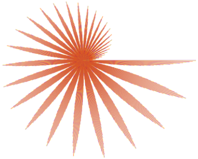
Recurrence function of Sturmian sequences. A probabilistic study

Pablo Rotondo

Universidad de la República, Uruguay

Ongoing work with

Valérie Berthé, Eda Cesaratto, Brigitte Vallée, and Alfredo Viola



AofA'15, 8–12 June, 2015.

Study in combinatorics of words.

Main aim: description of the finite factors of an infinite word u

- How many factors of length n? $\longrightarrow$ Complexity
- What are the gaps between them? $\longrightarrow$ Recurrence

Very easy when the word is eventually periodic !

Study in combinatorics of words.

Main aim: description of the finite factors of an infinite word u

– How many factors of length n? $\longrightarrow$ Complexity

– What are the gaps between them? $\longrightarrow$ Recurrence

Very easy when the word is eventually periodic !

Sturmian words:

the “simplest” binary infinite words that are not eventually periodic

Study in combinatorics of words.

Main aim: description of the finite factors of an infinite word u

- How many factors of length n? $\longrightarrow$ Complexity
- What are the gaps between them? $\longrightarrow$ Recurrence

Very easy when the word is eventually periodic !

Sturmian words:

the "simplest" binary infinite words that are not eventually periodic

The recurrence function is widely studied for Sturmian words.

Classical study : for each fixed Sturmian word,

what are the extreme bounds for the recurrence function?

Study in combinatorics of words.

Main aim: description of the finite factors of an infinite word u

– How many factors of length n? $\longrightarrow$ Complexity

– What are the gaps between them? $\longrightarrow$ Recurrence

Very easy when the word is eventually periodic !

Sturmian words:

the "simplest" binary infinite words that are not eventually periodic

The recurrence function is widely studied for Sturmian words.

Classical study : for each fixed Sturmian word,

what are the extreme bounds for the recurrence function?

Here, in a convenient model,

we perform a probabilistic study:

For a "random" sturmian word, and for a given "position",

– what is the mean value of the recurrence?

– what is the limit distribution of the recurrence?

Plan of the talk

Complexity

$\mathcal{L}_u(n)$ denotes the set of factors of length n in u.

Definition

Complexity function of an infinite word $u \in \mathcal{A}^{\mathbb{N}}$

$$p_u\colon \mathbb{N} \to \mathbb{N}\,, \qquad p_u(n) = |\mathcal{L}_u(n)|\ .$$

Two simple facts: $\quad p_u(n) \le |\mathcal{A}|^n\,, \qquad p_u(n) \le p_u(n+1)\,.$

Important property

$u \in \mathcal{A}^{\mathbb{N}}$ is not eventually periodic

$$\iff p_u(n+1) > p_u(n)$$

$$\implies p_u(n) \ge n+1\,.$$

Recurrence

Definition (Uniform recurrence)

A word $u \in \mathcal{A}^{\mathbb{N}}$ is uniformly recurrent iff each finite factor appears infinitely often and with bounded gaps.

Recurrence

Definition (Uniform recurrence)

A word $u \in \mathcal{A}^{\mathbb{N}}$ is uniformly recurrent iff each finite factor appears infinitely often and with bounded gaps.

Definition (Recurrence function)

Let $u \in \mathcal{A}^{\mathbb{N}}$ be uniformy recurrent. The recurrence function of u is:

$$R_{\langle u \rangle}(n) = \inf \{ m \in \mathbb{N} : \text{any } w \in \mathcal{L}_u(m) \text{ contains all the factors } v \in \mathcal{L}_u(n) \} .$$

Recurrence

Definition (Uniform recurrence)

A word $u \in \mathcal{A}^{\mathbb{N}}$ is uniformly recurrent iff each finite factor appears infinitely often and with bounded gaps.

Definition (Recurrence function)

Let $u \in \mathcal{A}^{\mathbb{N}}$ be uniformy recurrent. The recurrence function of u is:

$$R_{\langle u\rangle}(n) = \inf \{m \in \mathbb{N} : \text{any } w \in \mathcal{L}_u(m) \text{ contains all the factors } v \in \mathcal{L}_u(n)\}.$$

The recurrence function gives a notion of the cost we have to pay to 'discover' the factors of u.

Recurrence

Definition (Uniform recurrence)

A word $u \in \mathcal{A}^{\mathbb{N}}$ is uniformly recurrent iff each finite factor appears infinitely often and with bounded gaps.

Definition (Recurrence function)

Let $u \in \mathcal{A}^{\mathbb{N}}$ be uniformy recurrent. The recurrence function of u is:

