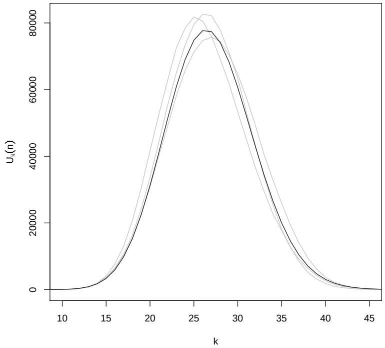
Asymptotic expansions for the profile of random trees



Henning Sulzbach

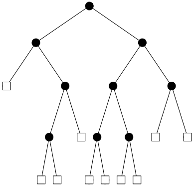
ALEA in Europe, Vienna, 10 October 2017

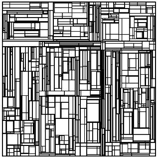
UNIVERSITY OF BIRMINGHAM

with Zakhar Kabluchko (Münster) and Alexander Marynych (Kyev)

Trees of interest

- data structures
- analysis of algo.
- real-world networks





Comparison-based: binary (m-ary) search trees, random recursive trees, preferential attachment trees

Multidimensional: quadtrees, K-d trees

Digital: digital search trees, tries

Trees are **flat** (i.e. logarithmic) and **wide**.

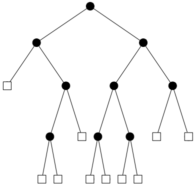
Quantities of interest

Global quantities:

- typical depths and distances,
- maximal depths and distances,
- total pathlength (sum over all node depths),
- mode and width.

Local quantities:

- degree distribution,
- fringe subtrees.



Put simply, the **profile**.

Outline

1. One-split branching random walks

2. Profile of binary search trees: a summary

3. Main result: an asymptotic profile expansion

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1. **One-split branching random walks**

2. Profile of binary search trees: a summary

3. Main result: an asymptotic profile expansion

The binary search tree

Input: numbers $0.6, 0.9, 0.3, 0.7, 0.5, 0.8, 0.1, 0.2$

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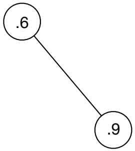

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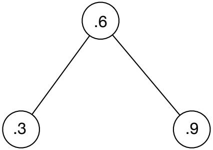

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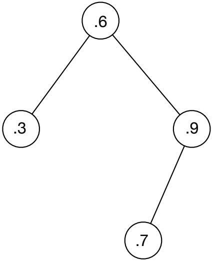

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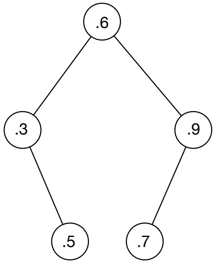

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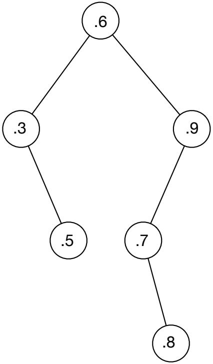

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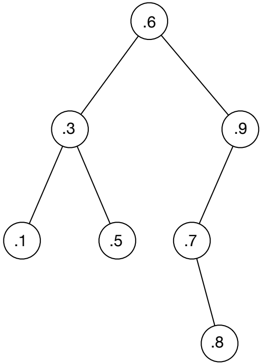

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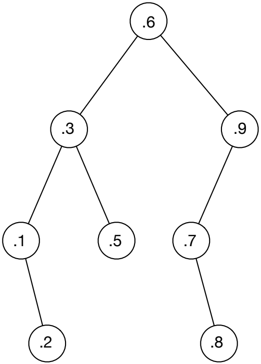

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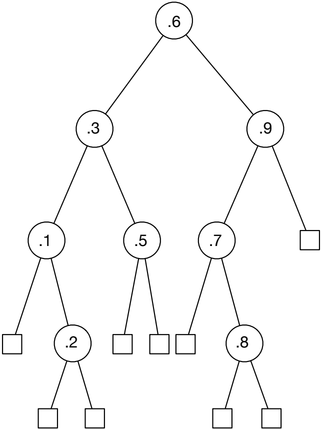
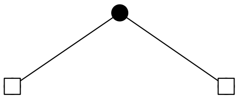


Model: Use iid unif$[0, 1]$ random variables $U_1, U_2, U_3, \ldots$

The binary search tree - a Markov chain



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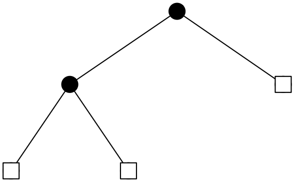

