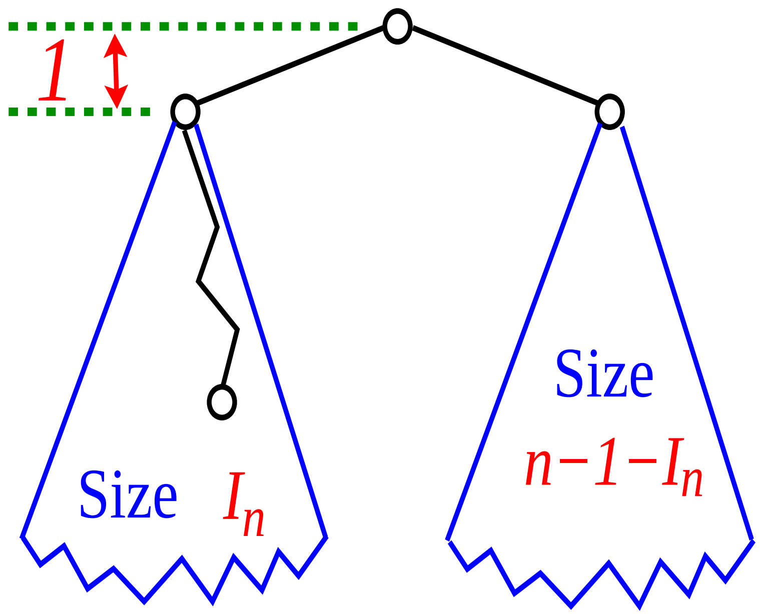

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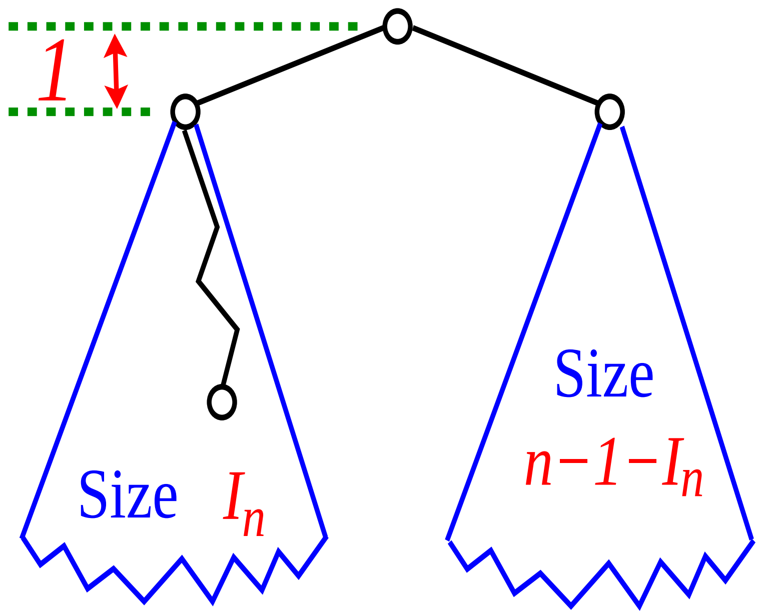
Internal path length



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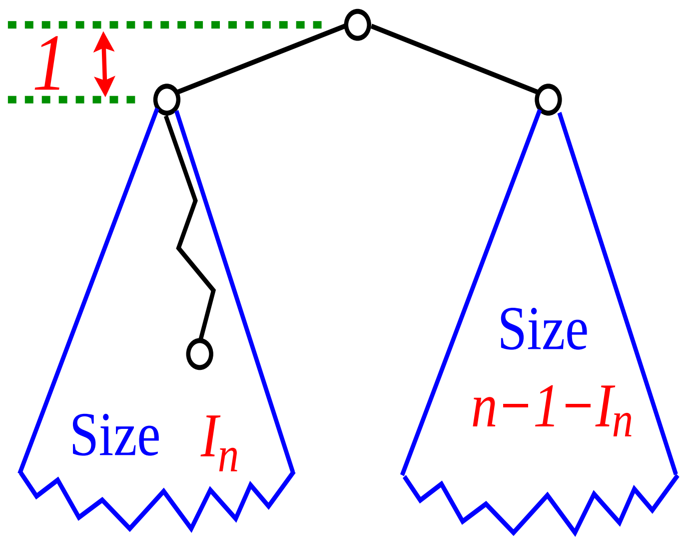
This needs a proof!

Internal path length Q_n



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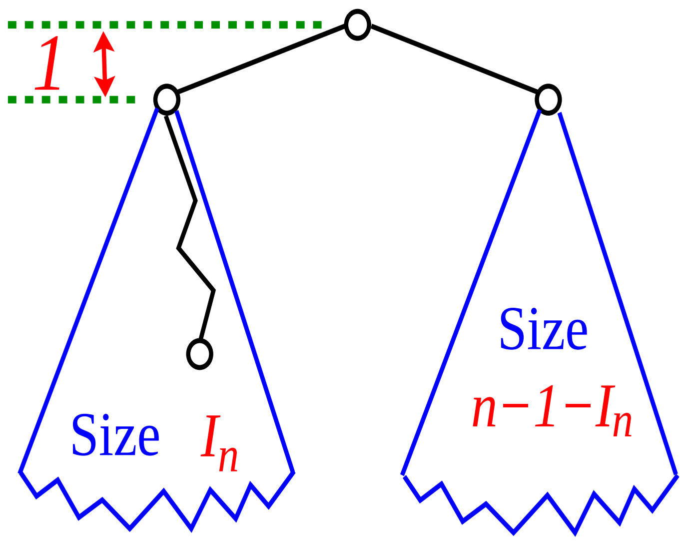

$$Q_n \stackrel{d}{=} Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1 =: Z_n,$$

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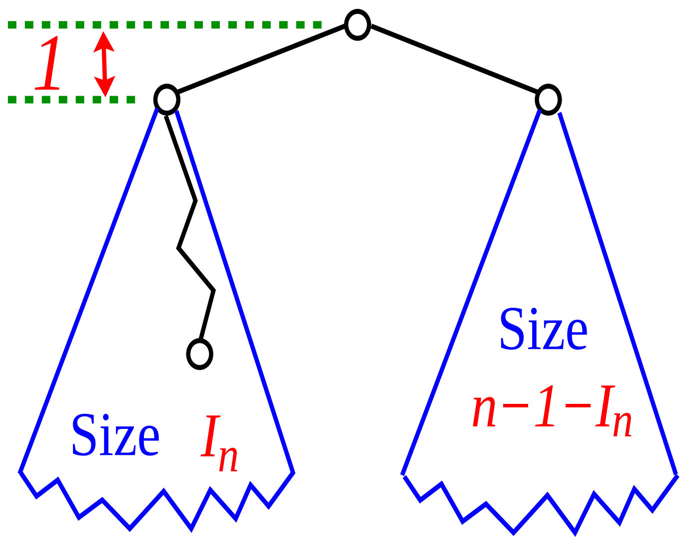

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]

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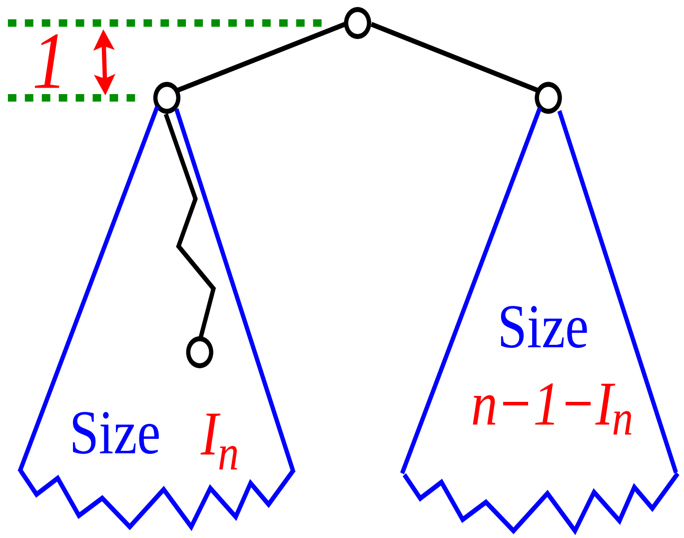

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Show: For all $j \in \mathbb{N}$, $k \in \{0, \dots, n-1\}$:

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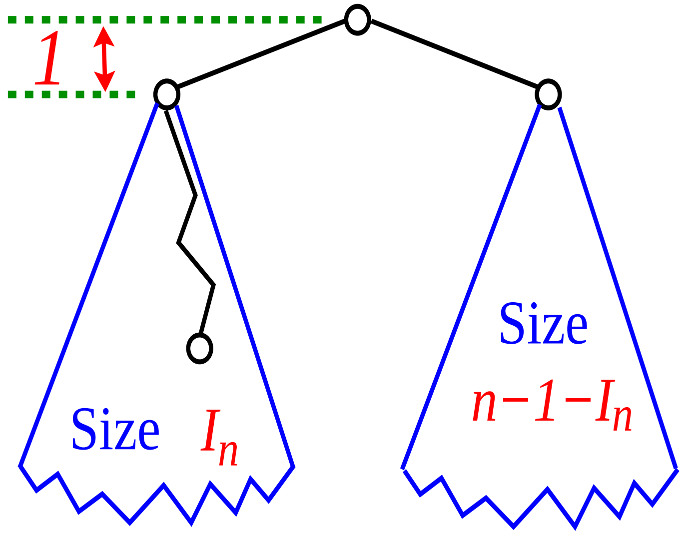

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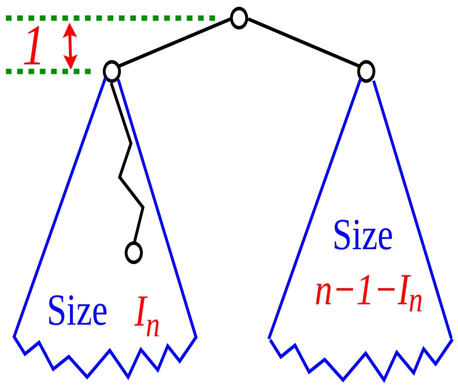
$$\mathbb{P}(Q_n = j \mid I_n = k) = \mathbb{P}(Z_n = j \mid I_n = k).$$

[Total probability theorem yields:

$$\begin{aligned} \mathbb{P}(Q_n = j) &= \sum_k \mathbb{P}(I_n = k)\,\mathbb{P}(Q_n = j \mid I_n = k) \\ &= \sum_k \mathbb{P}(I_n = k)\,\mathbb{P}(Z_n = j \mid I_n = k) = \mathbb{P}(Z_n = j).] \end{aligned}$$

Proof of the recurrence



To prove:

$$\mathbb{P}(Q_n = j \,|\, I_n = k)$$

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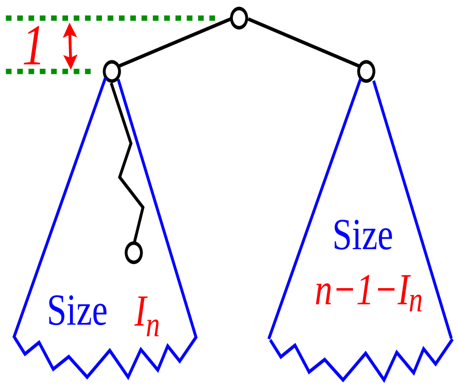

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a) $\mathcal{T}_1$ and $\mathcal{T}_2$ independent, b) $\mathcal{T}_1$ and $\mathcal{T}_2$ random BSB.

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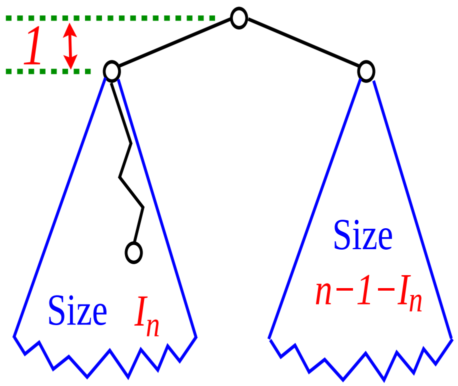

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$\pi = (5, 7, 3, 2, 6, 8, 1, 4)$

π equiprobable permutation in S_n.

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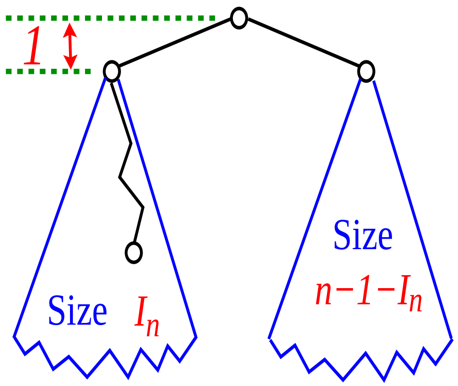

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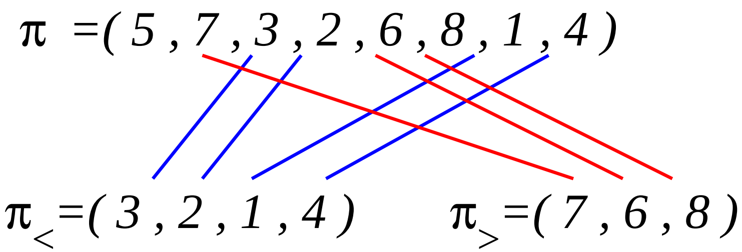

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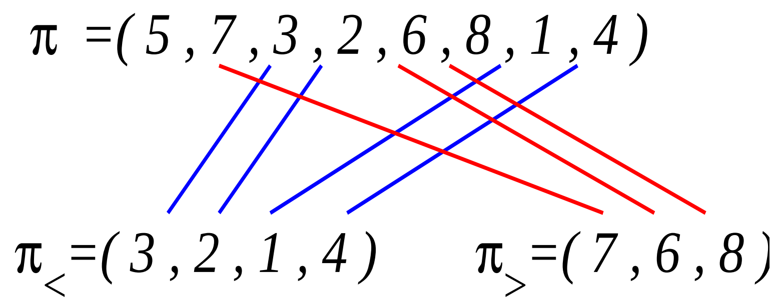
Construct $\pi_<$ and $\pi_>$

Proof of the recurrence II

π equiprobable in S_n.



Proof of the recurrence II

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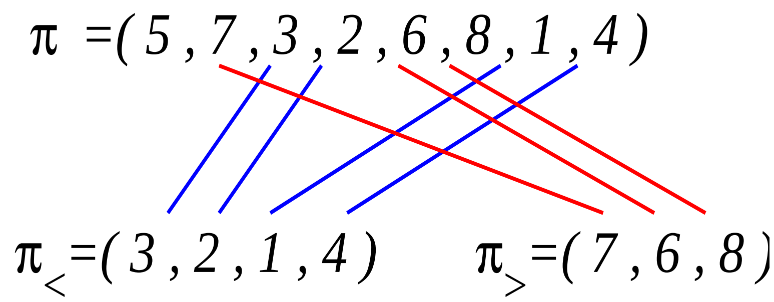
We show: Given $\pi_1 = k + 1$:



$\pi_<$ and $\pi_>$ independent and equiprobable on S_k and S_{n-1-k}. resp.

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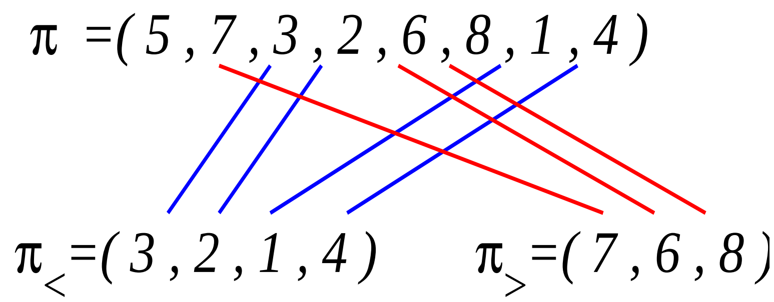

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For arbitrary $\sigma \in S_k$:

$$\mathbb{P}(\pi_< = \sigma \,|\, \pi_1 = k+1) = \frac{1}{1/n} \frac{\binom{n-1}{k}(n-1-k)!}{n!} = \frac{1}{k!}$$

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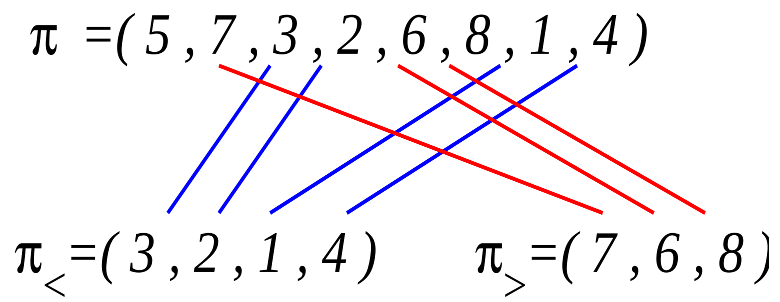

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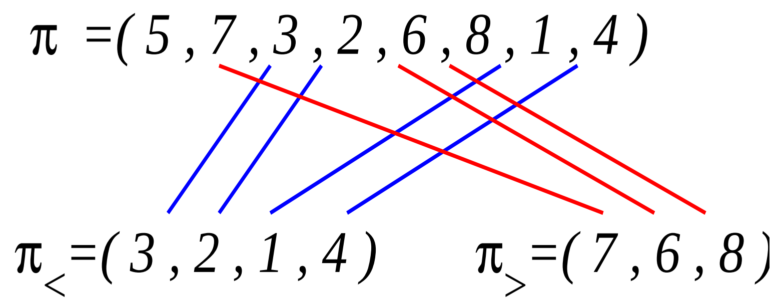

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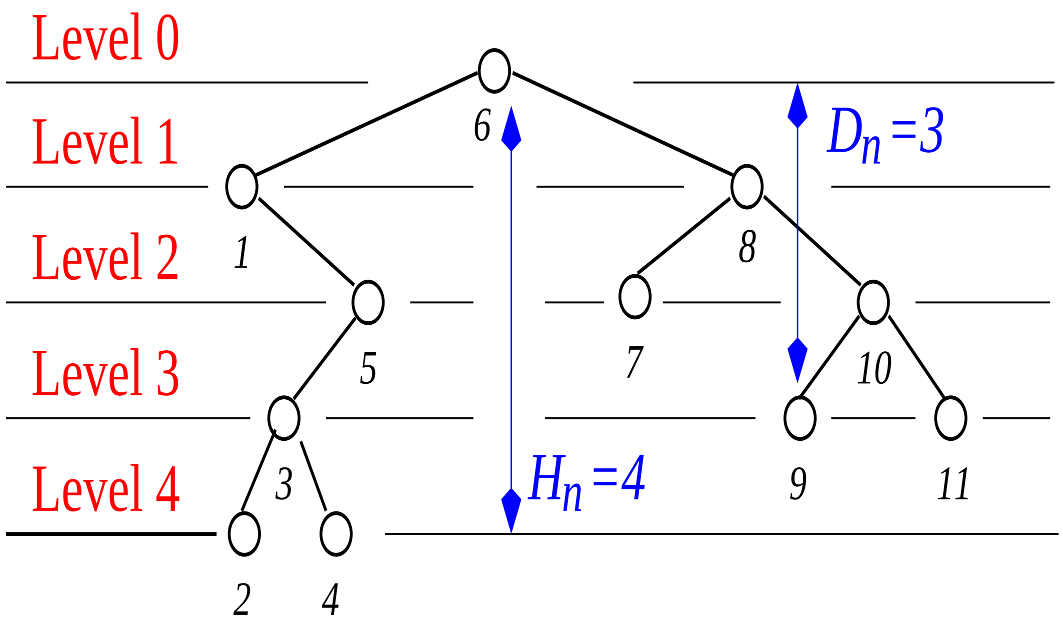
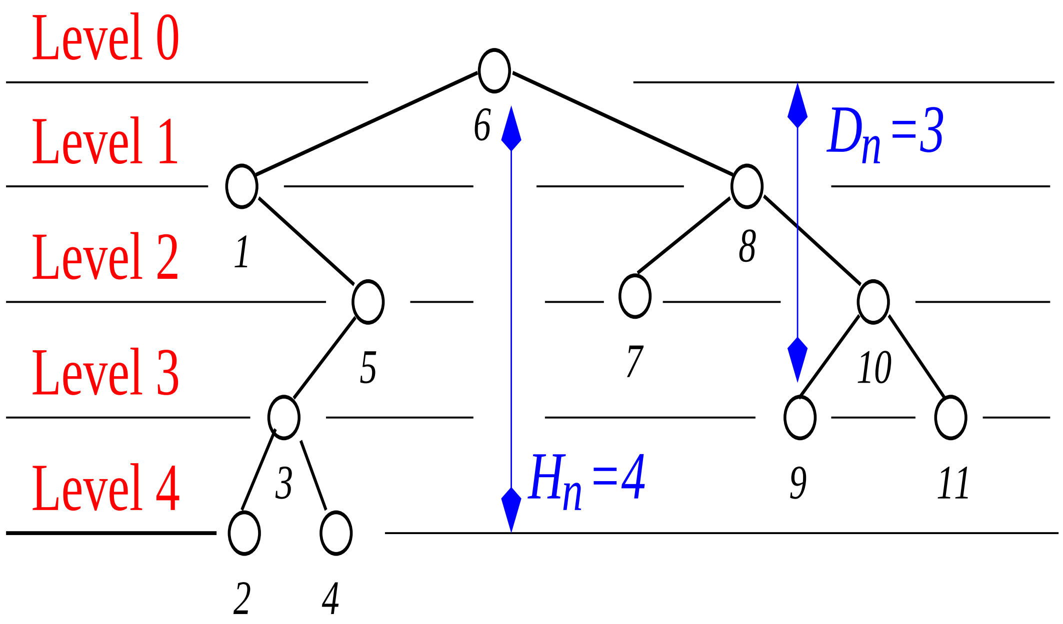
Second assertion similar.

