ASYMPTOTIC NORMALITY OF b-ADDITIVE FUNCTIONS ON POLYNOMIAL SEQUENCES IN THE GAUSSIAN NUMBER FIELD[1]

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Abstract. We consider the asymptotic behavior of b-additive functions f with respect to a base b of a canonical number system in the Gaussian number field. In particular, we get a normal limit law for $f(P(z))$ where $P(z)$ is a polynomial with integer coefficients. Our methods are exponential sums over the Gaussian number field as well as certain results from the theory of uniform distribution.

0. Notations

Throughout the paper we use the following notations: We write $e(z) = e^{2\pi i z}$; $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$, denote the set of complex numbers, real numbers, rational numbers, integers, positive integers, and positive integers including zero, respectively. $\mathbb{Q}(i)$ denotes the field of Gaussian numbers, and $\mathbb{Z}[i]$ the ring of Gaussian integers. We write $\mathrm{tr}(z)$ and $N(z)$ for the trace and the norm of z over $\mathbb{Q}$, and $\{z\}$ for the minimal distance of a real number z to the next integer. Furthermore, the largest integer less than or equal to a real number z is denoted by $[z]$. λ_n denotes the n-dimensional Lebesgue measure. V^T denotes the transposition of the matrix V. For a set A we denote its closure by $\bar{A}$ and its boundary by ∂A. Furthermore we use the symbol $f \ll g$ to mean that $f = \mathcal{O}(g)$ and $f \gg g$ to mean that $g = \mathcal{O}(f)$.

1. Introduction

Let $\nu_q(n)$ denote the sum of digits function of n in its q-adic representation for some integers $q \geq 2$ and $n \geq 0$. This function and related functions have been studied by several authors. In 1975 Delange [2] computed the average of $\nu_q(n)$:

$$\frac{1}{N}\sum_{n<N} \nu_q(n) = \frac{q-1}{2}\log_q N + \gamma_1(\log_q N),$$

where γ_1 is a continuous, nowhere differentiable and periodic function with period 1.

Higher moments were considered by Kirschenhofer [18] and independently by Kennedy and Cooper [17] who obtained a formula for the variance

$$\frac{1}{N}\sum_{n<N} \nu_q^2(n) - \frac{1}{N^2}\left(\sum_{n<N} \nu_q(n)\right)^2 = \left(\frac{q-1}{2}\right)^2 \log_q N + \gamma(\log_q N)$$

with a continuous fluctuation γ of period 1. Grabner, Kirschenhofer, Prodinger and Tichy [10] extended this result (dth moment for the case $q = 2$) and showed

$$\frac{1}{N}\sum_{n<N} \nu_2(n)^d = \frac{1}{2^d}(\log_2 N)^d + \sum_{i=0}^{d-1}(\log_2 N)^i \gamma_i(\log_2 N),$$

where the γ_i are again continuous fluctuations of period 1.

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In the literature there can also be found generalizations of these results to other than q-adic number systems. In particular, it is possible to extend the notion of q-adic number systems to number fields in a rather natural way. Since in the remaining part of this paper number systems in the Gaussian number field $\mathbb{Q}(i)$ play a prominent rôle, we recall their definition.

Definition 1.1. A pair $(b,\mathcal{N})$ with $b\in\mathbb{Z}[i]$ and $\mathcal{N}=\{0,1,\dots,|b|^2-1\}$ is called *canonical number system* if any $\gamma\in\mathbb{Z}[i]$ has a representation of the form

$$\gamma=c_0+c_1b+\cdots+c_hb^h,\quad c_h\neq 0\quad\text{if}\quad h\neq 0,$$

where $h\in\mathbb{N}_0$ and $c_j\in\mathcal{N}$ for $j=0,1,\dots,h$. b is called base and $\mathcal{N}$ is called set of digits of $(b,\mathcal{N})$. Furthermore, we define the *sum of digits function* by

$$\nu_b(\gamma)=c_0+c_1+\cdots+c_h.$$

Remark 1.1. Of course, the set of digits is uniquely determined by the base of a canonical number system. For the ring of Gaussian integers $\mathbb{Z}[i]$ the bases were characterized by Kátai and Szabó [16] who showed that the only bases are given by $b=-n\pm i$, where $n\in\mathbb{N}$. For generalizations to arbitrary number fields we refer to [14, 15, 19, 20].

Grabner, Kirschenhofer and Prodinger [9] and Thuswaldner [23] generalized Delange's result to canonical number systems in the Gaussian integers and to arbitrary canonical number systems, respectively. A treatment of the higher moments in the general case has been done recently by Gittenberger and Thuswaldner [8]. E.g. for the Gaussian integers we have

$$\begin{aligned}\frac{1}{N\pi+\mathcal{O}\left(\sqrt{N}\right)}&\sum_{|z|^2<N}(\nu_b(z))^d\\&=\left(\frac{|b|^2-1}{2}\right)^d\log_{|b|^2}^d N+\sum_{j=0}^{d-1}\log_{|b|^2}^j N\,\Phi_j(\log_{|b|^2}N)+\mathcal{O}\left(\sqrt{N}\log_{|b|^2}^d N\right),\end{aligned}$$

where $\Phi_0,\dots,\Phi_{d-1}$ are again continuous periodic fluctuations of period 1 and b is the base of a canonical number system in $\mathbb{Z}[i]$.

Let b be the base of a canonical number system in $\mathbb{Z}[i]$. Then obviously each $\gamma\in\mathbb{C}$ has a unique representation of the shape $\alpha_0+\alpha_1 b$ with $\alpha_0,\alpha_1\in\mathbb{R}$. Thus the mapping

$$\phi:\ \mathbb{C}\to\mathbb{R}^2;\quad \alpha_0+\alpha_1 b\mapsto(\alpha_0,\alpha_1)$$

is well defined. It turns out, that in order to simplify some computations it is convenient to use this embedding.

There also exist distributional results for the sum of digits function and related functions. For instance, Bassily and Kátai [1] studied the distribution of q-additive functions on polynomial sequences. Recall that a function f is said to be q-additive if $f(0)=0$ and

$$f(n)=\sum_{j\geq 0}f(a_j(n)q^j)\ \text{for}\ n=\sum_{j\geq 0}a_j(n)q^j,$$

where $a_j(n)\in E:=\{0,1,\dots,q-1\}$. A special q-additive function is the sum of digits function $\nu_q(n)$. Bassily and Kátai [1] showed the following theorem:

Theorem 1.1. *Let f be a q-additive function such that $f(cq^j)=\mathcal{O}(1)$ as $j\to\infty$ and $c\in E$. Furthermore let*

$$m_{k,q}:=\frac{1}{q}\sum_{c\in E}f(cq^k),\qquad \sigma_{k,q}^2:=\frac{1}{q}\sum_{c\in E}f^2(cq^k)-m_{k,q}^2,$$

and

$$M_q(x):=\sum_{k=0}^{N}m_{k,q},\qquad D_q^2(x)=\sum_{k=0}^{N}\sigma_{k,q}^2$$

with $N = \lfloor \log_q x \rfloor$*. Assume that* $\frac{D_q(x)}{(\log x)^{1/3}} \to \infty$ *as* $x \to \infty$ *and let* $P(x)$ *be a polynomial with integer coefficients, degree* r*, and positive leading term. Then, as* $x \to \infty$*,*

$$\frac{1}{x}\#\left\{n < x \,\middle|\, \frac{f(P(n)) - M_q(x^r)}{D_q(x^r)} < y\right\} \to \Phi(y),$$

where Φ *is the normal distribution function.*

Similar distribution results for the sum of digits function of number systems related to substitution automata were considered by Dumont and Thomas [5]. For number systems whose bases satisfy linear recurrences we refer to [3].

