STRATA OF RANDOM MAPPINGS – A COMBINATORIAL APPROACH

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Abstract. Consider the functional graph of a random mapping from an n–element set into itself. Then the number of nodes in the strata of this graph can be viewed as stochastic process. Using a generating function approach it is shown that a suitable normalization of this process converges weakly to local time of reflecting Brownian bridge.

1. Introduction

Let $\mathcal{F}_n$ denote the set of all mappings $\varphi : \{1, \dots, n\} \to \{1, \dots, n\}$ and assume that this set is equipped with the uniform distribution. Then a mapping $\varphi \in \mathcal{F}_n$ is usually called a random mapping. For our investigations it is convenient to represent random mappings by its functional graph G_φ, i.e. the graph consisting of the nodes $1, 2, \dots, n$ and of the edges $(i, \varphi(i))$, $i = 1, \dots, n$. It is easy to see that each component of such a graph consists of exactly one cycle of length ≥ 1 each point of which is the root of a labeled tree. Thus for each point $x \in G_\varphi$ there exists a unique path connecting x with the next cyclic point. The length of this path is called the distance of x to the cycle. The set of all points at a fixed distance r from the cycle is often called the r-th stratum of φ.

Let $L_n(r)$ denote the number of nodes in the r-th stratum of a random mapping $\varphi \in \mathcal{F}_n$. The behavior of this random variable for $n \to \infty$ has attracted the interest of many authors. Harris [14] showed that the number of cyclic points $L_n(0)/\sqrt{n}$ weakly converges to a Rayleigh distribution with mean value $\sqrt{\pi n/2}$. Mutafchiev [18] proved that this result is still true for $r = o(\sqrt{n}\,)$. The corresponding local limit theorem is derived in [8]. In case of $r \sim c\sqrt{n}$, $c > 0$, Mutafchiev's result is no longer true. Representations for the moments and the density of the limiting distribution for this case have been established by Proskurin [19]. Finally, it should be mentioned that a survey of several related random mapping characteristics as well as the relations to branching processes and random trees are contained in Kolchin's book [17].

Aldous and Pitman [4] studied the contour of a random mapping, i.e. the polygonal function obtained by traversing each tree of G_φ successively. They showed that the suitably rescaled contour process weakly converges to reflecting Brownian bridge (rBB), i.e. the process identical in law to $(|W(t) - tW(1)|, 0 \leq t \leq 1)$ where $W(t)$ is a one dimensional Brownian motion (BM) or roughly speaking rBB is a BM of length 1 reflected at 0 and conditioned to have zeros at 0 and 1. In view of the results in [9, 11] this suggests that the process $l_n(t) = n^{-1/2} L_n(t\sqrt{n}),\ t \geq 0$, where

$$L_n(t) = (\lfloor t \rfloor + 1 - t) L_n(\lfloor t \rfloor) + (t - \lfloor t \rfloor) L_n(\lfloor t \rfloor + 1), \quad \text{for non-integral } t \geq 0,$$

converges weakly to the local time process for rBB. In fact, we will prove

Theorem 1.1. *Let $B(t)$ denote reflecting Brownian bridge and $l(t)$ its local time, i.e.*

$$l(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 I_{[t, t+\varepsilon]}(B(s))\, ds$$

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Then we have

$$l_n(t) \xrightarrow{w} \frac{1}{2} l\left(\frac{t}{2}\right)$$

in $C[0,\infty)$, as $n \to \infty$.

What we have to do is to prove the weak convergence of the finite dimensional distributions (fdd's) and that the process is tight (see [6, Theorem 12.3] or [16, p. 63]). In order to do this we will proceed as follows: First we will calculate the limiting distribution of the fdd's of $l_n(t)$ using a generating function approach which is explained in the next section. Then we proceed with the computation of the fdd's of rBB local time by methods of Itô's excursion theory (see [15, 21]) and observe that those distributions coincide. We will also briefly indicate how the generating function approach could be used to obtain the local time distributions. Finally, the proof of tightness is presented.