Other BST recurrences



$$Q_n \stackrel{d}{=} Q^{(1)}_{I_n} + Q^{(2)}_{n-1-I_n} + n - 1$$

Other BST recurrences



$$Q_n \stackrel{d}{=} Q^{(1)}_{I_n} + Q^{(2)}_{n-1-I_n} + n - 1$$

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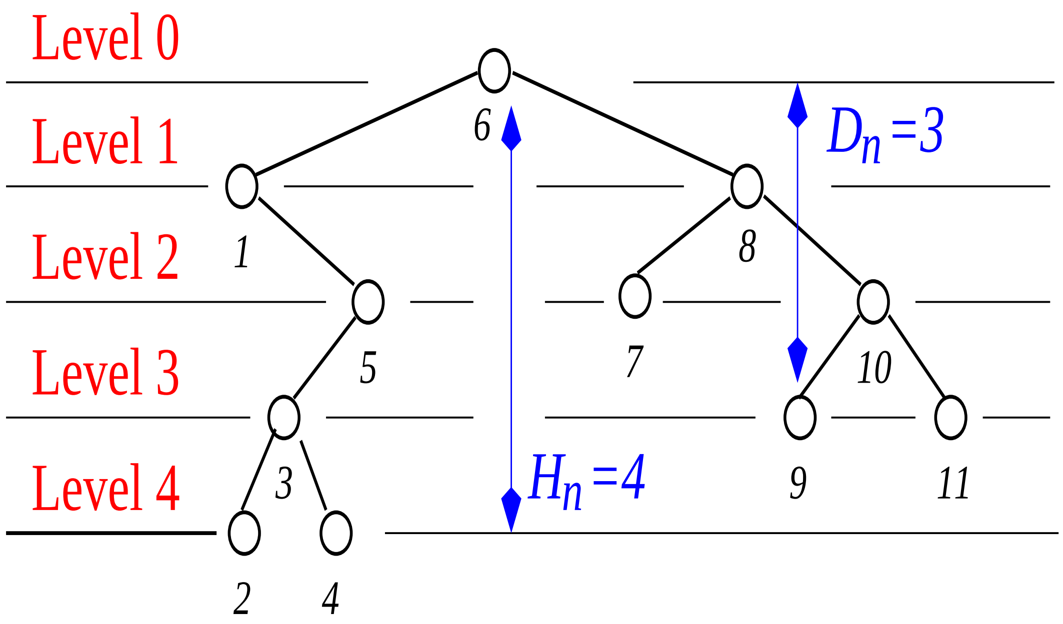

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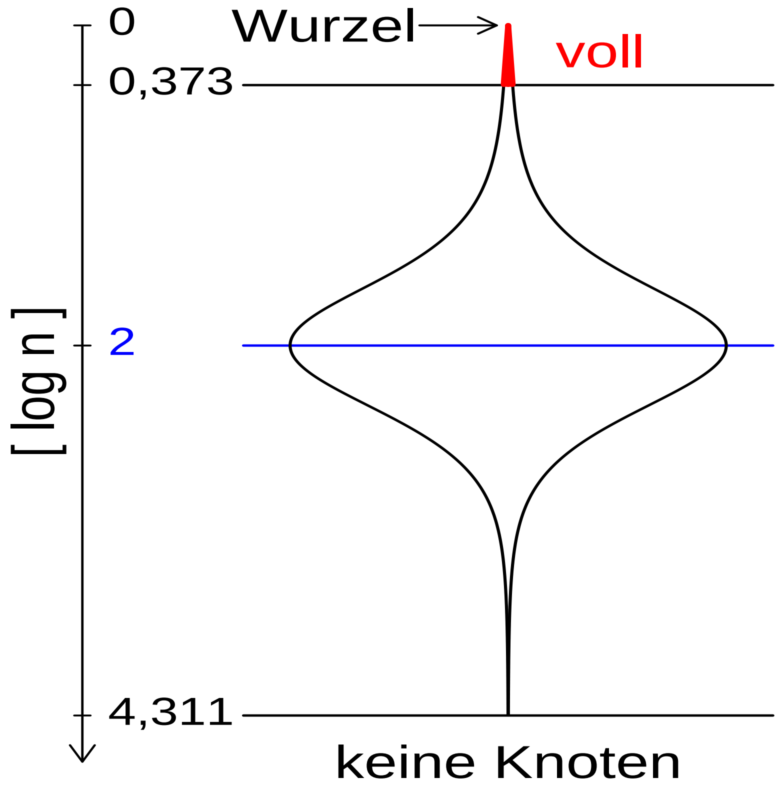
$$D_n \stackrel{d}{=} \mathbf{1}_{A_n} D_{I_n} + \mathbf{1}_{A_n^c} D_{n-1-I_n} + 1$$

The height



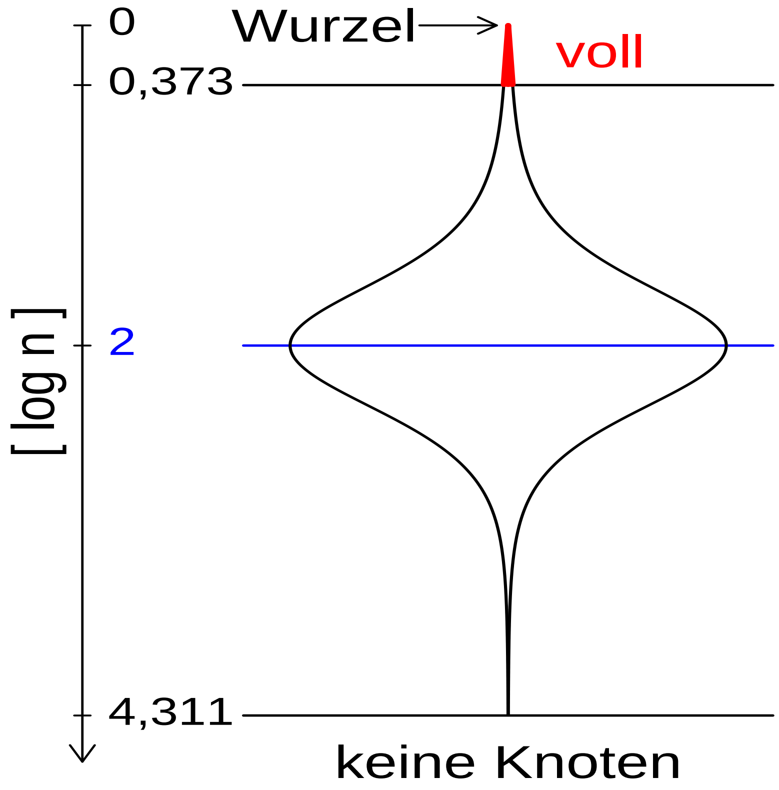
. . .

The height

height H_n:

$$\mathbb{E}\, H_n \sim \alpha_+ \log n,$$

$$\frac{H_n}{\log n} \xrightarrow{\mathbb{P}} \alpha_+$$



. . .

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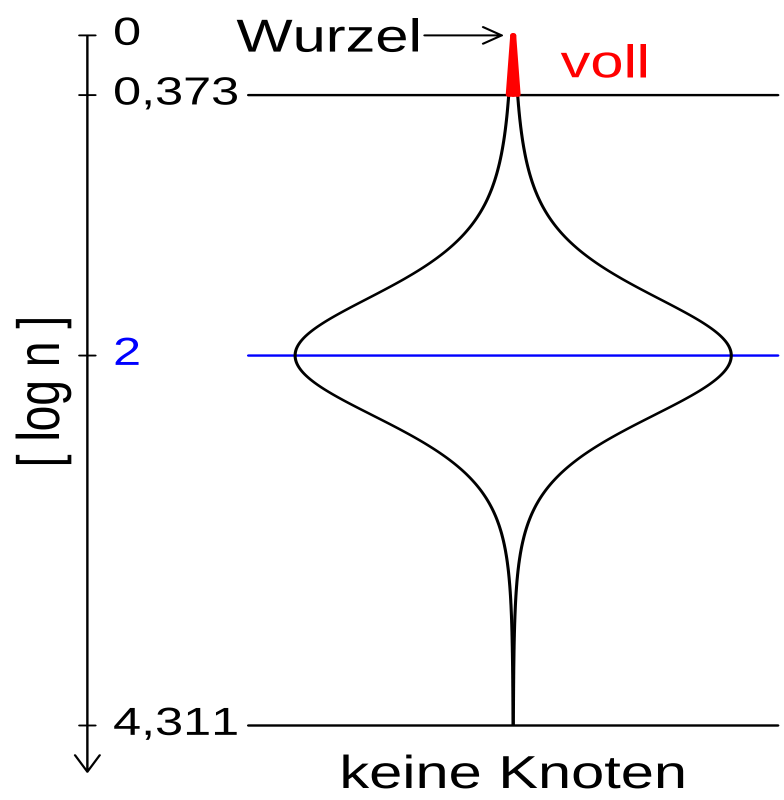
saturation level S_n:

$$\mathbb{E}\, S_n \sim \alpha_- \log n,$$

$$\frac{S_n}{\log n} \xrightarrow{\mathbb{P}} \alpha_-$$



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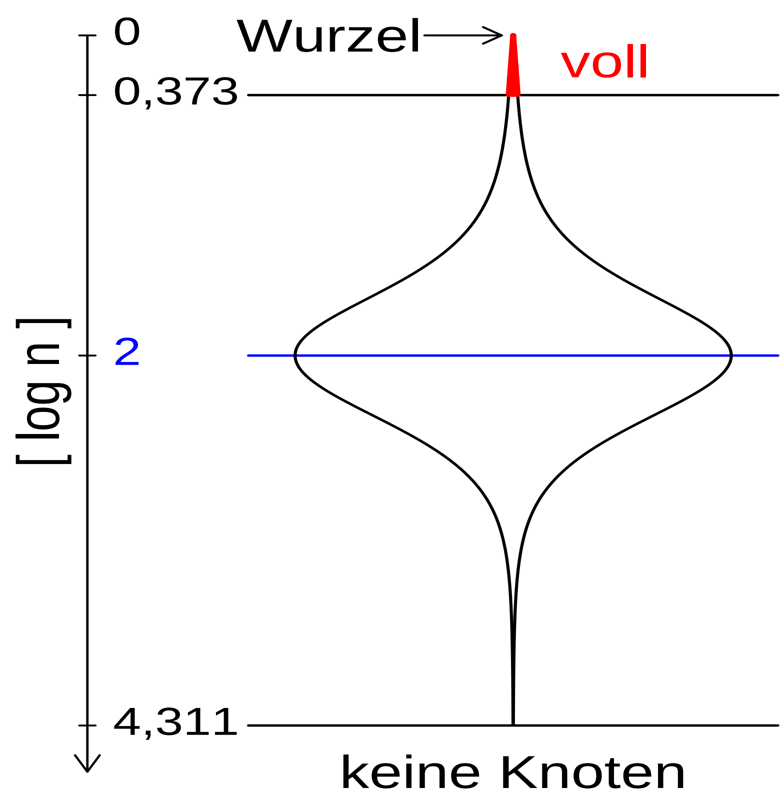

Here, $0 < \alpha_- < 2 < \alpha_+$ are the solutions of

$$\alpha \log\left(\frac{2e}{\alpha}\right) = 1, \quad \alpha_- \doteq 0,373, \quad \alpha_+ \doteq 4,311$$

Pittel ('84), Devroye ('86), Reed ('03), Drmota ('03),. . .

Expected internal path length

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$$Q_n \stackrel{d}{=} Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1, \qquad I_n \stackrel{d}{=} \mathrm{unif}\{0, \ldots, n-1\}.$$

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$$q_n := \mathbb{E}\, Q_n \;=\; \mathbb{E}\left[Q^{*}_{I_n} + Q^{**}_{n-1-I_n} + n - 1\right]$$

Expected internal path length

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$$\begin{aligned} q_n := \mathbb{E}\, Q_n &= \mathbb{E}\left[Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1\right] \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}\left[Q^*_k + Q^{**}_{n-1-k} + n - 1\right] \end{aligned}$$

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Expected internal path length

$$Q_n \stackrel{d}{=} Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1, \qquad I_n \stackrel{d}{=} \mathrm{unif}\{0, \ldots, n-1\}.$$

$$\begin{aligned} q_n := \mathbb{E}\, Q_n &= \mathbb{E}\left[Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1\right] \\ &= \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{E}\left[Q^*_k + Q^{**}_{n-1-k} + n - 1\right] \\ &= \frac{1}{n}\sum_{k=0}^{n-1} (q_k + q_{n-1-k} + n - 1) \\ &= n - 1 + \frac{2}{n}\sum_{k=0}^{n-1} q_k. \end{aligned}$$

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Solves easily:

$$q_n = 2(n+1)\mathcal{H}_n - 4n = 2n\log(n) + (2\gamma - 4)n + o(n).$$

[$\mathcal{H}_n := \sum_{i=1}^n 1/i$ harmonic numbers.]

Rescaling

$$Q_n \stackrel{d}{=} Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1, \qquad q_n = 2n \log(n) + cn + o(n).$$

Rescaling

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Scaling

$$Y_n := \frac{Q_n - q_n}{n}.$$

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Scaling

$$Y_n := \frac{Q_n - q_n}{n}.$$

Then

$$Y_n \stackrel{d}{=} \frac{1}{n}\left(Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1 - q_n\right)$$

Rescaling

$$Q_n \stackrel{d}{=} Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1, \qquad q_n = 2n\log(n) + cn + o(n).$$

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$$Y_n := \frac{Q_n - q_n}{n}.$$

Then

$$\begin{aligned} Y_n &\stackrel{d}{=} \frac{1}{n}\left(Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1 - q_n\right) \\ &= \frac{1}{n}\left(I_n \frac{Q^*_{I_n} \pm q_{I_n}}{I_n} + (n-1-I_n)\frac{Q^{**}_{n-1-I_n} \pm q_{n-1-I_n}}{n-1-I_n} + n - 1 - q_n\right) \end{aligned}$$

Rescaling

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Rescaling

$$Q_n \stackrel{d}{=} Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1, \qquad q_n = 2n\log(n) + cn + o(n).$$

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$$Y_n := \frac{Q_n - q_n}{n}.$$

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$$\begin{aligned} Y_n &\stackrel{d}{=} \frac{1}{n}\left(Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1 - q_n\right) \\ &= \frac{1}{n}\left(I_n \frac{Q^*_{I_n} \pm q_{I_n}}{I_n} + (n-1-I_n)\frac{Q^{**}_{n-1-I_n} \pm q_{n-1-I_n}}{n-1-I_n} + n - 1 - q_n\right) \\ &= \underbrace{\frac{I_n}{n}} Y^*_{I_n} + \underbrace{\frac{n-1-I_n}{n}} Y^{**}_{n-1-I_n} + b^{(n)} \end{aligned}$$

with

$$b^{(n)} = \frac{1}{n}\left(q_{I_n} + q_{n-1-I_n} - q_n + n - 1\right).$$

Rescaling

$Q_n \stackrel{d}{=} Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1, \quad q_n = 2n\log(n) + cn + o(n).$

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Then

$$\begin{aligned} Y_n &\stackrel{d}{=} \frac{1}{n}\left(Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1 - q_n\right) \\ &= \frac{1}{n}\left(I_n \frac{Q^*_{I_n} \pm q_{I_n}}{I_n} + (n-1-I_n)\frac{Q^{**}_{n-1-I_n} \pm q_{n-1-I_n}}{n-1-I_n} + n - 1 - q_n\right) \\ &= \underbrace{\frac{I_n}{n}}_{\to U} Y^*_{I_n} + \underbrace{\frac{n-1-I_n}{n}}_{\to 1-U} Y^{**}_{n-1-I_n} + b^{(n)} \end{aligned}$$

with

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Rescaling II

$$q_n = 2n\log(n) + cn + o(n).$$

$$b^{(n)} = \frac{1}{n}\left(q_{I_n} + q_{n-1-I_n} - q_n + n - 1\right).$$

Rescaling II

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$$\begin{aligned} b^{(n)} &= \frac{1}{n}\Big(q_{I_n} + q_{n-1-I_n} - q_n + n - 1\Big). \\ &= \frac{1}{n}\Big(2I_n\log(I_n) + cI_n + 2(n-1-I_n)\log(n-1-I_n) + c(n-1-I_n) \\ &\qquad - 2n\log(n) - cn + o(n) + n - 1\Big) \end{aligned}$$

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&\qquad - 2n\log(n) - cn + o(n) + n - 1\Big) \\
&= \frac{1}{n}\Big(2I_n\log(I_n) \qquad + 2(n-1-I_n)\log(n-1-I_n) \\
&\qquad - 2(I_n + (n-1-I_n) + 1)\log(n) + o(n) + n - 1\Big)
\end{aligned}$$

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&= 2\frac{I_n}{n}\log\left(\frac{I_n}{n}\right) + 2\frac{n-1-I_n}{n}\log\left(\frac{n-1-I_n}{n}\right) + 1 + o(1)
\end{aligned}$$

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&\qquad - 2(I_n + (n-1-I_n) + 1)\log(n) + o(n) + n - 1\Big) \\
&= 2\frac{I_n}{n}\log\left(\frac{I_n}{n}\right) + 2\frac{n-1-I_n}{n}\log\left(\frac{n-1-I_n}{n}\right) + 1 + o(1) \\
&\to 2U\log(U) + 2(1-U)\log(1-U) + 1 =: g(U)
\end{aligned}$$