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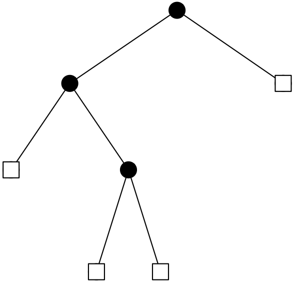

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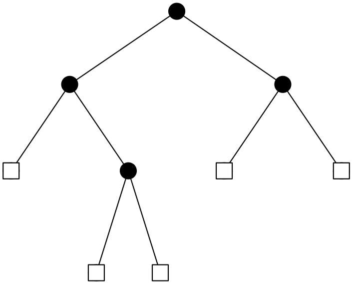

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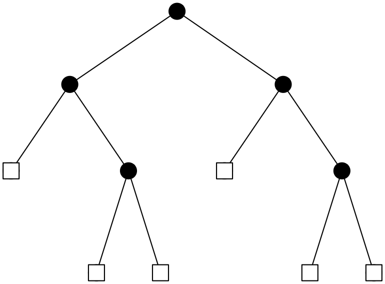

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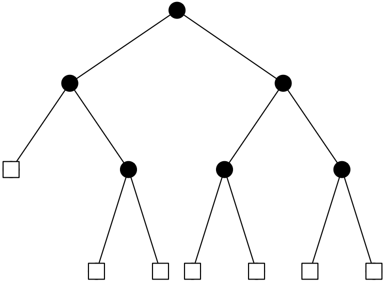

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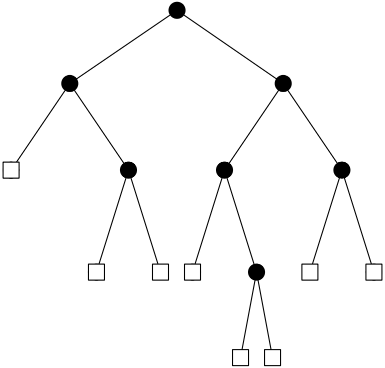

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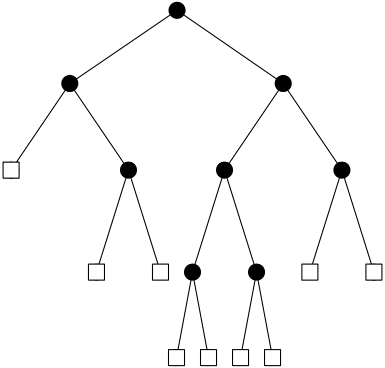

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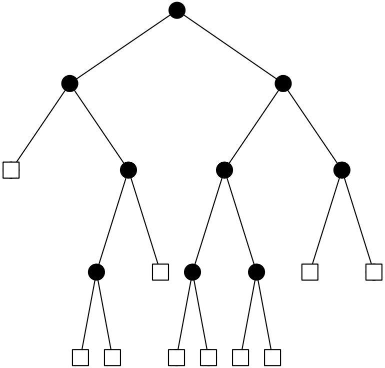

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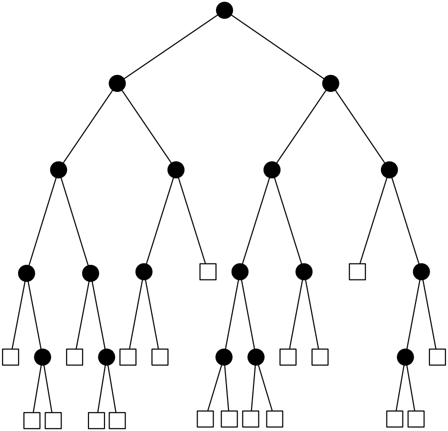

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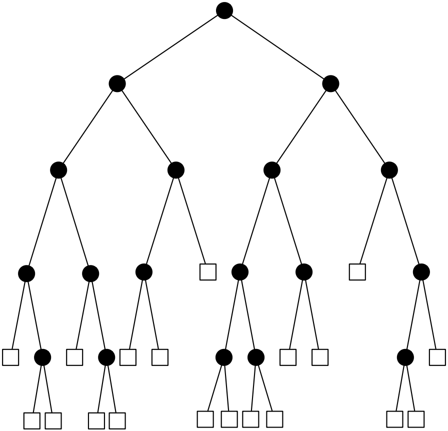