In this paper we will extend the above result of Bassily and Kátai to canonical number systems in $\mathbb{Z}[i]$. The concept of q-additivity is extendible to these number systems in an obvious way:

Definition 1.2. Let $(b, \mathcal{N})$ be a canonical number system in $\mathbb{Z}[i]$. A function f is called *b-additive* if $f(0) = 0$ and

$$f(\gamma) = \sum_{j \geq 0} f(a_j(\gamma) b^j) \text{ for } \gamma = \sum_{j \geq 0} a_j(\gamma) b^j \quad (a_j(\gamma) \in \mathcal{N}).$$

After these preparations we state our main result:

Theorem 1.2. *Let* f *be a* b*-additive function such that* $f(cb^j) = \mathcal{O}(1)$ *for* $j \in \mathbb{N}$ *and* $c \in \mathcal{N}$*. Furthermore let*

$$m_k := \frac{1}{|b|^2} \sum_{c \in \mathcal{N}} f(cb^k), \qquad \sigma_k^2 := \frac{1}{|b|^2} \sum_{c \in \mathcal{N}} f^2(cb^k) - m_k^2,$$

and

$$M(N) := \sum_{k=0}^{L} m_k, \qquad D^2(N) = \sum_{k=0}^{L} \sigma_k^2$$

with $L = \left\lfloor \log_{|b|} N \right\rfloor$*. Assume that* $\frac{D(N)}{(\log N)^{1/3}} \to \infty$ *as* $N \to \infty$ *and let* $P(z) = p_r z^r + \cdots + p_1 z + p_0$ *be a polynomial with coefficients in* $\mathbb{Z}[i]$*. Then, as* $N \to \infty$*,*

$$\frac{1}{\#\{z \,|\, |z|^2 < N\}} \#\left\{|z|^2 < N \,\middle|\, \frac{f(P(z)) - M(N^r)}{D(N^r)} < y\right\} \to \Phi(y),$$

where Φ *is the normal distribution function and* z *runs over the Gaussian integers.*

Corollary 1.1. *Since* $\nu_b(z)$ *fulfills all the conditions posed upon the* b*-additive function* f *in the theorem, we have*

$$\frac{1}{\#\{z \,|\, |z|^2 < N\}} \#\left\{|z|^2 < N \,\middle|\, \frac{\nu_b(P(z)) - M(N^r)}{D(N^r)} < y\right\} \to \Phi(y).$$

The paper is organized as follows: In the next section we extend some results of Hua [12] on exponential sums to the Gaussian number field. Section 3 is devoted to the construction of an Urysohn function for a certain domain related to the fundamental domain of the number system which will allow us to keep track of certain digits in a digit expansion. We will analyze some properties of the Fourier series of this function. Since we cannot avoid some errors arising in the region where the Urysohn function attains values in $(0, 1)$, we have to analyze the number of hits in this region for the polynomial sequence under consideration. This will be done in Section 4 by means of the Erdős-Turán-Koksma inequality. In Section 5 we will derive a proposition giving the crucial distributional result which will allow us to reduce our problem to the considerably simpler case $P(z) = z$ and to complete the proof of Theorem 1.2. This is done in the last section.

2. Exponential Sums Over Number Fields

In this section we establish a result on exponential sums of polynomials over the number field $\mathbb{Q}(i)$. Before we state this result, we list some lemmas which will be needed in its proof. We start with estimates for exponential sums of a simple type.

Lemma 2.1. *Let $h,q\in\mathbb{Z}[i]$ and define the square $D_\nu := \{z=a+bi\in\mathbb{Z}[i] \mid -\nu\le a,b\le\nu\}$. If we set $h/q=r+si$ and*

$$V:=\sum_{z\in D_\nu} e\left(\operatorname{tr}\left(\frac{h}{q}z\right)\right),$$

then the estimate

$$V\le \min\left(4\nu^2, \frac{\nu}{\{2r\}}, \frac{\nu}{\{2s\}}, \frac{1}{4\{2r\}\{2s\}}\right)$$

holds.

Proof. Let $z=a+bi$. It is easy to see that

$$V=\sum_{a+bi\in D_\nu} e(2(ra-sb)) = \sum_{a=-\nu}^{\nu} e(2ra)\sum_{b=-\nu}^{\nu} e(-2sb).$$

Using the estimate (cf. Hua [12, Lemma 1.8])

$$\left|\sum_{k=K_1}^{K_2} e(k\alpha)\right| \le \min\left(K_2-K_1, \frac{1}{2\{\alpha\}}\right)$$

we derive

$$|V|\le \min\left(2\nu, \frac{1}{2\{2r\}}\right)\min\left(2\nu, \frac{1}{2\{2s\}}\right)$$

and the result follows. □

With help of this result we derive a corresponding result for open discs. In the following, the summation variable z always runs over the Gaussian integers.

Lemma 2.2. *Let $h,q\in\mathbb{Z}[i]$ and*

$$S=\sum_{|z|^2<N} e\left(\operatorname{tr}\left(\frac{h}{q}z\right)\right).$$

If we set $h/q=r+si$ then the estimate

$$S\ll (\log N)^{\sigma_1}\min\left(\frac{N}{(\log N)^{\sigma_1}}, \frac{\sqrt{N/(\log N)^{\sigma_1}}}{\{2r\}}, \frac{\sqrt{N/(\log N)^{\sigma_1}}}{\{2s\}}, \frac{1}{\{2r\}\{2s\}}\right)+\frac{N}{(\log N)^{\sigma_1/2}}$$

holds for each positive real number σ_1.

Proof. This result follows easily from Lemma 2.1. We tesselate the open disc $|z|^2<N$ by squares of side length $\sqrt{N/(\log N)^{\sigma_1}}$. There are $\mathcal{O}\left((\log N)^{\sigma_1}\right)$ such squares in this open disc, which do not intersect its boundary. The contribution C_I of these squares can be estimated with help of Lemma 2.1 by

$$C_I\ll (\log N)^{\sigma_1}\min\left(\frac{N}{(\log N)^{\sigma_1}}, \frac{\sqrt{N/(\log N)^{\sigma_1}}}{\{2r\}}, \frac{\sqrt{N/(\log N)^{\sigma_1}}}{\{2s\}}, \frac{1}{\{2r\}\{2s\}}\right).$$

Since the squares intersecting the boundary can be covered by an annulus of width $\mathcal{O}\left(\sqrt{N}/(\log N)^{\sigma_1/2}\right)$, the contribution C_B of these squares can be estimated by

$$C_B\ll \frac{N}{(\log N)^{\sigma_1/2}}.$$

This yields the result. □

Remark 2.1. The same reasoning easily shows that the estimate in Lemma 2.2 remains valid if the range of summation has the shape $z\in\bigcap_{j=1}^{J}\left(a_j+\{y\in\mathbb{Z}[i]: |y|^2\le c_jN\}\right)$ with $a_j\in\mathbb{Z}[i]$ and $c_j>0$.