Rescaling: Summary

$Q_n \stackrel{d}{=} Q^*_{I_n} + Q^{**}_{n-1-I_n} + n - 1, \quad q_n = 2n\log(n) + cn + o(n).$
Scaling

$$Y_n := \frac{Q_n - q_n}{n}.$$

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with

$$g(U) = 2U\log(U) + 2(1-U)\log(1-U) + 1$$

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Hence, this suggests

$$Y_n \to Y$$

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with

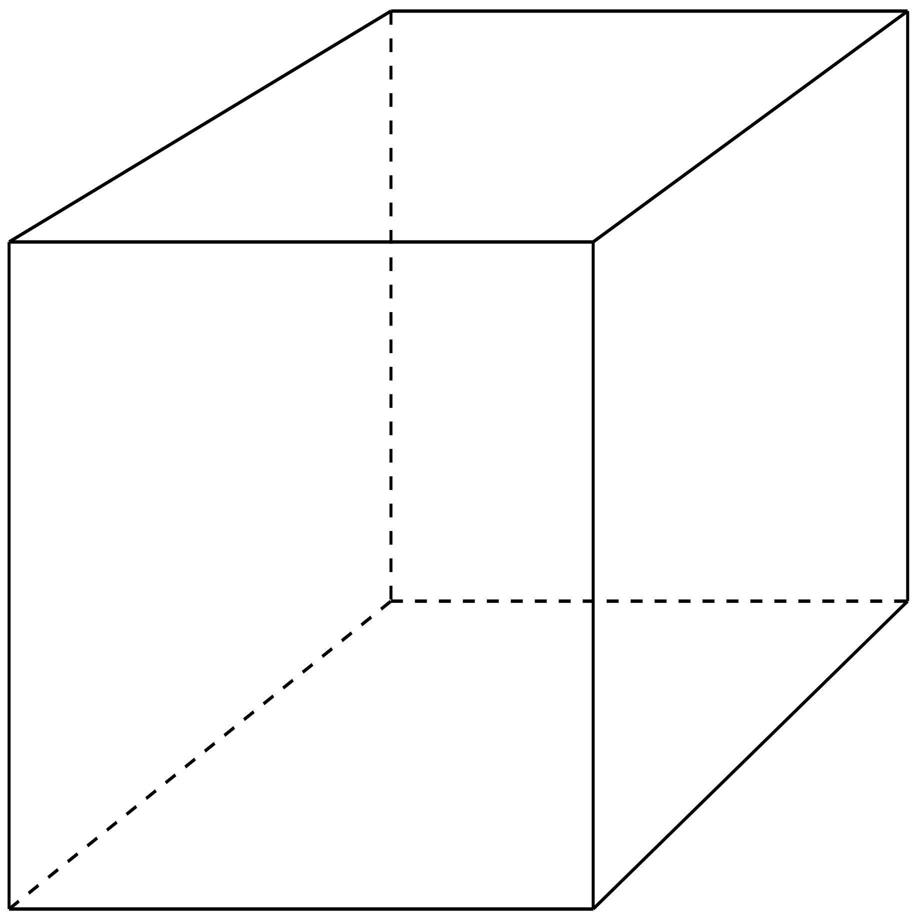
$$g(U) = 2U\log(U) + 2(1-U)\log(1-U) + 1$$

Hence, this suggests

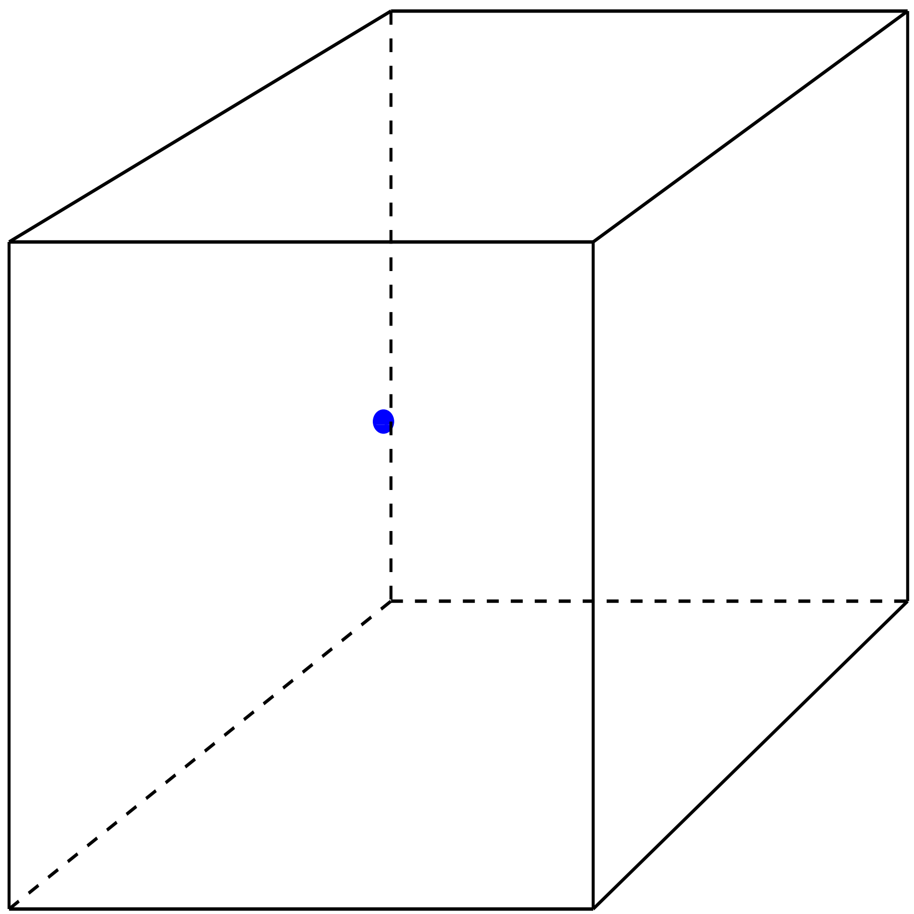
$$Y_n \to Y \stackrel{d}{=} UY^* + (1-U)Y^{**} + g(U),$$

with Y^*, Y^{**}, U independent, $Y \stackrel{d}{=} Y^* \stackrel{d}{=} Y^{**}$.

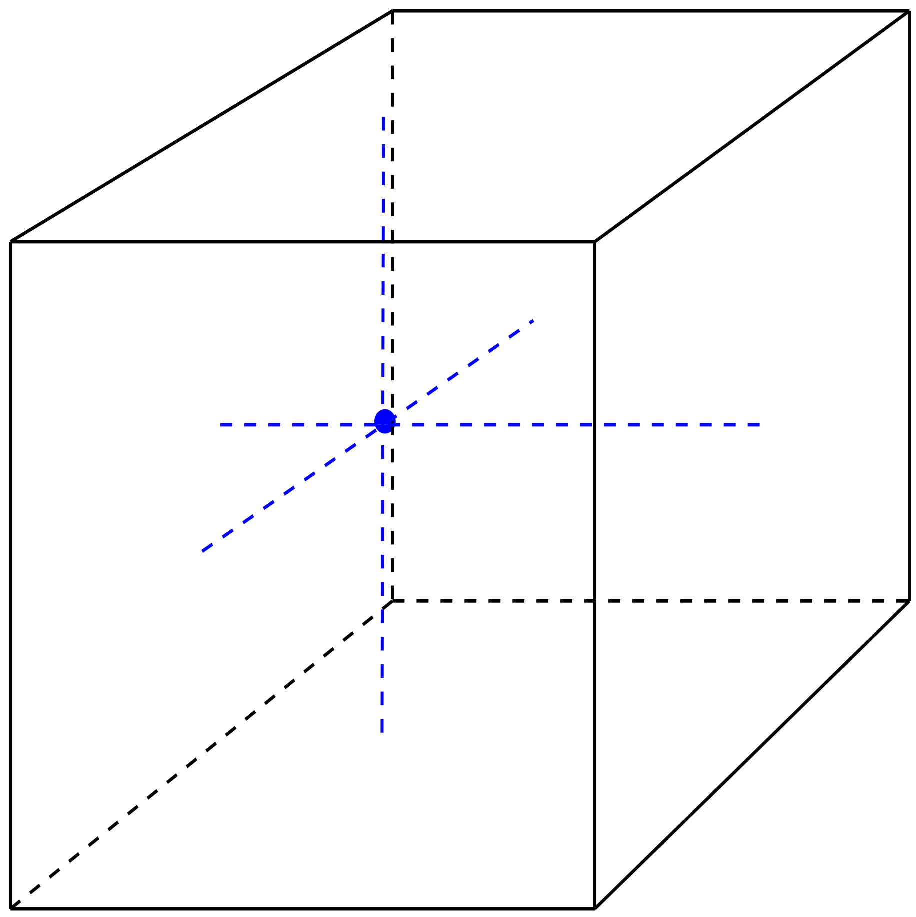
Quadtrees: higher dimensions



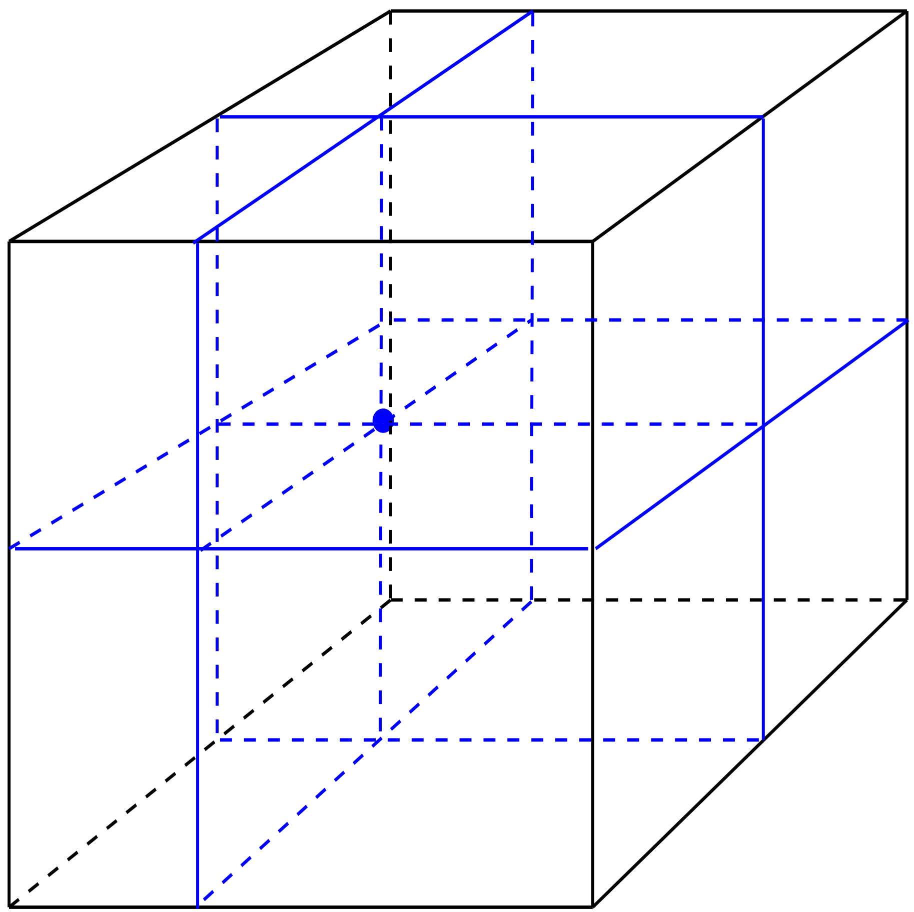
Quadtrees: higher dimensions



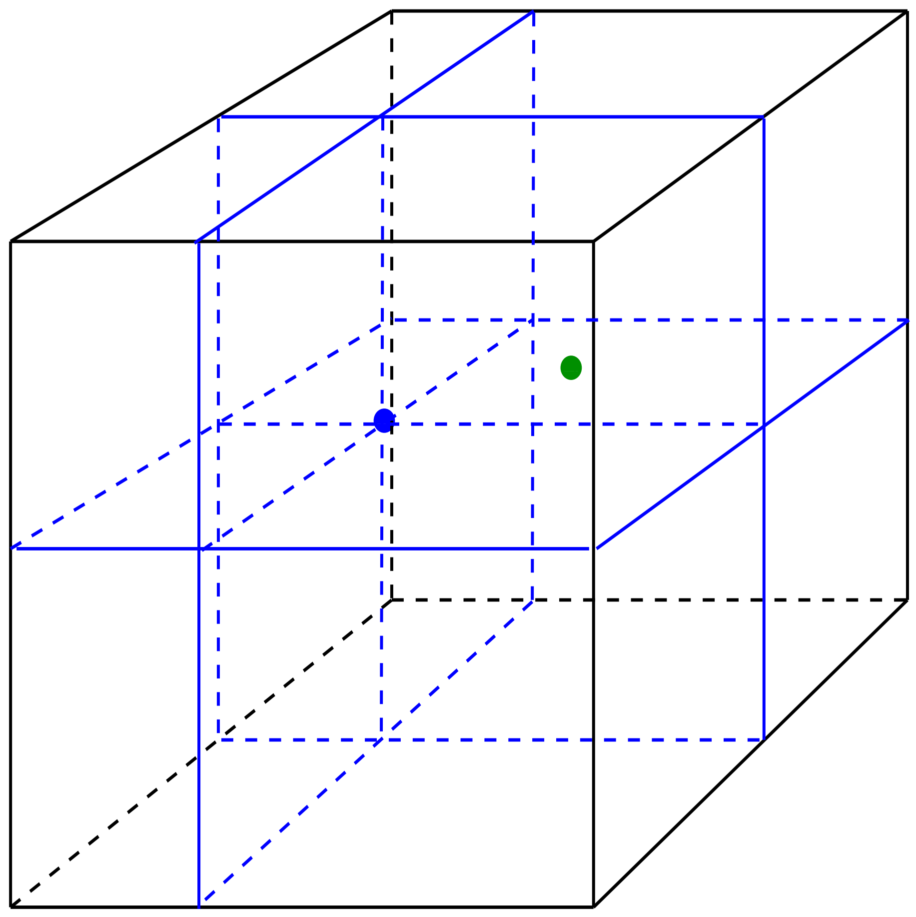
Quadtrees: higher dimensions



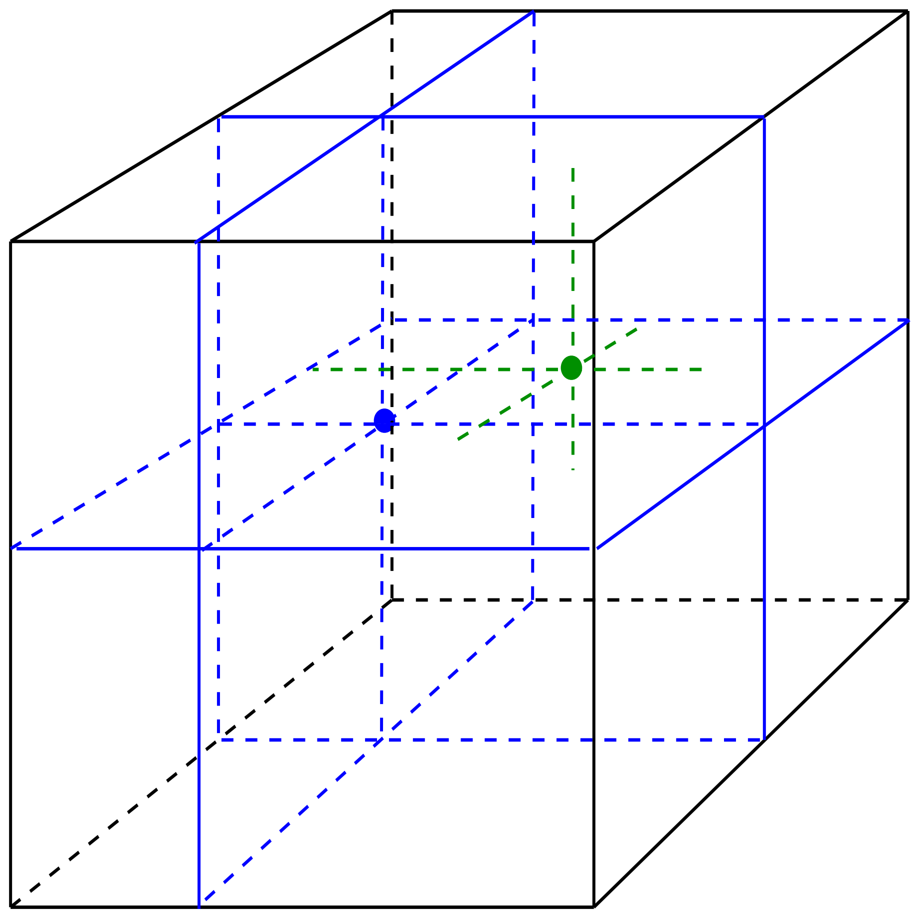
Quadtrees: higher dimensions



Quadtrees: higher dimensions



Quadtrees: higher dimensions



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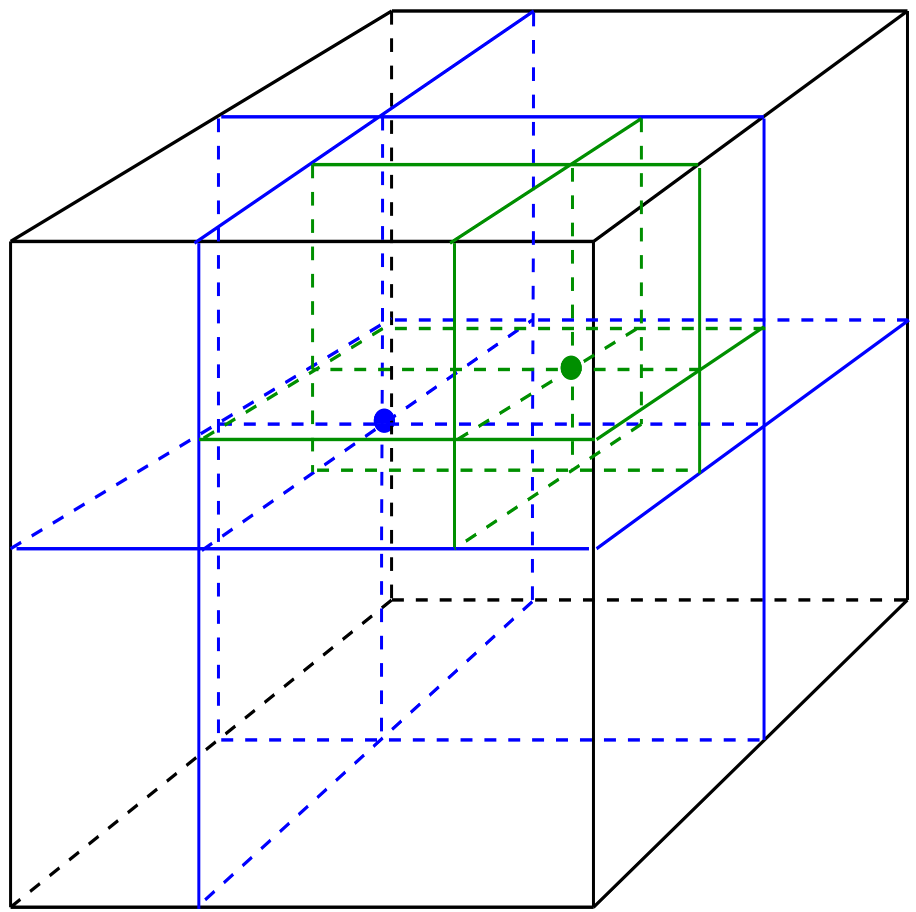