Remark . Note that our method also allows us to reprove [4, Theorem 8] in a similar way as it has been done the analogous problem for random trees (see e.g. [11] where a combinatorial approach is used to extend a result of [13] and to reprove parts of the results of [1, 2, 3]).

2. Preliminaries

Let b_{kmn} denote the number of all functional graphs in $\mathcal{F}_n$ where $L_n(k) = m$. As we are considering the uniform probability model, we have

$$\mathbf{P}\left\{L_n(k) = m | T \in \mathcal{A}_n\right\} = \frac{b_{nm,k}}{n^n}. \tag{2.1}$$

Furthermore the bivariate GF of $b_{nm,k}$ is given by

$$b_k(z,u) = \sum_{n,m\ge 0} b_{nm,k} u^m \frac{z^n}{n!} = \frac{1}{1-a_k(z,u)} \quad \text{with} \quad a_k(z,u) = y_k(z, ua(z))$$

where

$$y_0(z,u) = u$$
$$y_{i+1}(z,u) = ze^{y_i(z,u)}, \quad i \ge 0,$$

and $a(z)$ is the well-known tree function given by its functional equation $a(z) = z\exp(a(z))$. This follows immediately from the combinatorial setup (details see [9]). Hence the characteristic function of $n^{-1/2}L_n(k)$ is

$$\phi_{kn}(t) = \frac{n!}{n^n}[z^n]\left(1 - y_k\left(z, e^{it/\sqrt{n}}a(z)\right)\right)^{-1}$$

and that of $\left(n^{-1/2}L_n(k_1), \dots, n^{-1/2}L_n(k_p)\right)$ is given by

$$\phi_{k_1\cdots k_p n}(t_1,\dots,t_p) = \\ \frac{n!}{n^n}[z^n]\left[1 - y_{k_1}\left(z, e^{it_1/\sqrt{n}} y_{k_2-k_1}\left(z, \dots y_{k_p-k_{p-1}}\left(z, e^{it_p/\sqrt{n}}a(z)\right)\dots\right)\right]^{-1}. \tag{2.2}$$

Thus in order to prove Theorem 1.1 we have to show

$$\bar\phi_{\kappa_1/2,\dots,\kappa_p/2}(t_1/2,\dots,t_p/2) = \phi_{\kappa_1\cdots\kappa_p}(t_1,\dots,t_p) \tag{2.3}$$

where $\bar\phi_{\kappa_1,\dots,\kappa_p}(t_1,\dots,t_p)$ is the characteristic function of the joint distribution of $(l(\kappa_1),\dots,l(\kappa_p))$ and $\phi_{\kappa_1\dots\kappa_p}(t_1,\dots,t_p) = \lim_{n\to\infty}\phi_{k_1\cdots k_p n}(t_1,\dots,t_p)$. For extracting the coefficient in (2.2) we will use Cauchy's integral formula and singularity analysis in the sense of Flajolet and Odlyzko [10]. Thus we need some information about the local behaviour of the involved functions:

Lemma 2.1. *Let $z = e^{-1}\left(1+\frac{x}{n}\right)$. Furthermore assume that $|u - a(z)| = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ and $\frac{x}{n} \to 0$ in such a way that $|\arg(-x)| < \pi$ and*

$$\left|1 - \sqrt{\frac{-x}{n}}\right| \le 1 + \frac{C}{\sqrt{n}}$$

are satisfied. Then $y_k(z,u)$ admits the local representation

$$y_k(z,u) = a(z) + \frac{2\sqrt{-x/n}(u-a(z))a(z)^k}{\sqrt{-x/n}(1+a(z)^k) + \frac{1-u}{\sqrt{2}}(1-a(z)^k) + \mathcal{O}(|x|/n)},$$

uniformly for $k = \mathcal{O}(\sqrt{n})$.