$$X_n(k) = \#\{\text{nodes with depth } k\}, \quad k \geq 0,$$

$$U_n(k) = \#\{\text{boxes with depth } k\}, \quad k \geq 0.$$

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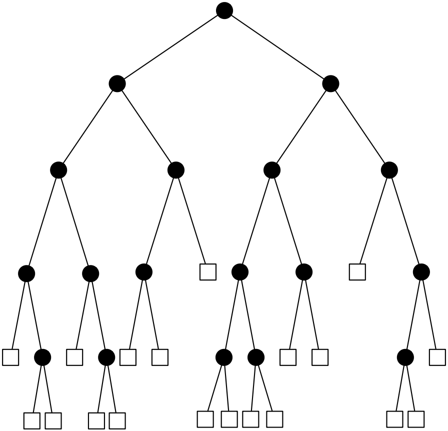
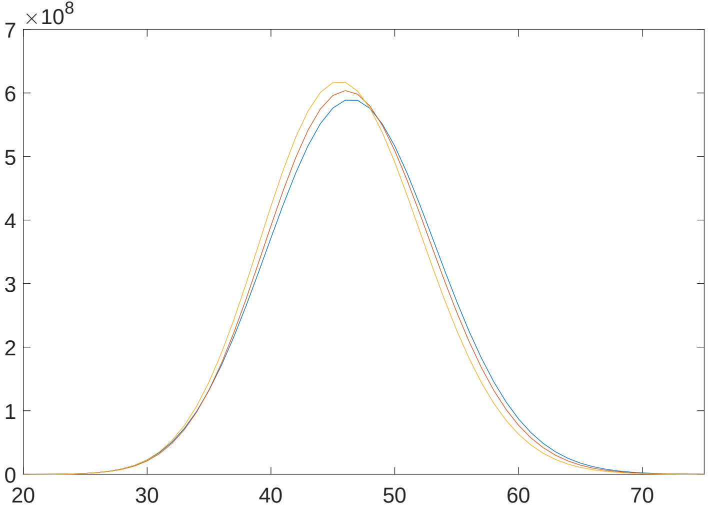


$$X_n = (1, 2, 4, 6, 5, 0, 0, \ldots)$$

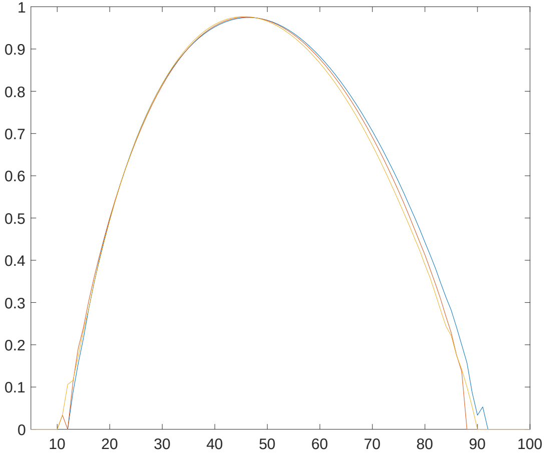
$$U_n = (0, 0, 0, 2, 7, 10, 0, \ldots)$$

The binary search tree - three simulations



$n = 10^{10}$, heights between 87 and 91.

The binary search tree - Logplot



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The random recursive tree

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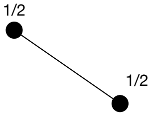

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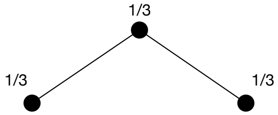

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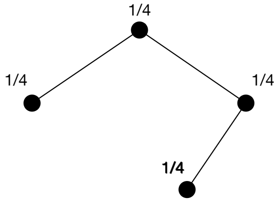

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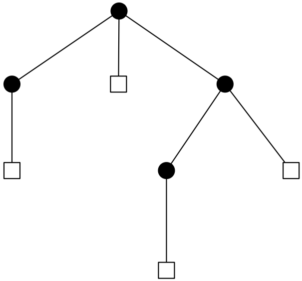

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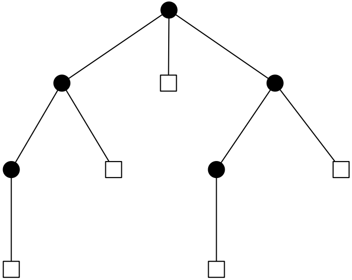