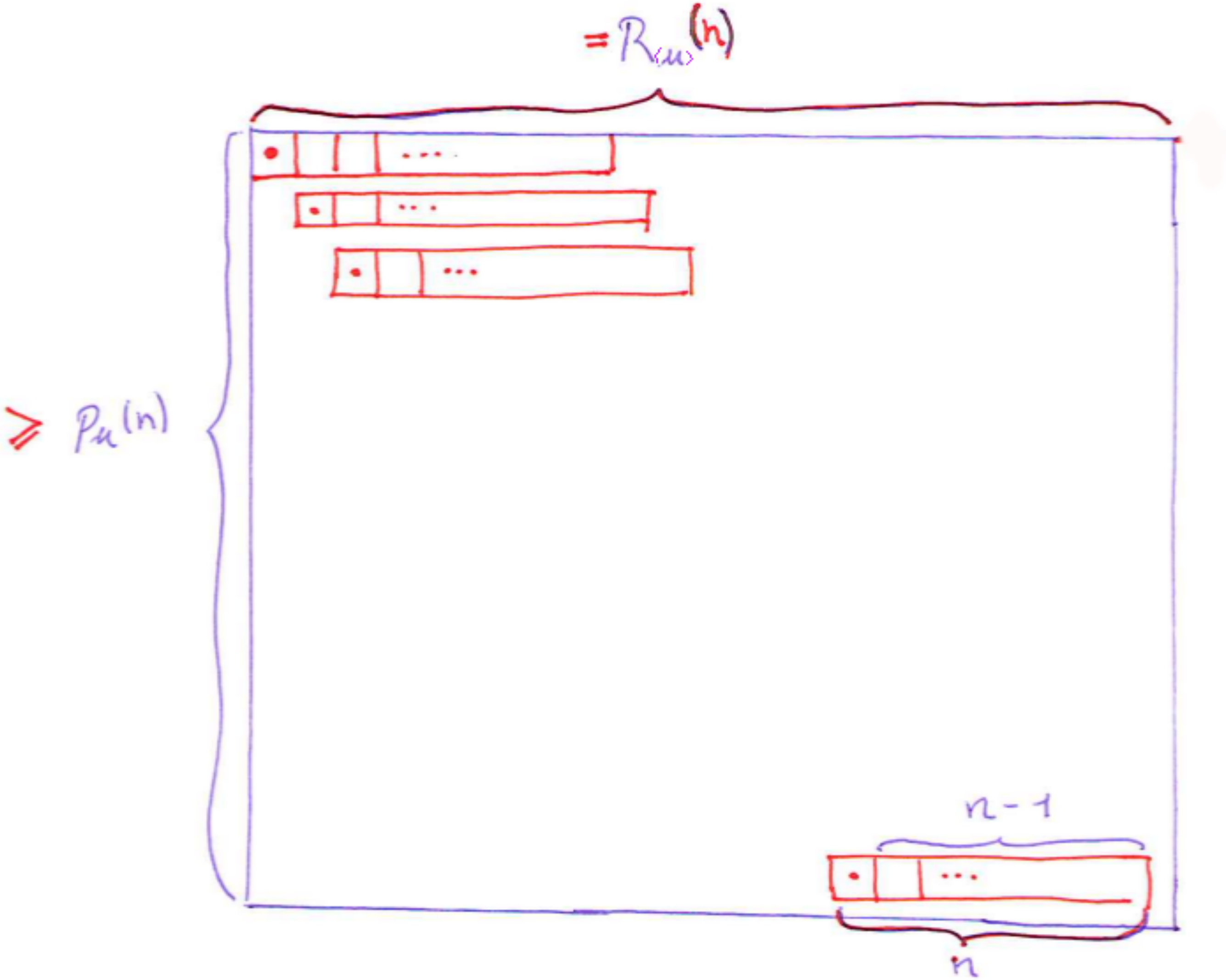
$$R_{\langle u \rangle}(n) = \inf \{m \in \mathbb{N} : \text{any } w \in \mathcal{L}_u(m) \text{ contains all the factors } v \in \mathcal{L}_u(n)\}.$$

The recurrence function gives a notion of the cost we have to pay to 'discover' the factors of u.

A noteworthy inequality between the two functions,

the complexity function and the recurrence function

$$R_{\langle u \rangle}(n) \geq p_u(n) + n - 1.$$



- All $p_u(n)$ factors of length n appear somewhere in our window of length $R_{\langle u \rangle}(n)$.
- We have $p_u(n)$ "starting points" + $(n-1)$ characters to finish the last factor $\Rightarrow$ $R_{\langle u \rangle}(n) \geq p_u(n) + n - 1$.

Sturmian words

These are the "simplest" words that are not eventually periodic.

Sturmian words

These are the "simplest" words that are not eventually periodic.

Definition

A word $u \in \{\mathbf{0}, \mathbf{1}\}^{\mathbb{N}}$ is Sturmian iff $p_u(n) = n + 1$ for each $n \geq 0$.

Sturmian words

These are the "simplest" words that are not eventually periodic.

Definition

A word $u \in \{\mathbf{0},\mathbf{1}\}^{\mathbb{N}}$ is Sturmian iff $p_u(n) = n+1$ for each $n \geq 0$.

Explicit construction

Associate with a pair (α, β) the two sequences

$$\underline{u}_n = \lfloor \alpha\,(n+1) + \beta \rfloor - \lfloor \alpha\, n + \beta \rfloor$$

$$\overline{u}_n = \lceil \alpha\,(n+1) + \beta \rceil - \lceil \alpha\, n + \beta \rceil$$

and the two words $\underline{S}(\alpha, \beta)$ and $\overline{S}(\alpha, \beta)$ produced in this way.

Sturmian words

These are the "simplest" words that are not eventually periodic.

Definition

A word $u \in \{\mathbf{0},\mathbf{1}\}^{\mathbb{N}}$ is Sturmian iff $p_u(n) = n+1$ for each $n \geq 0$.

Explicit construction

Associate with a pair (α, β) the two sequences

$$\underline{u}_n = \lfloor \alpha\,(n+1) + \beta \rfloor - \lfloor \alpha\, n + \beta \rfloor$$

$$\overline{u}_n = \lceil \alpha\,(n+1) + \beta \rceil - \lceil \alpha\, n + \beta \rceil$$

and the two words $\underline{S}(\alpha, \beta)$ and $\overline{S}(\alpha, \beta)$ produced in this way.

A word u is Sturmian iff there are $\alpha, \beta \in [0, 1[$, with α irrational, such that $u = \underline{S}(\alpha, \beta)$ or $u = \overline{S}(\alpha, \beta)$.

Recurrence of Sturmian words

Property

Let u be a Sturmian word of the form $\underline{S}(\alpha, \beta)$ or $\overline{S}(\alpha, \beta)$. Then

- u is uniformly recurrent
- $R_{\langle u \rangle}(n)$ only depends on α, and it is written as $R_{\alpha}(n)$.

