Asymptotics of the Coefficients of Bivariate Analytic Functions with Algebraic Singularities

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AofA'15

Overview

- Goal: Starting with the closed form for a generating function $F(\mathbf{z})$, approximate $[\mathbf{z}^{\mathbf{r}}]\, F(\mathbf{z})$ as $\mathbf{r} \to \infty$.
  - The coefficients $[\mathbf{z}^{\mathbf{r}}]F(\mathbf{z})$ count something useful.

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- Look at F with algebraic singularities.
  - The branch cuts will cause problems!
- Multivariate! Use the method from Pemantle and Wilson's book.
  - Can't use residues here.

The Procedure in One Dimension

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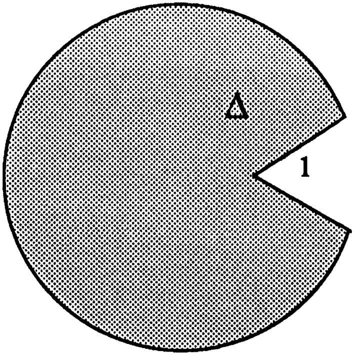
- Expand $\mathcal{C}$ until it gets stuck on a singularity of $F(z)$. Away from the singularity, expand beyond it.
- The z^{-n} term forces decay away from the singularity. So, analyze the integrand near the singularity.

Univariate Algebraic Singularity Example

- Flajolet-Odlyzko paper from 1990: Insist that $F(z) = O(|1 - z|^{\alpha})$ as $z \to 1$. Also, assume that F has no singularities except for $z = 1$ in the region pictured below:



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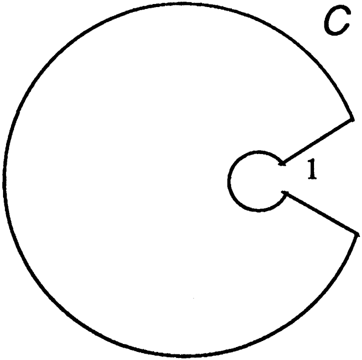

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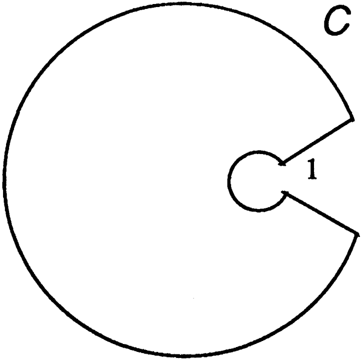
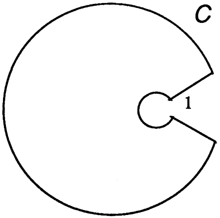


- Analyze each part separately.

Univariate Algebraic Singularity Example

- Since $F(z) = O(|1-z|^{\alpha})$, we'll compare the integrals,

$$\int_{\mathcal{C}} F(z) z^{-n-1} dz \quad \text{and} \quad \int_{\mathcal{C}} |1-z|^{\alpha} z^{-n-1} dz$$



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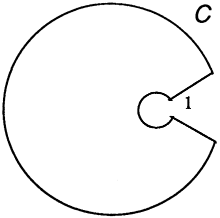

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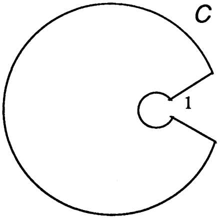

- The conclusion: $[z^n]F(z) = O\left(n^{-\alpha-1}\right)$
- Different assumptions about F near $z=1$ lead to different conclusions about the coefficients. ("Transfer Theorems.")

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- In 1996, Hwang used a probability framework and large deviation theorems to analyze a class of bivariate generating functions, again using FO results.
- Here, we'll use the multivariate Cauchy integral formula. Because there are branch cuts now, we'll rely on specific contour deformations instead of residues.

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- Use the Multivariate Cauchy Integral Formula,

$$[\mathbf{z}^{\mathbf{r}}]\, F(\mathbf{z}) = \left(\frac{1}{2\pi i}\right)^d \int_T F(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\, d\mathbf{z}$$

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- In order to take advantage of the decay of $\mathbf{z}^{-\mathbf{r}}$, we aim to expand T – but how?

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Step One: Critical Points

- Today, we'll start with $F = H(x,y)^{-\beta}$ for some $\beta \in \mathbb{R}, \beta \notin \mathbb{Z}_{\leq 0}$, and we'll estimate $[x^r y^s]H(x,y)^{-\beta}$ as $r, s \to \infty$ with $\frac{r}{s} \approx \lambda$.