Lemma 2.3. *Let $h, q \in \mathbb{Z}[i]$ with $|q| > 2$ and $(h,q) = 1$ and*

$$S = \sum_{|z|^2<N} e\left(\operatorname{tr}\left(\frac{h}{q}z\right)\right).$$

Then

$$|S| \ll \sqrt{N}|q|.$$

Proof. It is easy to see that there exists a residue system R modulo q with

$$R \subset \{z \in \mathbb{Z}[i] \mid |z| \leq 2|q|\}. \tag{2.1}$$

Suppose we tesselate the open disc $K_N := \{z \mid |z|^2 < N\}$ with translates of R. Let T be this tesselation. Now define

$$\begin{aligned} E_N &:= \{R \in T \mid R \subset K_N\}, \\ F_N &:= \{R \in T \mid R \not\subset K_N\}. \end{aligned}$$

Since $|q| > 2$ and the different of $\mathbb{Q}(i)$ is $2 \cdot \mathbb{Z}[i]$, we have by Hua [11, Theorem 3]

$$\sum_{z \in R} e\left(\operatorname{tr}\left(\frac{h}{q}z\right)\right) = 0 \quad \text{for } R \in E_N.$$

Thus

$$S := \sum_{R \in F_N} \sum_{z \in R \cap K_N} e\left(\operatorname{tr}\left(\frac{h}{q}z\right)\right).$$

By (2.1), this sum has at most $\mathcal{O}\left(\sqrt{N}|q|\right)$ summands. This implies the result. □

Next we give a lemma that will help us to reduce the degree of the polynomial in an exponential sum. The rational version of it has been proved in [12]. Since the $\mathbb{Q}(i)$ version given here can be proved in exactly the same way, we omit the proof. Adapting Hua's [12] notation to the present situation let the symbol $\sum_x^{c'}$ denote the sum over all integers in a set of the form $\bigcap_{j=1}^{J}\left(a_j + \{y : |y|^2 \leq c_j N\}\right)$ with $a_j \in \mathbb{Z}[i]$ and $0 < c_j < c'$. In this context the exact values of a_j, c_j and c' are not important. For details we refer to Hua [12, Lemma 3.3 and 3.4] and Vinogradov [24, p. 185].

Lemma 2.4. *(cf.* [12, Lemma 3.3 and 3.4]*) Let $f(x) = \sum_{j=0}^{k} a_j x^j$ be a polynomial of degree k and set*

$$S := \sum_{|z|^2<N} e(\operatorname{tr}(f(z))).$$

Then we have the estimate

$$|S|^{2^{k-1}} \leq cN^{2^{k-1}-k}\left|\sum_{y_1}^{c'} \cdots \sum_{y_{k-1}}^{c'} \sum_{y_k}^{c'} e\left(\operatorname{tr}\left(y_1 \cdots y_{k-1}(k!a_k y_k + \beta)\right)\right)\right|$$

with certain computable numbers c and β.

Let $d_k(z)$ be the number of representations of z as a product of k nonzero Gaussian integers. It is well-known that

$$\sum_{|z|^2<N} d_k(z) \ll N(\log N)^{k-1}$$

(cf. Narkiewicz [22, p. 514]). From this result we easily deduce the following lemma (cf. Hua [12, Lemma 6.1]).

Lemma 2.5. *For $\sigma_2 \geq 2^{3k} - 1$ the estimate*

$$\sum_{|z|^2<N}{}' \, d_k(z) = \mathcal{O}\left(N(\log N)^{-\sigma_2}\right)$$

holds. Here the prime (') indicates that the sum is taken over all z in the range of summation, for that

$$(\log N)^{\sigma_2} \leq c d_k(z).$$

Next we prove a version of Weyl's Lemma (cf. [12, Lemma 3.5]).

Lemma 2.6. *Let $h, q \in \mathbb{Z}[i]$, with $(h,q) = 1$ and let*

$$G(M) := \sum_{|z|^2<M} g(z)$$

where

$$g(z) := (\log N)^{\sigma_1} \min\left(\frac{N}{(\log N)^{\sigma_1}}, \frac{\sqrt{N/(\log N)^{\sigma_1}}}{\{2r\}}, \frac{\sqrt{N/(\log N)^{\sigma_1}}}{\{2s\}}, \frac{1}{\{2r\}\{2s\}}\right),$$

with $r = \Re(hz/q)$, $s = \Im(hz/q)$ and $\sigma_1 > 0$. Then

$$G(M) \ll \left(\frac{M}{|q|^2} + 1\right)\left(N|q| + |q|^2 \log^{2+\sigma_1} N\right).$$

Proof. Let T_0 be a set of complete residue systems $\bmod\, q$ that form a tiling of $\mathbb{Z}^2$, such that each $R \in T_0$ is a translate of $R_0 = Z[i] \cap q\{\alpha + \beta i \mid 0 \leq \alpha, \beta \leq 1\}$. Let T be the set of all $R \in T_0$ having nonempty intersection with $|z|^2 < M$. Then we can write

$$G(M) \leq \sum_{R\in T} G_R$$

with

$$G_R = \sum_{z\in R} g(z).$$

Note that since $(h,q) = 1$ we have

$$\sum_{z\in R} g(hz) = G_R$$

and thus we may w.l.o.g. assume $h = 1$.

We want to approximate the sum G_R by an integral and will use the Koksma-Hlawka inequality to estimate the error caused by this approximation (cf. [4, Theorem 1.14]). To this end we need the star discrepancy (see [4, p. 5] for a definition) D_R^* of the lattice induced by R, which is easily seen to be $D_R^* = \mathcal{O}(|q|)$. Moreover, we use the notion of bounded variation in the sense of Hardy and Krause $V^{(k)}(g)$, whose definition can also be found in [4, p. 10]. For $g(z)$ we easily derive $V^{(2)}(g) = \mathcal{O}(N)$. After these preparations the Koksma-Hlawka inequality yields

$$E_R := \left| \int_{Q\cdot[0,1]^2} g(\phi^{-1}(x,y))\, dx\, dy - G_R \right| \leq D_R^* V^{(2)}(g) = \mathcal{O}(N|q|),$$

where Q is the matrix corresponding to a multiplication with q in $\mathbb{Z}[i]$. Summing up over all residue systems contained in T and taking into account the residue systems intersecting the boundary of $|z|^2 < M$ we obtain

$$\sum_{R\in T} E_R = \mathcal{O}\left(\left(\frac{M}{|q|^2} + 1\right) N|q|\right). \tag{2.2}$$

It remains to estimate the integral

$$I := \int_{Q\cdot[0,1]^2} g(\phi^{-1}(x,y))\, dx\, dy = \int_{Q\cdot[0,1]^2} \tilde{g}((x,y)Q^{-1})\, dx\, dy,$$

where

$$\tilde{g}(x,y)=(\log N)^{\sigma_1}\min\left(\frac{N}{(\log N)^{\sigma_1}},\frac{\sqrt{N/(\log N)^{\sigma_1}}}{\{2x\}},\frac{\sqrt{N/(\log N)^{\sigma_1}}}{\{2y\}},\frac{1}{\{2x\}\{2y\}}\right).$$

Using the transformation formula and splitting the range of integration according to the values of the function $\{\cdot\}$ we get

$$\begin{aligned}
I &= |q|^2\left(\int_0^1\min\left(\sqrt{\frac{N}{(\log N)^{\sigma_1}}},\frac{1}{\{2a\}}\right)da\right)^2(\log N)^{\sigma_1}\\
&= 4|q|^2\left(\int_0^{\frac14}\min\left(\sqrt{\frac{N}{(\log N)^{\sigma_1}}},\frac{1}{2a}\right)da\right.\\
&\quad\left.+\int_{\frac14}^{\frac12}\min\left(\sqrt{\frac{N}{(\log N)^{\sigma_1}}},\frac{1}{(1-2a)}\right)da\right)^2(\log N)^{\sigma_1}\\
&= 4|q|^2\left(\sqrt{\frac{N}{(\log N)^{\sigma_1}}}\int_0^{\frac{1}{2\cdot\sqrt{\frac{N}{(\log N)^{\sigma_1}}}}}da+\int_{\frac{1}{2\cdot\sqrt{\frac{N}{(\log N)^{\sigma_1}}}}}^{\frac14}\frac{1}{2a}\,da\right.\\
&\quad\left.+\int_{\frac14}^{\frac12-\frac{1}{2\cdot\sqrt{\frac{N}{(\log N)^{\sigma_1}}}}}\frac{1}{(1-2a)}\,da+\sqrt{\frac{N}{(\log N)^{\sigma_1}}}\int_{\frac12-\frac{1}{2\cdot\sqrt{\frac{N}{(\log N)^{\sigma_1}}}}}^{\frac12}da\right)^2(\log N)^{\sigma_1}\\
&= \mathcal{O}\left(|q|^2\log^{2+\sigma_1}N\right).
\end{aligned}$$