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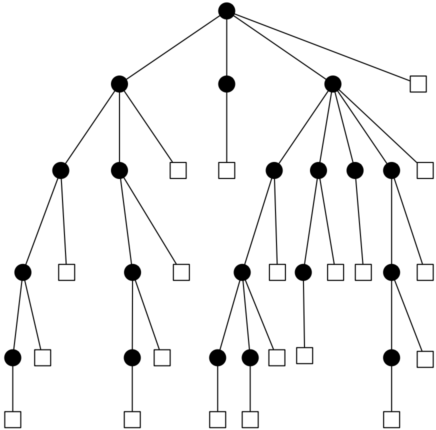

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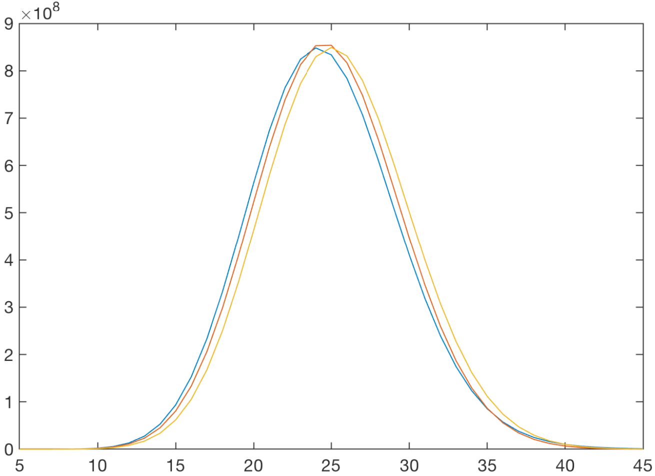

$n = 10^{10}$, heights between 57 and 62.

The plane-oriented recursive tree

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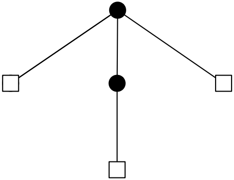

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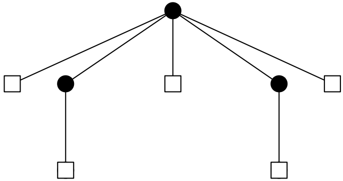

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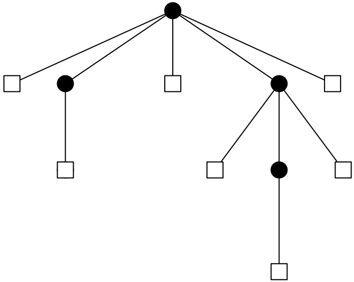

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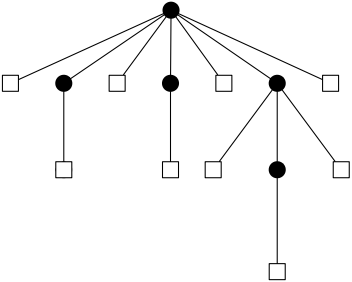

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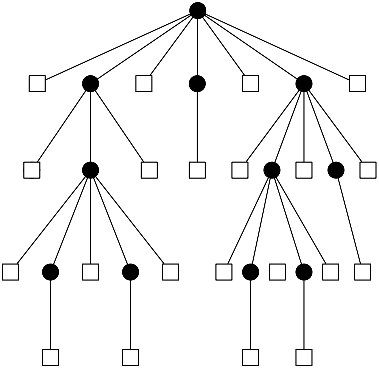

weight of v: $1 + d_v$

degree profile: j^{-2}

One-split branching random walks

Input: random point process ζ on $\mathbb{Z}$

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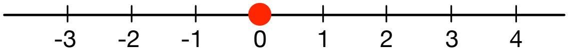
$$Z_0(k) = \delta_{0,k}$$



$$Z_0 = (\ldots, 0, 0, 1_*, 0, 0, \ldots)$$

Assumptions:

- $1 \leq \zeta(\mathbb{Z}) \leq C, \mathbb{P}(\zeta(\mathbb{Z}) > 1) > 0$ and ζ has bounded support,
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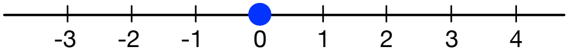

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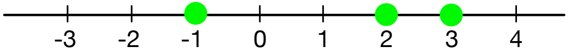

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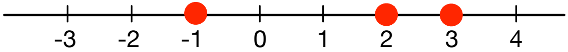

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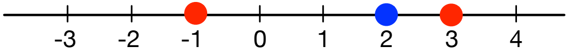