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- Let $\mathcal{V} := \{(x,y) : H(x,y) = 0\}$ be the singular variety. We want to find the right points on $\mathcal{V}$ before expanding T.
- We'll restrict to smooth critical points: that is, critical points where $\mathcal{V}$ is a smooth manifold. From Pemantle and Wilson's 2013 book, these points satisfy the following conditions:

$$\begin{aligned} H &= 0 \\ ry\frac{\partial H}{\partial y} &= sx\frac{\partial H}{\partial x} \\ \nabla H &\neq 0 \end{aligned}$$

Despite seeming unmotivated, we don't need more than to assume these equations hold.

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  This approximates the log magnitude of $x^{-r}y^{-s}$.
- As we expand T in an attempt to minimize the maximum of h, the topology of T changes only at the critical points of h restricted to $\mathcal{V}$.
- In the smooth critical point case, this boils down to $H = 0$ and $\nabla_{\log} H || \hat{r}$.

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- We'll call our unique strictly minimal critical point (p, q).

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- Roughly speaking, we'll expand the y component of the torus until it becomes the circle $|y| = q$. In the x component, we'll use the Flajolet-Odlyzko contour near the critical point.

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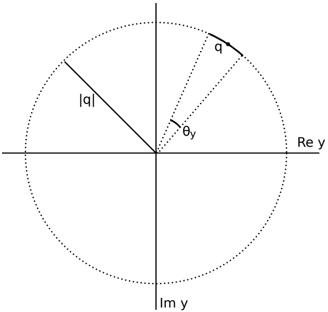
- Because we are assuming one minimal critical point, we can expand T beyond the critical point away from (p, q), which leads to exponentially faster decay for $x^{-r}y^{-s}$. Thus, we only care about the quasi-local contour near (p, q).

Step Two: The Contour

- First, we expand the y circle in T:



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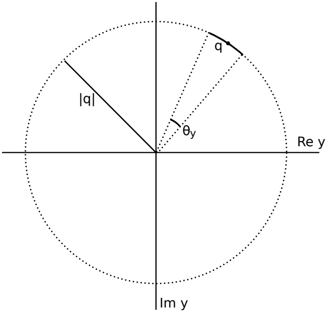
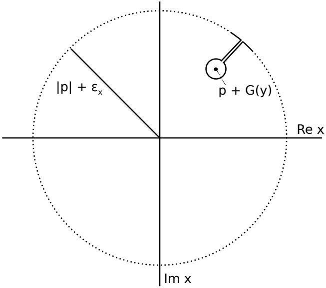


- By the Implicit Function Theorem, for each y on the arc near q, there is a $G(y)$ such that $H(p + G(y), y) = 0$.

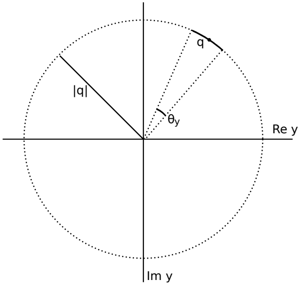
Step Two: The Contour

- Now, for each y in the arc near q, we expand x so it wraps around $p + G(y)$:



Call this quasi-local contour $\mathcal{C}^*$.

Step Two: The Contour – Problems



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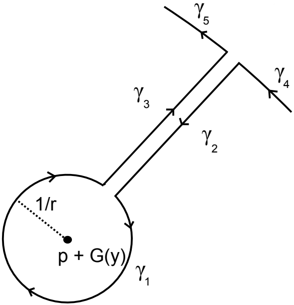

Close-up of the x contour

- We must connect this quasi-local contour to the rest of the torus.

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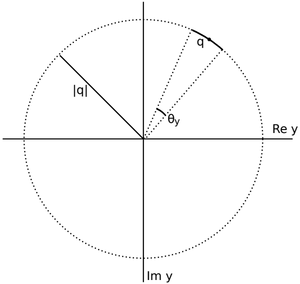

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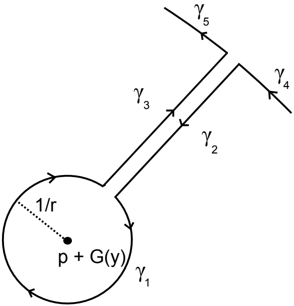

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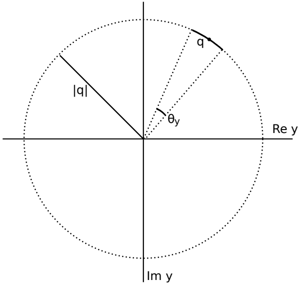

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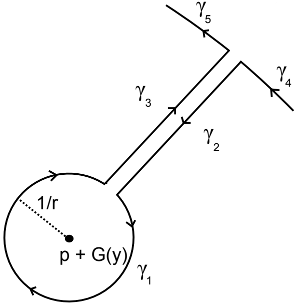

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- We've ignored branch cuts.