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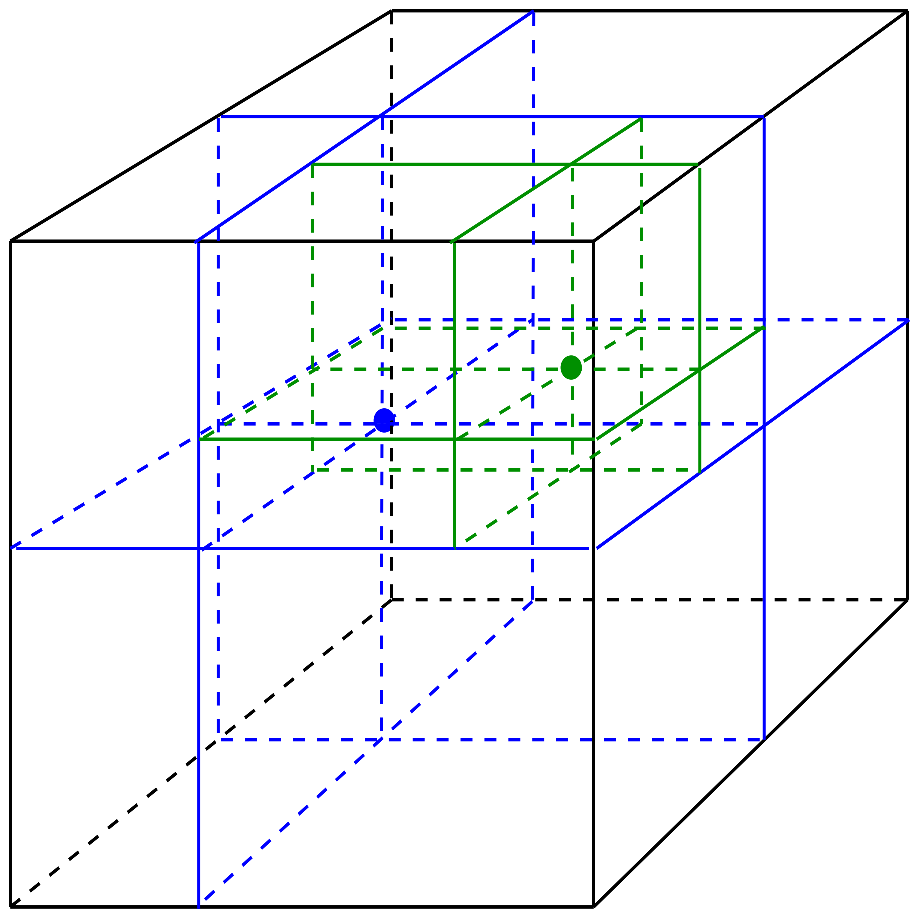

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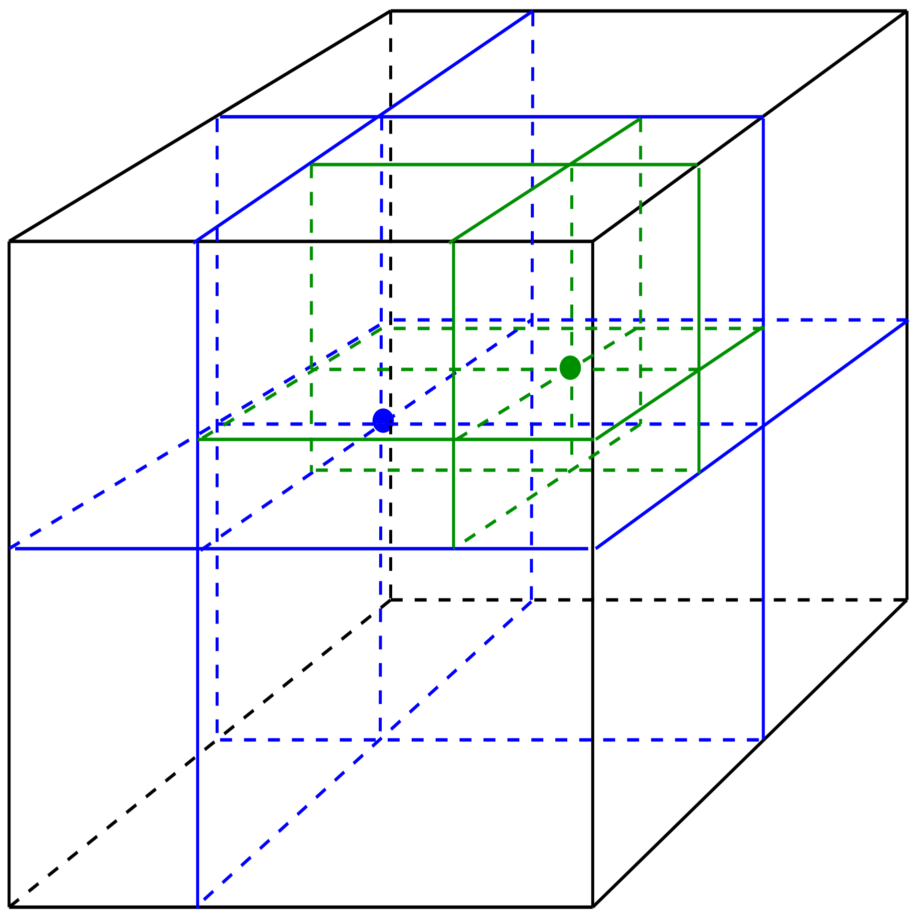
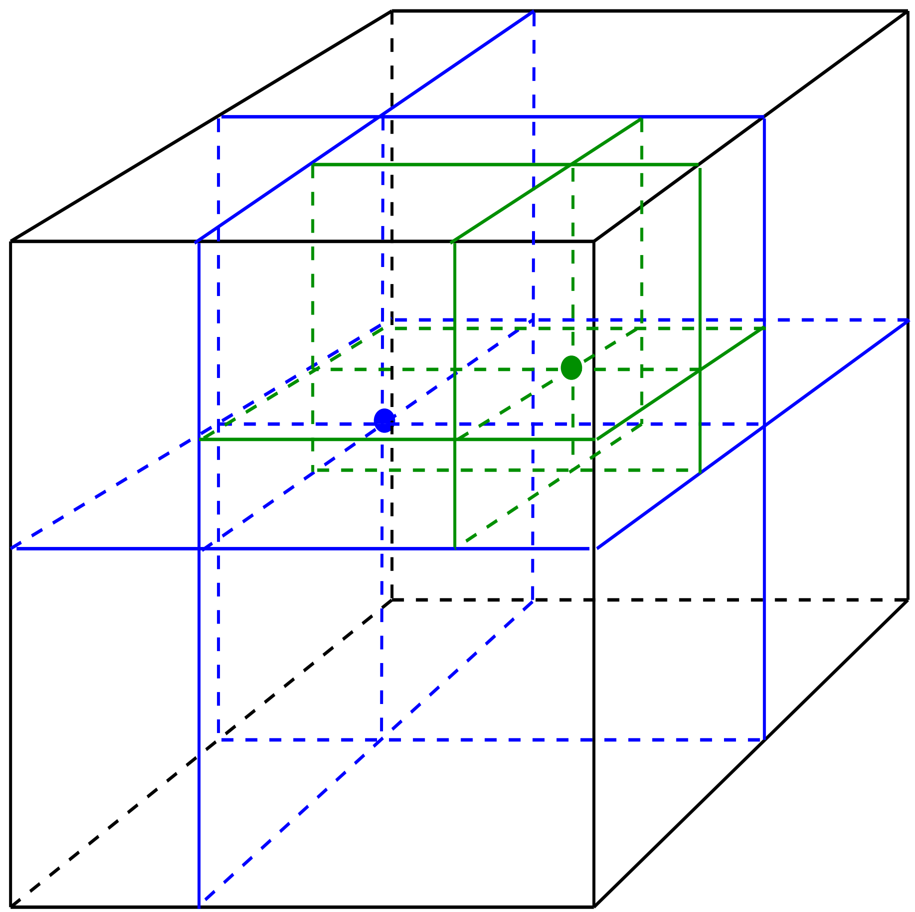


Data:

$U_1, \ldots, U_n$ i.i.d. unif$[0,1]$

Quadtrees: higher dimensions



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$(\langle U_1 \rangle_1, \ldots, \langle U_1 \rangle_{2^d})$ volumes of quadrants

Sizes of subtrees:

$I^{(n)} \stackrel{d}{=} M(n-1; \langle U_1 \rangle_1, \ldots, \langle U_1 \rangle_{2^d}),$

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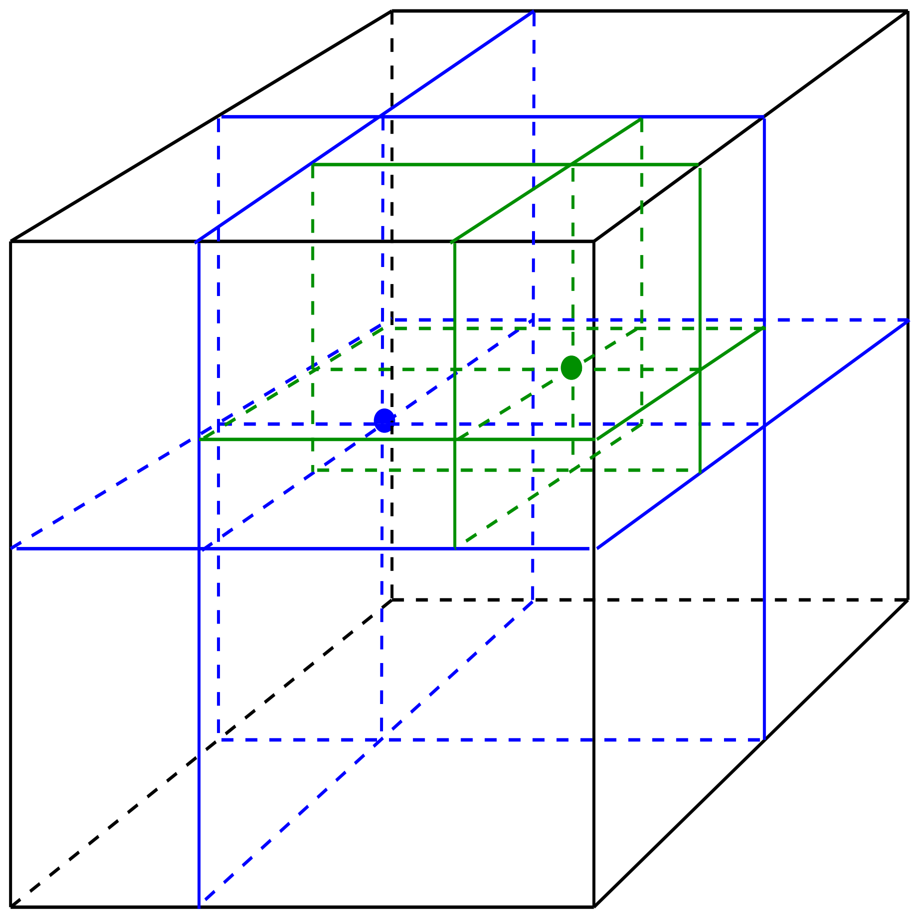

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$$I^{(n)} \stackrel{d}{=} M(n-1; \langle U_1\rangle_1, \ldots, \langle U_1\rangle_{2^d}),$$

Number of leaves:

$$X_n \stackrel{d}{=} \sum_{r=1}^{2^d} X^{(r)}_{I_r^{(n)}} + 1, \quad n \geq m.$$

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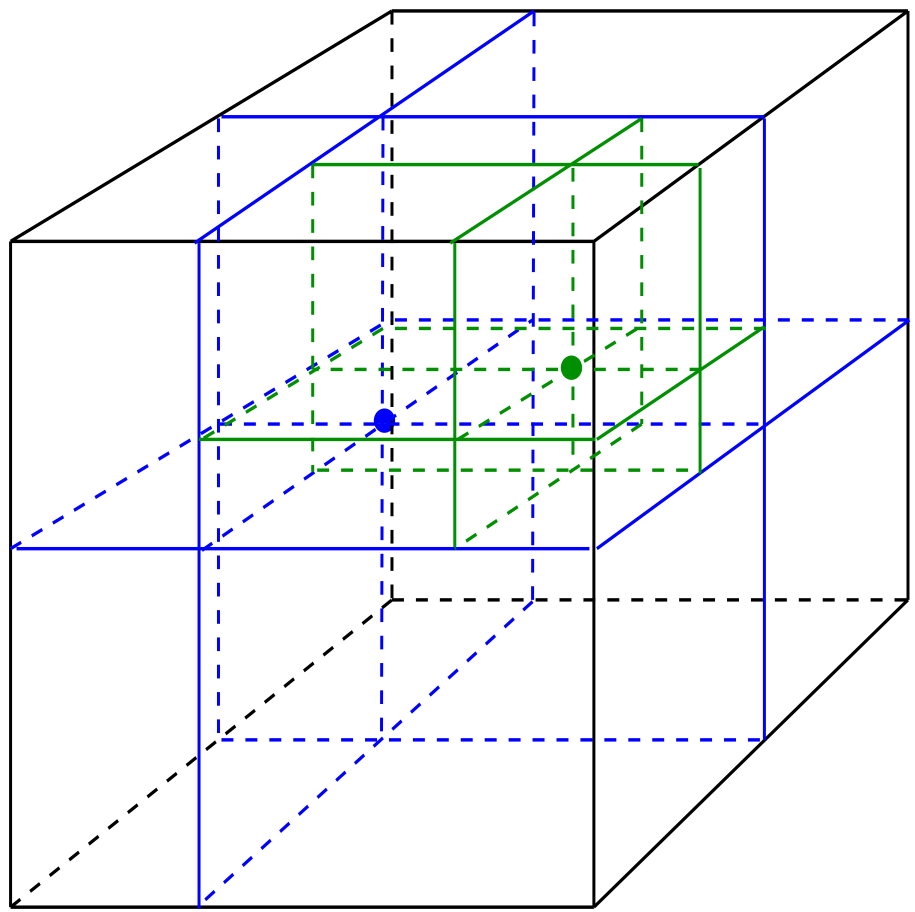

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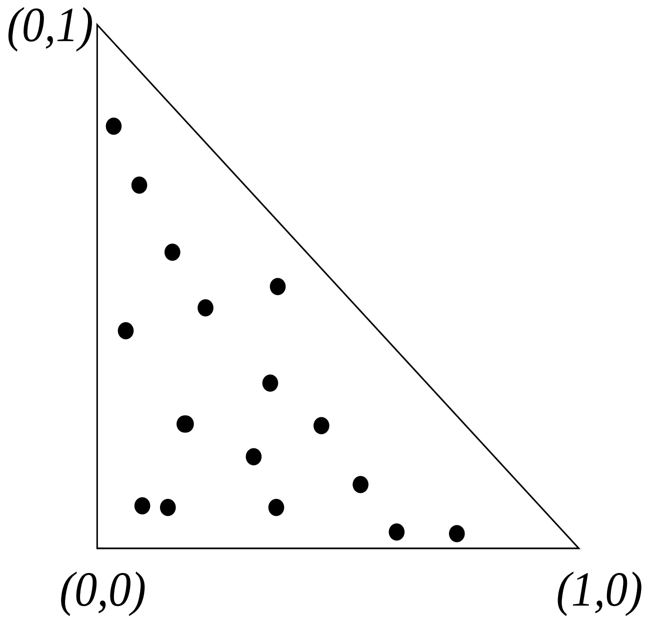
$$X_0 = 0, X_1 = 1.$$

Maxima in right triangles

Data: $u_1, \ldots, u_n$ indep. unif. in right triangle



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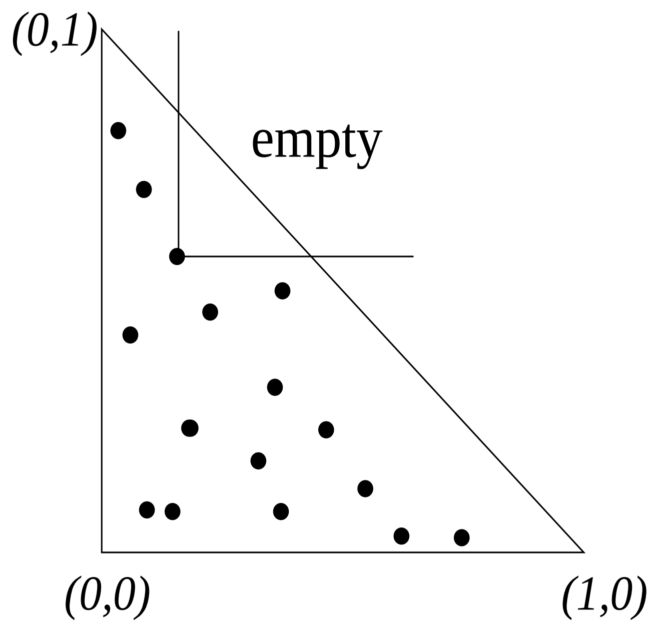
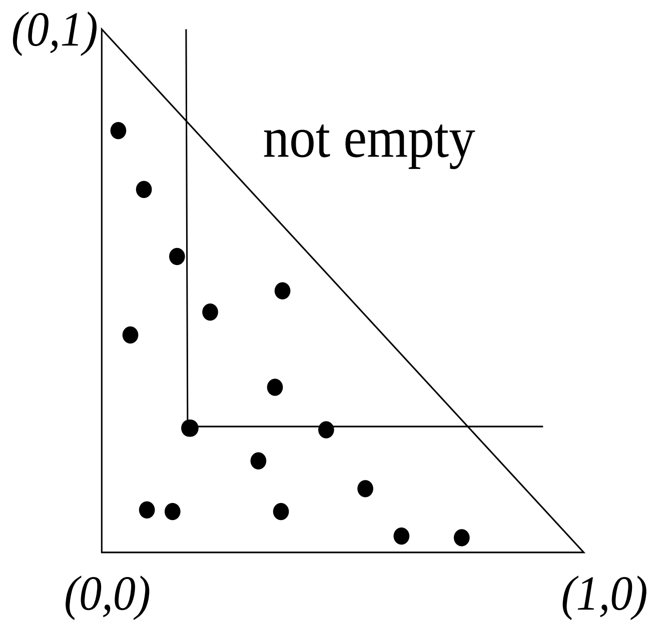


Maxima in right triangles

Data: $U_1, \ldots, U_n$ indep. unif. in right triangle



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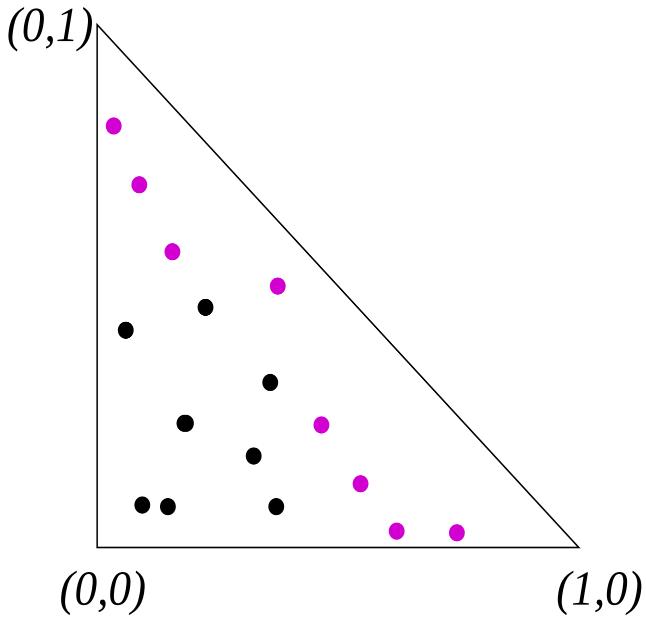

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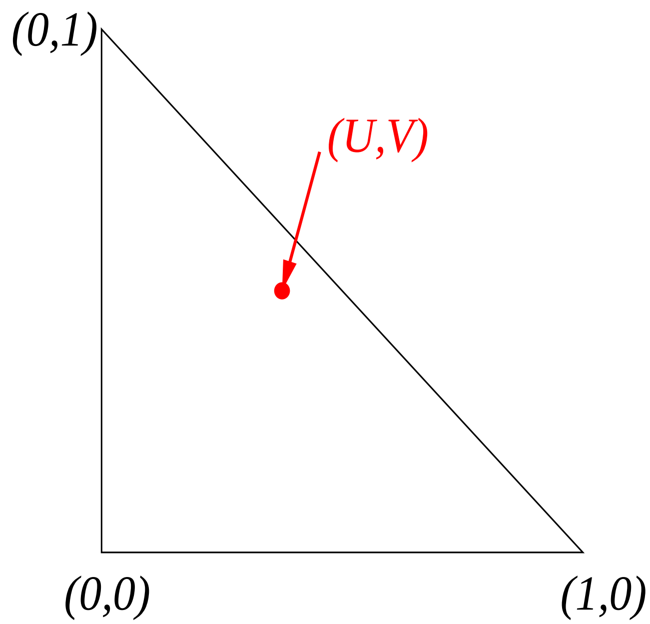

(U, V) has maximal sum of coordinates.

