Exercise 1: The exponential law

Recall that a random variable X follows the exponential distribution of parameter $a > 0$ if and only if, for all $x \geq 0$,

$$\mathbb{P}(X \geq x) = \int_x^\infty a\mathrm{e}^{-ax}\,dx.$$

Let $X, X_1, \dots, X_n$ be i.i.d. random variables exponentially distributed of parameter 1.

(a) What is the distribution of $\min_{i=1..n} X_i$?

(b) What is the probability that $\min_{i=1..n} X_i = X_1$?

(c) Show that for all $0 \leq x < y$,

$$\mathbb{P}(X \geq y \,|\, X \geq x) = \mathbb{P}(X \geq y - x).$$

(We say the the exponential distribution "lacks memory".)

Exercise 2: The Yule process

Recall that the Yule process of parameter η is characterised as follows: Let τ be an exponential random variable of parameter η, then $Y(t) = 1$ for all $t < \tau$, and for all $t \geq \tau$, $Y_t = Y^{(1)}_{t-\tau} + Y^{(2)}_{t-\tau}$ where $Y^{(1)}$ and $Y^{(2)}$ are two independent copies of Y.

Let $(Y_t : t \geq 0)$ be a Yule process with rate η.

(a) Let $a > 0$ and show that $(Y_{at} : t \geq 0)$ is a Yule process with rate $a\eta$.

(b) Show that $(\mathrm{e}^{-\eta t}Y_t : t \geq 0)$ is a martingale.

(c) Infer that there exists a random variable ξ such that, almost surely,

$$\lim_{t\to\infty} \mathrm{e}^{-\eta t}Y_t = \xi.$$

(d) Show that ξ is exponentially distributed with parameter one.

(e) Show that $\sup_{t\geq 0} \mathbb{E}\mathrm{e}^{-2\eta t}Y_t^2 < \infty$.

Solution:

(a) Fix $a > 0$ and let $\widehat{Y}_t := Y_{at}$. Then, for all $t < \tau/a$, we have $\widehat{Y}_t = 1$, and for all $t \geq \tau/a$,

$$\widehat{Y}_t = Y_{at} = Y^{(1)}_{at-\tau} + Y^{(2)}_{at-\tau} = \widehat{Y}^{(1)}_{a(t-\tau/a)} + \widehat{Y}^{(2)}_{a(t-\tau/a)},$$

where $\widehat{Y}^{(1)}$ and $\widehat{Y}^{(2)}$ are two independent copies of $\widehat{Y}$. Note that the random variable $\hat{\tau} = \tau/a$ is exponentially distributed of parameter $a\eta$, implying that $\widehat{Y}$ is indeed a Yule process of parameter $a\eta$.

(b) Let us first calculate the expectation of Y_t for all $t \geq 0$. Using that, by definition,

$$Y_t = \mathbf{1}_{t<\tau} + \left(Y^{(1)}_{t-\tau} + Y^{(2)}_{t-\tau}\right)\mathbf{1}_{t>\tau},$$

where τ is exponentially distributed of parameter η, we get that

$$\mathbb{E}Y_t = e^{-\eta t} + \int_0^\infty 2\mathbb{E}Y_{t-u}\,\eta e^{-\eta u}\,\mathrm{d}u = e^{-\eta t} + 2\eta e^{-\eta t}\int_0^\infty \mathbb{E}Y_s\,e^{\eta s}\,\mathrm{d}s.$$

Thus, if we denote by $y(t) = e^{\eta t}\mathbb{E}Y_t$, we get, for all $t \geq 0$, $y'(t) = 2\eta y(t)$, implying that, since $y(0) = 1$, $y(t) = e^{2\eta t}$ for all $t \geq 0$. Thus, for all $t \geq 0$, $\mathbb{E}Y_t = e^{\eta t}$.

For all $s, t \geq 0$, using the Markov property,

$$\mathbb{E}\left[Y_{t+s} \,|\, \mathcal{F}_s\right] = \mathbb{E}\left[\sum_{i=1}^{Y_s} Y_t^{(i)} \,\middle|\, \mathcal{F}_s\right],$$

where the $Y^{(i)}$ are independent copies of Y, independent of Y_s. Thus,

$$\mathbb{E}\left[Y_{t+s} \,|\, \mathcal{F}_s\right] = Y_s \mathbb{E} Y_t = Y_s e^{\eta t},$$

implying that $(e^{-\eta t} Y_t)_{t \geq 0}$ is indeed a martingale.

(c) The martingale $(e^{-\eta t} Y_t)_{t \geq 0}$ is non-negative and thus converges almost surely to a random variable ξ.

(d) We define the random variables T_i as the random distances between successive jump times of the Yule process $(Y_t)_{t\geq 0}$. Let $T_0 = 0$ and, for all $i \geq 1$, let $T_i = \inf\{s > T_{i-1} \colon Y_s = i+1\}$. Note that at time $T_1 + \cdots + T_{i-1}$, the Yule process is the sum of i independent copies of itself and each of them thus jumps after a random time of exponential law of parameter η. Thus, the time to wait before the next jump time is the minimum of these i random variables. T_i is thus exponentially distributed of parameter $i\eta$.