Recurrence of Sturmian words

Property

Let u be a Sturmian word of the form $\underline{S}(\alpha, \beta)$ or $\overline{S}(\alpha, \beta)$. Then

- u is uniformly recurrent
- $R_{\langle u \rangle}(n)$ only depends on α, and it is written as $R_{\alpha}(n)$.
- The sequence $(R_{\alpha}(n))$ only depends on the continuants of α.

Recurrence of Sturmian words

Property

Let u be a Sturmian word of the form $\underline{S}(\alpha, \beta)$ or $\overline{S}(\alpha, \beta)$. Then

- u is uniformly recurrent
- $R_{\langle u \rangle}(n)$ only depends on α, and it is written as $R_{\alpha}(n)$.
- The sequence $(R_{\alpha}(n))$ only depends on the continuants of α.

Reminder :
The continuant $q_k(\alpha)$ is the denominator of the k-th convergent of α.
It is obtained via the truncation at depth k of the CFE of α.

The sequence $(q_k(\alpha))_k$ is strictly increasing.

Recurrence of Sturmian words

Property

Let u be a Sturmian word of the form $\underline{S}(\alpha, \beta)$ or $\overline{S}(\alpha, \beta)$. Then

- u is uniformly recurrent
- $R_{\langle u \rangle}(n)$ only depends on α, and it is written as $R_{\alpha}(n)$.
- The sequence $(R_{\alpha}(n))$ only depends on the continuants of α.

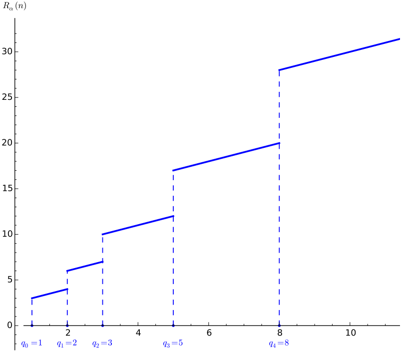
Reminder :
The continuant $q_k(\alpha)$ is the denominator of the k-th convergent of α.
It is obtained via the truncation at depth k of the CFE of α.
The sequence $(q_k(\alpha))_k$ is strictly increasing.

Theorem (Morse, Hedlund, 1940)

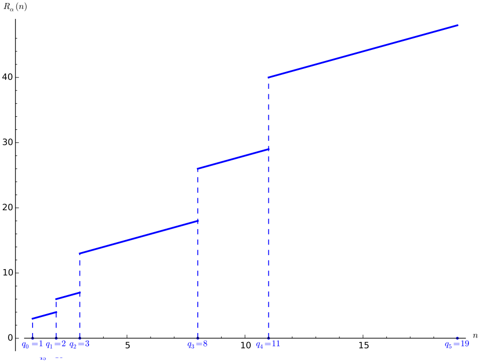
The recurrence function is piecewise affine and satisfies

$$R_{\alpha}(n) = n - 1 + q_{k-1}(\alpha) + q_k(\alpha)\,, \qquad \text{for } n \in [q_{k-1}(\alpha), q_k(\alpha)[.$$

Recurrence function for two Sturmian words



Recurrence function for $\alpha = \varphi^2$,
with $\varphi = (\sqrt{5}-1)/2$.



Recurrence function for $\alpha = 1/e$.

Recurrence function of Sturmian words: classical results.

Proposition

For any irrational $\alpha \in [0,1]$, *one has* $$\liminf \frac{R_\alpha(n)}{n} \leq 3\,.$$

Recurrence function of Sturmian words: classical results.

Proposition

For any irrational $\alpha \in [0,1]$, one has $$\liminf \frac{R_\alpha(n)}{n} \leq 3\,.$$

Proof: Take the sequence $n_k = q_k - 1$.

Recurrence function of Sturmian words: classical results.

Proposition

For any irrational $\alpha \in [0,1]$*, one has* $\liminf \frac{R_\alpha(n)}{n} \leq 3\,.$

Proof: Take the sequence $n_k = q_k - 1$.

Theorem

For almost any irrational α*, one has*

$$\limsup \frac{R_\alpha(n)}{n\,\log n} = \infty, \qquad \limsup \frac{R_\alpha(n)}{n\,(\log n)^c} = 0 \quad \text{for any } c > 1$$

Recurrence function of Sturmian words: classical results.

Proposition

For any irrational $\alpha \in [0,1]$, *one has* $\liminf \frac{R_\alpha(n)}{n} \leq 3\,.$

Proof: Take the sequence $n_k = q_k - 1$.

Theorem

For almost any irrational α, *one has*

$$\limsup \frac{R_\alpha(n)}{n\ \log n} = \infty, \qquad \limsup \frac{R_\alpha(n)}{n\ (\log n)^c} = 0 \quad \textit{for any } c > 1$$

Proof: Apply the Morse–Hedlund formula and Khinchin's Theorem.

Our point of view

Usual studies of $R_\alpha(n)$

- consider all possible sequences of indices n.
- give information on extreme cases.
- give results for almost all α.

Our point of view

Usual studies of $R_\alpha(n)$

- consider all possible sequences of indices n.
- give information on extreme cases.
- give results for almost all α.

Here:

- we study particular sequences of indices n depending on α, defined with their position on the intervals $[q_{k-1}(\alpha), q_k(\alpha)[$.
- we then draw α at random.
- we perform a probabilistic study.
- we then study the role of the position
  in the probabilistic behaviour of the recurrence function.

Subsequences with a fixed position

We work with particular subsequences of indices n

Given $\mu \in]0,1]$ the sequence

$$n_k^{\langle\mu\rangle}(\alpha) = q_{k-1}(\alpha) + \lfloor \mu \left(q_k(\alpha) - q_{k-1}(\alpha)\right) \rfloor$$

is the subsequence of position μ of α.