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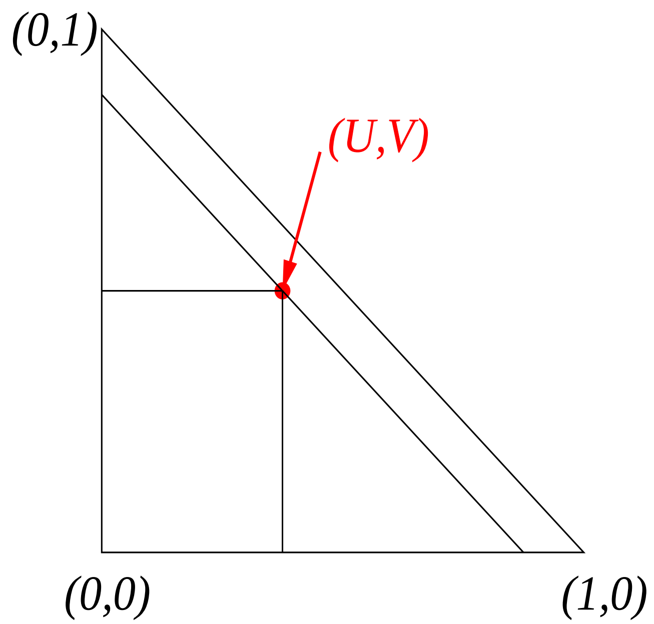

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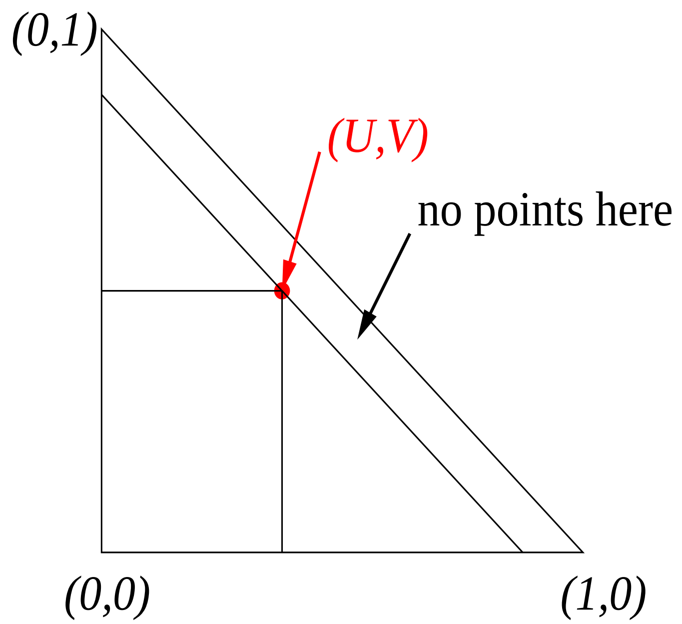

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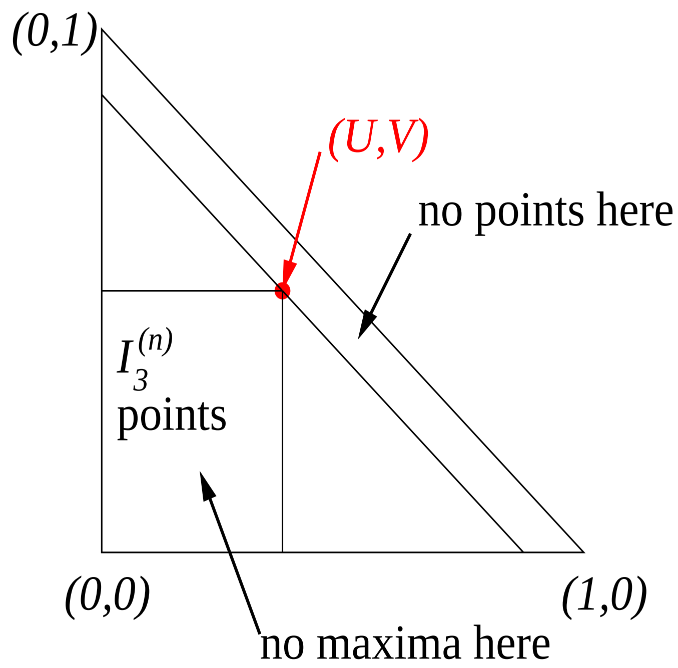

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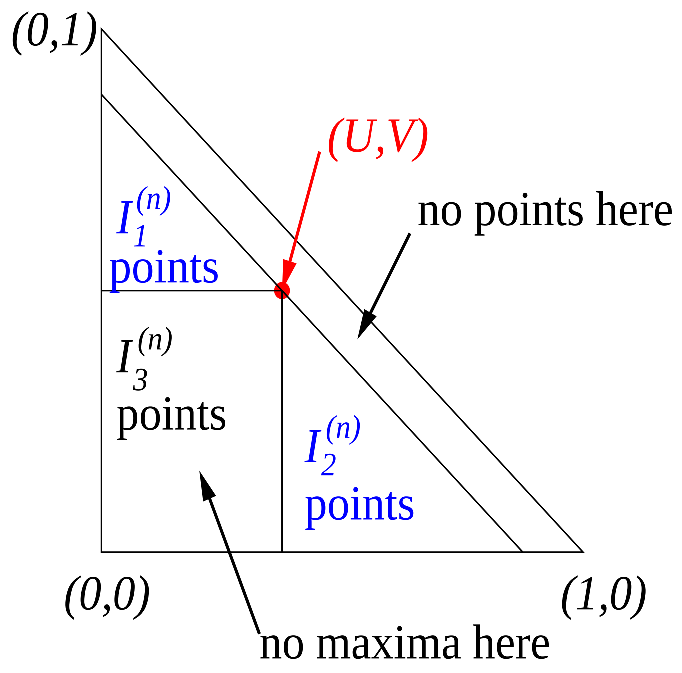

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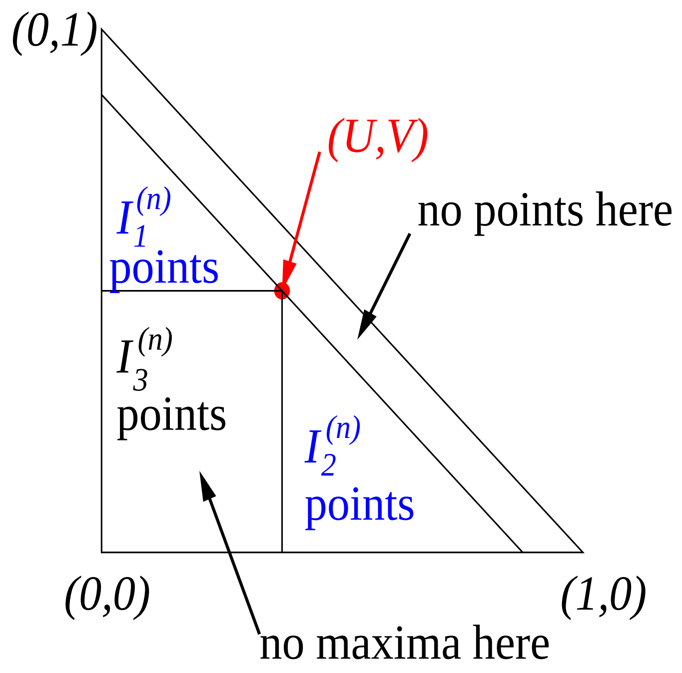
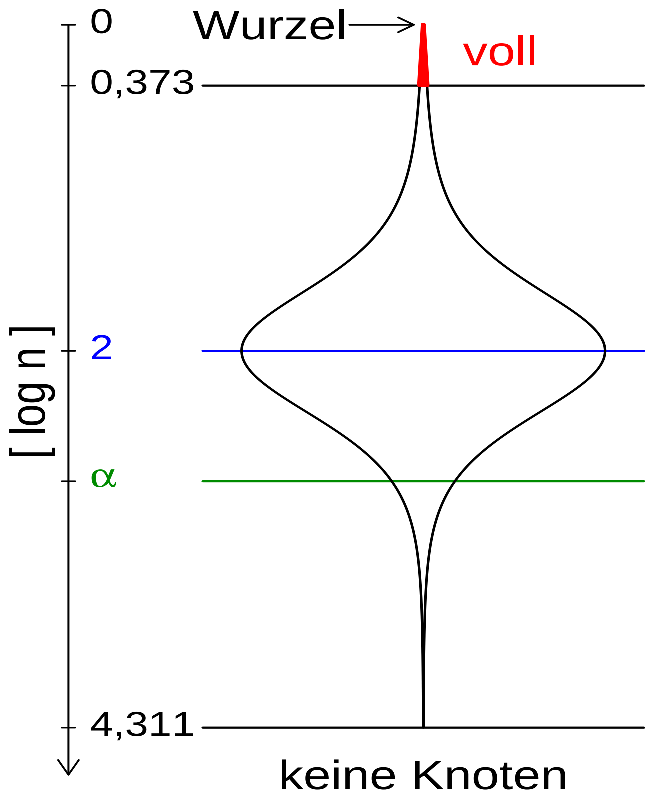


(U, V) has maximal sum of coordinates.

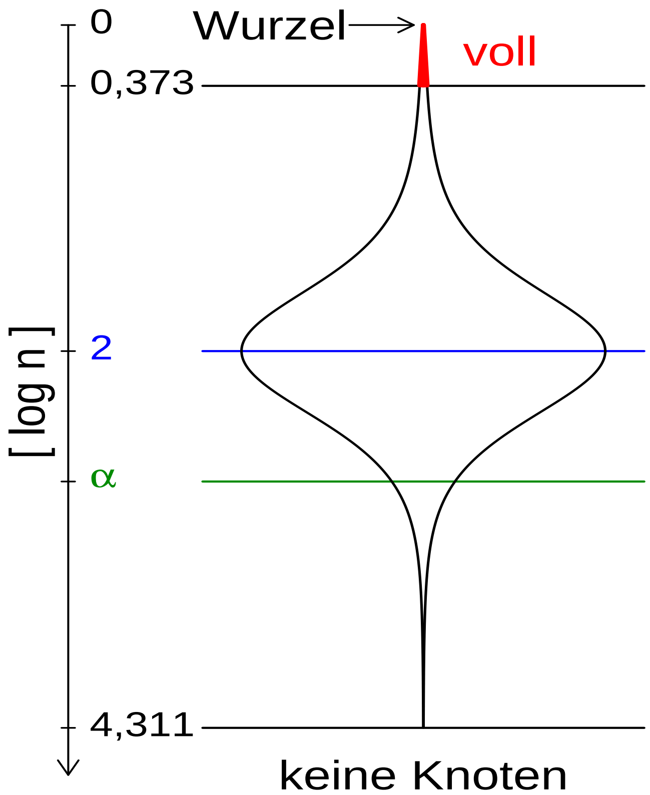
$$X_n \stackrel{d}{=} X^{(1)}_{I_1^{(n)}} + X^{(2)}_{I_2^{(n)}} + 1, \quad n \geq 2.$$

Profile in BST



Profile: $\Upsilon_{n,k}$ (external nodes)

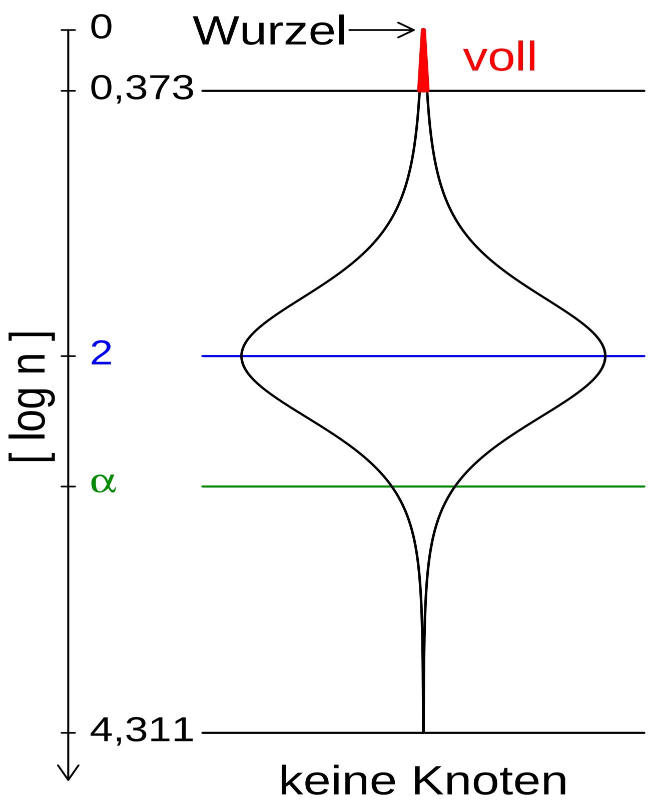
Profile in BST



Profile: $Y_{n,k}$ (external nodes)

Consider: $k \sim \alpha \log n$

Profile in BST

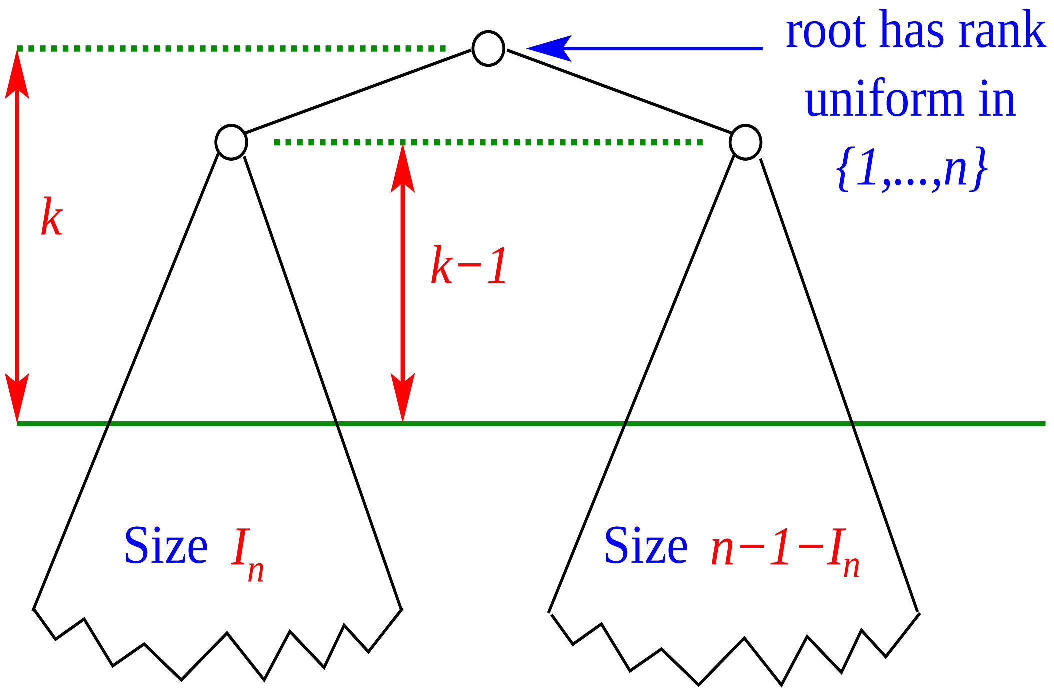


Profile: $Y_{n,k}$ (external nodes)

Consider: $k \sim \alpha \log n$

with $\alpha \in (\alpha_-, \alpha_+)$.

Recursive description of profile



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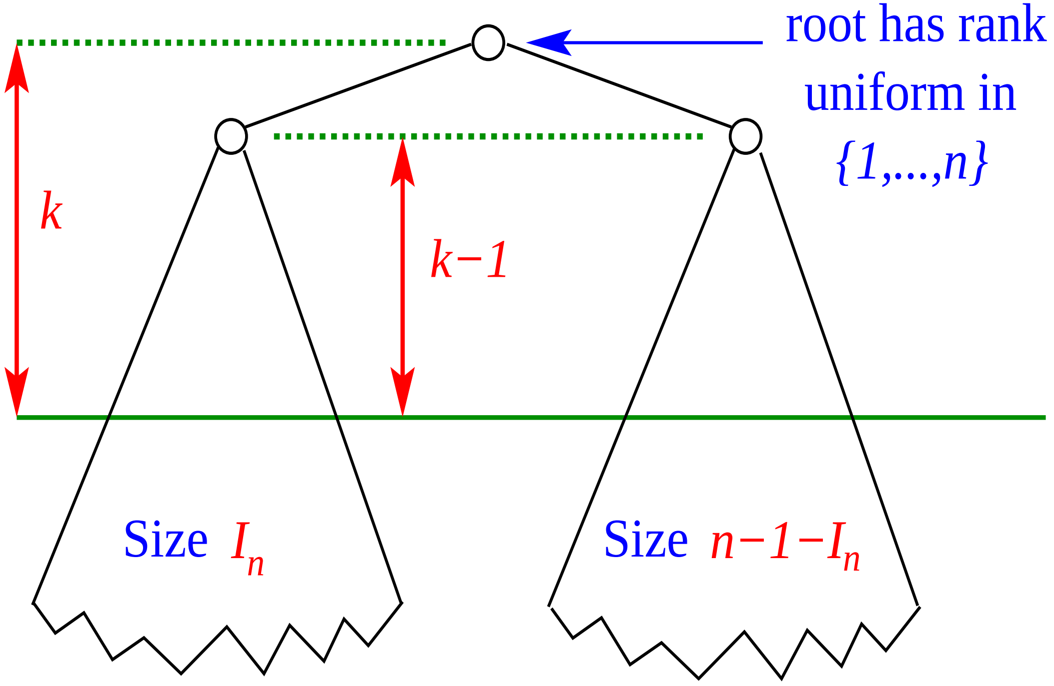

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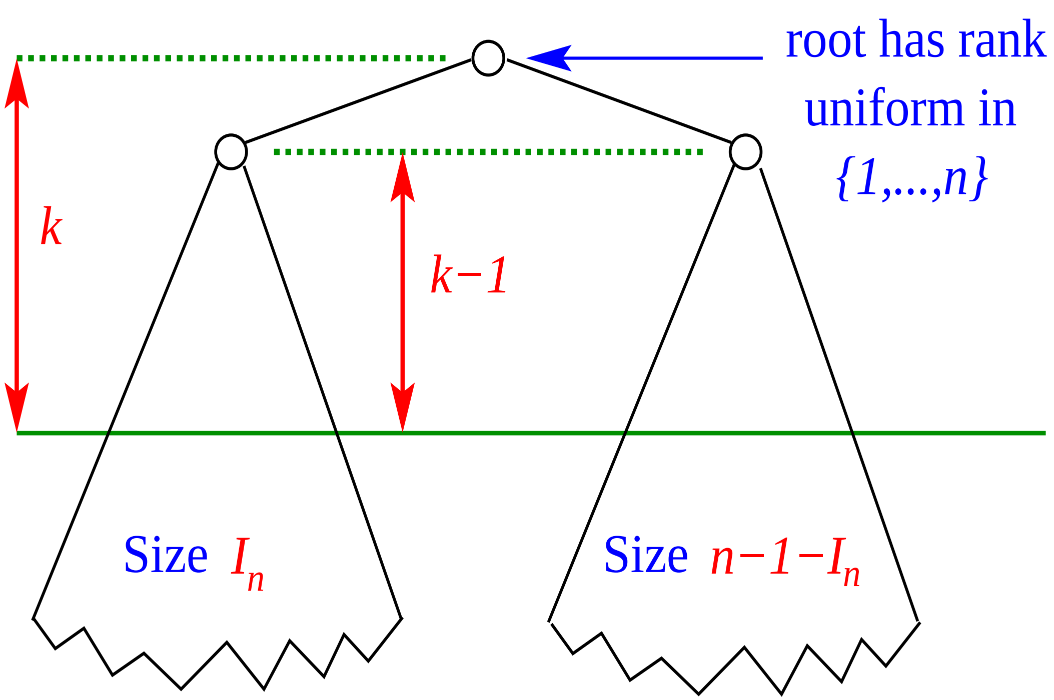