The Procedure

- Identify critical points: the singularities where T will become stuck.
- Expand T, and determine what it looks like near the critical points.
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- Analyze the remaining integral.

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- We'd like to approximate $H(x, y)$ as a one-dimensional function. It will help if

$$H(x, y) = \sum_{m,n \geq 0} a_{mn} x^m y^n$$

  with $a_{00} = a_{01} = a_{02} = 0$. This is enough to let us ignore y everywhere.

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- We'll choose the change of variables:

$$\begin{aligned} u &= x + \chi_1(y - q) + \chi_2(y - q)^2 \\ v &= y \end{aligned}$$

  χ_1 and χ_2 are constants in terms of the derivatives of H.

Step Three: Integrand – The Integral

- After applying the change of variables near (p, q), we have

$$\iint \tilde{H}(u, v)^{-\beta}(u - \chi_1(v - q) - \chi_2(v - q)^2)^{-r-1} v^{-s-1} \, \mathrm{d}u \mathrm{d}v$$

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- We want this instead:

$$\iint [H_x(p, q)(u - p)]^{-\beta} u^{-r-1} v^{-s-1} \left[1 - \frac{\chi_1 (v - q) + \chi_2 (v - q)^2}{p} \right]^{-r-1} \mathrm{d}u \mathrm{d}v$$

Then, we'd have a product integral.

Step Three: Integrand – Correction Factors

- We'll force what we want to be true:

$$\tilde{H}(u,v)^{-\beta}\left(u-\chi_1(v-q)-\chi_2(v-q)^2\right)^{-r-1}v^{-s-1}$$
$$=[H_x(p,q)\cdot(u-p)]^{-\beta}u^{-r-1}v^{-s-1}\left[1-\frac{\chi_1(v-q)+\chi_2(v-q)^2}{p}\right]^{-r-1}K(u,v)L(u,v)$$

Here, K and L are correction factors with the following definitions:

$$K(u,v):=\left(\frac{1-\frac{\chi_1(v-q)+\chi_2(v-q)^2}{u}}{1-\frac{\chi_1(v-q)+\chi_2(v-q)^2}{p}}\right)^{r-1}\quad\text{and}\quad L(u,v):=\left[\frac{\tilde{H}(u,v)}{H_x(p,q)(u-p)}\right]^{-\beta}$$

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- We show $K(u,v)$ and $L(u,v)=1+o(1)$ near (p,q). Away from (p,q), we show that the original integrand and the product integrand are both small.

The Procedure

- Identify critical points: the singularities where T will become stuck.
- Expand T, and determine what it looks like near the critical points.
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- Analyze the remaining integral.

Step Four: Evaluate – The u Integral

- $\int_{\mathcal{F}} [H_x(p, q) \cdot (u - p)]^{-\beta} u^{-r-1} \, \mathrm{d}u$

  Here, $\mathcal{F}$ is the u projection of the quasi-local contour. That is, it wraps around the critical point, p, like the Flajolet-Odlyzko contour.

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- This is just a binomial coefficient, using Cauchy's integral formula. After applying Stirling's approximation, we get:

  $$\frac{2\pi i}{\Gamma(\beta)} r^{\beta-1} p^{-r} \left\{ \left(-H_x(p,q)\right)^{-\beta} \right\}_P e^{-\beta(2\pi i \omega)}$$

Step Four: Evaluate – Branch Cut!

$$\frac{2\pi i}{\Gamma(\beta)} r^{\beta-1} p^{-r} \left\{ (-H_x(p,q))^{-\beta} \right\}_P e^{-\beta(2\pi i\omega)}$$

- We choose some branch cut of $\left\{x^{-\beta}\right\}_P$ so that $\left\{H(x,y)^{-\beta}\right\}_P$ agrees with the generating function near the origin.

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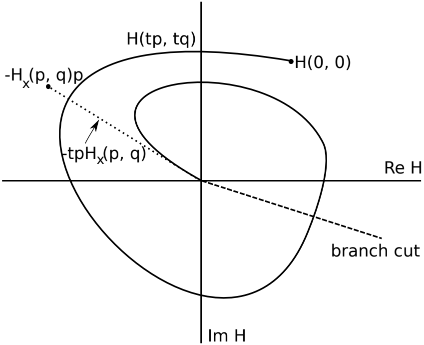
- As the torus expands towards (p,q), the image of $H(x,y)$ may wrap around the origin several times before hitting $H(p,q)$. We let ω count the number of times the image crosses over this branch cut.