Summing up over all residue systems and combining this with (2.2) we obtain the result. ☐

Proposition 2.1. *Let* $(h,q)=1$ *and*

$$f(x)=\frac{h}{q}x^k+\alpha_1x^{k-1}+\ldots+\alpha_{k-1}x+\alpha_k$$

where $(\log N)^{\sigma}\le|q|^2\le N^k(\log N)^{-\sigma}$. *Then we have*

$$S=\left|\sum_{|z|^2<N}e(\operatorname{tr}(f(z)))\right|=\mathcal{O}\left(N(\log N)^{-\sigma_0}\right)$$

with $\sigma\ge 2^{k+2}\sigma_0+2^{3(k+2)}$.

Proof. For $k=1$ we obtain, applying Lemma 2.3 and keeping in mind the upper bound for $|q|^2$,

$$S=\left|\sum_{|z|^2<N}e\left(\operatorname{tr}\left(\frac{h}{q}+\alpha_1\right)\right)\right|\le N(\log N)^{-\sigma/2}.$$

Suppose now that $k>1$. An application of Lemma 2.4 in combination with Lemma 2.2 and Remark 2.1 yields

$$\begin{aligned}
|S|^{2^{k-1}} &\le cN^{2^{k-1}-k}\sum_{y_1}^{c'}\cdots\sum_{y_{k-1}}^{c'}\left(\min\left(\frac{N}{(\log N)^{\sigma_1}},\frac{\sqrt{\frac{N}{(\log N)^{\sigma_1}}}}{\{2r\}},\frac{\sqrt{\frac{N}{(\log N)^{\sigma_1}}}}{\{2s\}},\frac{1}{\{2r\}\{2s\}}\right)(\log N)^{\sigma_1}\right.\\
&\quad\left.+\frac{N}{(\log N)^{\sigma_1/2}}\right)
\end{aligned}$$

where

$$r=\Re\left(k!\frac{h}{q}y_1\cdots y_{k-1}\right)\quad\text{and}\quad s=\Im\left(k!\frac{h}{q}y_1\cdots y_{k-1}\right).$$

Setting

$$\xi := k! y_1 \cdots y_{k-1} \tag{2.3}$$

we have $|\xi|^2 \le M = {c'}^k k! N^{k-1}$. For a fixed $\xi \neq 0$ the number of solutions of (2.3) is less than or equal to $d_{k-1}(\xi)$. For $\xi = 0$ the number of solutions of (2.3) is $\mathcal{O}\left(N^{k-2}\right)$. Thus we can apply Lemma 2.5 to obtain (note that the prime (') has the same meaning as in the statement of Lemma 2.5)

$$\begin{aligned}
|S|^{2^{k-1}} \ll\ & N^{2^{k-1}-k}\Bigg(N \sideset{}{'}\sum_{|\xi|^2\le M} d_{k-1}(\xi) \\
& +(\log N)^{\sigma_2+\sigma_1} \sum_{|\xi|^2\le M} \min\left(\frac{4N}{(\log N)^{\sigma_1}}, \frac{\sqrt{N/(\log N)^{\sigma_1}}}{\{2r\}}, \frac{\sqrt{N/(\log N)^{\sigma_1}}}{\{2s\}}, \frac{1}{4\{2r\}\{2s\}}\right) \\
& +NM(\log N)^{\sigma_2-\sigma_1/2} + N^{k-1}(\log N)^{\sigma_2}\Bigg) \\
\ll\ & N^{2^{k-1}-k}\Bigg(M(\log M)^{-\sigma_2} N \\
& +(\log N)^{\sigma_2+\sigma_1} \sum_{|\xi|^2\le M} \min\left(\frac{4N}{(\log N)^{\sigma_1}}, \frac{\sqrt{N/(\log N)^{\sigma_1}}}{\{2r\}}, \frac{\sqrt{N/(\log N)^{\sigma_1}}}{\{2s\}}, \frac{1}{4\{2r\}\{2s\}}\right) \\
& +NM(\log N)^{\sigma_2-\sigma_1/2} + N^{k-1}(\log N)^{\sigma_2}\Bigg).
\end{aligned}$$

Note that the last summand $N^{k-1}(\log N)^{\sigma_2}$ comes from the contributions of the case $\xi = 0$. Now we may apply Lemma 2.6 to the last sum. This yields

$$\begin{aligned}
|S|^{2^{k-1}} \ll\ & N^{2^{k-1}-k}\Bigg(N^k(\log N)^{-\sigma_2} + (\log N)^{\sigma_2}\left(\frac{M}{|q|^2}+1\right)\left(N|q| + |q|^2 \log^{2+\sigma_1} N\right) \\
& +NM(\log N)^{\sigma_2-\sigma_1/2} + N^{k-1}(\log N)^{\sigma_2}\Bigg).
\end{aligned}$$

Setting $\sigma_1 := 2^{k+1}\sigma_0 + 2^{3k+1} - 2$ and $\sigma_2 := 2^{k-1}\sigma_0 + 2^{3k} - 1$ we arrive at

$$|S|^{2^{k-1}} \ll N^{2^{k-1}}(\log N)^{-2^{k-1}\sigma_0}.$$

This proves the result. □

3. Approximations of the Fundamental Domain and the Fourier Series of an Urysohn Function

In this section we will prove some auxiliary results in order to generalize Lemma 5 of Bassily and Kátai [1]. Since the set of all numbers having integer part zero in their b-adic representation has a rather complicated shape, the proof will be much more involved than the proof of the q-adic analogue. This set is defined by

$$\mathcal{F}' = \left\{ z \in \mathbb{C} \mid z = \sum_{\ell=1}^{\infty} \varepsilon_\ell(z) b^{-\ell}, \varepsilon_\ell \in \mathcal{N} \right\}$$

and we call it the *fundamental domain* of the number system $(b, \mathcal{N})$. In our context it is convenient to work with the ϕ-embedding of $\mathcal{F}'$ in $\mathbb{R}^2$. We have

$$\mathcal{F} := \phi(\mathcal{F}') = \left\{ z \in \mathbb{R}^2 \mid z = \sum_{\ell=1}^{\infty} B^{-\ell} a_\ell,\ a_\ell \in \phi(\mathcal{N}) \right\}$$

with

$$B = \begin{pmatrix} 0 & 1-n^2 \\ 1 & -2n \end{pmatrix}.$$

It is well-known (cf. for instance [7, 21]) that one can approximate $\mathcal{F}$ with the help of the sets

$$Q_0 := \left\{ z \in \mathbb{R}^2 \mid \|z\|_\infty \le \frac{1}{2} \right\},$$

$$Q_k := \bigcup_{a\in\mathcal{N}} B^{-1}(Q_{k-1} + \phi(a)).$$

The approximation satisfies $d(\partial Q_k, \partial\mathcal{F}) \ll |b|^{-k}$ with respect to the Hausdorff metric $d(\cdot,\cdot)$. It is easy to see that the sets Q_k are connected sets and that they are the unions of $|\mathcal{N}|^k$ parallelograms. Moreover (cf. [7, 21]), there exists a μ with $1 < \mu < |b|^2$ such that $\mathcal{O}\left(\mu^k\right)$ of these parallelograms intersect the boundary of Q_k.