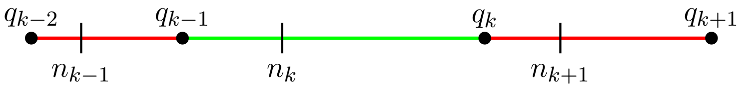


Figure: Sequence of indices n for $\mu = 1/3$.

We study

- the behaviour of

$$\frac{R_\alpha(n)}{n}, \quad n = n_k^{\langle\mu\rangle} = q_{k-1} + \lfloor \mu\,(q_k - q_{k-1}) \rfloor$$

  when n has a fixed position μ within $[q_{k-1}, q_k[$.

  Remark that $(n_k^{\langle\mu\rangle})_k$ is a sequence depending on $\alpha \in \mathcal{I}$.

- what happens when α is drawn uniformly from $\mathcal{I} = [0, 1]$.

We study

- the behaviour of

$$\frac{R_\alpha(n)}{n}, \quad n = n_k^{\langle\mu\rangle} = q_{k-1} + \lfloor \mu\,(q_k - q_{k-1}) \rfloor$$

  when n has a fixed position μ within $[q_{k-1}, q_k[$.

  Remark that $(n_k^{\langle\mu\rangle})_k$ is a sequence depending on $\alpha \in \mathcal{I}$.

- what happens when α is drawn uniformly from $\mathcal{I} = [0, 1]$.

We consider the sequence of random variables

$$S_k^{\langle\mu\rangle} = \frac{R_\alpha(n) + 1}{n} = 1 + \frac{q_{k-1} + q_k}{n}, \qquad n = n_k^{\langle\mu\rangle}.$$

For any fixed $\mu \in [0, 1]$, we perform an asymptotic study

- for expected values: $\lim_{k\to\infty} \mathbb{E}[S_k^{\langle\mu\rangle}]$
- for distributions : $\lim_{k\to\infty} \Pr[S_k^{\langle\mu\rangle} \in J]$

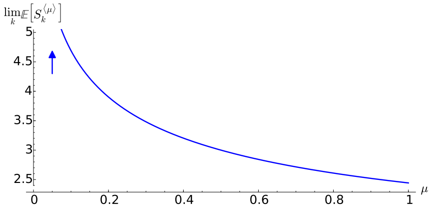
First result : Expectations

For each $\mu \in]0,1]$*, the sequence of random variables* $S_k^{\langle\mu\rangle}$ *satisfies*

$$\mathbb{E}[S_k^{\langle\mu\rangle}] = 1 + \frac{1}{\log 2}\frac{|\log \mu|}{1-\mu} + O\left(\frac{\varphi^{2k}}{\mu}\right) + O\left(\varphi^k \frac{|\log \mu|}{1-\mu}\right),$$

(for $k \to \infty$*). Here,* $\varphi = (\sqrt{5}-1)/2 \doteq 0.6180339\ldots$
and the constants of the O*-terms are uniform in* μ *and* k*.*

Remark: The result only holds for $\mu > 0$.



Limit of the expected value as a function of μ.

Second result : Distributions

For each $\mu \in [0, 1]$ *with* $\mu \neq 1/2$,
the sequence of random variables $S_k^{\langle\mu\rangle}$ *has a limit density*

$$s_\mu(x) = \frac{1}{\log 2\,(x-1)\ |2-\mu-x\,(1-\mu)|}\,\mathbf{1}_{I_\mu}(x)\,.$$

Here, I_μ *is the interval with endpoints* 3 *and* $1 + 1/\mu$.

Second result : Distributions

For each $\mu \in [0,1]$ *with* $\mu \neq 1/2$,
the sequence of random variables $S_k^{\langle\mu\rangle}$ *has a limit density*

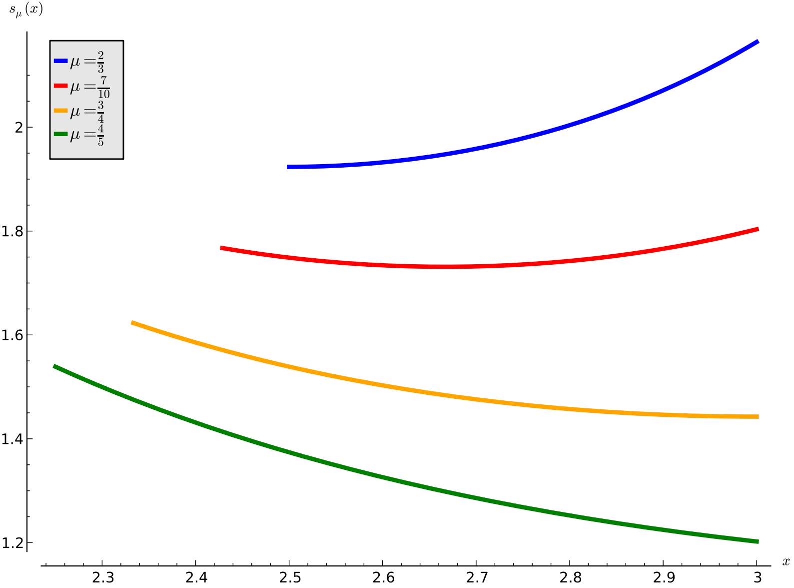
$$s_\mu(x) = \frac{1}{\log 2\,(x-1)\ |2-\mu-x\,(1-\mu)|}\,\mathbf{1}_{I_\mu}(x)\,.$$

Here, I_μ *is the interval with endpoints* 3 *and* $1+1/\mu$.
For all $b \geq \min\{3,\ 1+\frac{1}{\mu}\}$

$$\Pr\left[S_k^{\langle\mu\rangle} \leq b\right] = \int_0^b s_\mu(x)dx + \frac{1}{b}\,O\left(\varphi^k\right)\,.$$

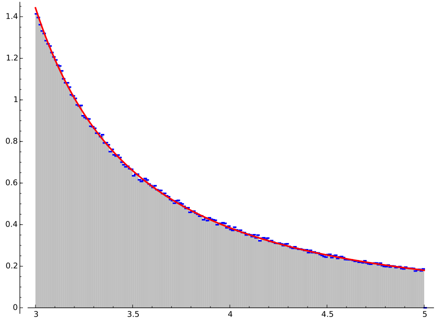
where the constant of the O*-term is uniform in* b *and* k.
When $|\mu - 1/2| \geq \epsilon$ *for a fixed* $\epsilon > 0$, *it is also uniform in* μ.