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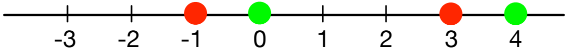

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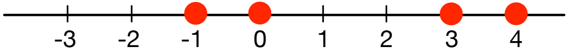

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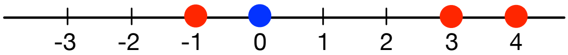

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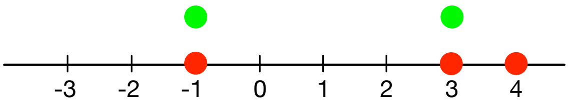

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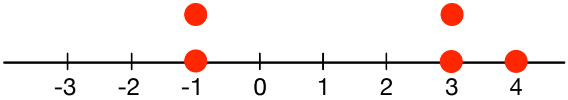
Input: random point process ζ on $\mathbb{Z}$

$Z_n(k)$: $\#$ of particles at k at time n

$$Z_0(k) = \delta_{0,k}$$



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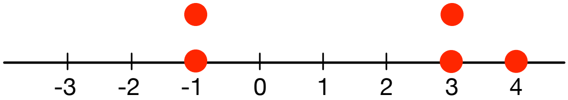
$$Z_0(k) = \delta_{0,k}$$



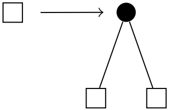
$$Z_3 = (\dots, 0, 2, 0_*, 0, 0, 2, 1, 0 \dots)$$

Assumptions:

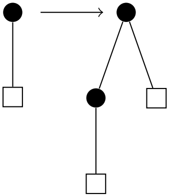
- $1 \le \zeta(\mathbb{Z}) \le C, \mathbb{P}(\zeta(\mathbb{Z}) > 1) > 0$ and ζ has bounded support,
- $\mathbb{P}(\zeta(c\mathbb{Z}) < \zeta(\mathbb{Z})) > 0$ for all $c \ge 2$. (wlog)

One-split branching random walks

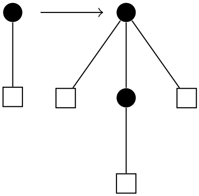
BST: $\zeta = (\ldots, 0, 0_*, 2, 0, \ldots) = 2\delta_1$



RRT: $\zeta = (\ldots, 0, 1_*, 1, 0, \ldots) = \delta_0 + \delta_1$



PORT: $\zeta = (\ldots, 0, 2_*, 1, 0, \ldots) = 2\delta_0 + \delta_1$



Note: ζ is deterministic.

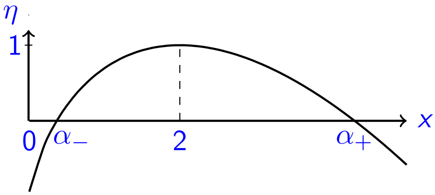
Outline

1. One-split branching random walks

2. **Profile of binary search trees: a summary**

3. Main result: an asymptotic profile expansion

Binary search tree - a rough picture



$$\eta(x) = x - x\log(x/2) - 1,$$
$$\alpha_- = 0.37\ldots, \alpha_+ = 4.31\ldots,$$
$$\eta(2) = 1.$$

For $k = \alpha \log n + o(\log n)$, as $n \to \infty$,

$$U_n(k) = n^{\eta(\alpha)+o(1)}, \quad \alpha_- < \alpha < \alpha_+.$$

As $n \to \infty$,

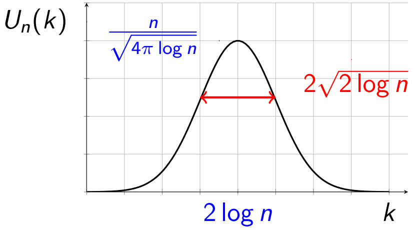
$$\frac{D_n - 2\log n}{\sqrt{2\log n}} \xrightarrow{d} \mathcal{N}$$

and

$$\text{Height} \sim \alpha_+ \log n, \quad \text{Fill-up level} \sim \alpha_- \log n.$$

DEVROYE '86 -'88

Profile - central regime



Recall: As $n \to \infty$,

$$\frac{D_n - 2\log n}{\sqrt{2\log n}} \xrightarrow{d} \mathcal{N}.$$

With

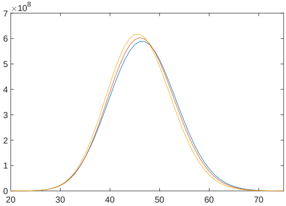
$$x_n(k) := \frac{k - 2\log n}{\sqrt{2\log n}},$$

uniformly over $k \in \mathbb{N}$, almost surely and in mean,

$$U_n(k) = \frac{n}{\sqrt{2\pi \cdot 2\log n}} \cdot e^{-\frac{1}{2}x_n^2(k)} \cdot (1 + o(1)).$$