Following Bassily and Kátai [1, Lemma 5] we will need for each $a \in \mathcal{N}$ a function that lets us keep track of a certain position in a digital expansion. Therefore we define an Urysohn function f_a for the domain

$$\mathcal{F}_a = B^{-1}(\mathcal{F} + \phi(a)),$$

i.e., that subdomain containing the numbers whose fractional parts start with the digit a. To this matter we need tubes $P_{k,a}$ with the following properties.

Lemma 3.1. *For all $a \in \mathcal{N}$ and all $k \in \mathbb{N}$ there exists an axe-parallel tube $P_{k,a}$ with the following properties:*

- *$\partial\mathcal{F}_a \subset P_{k,a}$ for all $k \in \mathbb{N}$.*
- *$\lambda_2(P_{k,a}) = \mathcal{O}\left(\mu^k/|b|^{2k}\right)$.*
- *$P_{k,a}$ consists of $\mathcal{O}\left(\mu^k\right)$ axe-parallel rectangles, each of which has Lebesgue measure $\mathcal{O}\left(|b|^{-2k}\right)$.*

Proof. We construct a tube $P_{k,a}$ that has the required properties. Let $Q_{k,a} := B^{-1}(Q_k + \phi(a))$. Then the family $Q_{k,a}$ has the same properties with respect to $\mathcal{F}_a$ as the family Q_k has with respect to $\mathcal{F}$. Thus the boundary $\partial Q_{k,a}$ of $Q_{k,a}$ is a polygon $\Pi'_{k,a}$. Let $\mathcal{R}_{k,a}$ be the family of the $|\mathcal{N}|^k$ parallelograms that result in $Q_{k,a}$. By the remarks at the beginning of the present section, $\mathcal{O}\left(\mu^k\right)$ of the elements of $\mathcal{R}_{k,a}$ have nonempty intersection with $\partial Q_{k,a}$. Thus the number of edges of $\Pi'_{k,a}$ is bounded by $\mathcal{O}\left(\mu^k\right)$. Since each element of $\mathcal{R}_{k,a}$ has diameter $c|b|^{-k}$ with some absolute constant c, we conclude that the length of $\Pi'_{k,a}$ is $\mathcal{O}\left(\mu^k|b|^{-k}\right)$. From this polygon we construct a polygon $\Pi_{k,a}$ with axe parallel sides in the following way: Let $E_{\Pi'_{k,a}}$ be the set of edges of $\Pi'_{k,a}$. Then define

$$\Pi_{k,a} := \bigcup_{\substack{\overline{(\alpha_1,\alpha_2)(\beta_1,\beta_2)} \in E_{\Pi'_{k,a}} \\ \alpha_2\le\beta_2}} \left(\overline{(\alpha_1,\alpha_2)(\beta_1,\alpha_2)} \cup \overline{(\beta_1,\alpha_2)(\beta_1,\beta_2)}\right).$$

Note that the length, the number of edges, and the maximal distance from $\mathcal{F}_a$ are comparable for $\Pi'_{k,a}$ and $\Pi_{k,a}$. Thus all estimates we gave for $\Pi'_{k,a}$ also hold for $\Pi_{k,a}$.

Now, since $d(\Pi_{k,a}, \partial\mathcal{F}_a) < c'|b|^{-k}$, we conclude that the tube

$$P_{k,a} := \left\{ z \in \mathbb{R}^2 \mid \|z - \Pi_{k,a}\|_\infty \le 2c'|b|^{-k} \right\}$$

has the properties required in the statement of the present lemma. □

For the remaining part of this paper we fix to each pair (k, a) a polygon $\Pi_{k,a}$ and the corresponding tube $P_{k,a}$ having the properties stated in Lemma 3.1.

Denote by $I_{k,a}$ the set of all points inside $\Pi_{k,a}$. Now we define f_a by

$$f_a(x,y) = \frac{1}{\Delta^2} \int_{-\Delta/2}^{\Delta/2} \int_{-\Delta/2}^{\Delta/2} \psi_a(x+x_1, y+y_1)\, dx_1\, dy_1, \tag{3.1}$$

where $\Delta = 2c'|b|^{-k}$ and

$$\psi_a(x,y) = \begin{cases} 1 & \text{if } (x,y) \in I_{k,a} \\ 1/2 & \text{if } (x,y) \in \Pi_{k,a} \\ 0 & \text{otherwise.} \end{cases}$$

Thus f_a is the desired Urysohn function which equals 1 for $z \in I_{k,a} \setminus P_{k,a}$, 0 for $z \in \mathbb{R}^2 \setminus (I_{k,a} \cup P_{k,a})$, and linear interpolation in between. Our next task is to give estimates for the Fourier coefficients of this function.

Lemma 3.2. *Let $f_a(x,y) = \sum_{n_1,n_2 \in \mathbb{Z}} c_{n_1 n_2} e(n_1 x + n_2 y)$ be the Fourier expansion of f_a. Then for the Fourier coefficients $c_{n_1 n_2}$ we get the estimates*

$$\begin{aligned} c_{n_1 n_2} &= \mathcal{O}\left(\frac{\mu^k}{\Delta^2 n_1^2 n_2^2}\right) \quad (n_1, n_2 \neq 0), \\ c_{n_1 0} &= \mathcal{O}\left(\frac{\mu^k}{\Delta n_1^2}\right) \quad (n_1 \neq 0), \\ c_{0 n_2} &= \mathcal{O}\left(\frac{\mu^k}{\Delta n_2^2}\right) \quad (n_2 \neq 0), \\ c_{00} &= \frac{1}{|b|^2}. \end{aligned}$$

Proof. If $\Pi_{k,a}$ is not rectangular, then the domain $I_{k,a}$ can be split into finitely many rectangles with axe-parallel edges. By the construction of $\Pi_{k,a}$ this can be done in a way, such that not more than $\mathcal{O}(\mu^k)$ of these rectangles intersect the boundary $\Pi_{k,a}$ of $I_{k,a}$. Suppose first that $I_{k,a}$ consists only of one rectangle with lower left vertex (α_1, β_1) and upper right vertex (α_2, β_2). Then elementary calculations yield

$$\begin{aligned} c_{n_1 n_2} &= \frac{\left(e\left(\frac{n_1\Delta}{2}\right) - e\left(-\frac{n_1\Delta}{2}\right)\right)\left(e\left(\frac{n_2\Delta}{2}\right) - e\left(-\frac{n_2\Delta}{2}\right)\right)}{16\pi^4 \Delta^2 n_1^2 n_2^2} (e(n_1\alpha_1) - e(n_1\alpha_2))(e(n_2\beta_1) - e(n_2\beta_2)) \\ &= \mathcal{O}\left(\frac{1}{\Delta^2 n_1^2 n_2^2}\right) \quad (n_1, n_2 \neq 0), \\ c_{n_1 0} &= \frac{e\left(\frac{n_1\Delta}{2}\right) - e\left(-\frac{n_1\Delta}{2}\right)}{4\pi^2 \Delta n_1^2} (e(n_1\alpha_1) - e(n_1\alpha_2))(\beta_1 - \beta_2) = \mathcal{O}\left(\frac{1}{\Delta n_1^2}\right) \quad (n_1 \neq 0), \qquad (3.2) \\ c_{0 n_2} &= \frac{e\left(\frac{n_2\Delta}{2}\right) - e\left(-\frac{n_2\Delta}{2}\right)}{4\pi^2 \Delta n_2^2} (e(n_1\beta_1) - e(n_1\beta_2))(\alpha_1 - \alpha_2) = \mathcal{O}\left(\frac{1}{\Delta n_2^2}\right) \quad (n_2 \neq 0), \\ c_{00} &= (\alpha_1 - \alpha_2)(\beta_1 - \beta_2). \end{aligned}$$