Limit density s_μ



Limit density for $\mu = 1/4$

Interval	Empirical Pr	Asymptotic Pr
$[3.0, 3.0]$	0.0	0.0
$[3.0, 3.5]$	$0.\mathbf{485}237$	$0.\mathbf{485}4\ldots$
$[3.0, 4.0]$	$0.\mathbf{73}7139$	$0.\mathbf{73}69\ldots$
$[3.0, 4.5]$	$0.\mathbf{893}511$	$0.\mathbf{893}1\ldots$
$[3.0, 5.0]$	1.0	1.0



In blue, the scaled histogram for $k = 25$, bin-width $\delta = 1/10$, obtained with 10^6 samples.

In red, the graph of the limit distribution $s_{1/4}(x) = \dfrac{1}{\log 2} \dfrac{4}{(x-1)(3x-7)}$.

Four steps in the proof

i) We drop the integer part in $S_k^{\langle\mu\rangle}$ getting

$$\tilde{S}_k^{\langle\mu\rangle} = 1 + \frac{q_k + q_{k-1}}{q_{k-1} + \mu\,(q_k - q_{k-1})}\,,$$

which depends only on $\frac{q_{k-1}}{q_k}$. Indeed

$$\tilde{S}_k^{\langle\mu\rangle} = f_\mu\left(\frac{q_{k-1}}{q_k}\right), \qquad \text{with} \qquad f_\mu(x) = 1 + \frac{1+x}{x + \mu\,(1-x)}\,.$$

Four steps in the proof

i) We drop the integer part in $S_k^{\langle\mu\rangle}$ getting

$$\tilde{S}_k^{\langle\mu\rangle} = 1 + \frac{q_k + q_{k-1}}{q_{k-1} + \mu\,(q_k - q_{k-1})}\,,$$

which depends only on $\frac{q_{k-1}}{q_k}$. Indeed

$$\tilde{S}_k^{\langle\mu\rangle} = f_\mu\left(\frac{q_{k-1}}{q_k}\right), \qquad \text{with} \qquad f_\mu(x) = 1 + \frac{1+x}{x+\mu\,(1-x)}\,.$$

ii) The expected value and the distribution of $\tilde{S}_k^{\langle\mu\rangle}$ are expressed with the k–th iterate of the Perron-Frobenius operator **H**.

Four steps in the proof

i) We drop the integer part in $S_k^{\langle\mu\rangle}$ getting

$$\tilde{S}_k^{\langle\mu\rangle} = 1 + \frac{q_k + q_{k-1}}{q_{k-1} + \mu\,(q_k - q_{k-1})}\,,$$

which depends only on $\frac{q_{k-1}}{q_k}$. Indeed

$$\tilde{S}_k^{\langle\mu\rangle} = f_\mu\left(\frac{q_{k-1}}{q_k}\right), \qquad \text{with} \qquad f_\mu(x) = 1 + \frac{1+x}{x + \mu\,(1-x)}\,.$$

ii) The expected value and the distribution of $\tilde{S}_k^{\langle\mu\rangle}$ are expressed with the k–th iterate of the Perron-Frobenius operator **H**.

iii) The asymptotics for $k \to \infty$ is obtained by using the spectral properties of **H**, when acting on the space of functions of bounded variation.

Four steps in the proof

i) We drop the integer part in $S_k^{\langle\mu\rangle}$ getting

$$\tilde{S}_k^{\langle\mu\rangle} = 1 + \frac{q_k + q_{k-1}}{q_{k-1} + \mu\,(q_k - q_{k-1})}\,,$$

which depends only on $\frac{q_{k-1}}{q_k}$. Indeed

$$\tilde{S}_k^{\langle\mu\rangle} = f_\mu\left(\frac{q_{k-1}}{q_k}\right), \qquad \text{with} \qquad f_\mu(x) = 1 + \frac{1+x}{x + \mu\,(1-x)}\,.$$

ii) The expected value and the distribution of $\tilde{S}_k^{\langle\mu\rangle}$ are expressed with the k–th iterate of the Perron-Frobenius operator **H**.

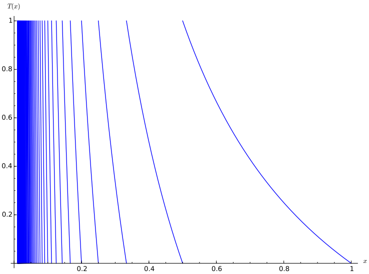
iii) The asymptotics for $k \to \infty$ is obtained by using the spectral properties of **H**, when acting on the space of functions of bounded variation.

iv) Finally we return from $\tilde{S}_k^{\langle\mu\rangle}$ to $S_k^{\langle\mu\rangle}$.