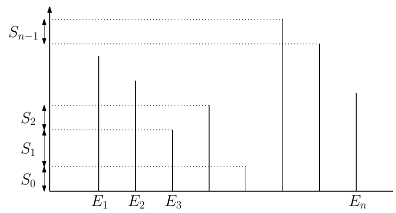

FIGURE 1 – *The k i.i.d. random variables $E_1, \ldots, E_k$ are represented by the length of the vertical sticks. The T_i are independent random variables exponentially distributed, of respective parameters $i\eta$.*

For all $t \geq 0$ and $k \in \mathbb{N}$, we have

$$\mathbb{P}(Y_t \geq k) = \mathbb{P}(T_1 + T_2 + \cdots + T_k \leq t) = \mathbb{P}\left(\max(E_1, \ldots, E_k) \leq t\right),$$

where $(E_i)_{i\geq 1}$ is a sequence of i.i.d. exponential random variables of parameter η (see Figure 1 for an explanation of the last equality). Thus, $\mathbb{P}(Y_t \geq k) = (1 - e^{-\eta t})^k$ for all integers k and for all $t \geq 0$. It implies that

$$\mathbb{P}(e^{-\eta t} Y_t \geq x) = \mathbb{P}(Y_t \geq x e^{\eta t}) = (1 - e^{-\eta t})^{\lfloor x e^{\eta t} \rfloor} \to e^{-x},$$

when t goes to infinity, which concludes the proof.

(e) Using again the fact that $Y_t = \mathbf{1}_{t<\tau} + \left(Y^{(1)}_{t-\tau} + Y^{(2)}_{t-\tau}\right)\mathbf{1}_{t>\tau}$, we get that (we skip the details since it is very similar to the calculation of Y_t in the solution of (b))

$$y_2(t) := e^{\eta t} \mathbb{E} Y_t^2 = 1 + \int_0^t \left(2\mathbb{E} Y_s^2 + 2\big(\mathbb{E} Y_s\big)^2\right) \eta e^{\eta s} \mathrm{d}s = \frac{1}{3} + \frac{2e^{3\eta t}}{3} + 2\eta \int_0^t y_2(s)\, \mathrm{d}s,$$

because $\mathbb{E} Y_s = e^{\eta s}$ for all $s \geq 0$. We thus get that

$$y_2'(t) = 2\eta y_2(t) + 2e^{3\eta t}, \qquad \text{and } y_2(0) = 1.$$

Solving this equation gives

$$y_2(t) = e^{\eta t} \mathbb{E} Y_t^2 = e^{2\eta t} \left(1 + \frac{2}{\eta}\left(e^{\eta t} - 1\right)\right),$$

and thus

$$e^{-2\eta t} \mathbb{E} Y_t^2 = e^{-\eta t} \left(1 + \frac{2}{\eta}\left(e^{\eta t} - 1\right)\right) \to \frac{2}{\eta},$$

when t goes to infinity, which implies the result.

Exercise 3: Scale-free property of the BB tree

Let us denote by

$$\Theta_t := \frac{1}{M(t)} \sum_{n=1}^{M(t)} \delta_{Z_n(t)}$$

is the empirical distribution of degrees in the Bianconi and Barabási continuous time tree at time t.

(a) Show that under Assumption 1 we have

$$\lim_{t\to\infty} \Theta_t = \nu \qquad \text{almost surely,}$$

where

$$\nu(k) = \int_0^1 \frac{\lambda^\star}{kx+\lambda^\star} \prod_{i=1}^{k-1} \frac{ix}{ix+\lambda^\star}\, d\mu(x).$$

(b) Show that $\lambda^\star \in (1,2)$ and that ν is a probability measure

(c) Show that $\nu(k) = k^{-(1+\lambda^\star)+o(1)}$ and hence the power law exponent ranges between the values 2 and 3, which is sometimes referred to as the supercritical regime.

Solution:

(a) Combining examples (2) and (5) and applying Theorem 3.1 we get that

$$\lim_{t\uparrow\infty} \Theta_t(k) = \lambda^\star \int_0^\infty e^{-\lambda^\star t}\, \mathbb{P}(Y(t)=k)\, dt,$$

where $(Y(t)\colon t>0)$ is a Yule process with random parameter X. We use the notation of the first exercise to write

$$\mathbb{P}(Y(t)=k) = \mathbb{P}(T_1+\cdots+T_{k-1}<t) - \mathbb{P}(T_1+\cdots+T_k<t),$$

where T_j is exponential with parameter jX. Now

$$\int_0^\infty \mathsf{e}^{-\lambda^\star t}\, \mathbb{P}(T_1+\cdots+T_k<t)\, dt = \mathbb{E}\int_{T_1+\cdots+T_k}^\infty \mathsf{e}^{-\lambda^\star t}\, dt = \frac{1}{\lambda^\star}\int d\mu(x) \prod_{i=1}^k \mathbb{E}\big[\mathsf{e}^{-\lambda^\star T_i}\big| X=x\big]$$
$$= \frac{1}{\lambda^\star}\int d\mu(x) \prod_{i=1}^k \frac{1}{1+\frac{\lambda^\star}{ix}}.$$