Suppose first that $n_1, n_2 \neq 0$. From the shape of (3.2) it is clear that the contribution of each rectangle to the Fourier coefficient of f_a is determined by its vertices: Observe that those coefficients have the shape

$$c_{n_1 n_2} = C(n_1, n_2) \sum_{(a_1, a_2)} \operatorname{sgn}((a_1, a_2)) e(n_1 a_1 + n_2 a_2),$$

where the sum runs over all vertices (a_1, a_2) of the rectangular subdomains and the sign of a vertex is negative if (a_1, a_2) is the upper left or the lower right vertex of a rectangle and positive otherwise. Now consider the rectangles of the above described tiling. Then the contribution of these rectangles is the sum of the contributions of each of its vertices. Thus one easily checks that these contributions cancel, unless the rectangle vertex under discussion coincides with a vertex of $\Pi_{k,a}$. Hence to each vertex $v = (\alpha, \beta)$ of $\Pi_{k,a}$ there corresponds a contribution

$$c_{n_1 n_2}(v) = \pm \frac{\left(e\left(\frac{n_1\Delta}{2}\right) - e\left(-\frac{n_1\Delta}{2}\right)\right)\left(e\left(\frac{n_2\Delta}{2}\right) - e\left(-\frac{n_2\Delta}{2}\right)\right)}{16\pi^4 \Delta^2 n_1^2 n_2^2} (e(n_1\alpha + n_2\beta)).$$

Since, by the construction of the Polygon $\Pi_{k,a}$ we have $\mathcal{O}(\mu^k)$ such vertices, the result follows for this case. The cases $c_{n_1 0}$ and $c_{0 n_2}$ can be treated in a similar way. It is easy to see that c_{00} is equal to the Lebesgue measure of $\mathcal{F}_a$, which is $|b|^{-2}$. $\square$

For certain pairs (n_1, n_2) it turns out that the corresponding Fourier coefficient $c_{n_1 n_2}$ vanishes. The next lemma provides a characterization of these pairs.

Lemma 3.3. *Suppose that for the pair $(n_1, n_2) \neq (0,0)$ the condition*

$$b \mid (\bar{b}n_1 - n_2) \tag{3.3}$$

holds. Then we have $c_{n_1 n_2} = 0$ for the corresponding Fourier coefficient.

Proof. Suppose first that $n_1, n_2 \neq 0$. We are dealing with the Urysohn function for a domain with boundary $\Pi_{k,a}$ and as in the proof of Lemma 3.2 we tile the domain into rectangular subdomains. As in the proof of Lemma 3.2 we see that all these contributions cancel, apart from those rectangle vertices that coincide with the vertices of $\Pi_{k,a}$.

Now let us examine the shape of $\Pi_{k,a}$. Due to the fact that the translates $\bar{I}_{k,a}+\phi(z)$ $(z \in b^{-1}\mathbb{Z}[i])$ of $\bar{I}_{k,a}$ form a tiling of $\mathbb{R}^2$, for each vertex v of $\Pi_{k,a}$ there exists an $a' \in b^{-1}\mathbb{Z}[i]$ such that $v - \phi(a') \in \Pi_{k,a}$. Hence each vertex has a *corresponding* vertex in $\Pi_{k,a}$ (in case of triple or quadruple points, i.e., the points belonging to three or four translates, respectively, there are two or three corresponding vertices, respectively). This induces a partitioning R of the set of vertices. Observe that the rectangular tiling of $I_{k,a}$ can be done in such a way that each vertex v belongs to four different rectangles which can be classified into four types according to their relative position to v $(R_1(v), \dots, R_4(v)$, ordered clockwise starting with upper left). Of course, these rectangles are not all contained in $I_{k,a}$, but to each class ρ of R there correspond exactly four rectangles $R_1(\rho), \dots, R_4(\rho)$ that are contained in $I_{k,a}$, one of each type.

Now let us consider the contributions to the Fourier coefficients of corresponding points $v_1 = (\alpha_1, \alpha_2)$ and $v_2 = (\beta_1, \beta_2)$ (triple and quadruple points can be treated analogously). We want to show that in presence of condition (3.3) the Fourier coefficient $c_{n_1 n_2}$ vanishes. To this matter it suffices to show that the contribution of each class ρ of R is zero. Due to the above considerations we have to show that

$$e(n_1 a_1 + n_2 a_2) = 1 \tag{3.4}$$

for $(a_1, a_2) = \phi((\nu_1 + \nu_2 i)b^{-1})$ with $\nu_1 + \nu_2 i \in \mathbb{Z}[i]$, since in this case the contributions of $R_1(\rho)$ and $R_3(\rho)$ cancel with the contributions of $R_2(\rho)$ and $R_4(\rho)$ in the same way as for vertices not coinciding with a vertex of $\Pi_{k,a}$. By (3.3) there exist $c + di \in \mathbb{Z}[i]$ with $(-n-i)n_1 - n_2 = (-n+i)(c+di)$. Comparing real and imaginary parts and inserting into (3.4) gives $n_1 a_1 + n_2 a_2 = -\nu_1 d - \nu_2 c \in \mathbb{Z}$ and we are done for the case where $n_1, n_2 \neq 0$.

Next we deal with the case $(n_1, 0)$, where $n_1 \neq 0$. In this case the Fourier coefficients are of the shape

$$c_{n_1 0} = C(n_1) \sum_{(a_1, a_2)} \operatorname{sgn}((a_1, a_2)) e(n_1 a_1) a_2,$$

where the sum runs over all vertices of the rectangle subdivision of $I_{k,a}$. As in the first case one easily checks that the contributions corresponding to rectangle vertices not coinciding with the vertices of $\Pi_{k,a}$ vanish. Thus let us consider the contributions at the vertices of $\Pi_{k,a}$. Arguing in the same way as above, yields that each class ρ of vertices corresponding to the vertex, say (a_1, a_2), gives a contribution

$$c_{n_1 0}(\rho) = \pm C' e(a_1 n_1) \neq 0.$$

But each vertex of $\Pi_{k,a}$ belongs to a horizontal edge of $\Pi_{k,a}$. It is easy to check that the class ρ' corresponding to the vertex (a_1, a_2') situated on the other end of this edge gives a contribution of the shape

$$c_{n_1 0}(\rho') = \mp C' e(a_1 n_1).$$

Since these two contributions have opposite signs, they cancel and we have shown the result also for this case. The case $(0, n_2)$ can be treated in an analogous way. $\square$

4. An Application of the Erdős-Turán-Koksma Inequality

Before we can prove our key proposition, we have to ensure that a certain sequence, connected with the polynomial $P(z)$ does not meet the tube $P_{k,a}$ too often. Precisely, we want to get an estimate for the quantities

$$F_j := \# \left\{ z \in \mathbb{Z}[i] \,\middle|\, |z|^2 < N \text{ and } \phi\left(\frac{P(z)}{b^{j+1}}\right) \in \bigcup_{a\in\mathcal{N}} P_{k,a} \bmod B^{-1}\mathbb{Z}^2 \right\}. \tag{4.1}$$

To this matter we use the two dimensional version of the Erdős-Turán-Koksma Inequality (cf. [4, Theorem 1.21]).