The Euclidean dynamical system

The Gauss map $T : [0,1] \to [0,1]$

$$T(x) = \left\{ \frac{1}{x} \right\} = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor .$$



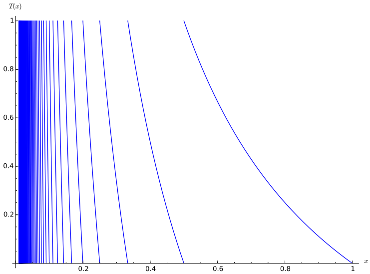
The inverse branches of T are:

$$\mathcal{H} = \left\{ h_m \colon x \mapsto \frac{1}{m+x} \quad : \quad m \geq 1 \right\} .$$

The Euclidean dynamical system

The Gauss map $T : [0,1] \to [0,1]$

$$T(x) = \left\{ \frac{1}{x} \right\} = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor .$$



The inverse branches of T are:

$$\mathcal{H} = \left\{ h_m \colon x \mapsto \frac{1}{m+x} \quad : \quad m \geq 1 \right\} .$$

The inverse branches of T^k are:

$$\mathcal{H}^k = \{ h_{m_1, m_2, \ldots m_k} = h_{m_1} \circ h_{m_2} \circ \ldots \circ h_{m_k} \quad : \quad m_1, \ldots, m_k \geq 1 \} .$$

The LFT $h_{m_1,\ldots,m_k} \in \mathcal{H}^k$ is expressed with continuants

$$h_{m_1,\ldots,m_k}(x) = \cfrac{1}{m_1 + \cfrac{1}{\ddots + \cfrac{1}{m_k + x}}} = \frac{p_{k-1}\,x + p_k}{q_{k-1}\,x + q_k}\,,$$

The LFT $h_{m_1,\ldots,m_k} \in \mathcal{H}^k$ is expressed with continuants

$$h_{m_1,\ldots,m_k}(x) = \cfrac{1}{m_1 + \cfrac{1}{\ddots + \cfrac{1}{m_k + x}}} = \frac{p_{k-1}\,x + p_k}{q_{k-1}\,x + q_k}\,,$$

and satisfies the mirror property

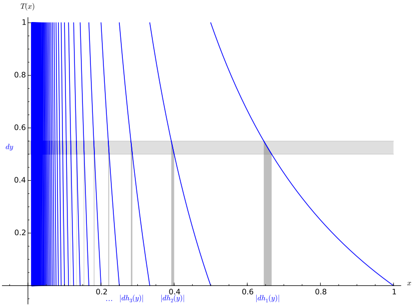
$$h_{m_k,\ldots,m_1}(x) = \cfrac{1}{m_k + \cfrac{1}{\ddots + \cfrac{1}{m_1 + x}}} = \frac{p_{k-1}\,x + q_{k-1}}{p_k\,x + q_k}\,.$$

The Perron-Frobenius operator **H**

If $g \in \mathcal{C}^0(\mathcal{I})$ is the density of α, what is the density of $T(\alpha)$?

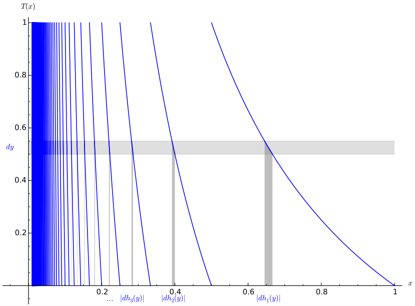
The Perron-Frobenius operator **H**

If $g \in \mathcal{C}^0(\mathcal{I})$ is the density of α, what is the density of $T(\alpha)$?



The Perron-Frobenius operator **H**

If $g \in \mathcal{C}^0(\mathcal{I})$ is the density of α, what is the density of $T(\alpha)$?



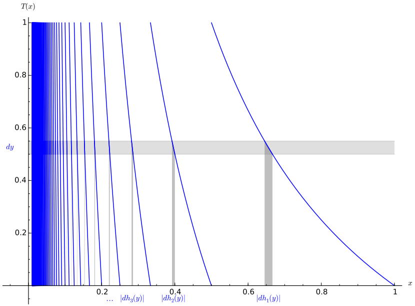
Answer: The density is

$$\mathbf{H}[g](x) = \sum_{h \in \mathcal{H}} \left| h'(x) \right| \, g\left(h(x)\right)$$

$$= \sum_{m=1}^{\infty} \frac{1}{(m+x)^2} \, g\left(\frac{1}{m+x}\right) .$$

The Perron-Frobenius operator $\mathbf{H}$

If $g \in \mathcal{C}^0(\mathcal{I})$ is the density of α, what is the density of $T(\alpha)$?



Answer: The density is

$$\mathbf{H}[g](x) = \sum_{h \in \mathcal{H}} \left| h'(x) \right| g\left(h(x)\right)$$
$$= \sum_{m=1}^{\infty} \frac{1}{(m+x)^2} \, g\left(\frac{1}{m+x}\right) .$$

For $k \geq 1$, the density of $T^k(\alpha)$ is given by the k-th iterate of $\mathbf{H}$

$$\mathbf{H}^k[g](x) = \sum_{h \in \mathcal{H}^k} \left| h'(x) \right| g\left(h(x)\right) .$$

$\mathbf{H}$ is called the Perron-Frobenius operator (or the density transform).

Evaluating at $x = 0$

$$\mathbf{H}^k[g](0) = \sum_{m_1,\ldots,m_k \geq 1} \frac{1}{q_k^2}\, g\left(\frac{p_k}{q_k}\right) \, .$$

Evaluating at $x = 0$

$$\mathbf{H}^k[g](0) = \sum_{m_1,\ldots,m_k \geq 1} \frac{1}{q_k^2}\, g\left(\frac{p_k}{q_k}\right).$$

As the sum is over all k-tuples, we apply the mirror property, and

$$\mathbf{H}^k[g](0) = \sum_{m_1,\ldots,m_k \geq 1} \frac{1}{q_k^2}\, g\left(\frac{q_{k-1}}{q_k}\right).$$

Expressions in terms of the operator $\mathbf{H}$.

Three main facts:

- The intervals $h(\mathcal{I})$ for $h \in \mathcal{H}^k$ form a partition of $(0, 1)$

Expressions in terms of the operator $\mathbf{H}$.