Hwang '95, Chauvin, Drmota and Jabbour-Hattab '01

Width and mode



$$W_n := \max\{U_n(k) : k \geq 1\}$$
$$m_n := \max\{k : U_n(k) = W_n\}$$

$$W_n = \frac{n}{\sqrt{4\pi \log n}} \cdot (1 + o(1))$$

Open: Limit theorem for W_n

The sequence

$$(m_n - 2\log n)_{n\geq 1}$$

is tight.

Devroye and Hwang '06

Open: Limit theorem for $m_n - 2\log n$

Profile - limit theorem

Theorem (Hwang '95)

For $C > 0$, *uniformly in* $0 \leq k \leq C \log n$, *as* $n \to \infty$,

$$\mathbb{E}\left[U_n(k)\right] \sim \frac{1}{\Gamma(\alpha_k) \cdot \sqrt{2\pi\alpha_k}} \cdot \frac{n^{\eta(\alpha_k)}}{\sqrt{\log n}}, \quad \alpha_k = \frac{k}{\log n}.$$

Theorem (Chauvin, Klein, Marckert and Rouault '05)

There exists a random analytic function X *on a complex domain* G *with* $(\alpha_-, \alpha_+) \subseteq G$ *with* $\mathbb{E}\left[X(\alpha)\right] = 1$ *and* $X > 0$ *on* (α_-, α_+)*:*

$$\sup_{\alpha_k \in (\overline{\alpha_-}, \overline{\alpha_+})} \left| \frac{U_n(k)}{\mathbb{E}\left[U_n(k)\right]} - X(\alpha_k) \right| \xrightarrow{a.s.} 0.$$

The special regimes

The limit $X(\alpha)$ is random if $\alpha \notin \{1,2\}$.

Theorem (FUCHS, HWANG AND NEININGER '06)

Let $c \in \{1,2\}$. For $k = c \log n + c_n$ with $c_n = o(\log n)$ and $|c_n| \to \infty$, we have

$$U_n(k)^* \xrightarrow{d} (X'(c))^*.$$

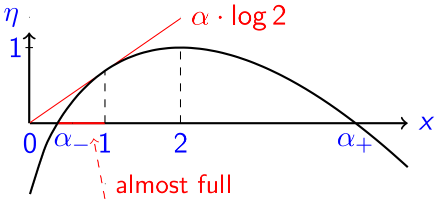
$(U_n(k)^*)_{n\geq 1}$ does not converge in distribution if $c_n = O(1)$.

For $P_n := \sum_k k \cdot U_n(k)$:

$$P_n^* \xrightarrow{a.s.} (X'(2))^*.$$

RÉGNIER '89, RÖSLER '91

The internal profile



$$X_n(k) = n^{\eta(\alpha)+o(1)}, \qquad 1 < \alpha < \alpha_+$$

$$2^k - X_n(k) = n^{\eta(\alpha)+o(1)}, \quad \alpha_- < \alpha < 1.$$

Analogous mean expansions and limit theorems for

$$X_n(k) \qquad \text{for } \frac{k}{\log n} \in (\overline{1}, \overline{\alpha_+}),$$

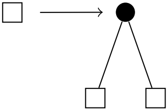
$$2^k - X_n(k) \quad \text{for } \frac{k}{\log n} \in (\overline{\alpha_-}, \overline{1}).$$

Hwang '95, Chauvin, Drmota and Jabbour-Hattab '01

Techniques and references

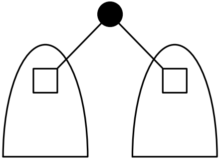
FORWARD

- Jabbour-Hattab '01
- Chauvin, Drmota and Jabbour-Hattab '01
- Chauvin, Klein, Marckert and Rouault '05
- Katona '05
- Labarbe '08
- Schopp '10
- Mailler and Marckert '17



BACKWARD

- Drmota and Hwang '04
- Drmota and Hwang '05
- Fuchs, Hwang and Neininger '06
- Devroye and Hwang '06
- Hwang '07
- Drmota, Janson Neininger '08



Outline

1. One-split branching random walks

2. Profile of binary search trees: a summary

3. **Main result: an asymptotic profile expansion**

Classical Chebyshev-Edgeworth-Cramér expansion

Let $Z_1, Z_2, \ldots$ be iid integer random variables with

- $\mathbb{E}\left[e^{tZ_1}\right] < \infty$ in a neighbourhood of 0,
- $\mathbb{E}[Z_1] = 0, \mathrm{Var}(Z_1) = 1$,
- Z_1 is not concentrated on a non-trivial sublattice.