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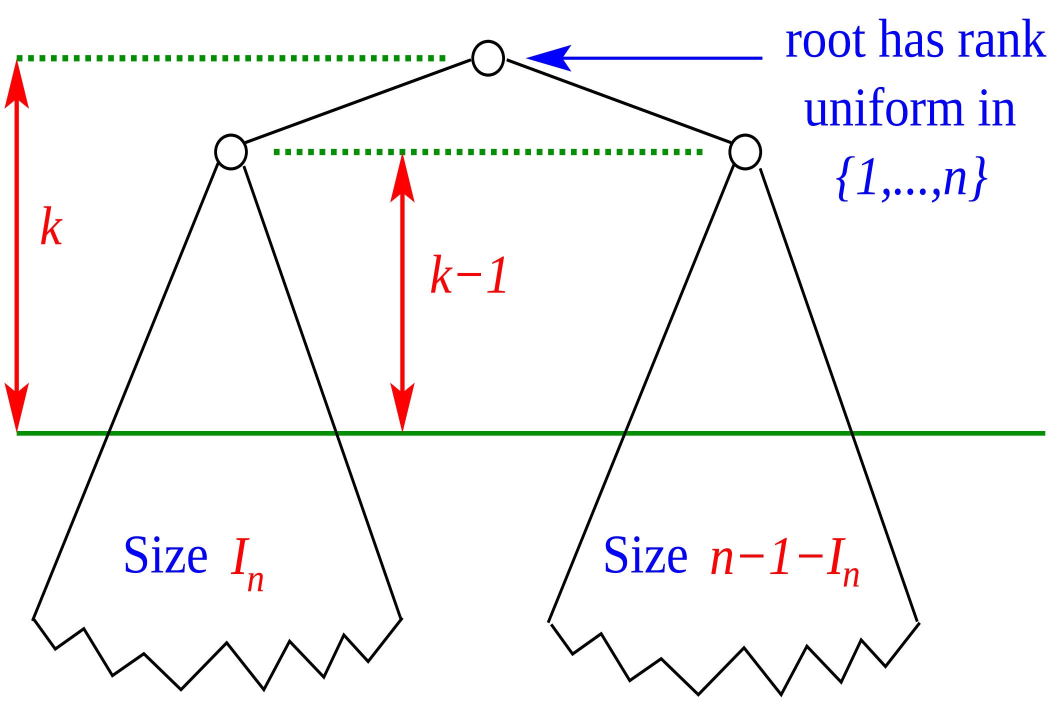

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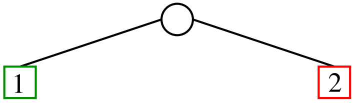

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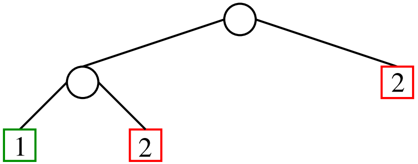

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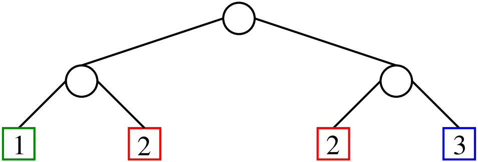

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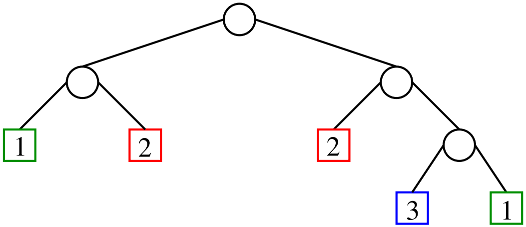

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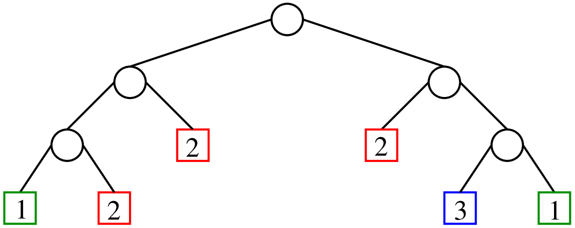

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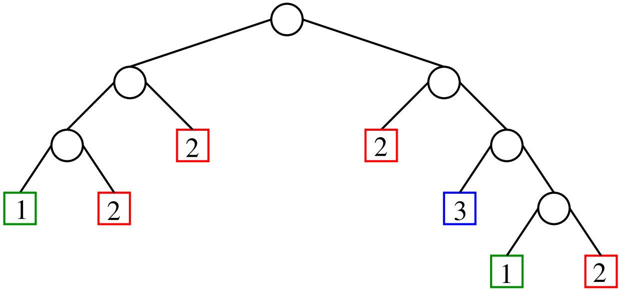

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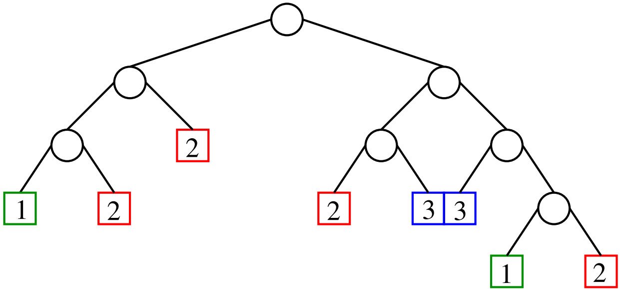

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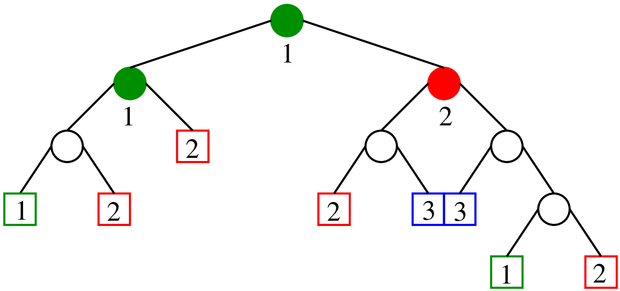
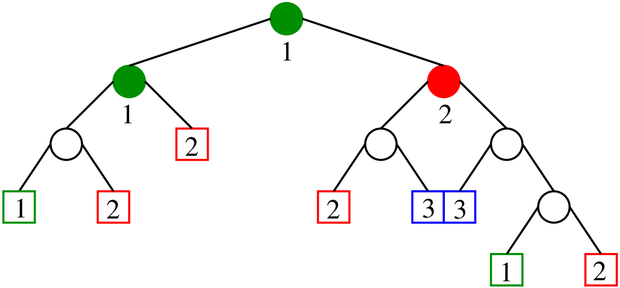


Recurrences



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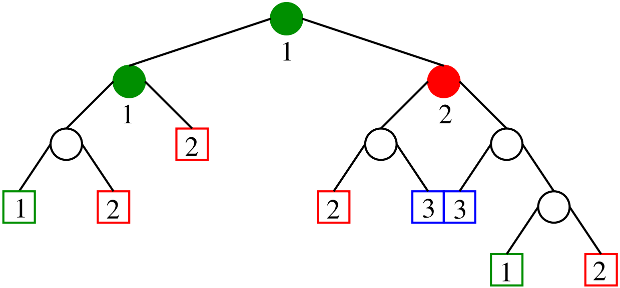

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I_n: uniform on $\{0, \ldots, n-1\}$ and $J_n = n - 1 - I_n$.

General recursion

$$X_n \stackrel{d}{=} \sum_{r=1}^{K} A_r(n) X^{(r)}_{I_r^{(n)}} + b_n, \qquad n > n_0.$$

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- $K \geq 1$ Number of subproblems (also $K = K_n$).
- $X_n^{(r)} \stackrel{d}{=} X_n$ (recursive).
- $I^{(n)} = (I_1^{(n)}, \dots, I_K^{(n)})$ Sizes of subproblems.
- $\left(X_n^{(1)}\right), \dots, \left(X_n^{(K)}\right), (A_1(n), \dots, A_K(n), b_n, I^{(n)})$ independent.

Contraction method

Rösler (1991, 1992)
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$$A_r^{(n)} = \frac{\sigma(I_r^{(n)})}{\sigma(n)} A_r(n),$$

$$b^{(n)} = \frac{1}{\sigma(n)}\Big(b_n - \mu(n) + \sum_{r=1}^{K} A_r(n)\mu(I_r^{(n)})\Big).$$

Convergence

Idea:

$$\begin{array}{ccccccc} Y_n & \stackrel{d}{=} & \sum_{r=1}^{K} & A_r^{(n)} & Y_{I_r^{(n)}}^{(r)} & + & b^{(n)} \\ \downarrow & & & \downarrow & \downarrow & & \downarrow \\ Y & \stackrel{d}{=} & \sum_{r=1}^{K} & A_r^{*} & Y^{(r)} & + & b^{*} \end{array}$$

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Limit map:

$$\begin{aligned} T : \mathcal{M} &\to \mathcal{M} \\ \nu &\mapsto \mathcal{L}\Big(\sum_{r=1}^{K} A_r^{*} Z^{(r)} + b^{*}\Big) \end{aligned}$$

with $(A_1^*, \dots, A_K^*, b^*)$, $Z^{(1)}, \dots, Z^{(K)}$ independent, $Z^{(r)} \overset{d}{=} \nu$.

The minimal ℓ_p metric

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Definition: The minimal ℓ_p metric ($p \geq 1$ fixed) is given by

$$\begin{aligned}
\ell_p : \mathcal{M}_p \times \mathcal{M}_p \quad &\to \quad [0, \infty) \\
(\mu, \nu) \quad &\mapsto \quad \inf\{\|X - Y\|_p : \mathcal{L}(X) = \mu, \mathcal{L}(Y) = \nu\}
\end{aligned}$$

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Well-known fact: For a unif$[0,1]$ r.v. U we have

$$\mathcal{L}\left(F_X^{-1}(U)\right) = \mathcal{L}(X).$$

The minimal ℓ_p metric — optimal couplings

2^{nd} step: Use the same unif$[0,1]$ r.v. U for both, i.e.

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Definition: A vector (X, Y) with $\mathcal{L}(X) = \mu$, $\mathcal{L}(Y) = \nu$ and

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Theorem: Optimal coupling do always exist. For $\mu, \nu \in \mathcal{M}_p$ optimal couplings are given by (4).

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Theorem: Optimal coupling do always exist. For $\mu, \nu \in \mathcal{M}_p$ optimal couplings are given by (4).

Corollary: We have

$$\ell_p(\mu, \nu) = \left(\int_0^1 \left| F_\mu^{-1}(u) - F_\nu^{-1}(u) \right|^p du \right)^{1/p}.$$

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Hence

$$\ell_p(\mu, \nu) = \|X - Y\|_p \le \|X - Z\|_p + \|Z - Y\|_p = \ell_p(\mu, \nu) + \ell_p(\nu, \rho). \quad \clubsuit$$

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$\Rightarrow \mu_n \xrightarrow{\ell_p} \mathcal{L}(X) \in \mathcal{M}_p$. ♣.

The minimal ℓ_p metric — properties

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Lipschitz continuity on $(\mathcal{M}_\mathrm{p}, \ell_\mathrm{p})$

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Theorem: Assume that $(A_1, \ldots, A_k, b)$ are L^p-integrable r.v.,

$$\begin{aligned} T : \mathcal{M}_p &\to \mathcal{M}_p \\ \mu &\mapsto \mathcal{L}\left(\sum_{r=1}^{K} A_r Z^{(r)} + b\right), \end{aligned}$$

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Lipschitz on $(\mathcal{M}_2(0), \ell_2)$

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Theorem: Assume $(A_1, \ldots, A_k, b)$ are L^2-integr. r.v. with $\mathbb{E}\, b = 0$,

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$$Y_n \stackrel{d}{=} \frac{I_n}{n} Y_{I_n} + \frac{n-1}{n}, \qquad \mathcal{L}(I_n) = \mathrm{unif}\{0, \dots, n-1\}$$

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We obtain $\Delta(n) \to 0$.

(E.g., for $p = 1$ show $\Delta(n) \le (C \log n)/n$ by induction.)

General theorem in $\mathcal{M}_p$

Let $(Y_n)_{n\geq 0}$ be L^p-integrable, $p \geq 1$, with (as before)

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$$\mathbb{E}\left[\mathbf{1}_{\{I_r^{(n)} \leq \ell\}} \left|A_r^{(n)}\right|^p\right] \xrightarrow{n\to\infty} 0, \quad \forall\, \ell \in \mathbb{N}, r = 1, \dots, K.$$

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$$\ell_p(\mathcal{L}(Y_n), \mathcal{L}(Y)) \xrightarrow{n\to\infty} 0,$$

where $\mathcal{L}(Y)$ is the fixed-point of T in $\mathcal{M}_p$.

General theorem in $\mathcal{M}_2(0)$

Let $(Y_n)_{n\geq 0}$ be L^2-integrable, with $\mathbb{E}\, Y_n = 0$ for all $n \geq 0$ and (as before)

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Application: Path length in BST

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\begin{aligned}
Y_n &\stackrel{d}{=} \frac{I_n}{n} Y^*_{I_n} + \frac{n-1-I_n}{n} Y^{**}_{n-1-I_n} + b^{(n)}, \\
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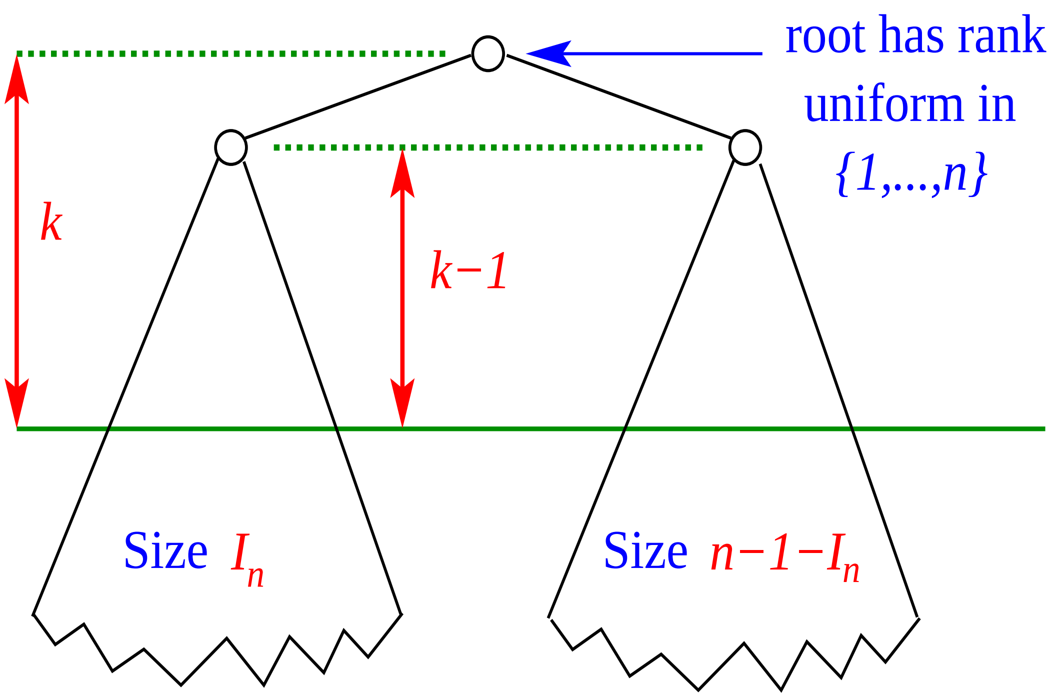
$$\mathbb{E}\left[(A_1^*)^2\right] + \mathbb{E}\left[(A_2^*)^2\right] = \mathbb{E}\, U^2 + \mathbb{E}\,(1-U)^2 = \frac{2}{3} < 1.$$

Recursive description of profile



$$Y_{n,k} \stackrel{d}{=} Y^{(1)}_{I_n,k-1} + Y^{(2)}_{n-1-I_n,k-1},$$

$Y^{(1)}_{0,k-1}, \ldots, Y^{(1)}_{n-1,k-1}, Y^{(2)}_{0,k-1}, \ldots, Y^{(2)}_{n-1,k-1}, I_n$ independent,

I_n uniform distributed on $\{0, \ldots, n-1\}$.

Mean of the profile

Thm: (Lynch 1965)

$$\mathbb{E}\, Y_{n,k} = \frac{2^k}{n!} S(n,k)$$

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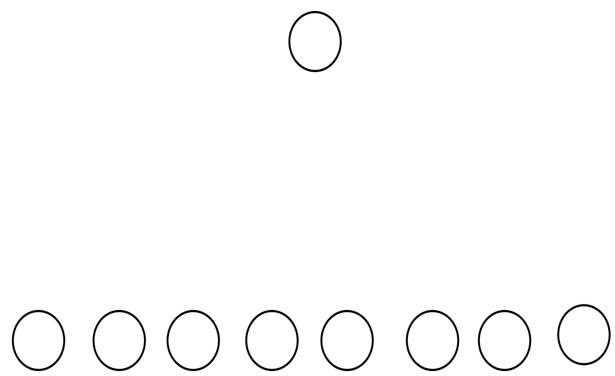

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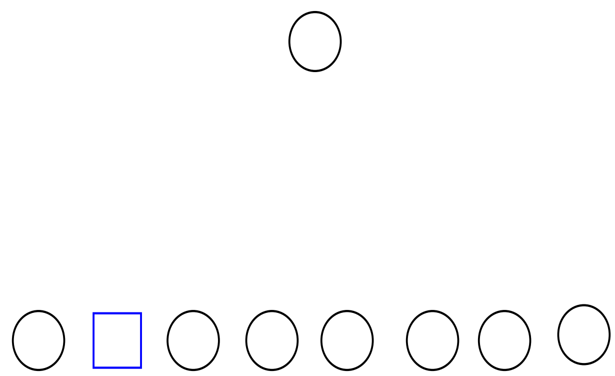

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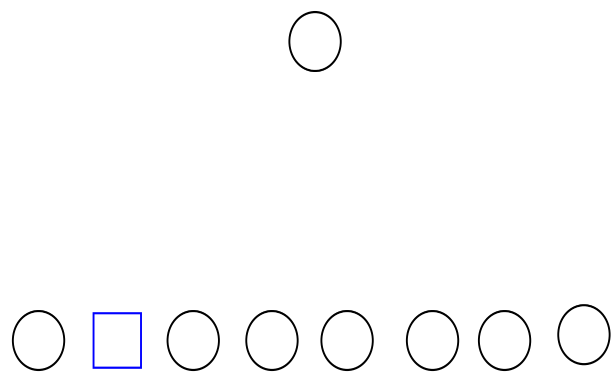

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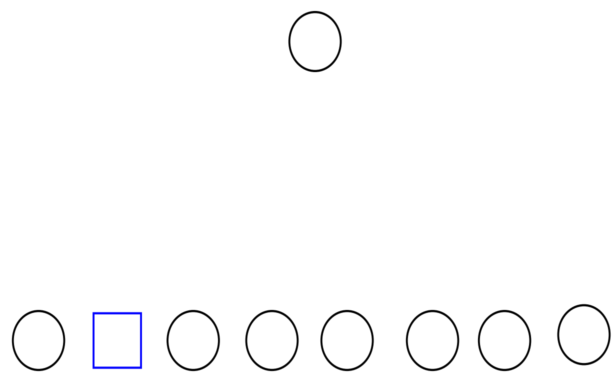