Lemma 4.1. *Let $x_1, \dots, x_L$ be points in the 2-dimensional real vector space $\mathbb{R}^2$ and H an arbitrary positive integer. Then the discrepancy $D_L(x_1, \dots, x_L)$ fulfills the inequality*

$$D_L(x_1, \dots, x_L) \ll \frac{2}{H+1} + \sum_{0 \le \|h\|_\infty \le H} \frac{1}{r(h)} \left| \frac{1}{L} \sum_{l=1}^{L} e(h \cdot x_l) \right|,$$

where $h \in \mathbb{Z}^2$ and $r(h) = \max(1, |h_1|) \cdot \max(1, |h_2|)$.

It will turn out that the exponential sum occurring in this inequality can be estimated with help of Proposition 2.1. In fact, we shall establish the following result.

Lemma 4.2. *Let F_j be defined as in (4.1) and let $\mu < |b|^2$ be as at the beginning of Section 3. Then*

$$F_j \ll \left(\frac{\mu}{|b|^2}\right)^k N + N(\log N)^{-\lambda}\mu^k,$$

for an arbitrary positive constant λ.

Proof. Since the discrepancy is defined as a supremum over certain rectangles, we subdivide the tube $P_{k,a}$ into a family of rectangles. Let R_a be one of these rectangles. Furthermore, let $x_z := \phi\left(\frac{P(z)}{b^{j+1}}\right)$ for each $z \in \mathbb{Z}[i]$ with $|z|^2 < N$. We want to derive an estimate for

$$F_j(R_a) := \# \left\{ z \in \mathbb{Z}[i] \,\middle|\, |z|^2 < N \text{ and } \phi\left(\frac{P(z)}{b^{j+1}}\right) \in R_a \bmod B^{-1}\mathbb{Z}^2 \right\}$$

It is clear that

$$F_j(R_a) \ll N\lambda_2(R_a) + N D_L(\{x_z\})$$

where $L = \pi N + \mathcal{O}\left(\sqrt{N}\right)$. Thus it remains to estimate the discrepancy of the point sequence $\{x_z\}$. Applying Lemma 4.1 yields

$$D_L(\{x_z\}) \ll \frac{2}{H+1} + \sum_{0 \le \|h\|_\infty \le H} \frac{1}{r(h)} \left| \frac{1}{L} \sum_{|z|^2 < N} e(h \cdot x_z) \right|. \tag{4.2}$$

Thus we have to estimate the exponential sum in (4.2). Let

$$\tau(z) := (\mathrm{tr}(z), \mathrm{tr}(bz))^T = \Xi\phi(z), \tag{4.3}$$

where $\Xi = VV^T$ and V is the Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 \\ b & \bar{b} \end{pmatrix}.$$

With these notations we have

$$h \cdot \phi\left(\frac{P(z)}{b^{j+1}}\right) = h\Xi^{-1}\tau\left(\frac{P(z)}{b^{j+1}}\right) = \mathrm{tr}\left(\sum_{l=1}^{2} \tilde{h}_l b^{l-1} \frac{P(z)}{b^{j+1}}\right) = \mathrm{tr}(r(z)),$$

where $(\tilde{h}_1, \tilde{h}_2)^T := h\Xi^{-1}$ and $r(z)$ is a certain polynomial. It is easy to see that $r(z)$ fulfills the conditions of Proposition 2.1. Thus we apply this proposition to (4.2) to derive the estimate

$$D_L(\{x_z\}) \ll \frac{2}{H+1} + \sum_{0 \le \|h\|_\infty \le H} \frac{1}{r(h)} (\log N)^{-\sigma_0}$$

for an arbitrary constant σ_0. Thus we arrive at

$$F_j(R_a) \ll N\lambda_2(R) + N(\log H)^2 (\log N)^{-\sigma_0} + \frac{N}{H+1}.$$

Now observe that (because of possible overlappings) $F_j \le \sum_{a\in\mathcal{N}} \sum_R F_j(R_a)$, where the second sum runs over all rectangles R, in which we subdivided $P_{k,a}$. By the properties of $P_{k,a}$ listed in Lemma 3.1 we conclude that

$$F_j \ll \left(\frac{\mu}{|b|^2}\right)^k N + \left(N(\log N)^{-\sigma_0} (\log H)^2 + \frac{N}{H+1}\right) \mu^k.$$

Setting $H := \exp\left((\log N)^{\sigma_0/4}\right)$ and $\lambda := \sigma_0/4$ the result follows. ☐

5. The Key Step

In this section we prove a proposition that will play a crucial rôle in the proof of our main result. Before we state this proposition, we state a lemma that gives sharp bounds for the length of the b-adic representation of a Gaussian integer. This result was first proved in a more general setting in [13] (cf. also [9]).

Lemma 5.1. *Let l be the length of the b-adic representation of $z \in \mathbb{Z}[i]$, i.e., the smallest number l, such that $z = \sum_{0\le j<l} a_j b^j$ with $a_j \in \mathcal{N}$. Then the estimate*

$$2\log_{|b|^2} |z| - c \le l \le 2\log_{|b|^2} |z| + c$$

holds for a certain absolute constant c.

With help of this lemma we can formulate the following result. As in the introduction we will denote the j-th digit of a number $z \in \mathbb{Z}[i]$ in its b-adic representation by $a_j(z)$.

Proposition 5.1. *Let $L = 2\log_{|b|^2} N + c$ be an upper bound for the maximal length of the b-adic representation of Gaussian integers z with $|z|^2 < N$. For*

$$L^{1/3} \le l_1 < l_2 < \cdots < l_h \le rL - L^{1/3}$$

we have, as $N \to \infty$,

$$\Theta := \#\{|z|^2 < N \mid a_{l_j}(P(z)) = b_j, j = 1, \ldots, h\} = \frac{\pi N}{|b|^{2h}} + \mathcal{O}\left(N(\log N)^{-\sigma'}\right),$$

uniformly for $b_j \in \mathcal{N}$ and l_j in the given range. z runs over the Gaussian integers and σ' is an arbitrary positive constant.

Proof. For $v \in \mathbb{R}^2$ let

$$t(v) = f_{b_1}\left(B^{-l_1-1}v\right) \cdots f_{b_h}\left(B^{-l_h-1}v\right)$$

(note that $\phi(bz) = B\phi(z)$). Furthermore, let

$$\mathcal{M} = \{M = (\mu_1, \ldots, \mu_h) \mid \mu_j = (m_{j1}, m_{j2}) \text{ with } m_{j1}, m_{j2} \in \mathbb{Z}; j = 1, \ldots, h\}.$$

Then a straight forward calculation yields

$$t(x,y) = \sum_{M\in\mathcal{M}} T_M e\left(\sum_{j=1}^h \mu_j B^{-l_j-1}(x,y)^T\right),$$

where $T_M = \prod_{i=1}^h c_{m_{i1}m_{i2}}$.