Then, with $S_n = Z_1 + \cdots + Z_n, x_n(k) = \frac{k}{\sqrt{n}}$ and $r \in \mathbb{N}_0$:

$$n^{\frac{r+1}{2}} \sup_{k \in \mathbb{Z}} \left| \mathbb{P}(S_n = k) - \frac{e^{-\frac{1}{2}x_n^2(k)}}{\sqrt{2\pi n}} \sum_{s=0}^{r} \frac{Q_s(x_n(k))}{n^{s/2}} \right| \to 0,$$

where Q_s is a polynomial of degree $3s$ expressed through the cumulants $\kappa_2, \ldots, \kappa_{s+2}$. $Q_0 = 1$ and

$$Q_1(x) = \frac{\kappa_3}{6}\mathrm{He}_3(x), \quad Q_2(x) = \frac{\kappa_4}{24}\mathrm{He}_4(x) + \frac{\kappa_3^2}{72}\mathrm{He}_6(x).$$

Profile expansion for the binary search tree

Theorem (Kabluchko, Marynych and S. '16)

Let $U_n(k)$ be the external profile of a sequence of random binary search trees. Set

$$x_n(k) = x_n(k;\alpha) = \frac{k - \alpha \log n}{\sqrt{\alpha \log n}}, \quad \alpha_k = \frac{k}{\log n}.$$

Fix $r \geq 0, K \subseteq (\alpha_-, \alpha_+)$ compact. Uniformly in $k \in \mathbb{N}$ and $\alpha \in K$

$$(\log n)^{\frac{r+1}{2}} \left| \frac{U_n(k)}{n^{\alpha - 1 - \alpha_k \cdot \log \alpha / 2}} - \frac{e^{-\frac{1}{2} x_n^2(k)}}{\sqrt{2\pi \cdot \alpha \log n}} \sum_{s=0}^{r} \frac{F_s(x_n(k);\alpha)}{(\log n)^{s/2}} \right| \xrightarrow{a.s.} 0,$$

where $F_s(x;\alpha)$ is a polynomial in x of degree $3s$ whose coefficients are linear combinations of

$$X(\alpha), \ldots, X^{(s)}(\alpha).$$

Profile expansion for the binary search tree

$$(\log n)^{\frac{r+1}{2}} \left| \frac{U_n(k)}{n^{\alpha-1-\alpha_k \cdot \log \alpha/2}} - \frac{e^{-\frac{1}{2}x_n^2(k)}}{\sqrt{2\pi \cdot \alpha \log n}} \sum_{s=0}^{r} \frac{F_s(x_n(k);\alpha)}{(\log n)^{s/2}} \right| \xrightarrow{a.s.} 0,$$

where $F_0(x;\alpha) = X(\alpha)$ and

$$F_1(x;\alpha) = \frac{X'(\alpha)}{\sqrt{\alpha}} x + \frac{X(\alpha)}{6\sqrt{\alpha}} \mathrm{He}_3(x),$$

$$F_2(x;\alpha) = \frac{X''(\alpha)}{2\alpha} \mathrm{He}_2(x) + \left(\frac{X(\alpha)}{24\alpha} + \frac{X'(\alpha)}{6\alpha} \right) \mathrm{He}_4(x) + \frac{X(\alpha)}{72\alpha} \mathrm{He}_6(x),$$

and the first Hermite polynomials are

$$\mathrm{He}_2(x) = x^2 - 1, \qquad \mathrm{He}_3(x) = x^3 - 3x,$$

$$\mathrm{He}_4(x) = x^4 - 6x^2 + 3, \qquad \mathrm{He}_6(x) = x^6 - 15x^4 + 45x^2 - 15.$$

External BST profile - central regime

Recall: For $k = 2\log n + c_n$ and $c_n = O(1)$, the sequence

$$\left(\frac{U_n(k) - \mathbb{E}\left[U_n(k)\right]}{\sqrt{\mathrm{Var}(U_n(k))}} \right)_{n\geq 1}$$

does **not** converge in distribution.