Three main facts:

- The intervals $h(\mathcal{I})$ for $h \in \mathcal{H}^k$ form a partition of $(0,1)$
- $\tilde{S}_k^{\langle\mu\rangle}$ is a step function, constant on each $h_{m_1,\ldots,m_k}(\mathcal{I})$,

$$\tilde{S}_k^{\langle\mu\rangle} = f_\mu\left(\frac{q_{k-1}}{q_k}\right)$$

Expressions in terms of the operator $\mathbf{H}$.

Three main facts:

- The intervals $h(\mathcal{I})$ for $h \in \mathcal{H}^k$ form a partition of $(0,1)$
- $\tilde{S}_k^{\langle\mu\rangle}$ is a step function, constant on each $h_{m_1,\ldots,m_k}(\mathcal{I})$,

$$\tilde{S}_k^{\langle\mu\rangle} = f_\mu\left(\frac{q_{k-1}}{q_k}\right)$$

- The length of the interval $h_{m_1,\ldots,m_k}(\mathcal{I})$ is

$$|h(0) - h(1)| = \frac{1}{q_k\ (q_k + q_{k-1})} = \frac{1}{q_k^2} \cdot \frac{1}{1 + \frac{q_{k-1}}{q_k}}$$

Expressions in terms of the operator $\mathbf{H}$.

Three main facts:

- The intervals $h(\mathcal{I})$ for $h \in \mathcal{H}^k$ form a partition of $(0, 1)$
- $\tilde{S}_k^{\langle\mu\rangle}$ is a step function, constant on each $h_{m_1,\ldots,m_k}(\mathcal{I})$,

$$\tilde{S}_k^{\langle\mu\rangle} = f_\mu\left(\frac{q_{k-1}}{q_k}\right)$$

- The length of the interval $h_{m_1,\ldots,m_k}(\mathcal{I})$ is

$$|h(0) - h(1)| = \frac{1}{q_k\ (q_k + q_{k-1})} = \frac{1}{q_k^2} \cdot \frac{1}{1 + \frac{q_{k-1}}{q_k}}$$

Then: $$\mathbb{E}\left[\tilde{S}_k^{\langle\mu\rangle}\right] = \sum_{m_1,\ldots,m_k \geq 1} \frac{1}{q_k^2} \frac{f_\mu(q_{k-1}/q_k)}{1 + (q_{k-1}/q_k)} = \mathbf{H}^k\left[\frac{f_\mu(x)}{1+x}\right](0)\,,$$

Expressions in terms of the operator $\mathbf{H}$.

Three main facts:

- The intervals $h(\mathcal{I})$ for $h \in \mathcal{H}^k$ form a partition of $(0,1)$
- $\tilde{S}_k^{\langle\mu\rangle}$ is a step function, constant on each $h_{m_1,\ldots,m_k}(\mathcal{I})$,

$$\tilde{S}_k^{\langle\mu\rangle} = f_\mu\left(\frac{q_{k-1}}{q_k}\right)$$

- The length of the interval $h_{m_1,\ldots,m_k}(\mathcal{I})$ is

$$|h(0) - h(1)| = \frac{1}{q_k\,(q_k + q_{k-1})} = \frac{1}{q_k^2} \cdot \frac{1}{1 + \frac{q_{k-1}}{q_k}}$$

Then: $$\mathbb{E}\left[\tilde{S}_k^{\langle\mu\rangle}\right] = \sum_{m_1,\ldots,m_k \geq 1} \frac{1}{q_k^2}\,\frac{f_\mu(q_{k-1}/q_k)}{1 + (q_{k-1}/q_k)} = \mathbf{H}^k\left[\frac{f_\mu(x)}{1+x}\right](0),$$

And $$\Pr\left[\tilde{S}_k^{\langle\mu\rangle} \in J\right] = \mathbb{E}\left[\mathbf{1}_J \circ \tilde{S}_k^{\langle\mu\rangle}\right] = \mathbf{H}^k\left[\frac{\mathbf{1}_J \circ f_\mu(x)}{1+x}\right](0)$$

Analytic properties of **H**

The operator **H** acts on the Banach space BV$(\mathcal{I})$ of functions of bounded variation,

$$\text{with norm} \quad \|f\|_{BV} = V_0^1(f) + \|f\|_1 \, .$$

Analytic properties of **H**

The operator **H** acts on the Banach space BV($\mathcal{I}$) of functions of bounded variation,

$$\text{with norm} \quad \|f\|_{BV} = V_0^1(f) + \|f\|_1 \,.$$

The following dominant spectral properties are well-known

- Dominant eigenvalue (simple) : $\lambda = 1$
- Dominant eigenfunction: $\psi(x) = \dfrac{1}{\log 2} \dfrac{1}{1+x}$.
- Dominant eigenmeasure for the adjoint: Lebesgue measure
- Subdominant spectral radius: φ^2 with $\varphi = (\sqrt{5}-1)/2$.

Analytic properties of $\mathbf{H}$

The operator $\mathbf{H}$ acts on the Banach space $\mathsf{BV}(\mathcal{I})$ of functions of bounded variation,

$$\text{with norm} \quad \|f\|_{BV} = V_0^1(f) + \|f\|_1 \,.$$

The following dominant spectral properties are well-known

- Dominant eigenvalue (simple) : $\lambda = 1$
- Dominant eigenfunction: $\psi(x) = \frac{1}{\log 2}\,\frac{1}{1+x}$.
- Dominant eigenmeasure for the adjoint: Lebesgue measure
- Subdominant spectral radius: φ^2 with $\varphi = (\sqrt{5}-1)/2$.

Then, for any $g \in \mathsf{BV}(\mathcal{I})$, the asymptotic estimate holds:

$$\mathbf{H}^k[g](x) = \frac{1}{\log 2}\,\frac{1}{1+x}\,\int_0^1 g(x)dx + O\left(\varphi^{2k}\,\|g\|_{BV}\right) \,.$$

Going back to the expectations and distributions.