Step Four: Evaluate – Branch Cut!



Here, $\omega = 1$.

Step 4: Evaluate – The v Integral

- $$\int_{\mathcal{G}} v^{-s-1} \left[1 - \frac{\chi_1(v-q) + \chi_2(v-q)^2}{p}\right]^{-r-1} \mathrm{d}v$$

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  Here, $\mathcal{G}$ is the v projection of the quasi-local contour. That is, it is an arc near q.
- This integral is a Fourier-Laplace type integral, and standard results give us that it is asymptotically

$$iq^{-s}\sqrt{\frac{2\pi}{-q^2 Mr}}$$

Here, M involves the derivatives of a phase function after rewriting the integrand. M is defined in terms of χ_1 and χ_2, and reflects the curvature of $\mathcal{V}$ at (p, q).

Step 4: Evaluate – The v Integral

- $\displaystyle\int_{\mathcal{G}} v^{-s-1}\left[1-\frac{\chi_1(v-q)+\chi_2(v-q)^2}{p}\right]^{-r-1}\mathrm{d}v$

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- Multiplying these two integral approximations together completes our procedure.

The Result

Theorem (G. 2015)

Let $H(x, y)$ be an analytic function with a single minimal critical point (p, q), where $\frac{\partial H}{\partial x}\Big|_{(x,y)=(p,q)} \neq 0$. Let $\beta \in \mathbb{R}, \beta \notin \mathbb{Z}_{\leq 0}$. Assume p, q, and $M \neq 0$. Then, as r and $s \to \infty$ with $\lambda = \dfrac{r + O(1)}{s}$,

$$[x^r y^s]H(x, y)^{-\beta} \sim \frac{r^{\beta - \frac{3}{2}} p^{-r} q^{-s} \left\{(-H_x(p, q)p)^{-\beta}\right\}_P e^{-\beta(2\pi i \omega)}}{\Gamma(\beta)\sqrt{-2\pi q^2 M}}$$

Here, M depends on the curvature of the zero set of H, and $\{x^{-\beta}\}_P$ is defined with a precise argument. (Some technical details are missing.)

Example

- The Grahams studied the cover polynomials of digraphs, and came up with the following generating function:

$$F(x, y) = \frac{1 - x(1 + y)}{\sqrt{1 - 2x(1 + y) - x^2(1 - y)^2}}$$

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- We can compute the solutions to this with a Gröbner basis in Maple:

```
gb := Basis([H, y * diff(H, y) - mu * x * diff(H, x)], plex(x, y));
```

Example Continued

- The first polynomial in the Gröbner basis is:

$$1 - 2\mu + \mu^2 + (-4 - 2\mu^2 + 6\mu)x + 2x^3 + (2\mu^2 - 4\mu + 3)x^2$$

Solve this for the three x solutions in terms of μ. These are the x components of the critical points.

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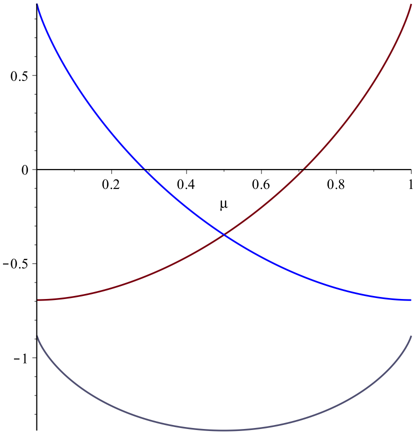
- We can use the second basis element to solve for y.

- We can plot the negative heights of the three critical point solutions. (That is, $-h = r\mathrm{Re}(\log x) + s\mathrm{Re}(\log y)$, the negative log magnitude of $x^{-r}y^{-s}$.)



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Example Continued

- The fact that one solution curve is below the others means that there is at most one minimal critical point for each μ. It is still computationally difficult to show that this critical point is minimal.
- We can apply the previous theorem using this one critical point to estimate the asymptotics of the coefficients.
- For example, when $\mu = \frac{1}{2}$, the unique minimal critical point is $(x, y) = \left(\frac{1}{4}, 1\right)$. If we choose $r = 70$, then $s = 35$, and the theorem says that the coefficient is approximately $3.65924 \cdot 10^{39}$. It is actually $3.59821 \cdot 10^{39}$. The ratio is 1.017.

Future Research

- More terms in the asymptotic expansion.

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- Extend to more variables.
- Broader class of algebraic singularities. (Not just $H^{-\beta}$.)
- Combine with other asymptotic techniques, like creative telescoping methods.

Thank you!