Obviously we have

$$\left|\Theta - \sum_{|z|^2<N} t(\phi(P(z)))\right| \le F_{l_1} + \cdots + F_{l_h}. \tag{5.1}$$

With the same notations as in (4.3) we get

$$\begin{aligned}\sum_{|z|^2<N} t(\phi(P(z))) &= \sum_{M\in\mathcal{M}} T_M \sum_{|z|^2<N} e\left(\sum_{j=1}^{h} \mu_j B^{-l_j-1}\phi(P(z))\right)\\ &= \sum_{M\in\mathcal{M}} T_M \sum_{|z|^2<N} e\left(\sum_{j=1}^{h} \mu_j B^{-l_j-1}\Xi^{-1}\tau(P(z))\right).\end{aligned}$$

Set

$$\tilde{\mu}_j = (\tilde{m}_{j1}, \tilde{m}_{j2}) := \mu_j B^{-l_j-1}\Xi^{-1}$$

and observe that

$$\begin{aligned}\tilde{\mu}_j\tau(P(z)) &= \sum_{i=1}^{2} \tilde{m}_{ji}\mathrm{tr}(b^{i-1}P(z))\\ &= \mathrm{tr}\left(\sum_{i=1}^{2} \tilde{m}_{ji} b^{i-1}P(z)\right)\end{aligned}$$

and thus

$$\sum_{|z|^2<N} t(\phi(P(z))) = \sum_{M\in\mathcal{M}} T_M \sum_{|z|^2<N} e(\mathrm{tr}(q(z)))$$

with a polynomial $q(z)$. Now we want to apply Proposition 2.1 to this sum. Hence we have to check if the leading coefficient of $q(z)$ satisfies the conditions of Proposition 2.1. In particular, we have

$$q(z) = \sum_{j=1}^{h}\sum_{i=1}^{2} \tilde{m}_{ji} b^{i-1}P(z).$$

It is an easy exercise to derive that the leading coefficient is

$$\frac{A_M}{H_M} := p_r \sum_{j=1}^{h} \frac{b^{-l_j-1}(\bar{b}m_{j1} - m_{j2})}{\bar{b} - b},$$

where $(A_M, H_M) = 1$ in $\mathbb{Z}[i]$. We will now characterize those M, for which $(\log N)^\sigma < H_M$. Assume that the vector μ_h satisfies the condition $b \not| (\bar{b}m_{h1} - m_{h2})$. Now we have

$$H_M p_r \sum_{j=1}^{h} \frac{b^{l_h-l_j}(\bar{b}m_{j1} - m_{j2})}{\bar{b} - b} = A_M b^{l_h+1}. \tag{5.2}$$

Let $b = p_1^{\varepsilon_1}\cdots p_g^{\varepsilon_g}$ be the prime factor decomposition of b. Then $p_t^{\varepsilon_t} \not| (\bar{b}m_{h1} - m_{h2})$ for some t and hence (5.2) implies $p_t^{l_h\varepsilon_t} \mid H_M$. Thus there exists an $\eta > 0$ such that $H_M \ge b^{\eta l_h}$. By the assumptions on l_h we conclude that $(\log N)^\sigma < H_M$. In case of $bm_{i1} - m_{i2} = 0$ for $i = s+1, \dots, h$ and $b \not| (\bar{b}m_{s1} - m_{s2})$ we can prove similarly that $H_M \ge b^{\eta l_s}$. If, on the other hand, there exists a $j \in \{1, \dots, h\}$, such that $b|\bar{b}m_{j1} - m_{j2}$, Lemma 3.3 implies that the corresponding $T_M = 0$.

Thus we have proved that

$$\sum_{M\in\mathcal{M}} T_M \sum_{|z|^2<N} e(\mathrm{tr}(q(z))) = {\sum_{M\in\mathcal{M}}}' T_M \sum_{|z|^2<N} e(\mathrm{tr}(q(z))),$$

where the prime (') indicates that we only sum over $M = 0$ and those $M \in \mathcal{M}$, for which the leading coefficient r_k/q_k of $q(z)$ has $q_k \gg (\log N)^\sigma$ for arbitrary σ. Since the inequality $q_k \ll N^r(\log N)^{-\sigma}$ is obvious, we can apply Proposition 2.1 to each of the inner sums corresponding to nonzero M. Thus they are bounded uniformly by $\mathcal{O}\left(N(\log N)^{-\sigma_0}\right)$, where σ_0 is an arbitrary positive constant.

Concerning the summand corresponding to $M = 0$ we remark, that the summands of the inner sum are all equal to 1. Since $c_{00} := |b|^{-2}$ by Lemma 3.2 the contribution corresponding to $M = 0$ is $\pi N/|b|^{2h} + \mathcal{O}\left(\sqrt{N}\right)$. Using (5.1) we arrive at

$$\Theta = \frac{\pi N}{|b|^{2h}} + \mathcal{O}\left(N(\log N)^{-\sigma_0} \sum_{M \neq 0} |T_M|\right) + \mathcal{O}\left(\sum_{j=1}^{h} F_{l_j}\right).$$

Setting $k = C \log\log N$ with some positive constant C, the result follows from Lemma 4.2 and the estimate

$$\sum_{M} |T_M| \ll \Delta^{-2h}$$

which is a consequence of Lemma 3.2. □

6. Proof of the Theorem

Now we are ready to give a proof of Theorem 1.2. To this matter set $A := [L^{1/3}]$ and $B := L - A$, where L is defined as in the statement of Proposition 5.1. Furthermore, define the function

$$f'(P(z)) = \sum_{j=A}^{B} f(a_j(P(z))b^j).$$

Since $f(cb^j) = \mathcal{O}(1)$, we conclude that $f'(P(z)) = f(P(z)) + \mathcal{O}\left(L^{1/3}\right)$. We also define the approximations

$$M'(N^r) := \sum_{j=A}^{B} m_j \quad \text{and} \quad {D'}^2(N^r) := \sum_{j=A}^{B} \sigma_j^2$$

for $M(N^r)$ and $D^2(N^r)$. It is obvious that $M'(N^r) - M(N^r) = \mathcal{O}\left(L^{1/3}\right)$ and ${D'}^2(N^r) - D^2(N^r) = \mathcal{O}\left(L^{1/3}\right)$. Summing up all these estimates we arrive at

$$\max_{|z|^2 < N} \left| \frac{f(P(z)) - M(N^r)}{D(N^r)} - \frac{f'(P(z)) - M'(N^r)}{D'(N^r)} \right| \to 0 \quad \text{for } N \to \infty$$

using the requirements upon $D(N)$ stated in the theorem. This means that it is enough to show that

$$\frac{1}{\#\{z \mid |z|^2 < N\}} \# \left\{ |z|^2 < N \;\middle|\; \frac{f'(P(z)) - M'(N^r)}{D'(N^r)} < y \right\} \to \Phi(y).$$

By the Fréchet-Shohat Theorem (cf. for instance [6, Lemma 1.43]) this holds if and only if the moments

$$\xi_k(N) := \frac{1}{\#\{z \mid |z|^2 < N\}} \sum_{|z|^2 < N} \left(\frac{f'(P(z)) - M'(N^r)}{D'(N^r)} \right)^k$$

converge to the moments of the normal law for $N \to \infty$. Instead of proving this, we compare $\xi_k(N)$ with

$$\eta_k(N) := \frac{1}{\#\{z \mid |z|^2 < N^r\}} \sum_{|z|^2 < N^r} \left(\frac{f'(z) - M'(N^r)}{D'(N^r)} \right)^k.$$

Proposition 5.1 now implies that

$$\xi_k(N) - \eta_k(N) \to 0 \quad \text{for} \quad N \to \infty.$$

Obviously, $\eta_k(N)$ are the moments of

$$\frac{f'(z) - M'(N^r)}{D'(N^r)} \quad (|z|^2 < N^r).$$

By Lemma 5.1 these are sums of independently identically distributed random variables (apart from $2c$ variables, which are not independent from the others; but since c is an absolute constant, they do not play any rôle). Thus it follows from the central limit theorem that their distribution converges to the normal law. Hence, also $\eta_k(N)$ converge to the moments of the normal law. This yields

$$\lim_{N\to\infty} \xi_k(N) = \lim_{N\to\infty} \eta_k(N) = \int x^k d\Phi.$$

Applying the Fréchet-Shohat Theorem again the other way round, the theorem follows.

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