With the expressions for the expectations and distributions,

$$\mathbb{E}\left[\tilde{S}_k^{\langle\mu\rangle}\right] = \mathbf{H}^k\left[\frac{f_\mu(x)}{1+x}\right](0), \qquad \Pr\left[\tilde{S}_k^{\langle\mu\rangle} \in J\right] = \mathbf{H}^k\left[\frac{\mathbf{1}_J \circ f_\mu(x)}{1+x}\right](0)$$

We apply the previous result to the "red" functions:

Going back to the expectations and distributions.

With the expressions for the expectations and distributions,

$$\mathbb{E}\left[\tilde{S}_k^{\langle\mu\rangle}\right] = \mathbf{H}^k\left[\frac{f_\mu(x)}{1+x}\right](0), \qquad \Pr\left[\tilde{S}_k^{\langle\mu\rangle} \in J\right] = \mathbf{H}^k\left[\frac{\mathbf{1}_J \circ f_\mu(x)}{1+x}\right](0)$$

We apply the previous result to the "red" functions:

- The first function belongs to $\mathrm{B}V(\mathcal{I})$

  only for $\mu \neq 0$, with a BV-norm $O(1/\mu)$.

Going back to the expectations and distributions.

With the expressions for the expectations and distributions,

$$\mathbb{E}\left[\tilde{S}_k^{\langle\mu\rangle}\right] = \mathbf{H}^k\left[\frac{f_\mu(x)}{1+x}\right](0), \qquad \Pr\left[\tilde{S}_k^{\langle\mu\rangle} \in J\right] = \mathbf{H}^k\left[\frac{\mathbf{1}_J \circ f_\mu(x)}{1+x}\right](0)$$

We apply the previous result to the "red" functions:

- The first function belongs to $\mathrm{B}V(\mathcal{I})$

  only for $\mu \neq 0$, with a BV-norm $O(1/\mu)$.

- The second function always belongs to $\mathrm{B}V(\mathcal{I})$,

  even for $\mu = 0$ with a bounded BV-norm wrt μ.

  The limit distribution

$$\lim_{k\to\infty} \Pr\left[\tilde{S}_k^{\langle\mu\rangle} \in J\right] = \frac{1}{\log 2}\int_0^1 \frac{\mathbf{1}_J \circ f_\mu(x)}{1+x}dx,$$

  is expressed with the inverse of f_μ in the interval I_μ.

Going back to the expectations and distributions.

With the expressions for the expectations and distributions,

$$\mathbb{E}\left[\tilde{S}_k^{\langle\mu\rangle}\right] = \mathbf{H}^k\left[\frac{f_\mu(x)}{1+x}\right](0), \qquad \Pr\left[\tilde{S}_k^{\langle\mu\rangle} \in J\right] = \mathbf{H}^k\left[\frac{\mathbf{1}_J \circ f_\mu(x)}{1+x}\right](0)$$

We apply the previous result to the "red" functions:

- The first function belongs to $\mathrm{B}V(\mathcal{I})$

  only for $\mu \neq 0$, with a BV-norm $O(1/\mu)$.

- The second function always belongs to $\mathrm{B}V(\mathcal{I})$,

  even for $\mu = 0$ with a bounded BV-norm wrt μ.

  The limit distribution

$$\lim_{k\to\infty} \Pr\left[\tilde{S}_k^{\langle\mu\rangle} \in J\right] = \frac{1}{\log 2}\int_0^1 \frac{\mathbf{1}_J \circ f_\mu(x)}{1+x}dx,$$

  is expressed with the inverse of f_μ in the interval I_μ.

Thus the asymptotics are obtained for $\tilde{S}_k^{\langle\mu\rangle}$. We then return to $S_k^{\langle\mu\rangle}$.

Possible extensions: variable μ

Possible extensions: variable μ

As our estimates are uniform wrt position μ, and index k,

thus making it possible to deal with a position depending on k.

$\Longrightarrow$ We then let $\mu_k \to 0$ as $k \to \infty$.

Possible extensions: variable μ

As our estimates are uniform wrt position μ, and index k,

thus making it possible to deal with a position depending on k.

$\Longrightarrow$ We then let $\mu_k \to 0$ as $k \to \infty$.

Theorem

For each $\tau \in [\varphi^2, 1[$, *considering* $\mu_k = \tau^k$ *we have*

$$\mathbb{E}_\alpha \left[\frac{R_\alpha(n)}{n} - \frac{12\ |\log \tau|}{\pi^2} \log n \right] = O(1)\,, \qquad \left(n = n_k^{\langle \mu_k \rangle}(\alpha) \right)$$

as $k \to \infty$, *where the constant depends on* τ.

Possible extensions: variable μ

As our estimates are uniform wrt position μ, and index k,
thus making it possible to deal with a position depending on k.
$\Longrightarrow$ We then let $\mu_k \to 0$ as $k \to \infty$.

Theorem

For each $\tau \in [\varphi^2, 1[$, considering $\mu_k = \tau^k$ we have

$$\mathbb{E}_\alpha \left[\frac{R_\alpha(n)}{n} - \frac{12\ |\log \tau|}{\pi^2} \log n \right] = O(1)\,, \qquad \Big(n = n_k^{\langle \mu_k \rangle}(\alpha) \Big)$$

as $k \to \infty$, where the constant depends on τ.

Theorem

If $b \in (0,1)$ and for each k we pick $\mu_k \in [0,1]$ uniformly, then

$$\lim_{k \to \infty} \mathbb{E}_{\alpha, \mu_k} \left[S_k^{\langle \mu_k \rangle} \middle| \mu_k \geq b^k \right] = 1 + \frac{\pi^2}{6}\,.$$