Corollary (Kabluchko, Marynych and S. '16)

Let $k = \lfloor 2\log n \rfloor + a$ *with* $a \in \mathbb{Z}$. *Then, as* $n \to \infty$,

$$\frac{(\log n)^{3/2}}{n}\left(U_n(k) - \mathbb{E}\left[U_n(k)\right]\right) - \frac{X'(2)}{4\sqrt{\pi}}\left(\{2\log n\} + a + 1/2\right)$$
$$\xrightarrow{a.s.} -\frac{\chi - \mathbb{E}\left[\chi\right]}{8\sqrt{\pi}},$$

where $\{x\} := x - \lfloor x \rfloor$ *and* $\chi = X''(2) - X'(2)^2$.

External BST profile - mode

Recall: $m_n - 2\log n, n \geq 1$ is a tight sequence.

Corollary (KABLUCHKO, MARYNYCH AND S. '16)

For all n sufficiently large, m_n takes its value(s) in the set

$$\{\lfloor 2\log n + X'(2) - 1/2 \rfloor, \lceil 2\log n + X'(2) - 1/2 \rceil\}.$$

For a set of asymptotic frequency 1, m_n is equal to the integer closest to

$$2\log n + X'(2) - 1/2.$$

The width - more periodicities

Recall: $W_n \sim \frac{n}{\sqrt{4\pi \log n}}$ almost surely.

Corollary (Kabluchko, Marynych and S. '16)

Let

$$\overline{W}_n := 4 \log n \left(1 - \frac{\sqrt{4\pi \log n} W_n}{n}\right).$$

Then,

$$\overline{W}_n - \theta_n^2 \xrightarrow{a.s.} \chi - \frac{1}{12},$$

where

$$\chi = X''(2) - X'(2)^2,$$
$$\theta_n = \min_{k \in \mathbb{Z}} \left| 2 \log n + X'(2) - 1/2 - k \right|.$$

Outline

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Discussion - the proof

Fourier inversion using

$$W_n(\lambda) = \sum_{k\in\mathbb{N}} U_n(k)\cdot e^{\lambda k}, \quad \lambda \in \mathbb{C}.$$

Then,

$$\mathbb{E}\left[W_n(\lambda)\right] = \frac{n^{2e^{\lambda}-1}}{\Gamma(2e^{\lambda})}\cdot(1+o(1)), \quad \Re(\lambda) > 0.$$

Brown and Shubert '84, Jabbour-Hattab '01

Theorem (Chauvin, Klein, Marckert, Rouault '05)

There exists a complex domain G with $(\log\frac{\alpha_-}{2}, \log\frac{\alpha_+}{2}) \subseteq G$ such that, almost surely, uniformly on compact sets $K \subseteq G$ with polynomial rate of convergence,

$$\frac{W_n(\lambda)}{\mathbb{E}\left[W_n(\lambda)\right]} \to W(\lambda),$$

and $X(\alpha) = W(\log\frac{\alpha}{2})$.

Biggins '77, '92

Discussion - generalisations

Analogous expansions for

- general profiles $A_n(k), k \in \mathbb{Z}, n \geq 1$ with

$$e^{-w_n \cdot \varphi(\lambda)} \cdot \sum_{k \in \mathbb{Z}} A_n(k) \cdot e^{\lambda k} \to \Psi(\lambda),$$

  with an analytic function Ψ, where
  - $w_n \to \infty$,
  - φ is strictly convex on $\mathbb{R}$,
  - the convergence is exponential in w_n on compact subsets of a domain close to the real axis,
  - $e^{-w_n \cdot \varphi(\theta)} \cdot \sum_{k \in \mathbb{Z}} A_n(k) \cdot e^{(\theta + i\eta)k} \to 0$ for $\varepsilon < |\eta| < \pi$ with exponential rate of convergence.
- the profile of one-split branching random walks,
- the expected profile if $\zeta(\mathbb{Z})$ is deterministic,
- standard lattice BRWs

Grübel and Kabluchko '15

Summary and conclusion

- **full** uniform asymptotic profile expansion,
- precise information on occupation numbers, mode and width can be extracted almost automatically,
- extends to more general profiles $A_n(k), k \in \mathbb{Z}, n \geq 1$ upon controlling

$$\sum_{k \in \mathbb{Z}}^{\infty} A_n(k) \cdot e^{\lambda k}.$$

- martingale-free trees? Split trees?

THANK YOU