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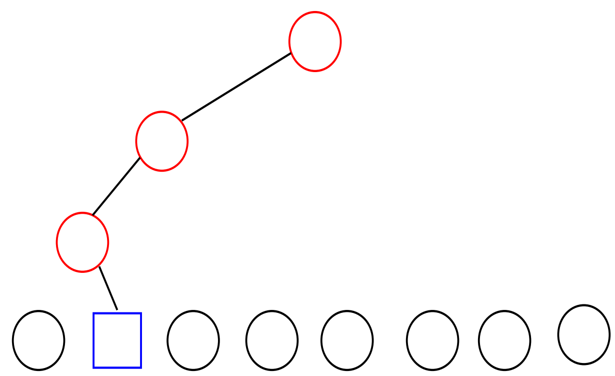

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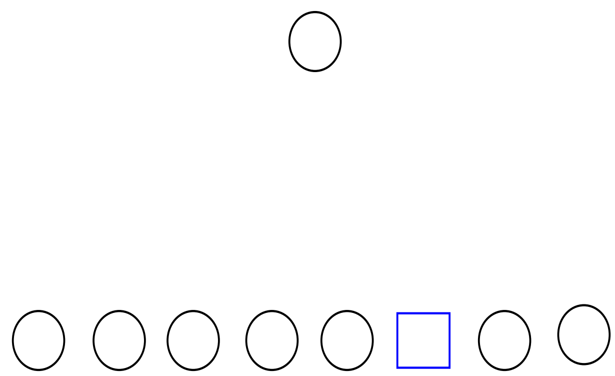

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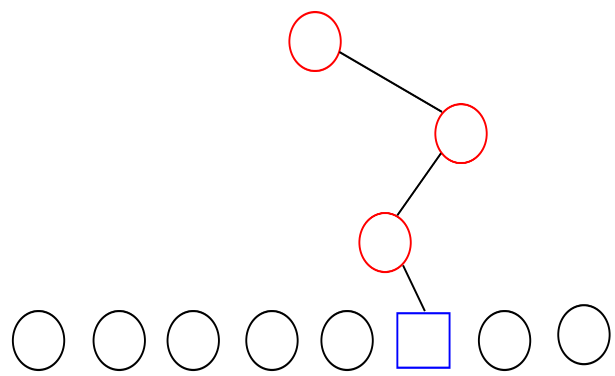

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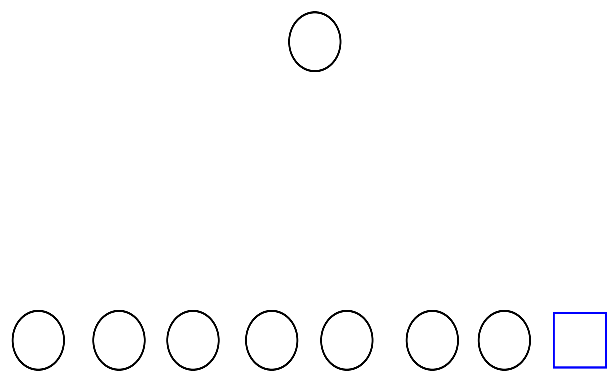

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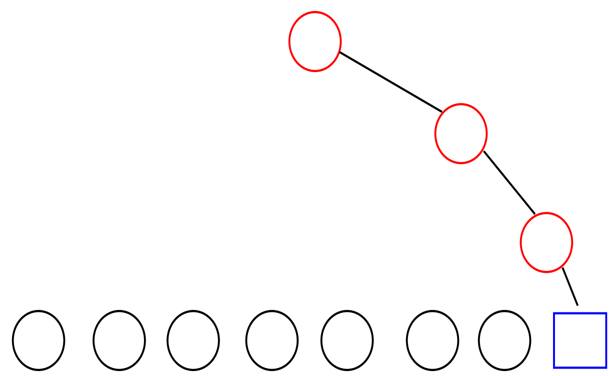

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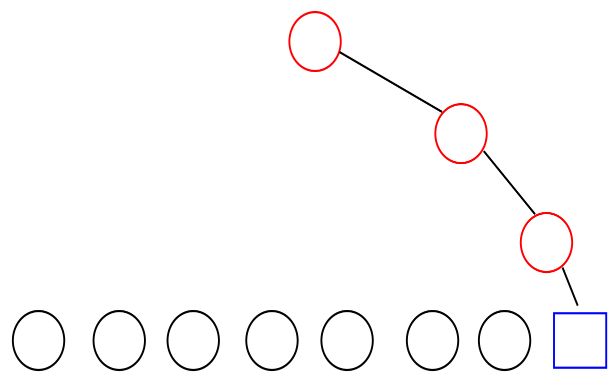

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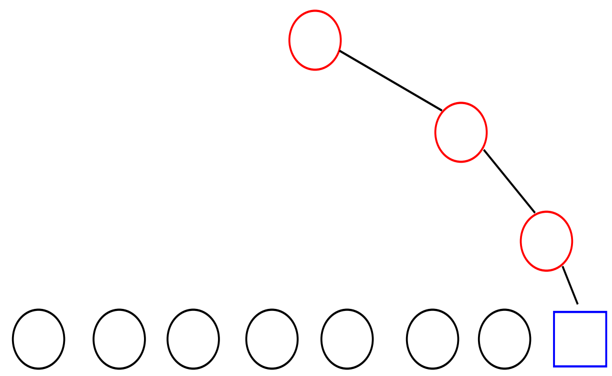
$\mathbb{P}(A_{2k})$



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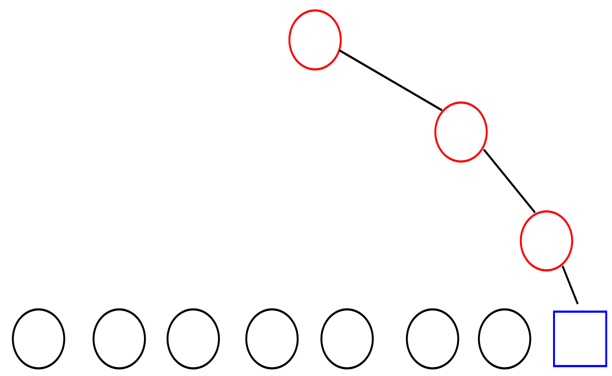

$(3, 2, 6, 1, 5, 7, 4)$

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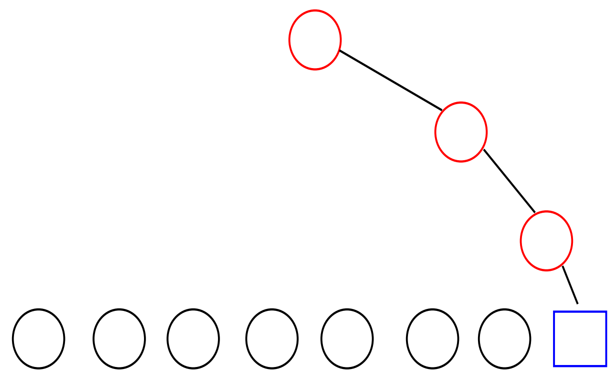

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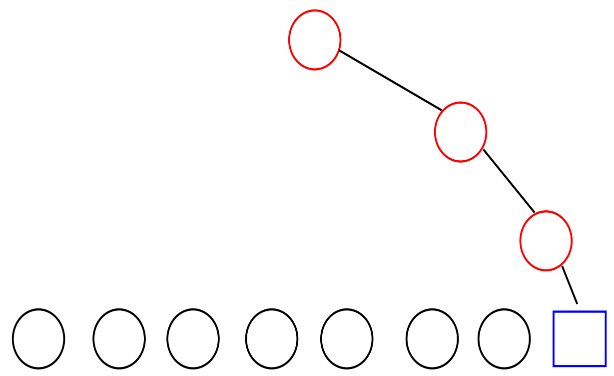

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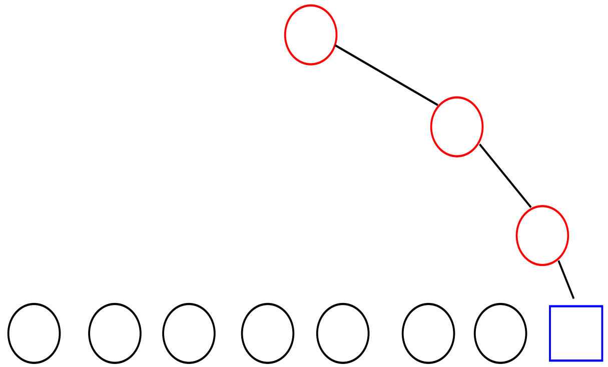

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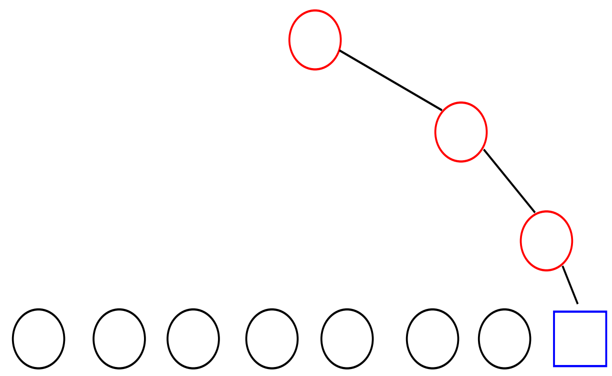

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$A_{2^k} = \{$Perm. has k up-records$\}$

Mean of profile

Hence:

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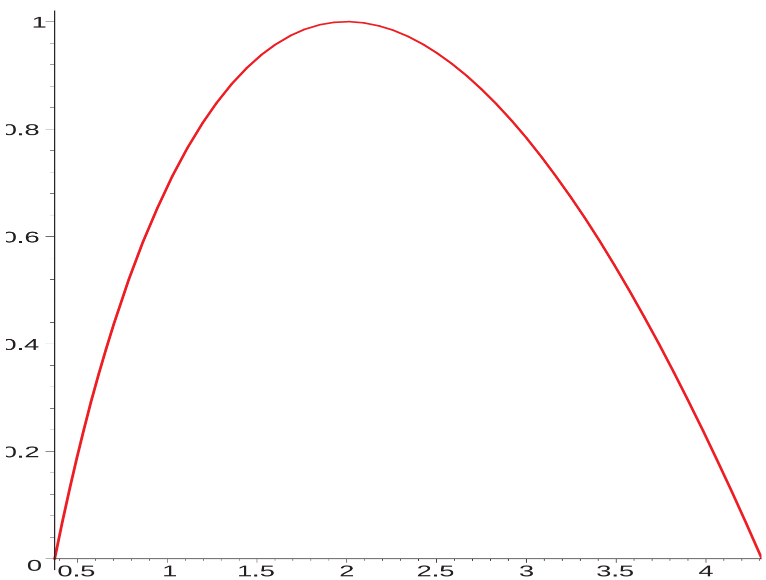
Asymptotic: For all $\beta > 0$ uniformly for $k \le \beta \log n$:

$$\mathbb{E}\, Y_{n,k} = \frac{(2\log n)^k}{k!\, n\Gamma\left(\frac{k}{\log n}\right)}(1+o(1)) \qquad \text{(H.-K. Hwang 1995)}$$

Asymptotic mean

$$\mathbb{E}\, Y_{n,k} = \frac{2^k}{n!} s(n,k) = \frac{(2\log n)^k}{\Gamma(k/\log n)\, k!\, n}\left(1 + O\left(\frac{1}{\log n}\right)\right)$$



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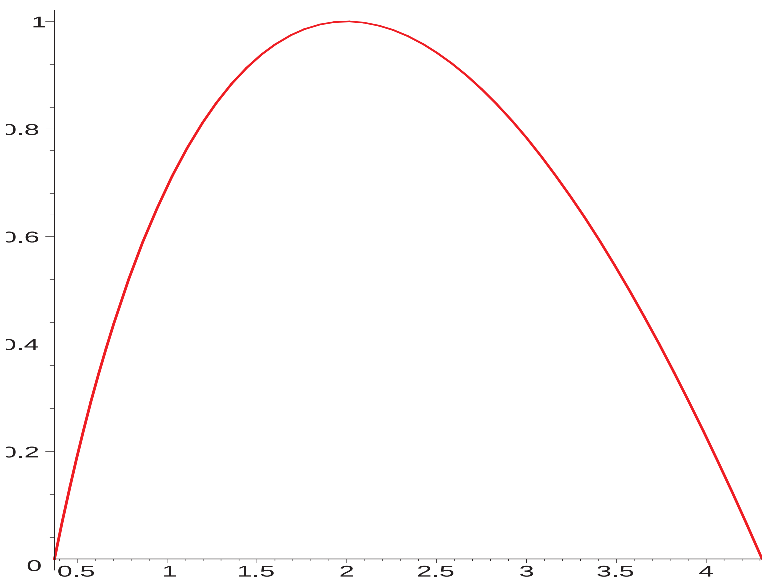
For $k \sim \alpha \log n$ we have

$$\frac{\log \mathbb{E}\, Y_{n,k}}{\log n} \to \lambda(\alpha) = \alpha - 1 - \alpha \log(\alpha/2)$$



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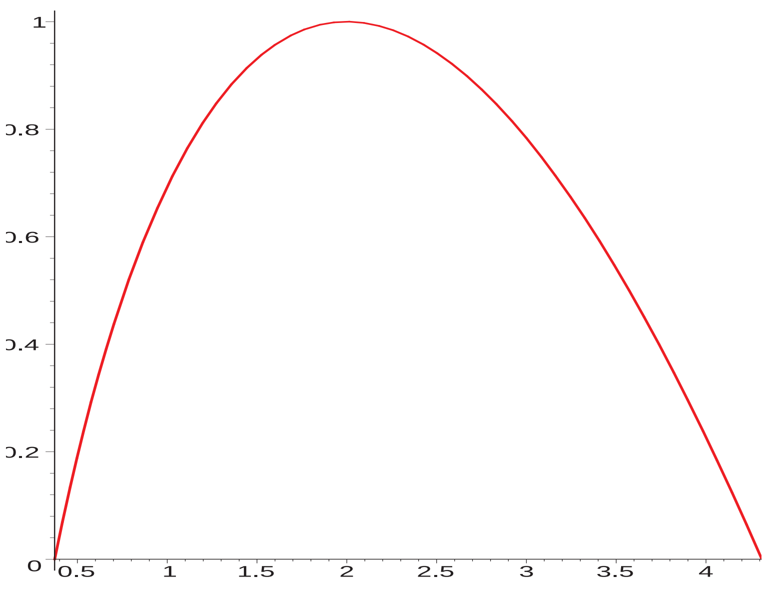

Most nodes expected at $k \sim 2\log n$:

$$\mathbb{E}\, Y_{n,\lfloor 2\log n \rfloor} \sim \frac{n}{\sqrt{4\pi \log n}}$$

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Expansion of $\mu_{n,k}$ yields (for $k \sim \alpha \log n$):

$$\frac{\mu_{I_n,k-1}}{\mu_{n,k}} \to \frac{\alpha}{2} U^{\alpha-1}, \qquad \frac{\mu_{n-1-I_n,k-1}}{\mu_{n,k}} \to \frac{\alpha}{2}(1-U)^{\alpha-1}$$

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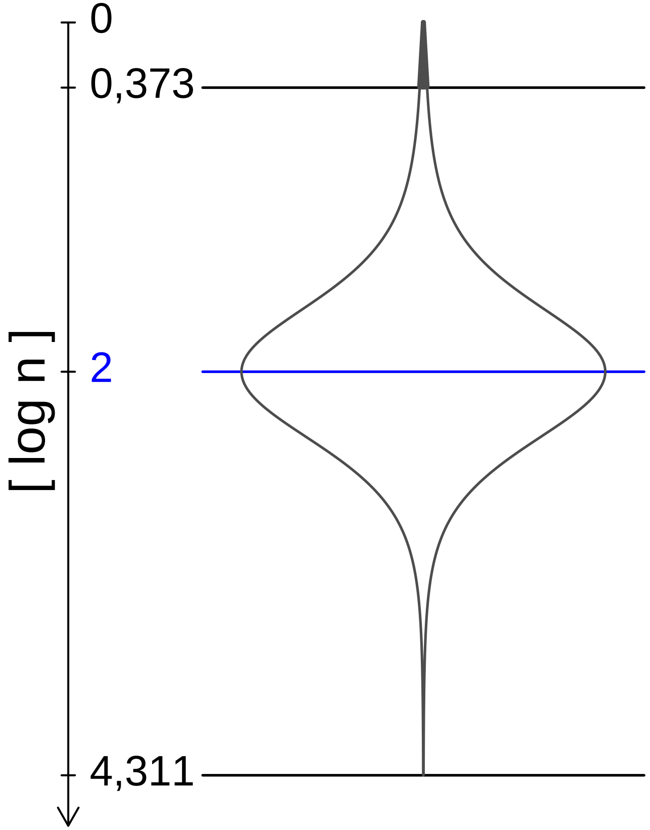
$\longrightarrow$ Limit laws with rate in ζ_s.

Techniques



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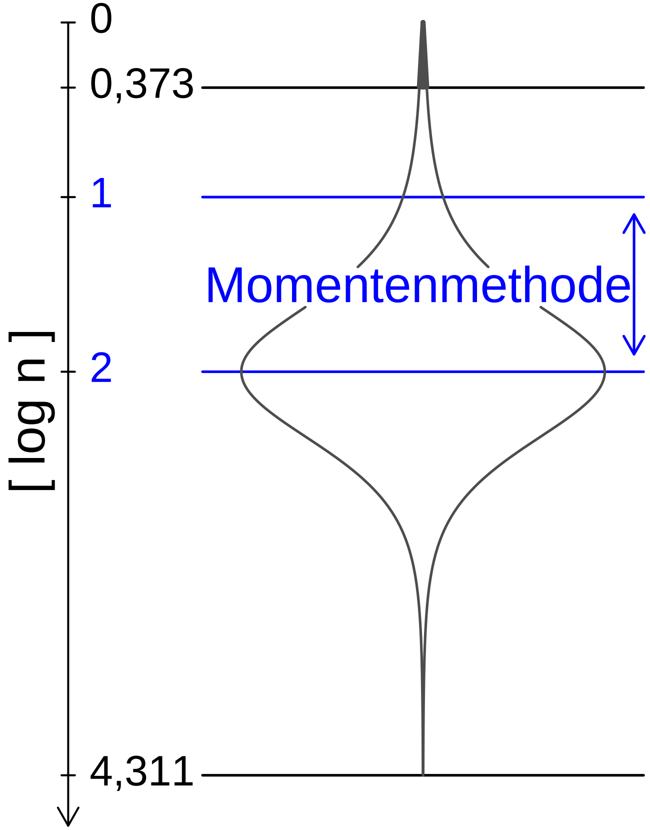

For $\alpha \in [1, 2]$:

moments method possible

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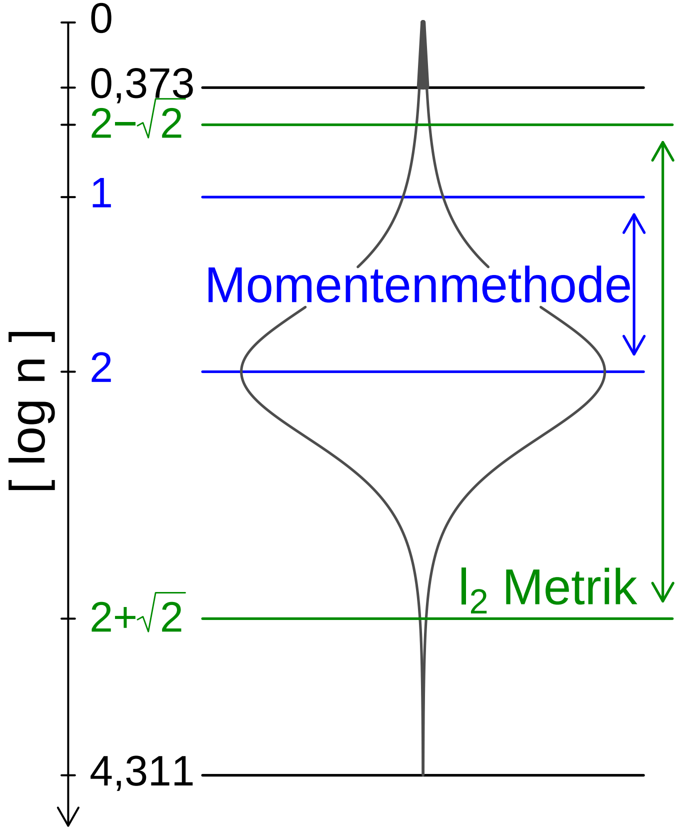