Hence,

$$\begin{aligned}\lambda^\star \int_0^\infty \mathsf{e}^{-\lambda^\star t}\, \mathbb{P}(Y(t)=k)\, dt &= \int d\mu(x) \Big(\prod_{i=1}^{k-1}\frac{ix}{ix+\lambda^\star} - \prod_{i=1}^{k}\frac{ix}{ix+\lambda^\star}\Big)\\ &= \int d\mu(x)\, \frac{\lambda^\star}{kx+\lambda^\star} \prod_{i=1}^{k-1}\frac{ix}{ix+\lambda^\star}.\end{aligned} \tag{1}$$

Observe that if ν is identified as a probability measure then the convergence holds automatically in the stronger total variation sense.

(b) $\lambda^\star$ is the unique solution of the equation

$$\int \frac{x}{\lambda-x}\, d\mu(x) = 1.$$

The left hand side is monotonically decreasing in λ and takes a value >1 for $\lambda=1$ and a value <1 for $\lambda=2$. Hence the solution lies in the interval $(1,2)$. Summing over $k=1,2,\dots$ in (1) shows that ν is a probability measure.

(c) Note that, for $k > n$,

$$\log \prod_{i=n}^{k-1} \frac{ix}{ix+\lambda^\star} = \sum_{i=n}^{k-1} \log \frac{1}{1+\frac{\lambda^\star}{ix}} = -(1+o_n(1)) \sum_{i=n}^{k-1} \frac{\lambda^\star}{ix} = -(1+o_n(1)) \frac{\lambda^\star}{x} \Big(\log \big({}^{k}\!/_{n} \big) + o_n(1) \Big).$$

We infer (without spelling out all details here) that for large k the main contribution to the integral comes from values of f close to one and that therefore

$$\int d\mu(x) \, \frac{\lambda^\star}{kx+\lambda^\star} \prod_{i=1}^{k-1} \frac{ix}{ix+\lambda^\star} = k^{-(1+\lambda^\star)+o_k(1)}.$$

Exercise 4: Size of the largest family

Show that, in distribution as $t \to \infty$,

$$\mathsf{e}^{-\gamma(t-T(t))} \max_{n\in\mathbb{N}} Z_n(t) \to W^{-\frac{\gamma}{\lambda^\star}},$$

where W is exponentially distributed with parameter $\Gamma(\alpha+1)\Gamma(1+\frac{\lambda^\star}{\gamma})(\lambda^\star)^{-\alpha}$

Solution:

We fix $x > 0$ and apply the vague convergence proved in Theorem 4.1 to the compact set

$$K := [-\infty, +\infty] \times [0, \infty] \times [x, \infty].$$

We get that

$$\sum_{n=1}^{M(t)} \mathbf{1}_K(\tau_n - T(t), t(1-X_n), \mathsf{e}^{-\gamma(t-T(t))} Z_n(t)) \Rightarrow \texttt{Poisson}\big(\textstyle\int_K d\zeta\big).$$

Hence

$$\mathbb{P}\Big(\mathsf{e}^{-\gamma(t-T(t))} \max_{n\in\{1,\dots,M(t)\}} Z_n(t) \geq x\Big) \to \mathbb{P}\Big(\texttt{Poisson}\big(\textstyle\int_K d\zeta\big) \geq 1\Big) = 1 - \exp\big(-\textstyle\int_K d\zeta\big). \qquad (2)$$

Integrating out gives

$$\begin{aligned} \int_K d\zeta &= \int_{-\infty}^{+\infty} \int_0^\infty \int_x^\infty \alpha f^{\alpha-1} \lambda^\star e^{\lambda^\star s} e^{-ze^{\gamma(s+f)}} e^{\gamma(s+f)} dz \; df \; ds \\ &= \int_0^\infty e^{-w} \int_0^\infty \alpha f^{\alpha-1} \int_{-\infty}^{\frac{1}{\gamma}\log\frac{w}{x} - f} \lambda^\star e^{\lambda^\star s} ds \; df \; dw \\ &= \Big(\int_0^\infty e^{-w} \Big(\frac{w}{x}\Big)^{\frac{\lambda^\star}{\gamma}} dw \Big) \Big(\int_0^\infty \alpha f^{\alpha-1} e^{-\lambda^\star f} df \Big) \\ &= \frac{\Gamma(\alpha+1)\Gamma(1+\frac{\lambda^\star}{\gamma})}{(\lambda^\star)^\alpha} \, x^{-\frac{\lambda^\star}{\gamma}}. \end{aligned}$$

Thus, the right hand side in (2) is $1 - \exp(-\Lambda x^{-\eta})$, for $\Lambda = \Gamma(\alpha+1)\Gamma(1+\frac{\lambda^\star}{\gamma})(\lambda^\star)^{-\alpha}$ and $\eta = \frac{\lambda^\star}{\gamma}$, which proves the statement.