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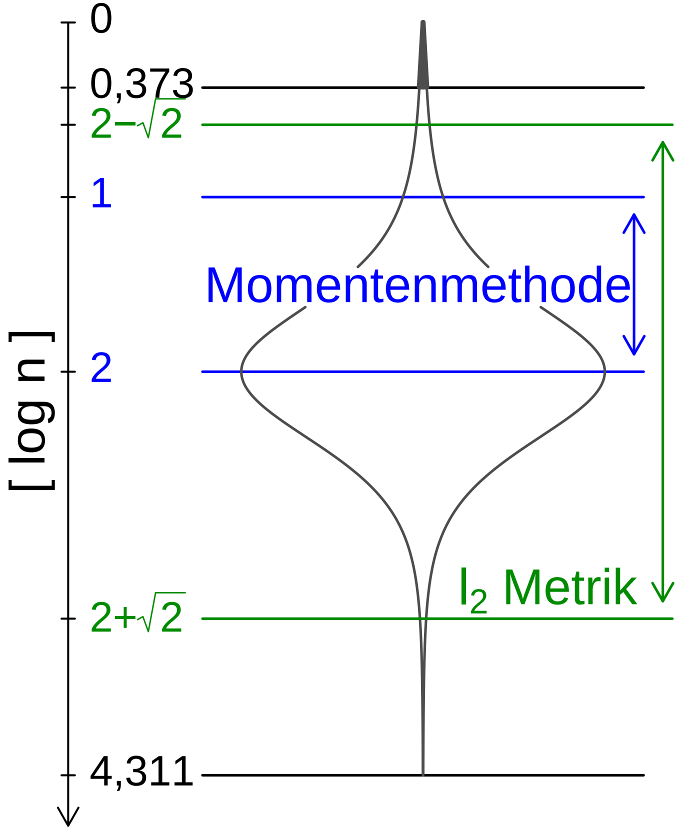

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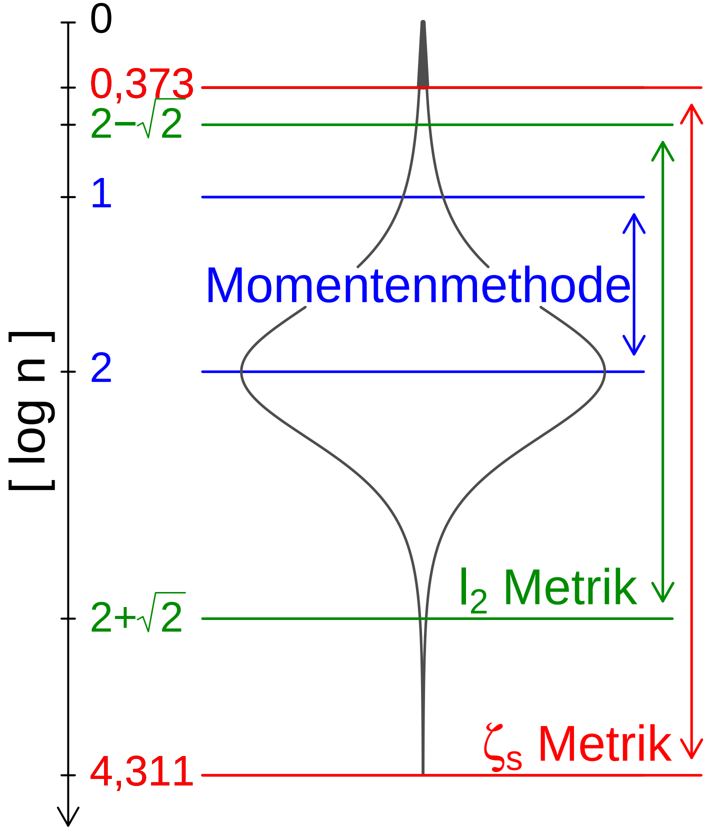

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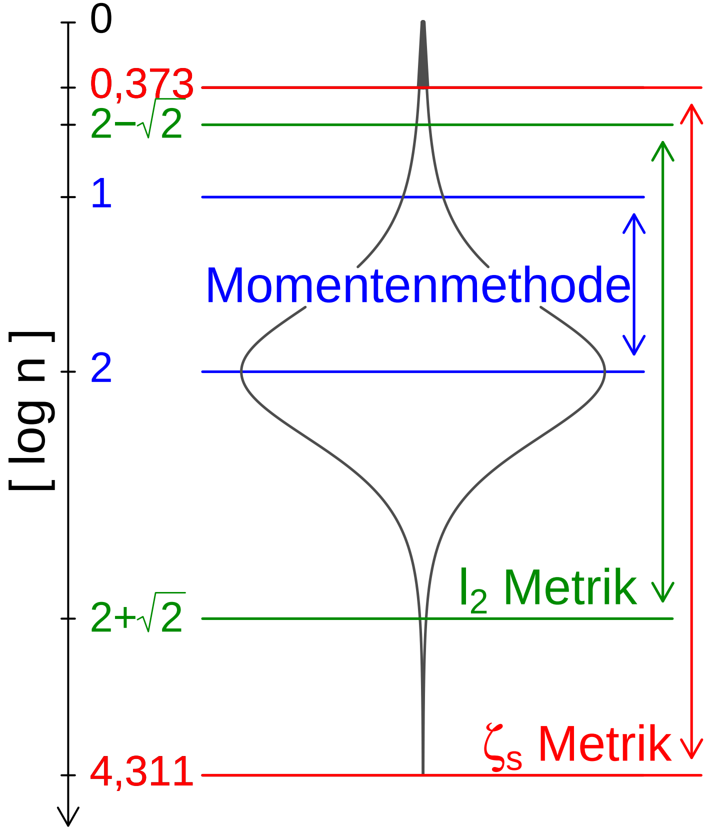

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For all $\alpha \in (\alpha_-, \alpha_+)$ there exists $s = s(\alpha)$:

$$\zeta_s\left(\frac{X_{n,k}}{\mathbb{E}\,X_{n,k}}, X_\alpha\right) = O\left(\left|\frac{k}{\log n} - \alpha\right| \vee \frac{1}{\log n}\right)$$

(Fuchs, Hwang, N. 2005)

Application: Central limit theorem

Let $W_1, W_2, \ldots$ be i.i.d., L^p-integrable, $p \geq 2$, with $\mathbb{E} W_1 = \mu$, $\mathrm{Var}(W_1) = \sigma^2$.

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$$\sum_{r=1}^{K} (A_r^*)^2 = 1 \qquad \text{almost surely.}$$

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A general theorem: Periodicities

$$X_n \stackrel{d}{=} \sum_{r=1}^{K} X^{(r)}_{I_r^{(n)}} + b_n, \quad n \geq n_0, \tag{1}$$

all r.v. L_2-integrable with conditions as before. Assume

$$\mathbb{E}[X_n] = f(n) + \Re(\gamma n^\lambda) + o(n^\sigma), \tag{2}$$

with a function $f : \mathbb{N}_0 \to \mathbb{R}$, $\gamma \in \mathbb{C} \setminus \{0\}$, and $\lambda = \sigma + \mathrm{i}\tau \in \mathbb{C}$ with $\sigma > 0$. We denote

$$A_r^{(n)} = \left(\frac{I_r^{(n)}}{n} \right)^\lambda, \quad r = 1, \ldots, K, \tag{3}$$

$$b^{(n)} = \frac{1}{n^\sigma} \left(b_n - f(n) + \sum_{r=1}^{K} f(I_r^{(n)}) \right). \tag{4}$$

A general theorem: Periodicities

Assume

$$(A_1^{(n)},\dots,A_K^{(n)}) \xrightarrow{\ell_2} (A_1^*,\dots,A_K^*) \quad \text{and} \quad \|b^{(n)}\|_2 \to 0, \tag{5}$$

and furthermore

$$\mathbb{E}\sum_{r=1}^{K}|A_r^*|^2 < 1. \tag{6}$$

Then,

$$\ell_2\left(\frac{X_n - f(n)}{n^\sigma}, \Re\left(e^{\mathrm{i}\tau \ln n}Y\right)\right) \to 0, \tag{7}$$

where $\mathcal{L}(Y)$ is the unique fixed point in $\mathcal{M}_2^{\mathbb{C}}(\gamma)$ of

$$T:\mathcal{M}^{\mathbb{C}} \to \mathcal{M}^{\mathbb{C}}, \quad \eta \mapsto \mathcal{L}\left(\sum_{r=1}^{K} A_r^* Z^{(r)}\right), \tag{8}$$

where $(A_1^*,\dots,A_K^*)$, $Z^{(1)},\dots,Z^{(K)}$ are independent and $\mathcal{L}(Z^{(r)}) = \eta$ for $r = 1,\dots,K$.

Applications

- # leaves in d-dim. quadtrees, $d \geq 9$
- # nodes in m-ary search trees, $m \geq 27$
- size of fragmentation trees
- composition of various urn models
- e.g. cyclic urns with at least 7 colors

Proof

Exercise:
The restriction of T to $\mathcal{M}_2^{\mathbb{C}}(\gamma)$ maps into $\mathcal{M}_2^{\mathbb{C}}(\gamma)$ and is Lipschitz in ℓ_2 with Lipschitz constant bounded by

$$\left(\mathbb{E}\sum_{r=1}^{K}|A_r^*|^2\right)^{1/2} < 1.$$

Comment: The proof can be given along the lines for the corresponding result above in $(\mathcal{M}_2(0), \ell_2)$.

Thus, (6) implies the existence of a unique fixed-point $\mathcal{L}(Y)$.

Proof

With $Y_0 := 0$ and

$$Y_n := \frac{X_n - f(n)}{n^\sigma}, \quad n \geq 0$$

we obtain

$$Y_n \stackrel{d}{=} \sum_{r=1}^{K} \left(\frac{I_r^{(n)}}{n} \right)^\sigma Y^{(r)}_{I_r^{(n)}} + b^{(n)}, \qquad n \geq n_0, \tag{9}$$

The fixed point property of $\mathcal{L}(Y)$ implies

$$\frac{1}{n^\sigma} \Re\left(n^\lambda Y\right) \stackrel{\mathrm{d}}{=} \frac{1}{n^\sigma} \Re\left(\sum_{r=1}^{K} n^\lambda A_r^* Y^{(r)} \right). \tag{10}$$

where $(A_1^*, \ldots, A_K^*)$, $Y^{(1)}, \ldots, Y^{(b)}$ are independent and $\mathcal{L}(Y^{(r)}) = \mathcal{L}(Y)$ for $r = 1, \ldots, K$.

Construction on one probability space

We may assume, e.g. by taking optimal couplings, that

$$\|A_r^{(n)} - A_r^*\|_2 = \left\| \left(\frac{I_r^{(n)}}{n} \right)^{\lambda} - A_r^* \right\|_2 \to 0, \quad (n \to \infty).$$

We choose $X_n^{(r)}$ as optimal couplings to $\Re(n^{\mathrm{i}\tau} X^{(r)})$ for $n \geq 0$ and $r = 1, \ldots, K$.

Clearly, we may assume that, as required, $X_n^{(r)}$, $r = 1, \ldots, K$, are independent of each other and of $(I^{(n)}, b_n)_n$.

Bounding the ℓ_2-distance

We denote, for $n \geq 1$,

$$\Delta(n) := \ell_2\left(\frac{Y_n - f(n)}{n^\sigma}, \Re\left(X e^{\mathrm{i}\tau \ln n}\right)\right) = \ell_2\left(X_n, \frac{1}{n^\sigma}\Re\left(n^\lambda X\right)\right).$$

Using (9) and (10) we obtain, for $n \geq n_0$,

$$\begin{aligned}
\Delta(n) &= \ell_2\left(\sum_{r=1}^{K}\left(\frac{I_r^{(n)}}{n}\right)^\sigma X^{(r)}_{I_r^{(n)}} + b^{(n)}, \frac{1}{n^\sigma}\Re\left(\sum_{r=1}^{K} n^\lambda A_r^* X^{(r)}\right)\right) \\
&\leq \left\|\sum_{r=1}^{K}\left(\left(\frac{I_r^{(n)}}{n}\right)^\sigma X^{(r)}_{I_r^{(n)}} - \frac{1}{n^\sigma}\Re\left(n^\lambda A_r^* X^{(r)}\right)\right)\right\|_2 + \left\|b^{(n)}\right\|_2 \\
&\leq \left\|\sum_{r=1}^{K}\left(\left(\frac{I_r^{(n)}}{n}\right)^\sigma X^{(r)}_{I_r^{(n)}} - \frac{1}{n^\sigma}\Re\left((I_r^{(n)})^\lambda X^{(r)}\right)\right)\right\|_2 + \left\|b^{(n)}\right\|_2 \\
&\quad + \left\|\sum_{r=1}^{K}\left(\frac{1}{n^\sigma}\Re\left((I_r^{(n)})^\lambda X^{(r)}\right) - \frac{1}{n^\sigma}\Re\left(n^\lambda A_r^* X^{(r)}\right)\right)\right\|_2.
\end{aligned}$$

Bounding the ℓ_2-distance

By (5) and (3) the second and third of the three latter summands tend to zero as $n \to \infty$. We abbreviate

$$W_r^{(n)} = \left(\frac{I_r^{(n)}}{n} \right)^{\sigma} X^{(r)}_{I_r^{(n)}} - \frac{1}{n^{\sigma}} \Re \left((I_r^{(n)})^{\lambda} X^{(r)} \right). \tag{11}$$

Hence, the latter estimate implies

$$\begin{aligned} \Delta(n) &\leq \left(\mathbb{E} \left(\sum_{r=1}^{K} W_r^{(n)} \right)^2 \right)^{1/2} + o(1) \\ &= \left(\mathbb{E} \sum_{r=1}^{K} (W_r^{(n)})^2 + \mathbb{E} \sum_{\substack{r,s=1 \\ r \neq s}}^{K} W_r^{(n)} W_s^{(n)} \right)^{1/2} + o(1). \end{aligned} \tag{12}$$

Bounding the ℓ_2-distance

Since $X_n^{(r)}$ and $\Re(n^{\mathrm{i}\tau} X^{(r)})$ are optimal couplings for all $n \geq 1$ and $r = 1, \ldots, K$ we obtain

$$\mathbb{E}(W_r^{(n)})^2 = \mathbb{E}\left[\left(\frac{I_r^{(n)}}{n}\right)^{2\sigma} \Delta^2(I_r^{(n)})\right]. \tag{13}$$

From (2) we obtain

$$\mathbb{E}[X_n] = \frac{1}{n^\sigma}\Re(\gamma n^\lambda) + R(n), \qquad n \geq 1,$$

with $R(n) \to 0$ as $n \to \infty$.

Bounding the ℓ_2-distance

Since $\mathbb{E}[X^{(r)}] = \gamma$ and by the independence conditions we obtain

$$\mathbb{E}W_r^{(n)} = \mathbb{E}[(I_r^{(n)}/n)^\sigma R(I_r^{(n)})]$$
$$\mathbb{E}[W_r^{(n)} W_s^{(n)}] = \mathbb{E}\left[\left(\frac{I_r^{(n)}}{n}\frac{I_s^{(n)}}{n}\right)^\sigma R(I_r^{(n)})R(I_s^{(n)})\right].$$

Splitting the latter integral into the events $\{I_r^{(n)} \le n_1 \text{ or } I_s^{(n)} \le n_1\}$ and $\{I_r^{(n)} > n_1 \text{ and } I_s^{(n)} > n_1\}$ for some $n_1 > 0$ we obtain, for every $n_1 > 0$,

$$|\mathbb{E}[W_r^{(n)} W_s^{(n)}]| \le \left(\frac{n_1}{n}\right)^\sigma \|R\|_\infty^2 + \sup_{n \ge n_1} R^2(n),$$

where $\|R\|_\infty := \sup_{n \ge n_1} |R(n)| < \infty$.

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$$|\mathbb{E}[W_r^{(n)} W_s^{(n)}]| \leq \left(\frac{n_1}{n}\right)^{\sigma} \|R\|_\infty^2 + \sup_{n \geq n_1} R^2(n),$$

From this we obtain first, letting $n \to \infty$, $\limsup_{n\to\infty} |\mathbb{E}[W_r^{(n)} W_s^{(n)}]| \leq \sup_{n \geq n_1} R^2(n)$, and then, letting $n_1 \to \infty$,

$$\mathbb{E}[W_r^{(n)} W_s^{(n)}] \to 0 \qquad \text{as } t \to \infty. \tag{14}$$

Now, (12), (13), and (14) imply, for $n > n_0$,

$$\Delta(n) \leq \left(\mathbb{E}\left[\sum_{r=1}^{K} \left(\frac{I_r^{(n)}}{n}\right)^{2\sigma} \Delta^2(I_r^{(n)})\right] + R_1(n)\right)^{1/2} + R_2(n), \tag{15}$$

with $R_1(n), R_2(n) \to 0$ as $n \to \infty$

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First step: Show $\|\Delta\|_\infty < \infty$.

Define $\Delta^*(n) := \sup_{0<j\leq n} \Delta(j)$.
We have $|R_1(n)| < 1$ and $|R_2(n)| < 1$ for $n \geq n_1$. Then with (15) we obtain, for $n \geq n_1$,

$$\Delta(n) \leq \left(\mathbb{E}\left[\sum_{r=1}^{K} \left(\frac{I_r^{(n)}}{n} \right)^{2\sigma} (\Delta^*)^2(n) \right] + 1 \right)^{1/2} + 1.$$

By (3), (5) and (6)

$$\mathbb{E} \sum_{r=1}^{K} (I_r^{(n)}/n)^{2\sigma} \leq \xi < 1, \quad n \geq n_2 > n_1.$$

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Thus, for all $n \geq n_2$ we obtain, with $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$,

$$\Delta(n) \leq \sqrt{\xi}\Delta^*(n) + 2,$$

and thus

$$\Delta^*(n) \leq \sqrt{\xi}\Delta^*(n) + 2 + \Delta^*(n_2),$$

which implies $\|\Delta\|_\infty \leq 2 + \Delta^*(n_2)/(1 - \sqrt{\xi}) < \infty$.

Second step: $\Delta(n) \to 0$.

$L := \limsup_{n\to\infty} \Delta(n) > 0$. Let $\varepsilon > 0$. There exists an $n_3 \geq n_2$ such that for all $n \geq n_3$ we have $\Delta(n) \leq L + \varepsilon$. Then (15) implies

Bounding the ℓ_2-distance

$$\begin{aligned}
&\Delta(n)\\
&\leq \left(\mathbb{E}\left[\sum_{r=1}^{K}\left(\frac{I_r^{(n)}}{n}\right)^{2\sigma}\left(\eta\{I_r^{(n)} < n_3\} + \eta\{I_r^{(n)} \geq n_3\}\right)\Delta^2(I_r^{(n)})\right] + R_1\right.\\
&\leq \left(\sum_{r=1}^{K}\left(\frac{n_3}{n}\right)^{2\sigma}\|\Delta\|_\infty^2 + \xi(L+\varepsilon)^2 + R_1(n)\right)^{1/2} + R_2(n).
\end{aligned}$$

Hence, $n \to \infty$ implies

$$L \leq \sqrt{\xi}(L+\varepsilon),$$

which if $L > 0$ is a contradiction if we choose ε small enough. Consequently, we have $L = 0$ yielding the assertion.