Proof. The proof is immediate by setting $\varphi(t) = e^t$, $\sigma = 1$ and $\tau = 1$ in [9, Lemma 2.1]. □

3. Convergence of the Finite Dimensional Distributions

In this section we will show the following two theorems:

Theorem 3.1. *Let $k_i = \kappa_i\sqrt{n}, i = 1,\dots,k$ where $0 < \kappa_1 < \cdots < \kappa_p$. Then the characteristic function $\phi_{\kappa_1\dots\kappa_p}(t_1,\dots,t_p) = \lim_{n\to\infty} \phi_{k_1\cdots k_p n}(t_1,\dots,t_p)$ of the limiting distribution of $\left(\frac{1}{\sqrt{n}}L_n(k_1),\dots,\frac{1}{\sqrt{n}}L_n(k_p)\right)$ satisfies*

$$\phi_{\kappa_1\dots\kappa_p}(t_1,\dots,t_p) = \frac{1}{2i\sqrt{\pi}}\int_\gamma f_{\kappa_1,\dots,\kappa_p}(x,t_1,\dots,t_p)\frac{e^{-x}}{\sqrt{-x}}\,dx, \tag{3.1}$$

where

$$\begin{aligned} &f_{\kappa_1,\dots,\kappa_p}(x,t_1,\dots,t_p)\\ &= \Psi_{\kappa_1}\left(x, \frac{it_1}{\sqrt{2}} + \tilde{\Psi}_{\kappa_2-\kappa_1}\left(\frac{it_2}{\sqrt{2}} + \tilde{\Psi}_{\kappa_3-\kappa_2}\left(\dots\tilde{\Psi}_{\kappa_{p-1}-\kappa_{p-2}}\left(x, \frac{it_{p-1}}{\sqrt{2}} + \tilde{\Psi}_{\kappa_p-\kappa_{p-1}}\left(x, \frac{it_p}{\sqrt{2}}\right)\right)\cdots\right)\right.\right.\end{aligned}$$

with

$$\Psi_\kappa(x,t) = \frac{\sqrt{-x}e^{-\kappa\sqrt{-x/2}} - t\sinh\left(\kappa\sqrt{-x/2}\right)}{\sqrt{-x}e^{\kappa\sqrt{-x/2}} - t\cosh\left(\kappa\sqrt{-x/2}\right)}$$

and

$$\tilde{\Psi}_\kappa(x,t) = \frac{t\sqrt{-x}e^{-\kappa\sqrt{-x/2}}}{\sqrt{-x}e^{\kappa\sqrt{-x/2}} - t\sinh\left(\kappa\sqrt{-x/2}\right)} \tag{3.2}$$

and γ is the Hankel contour[1] $\gamma_1\cup\gamma_2\cup\gamma_3$ defined by

$$\begin{aligned} \gamma_1 &= \left\{s\big||s| = 1 \text{ and } \Re s \le 0\right\},\\ \gamma_2 &= \left\{s\big|\Im s = 1 \text{ and } \Re s \ge 0\right\},\\ \gamma_3 &= \overline{\gamma}_2. \end{aligned} \tag{3.3}$$

Remark . Note that by means of the generating function approach we get only a proof of this theorem for integral k_i and thus a limit theorem for the step function process $L_n(\lfloor t\sqrt{n}\rfloor)/\sqrt{n}$. However, a direct application of the tightness inequality (Theorem 4.1) shows that the difference $L_n(\lfloor t\sqrt{n}\rfloor)/\sqrt{n} - l_n(t)$ converges to zero in probability and thus the theorem is correct as stated.

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[1] The names "Hankel contour", "Hankel integral", etc. originate from Hankel's representation of the Gamma function,

$$\frac{1}{2\pi i}\int_\gamma (-s)^{-\alpha}e^{-s}\,ds = \frac{1}{\Gamma(\alpha)},$$

and have become usual due to the quite frequent occurrence of integration contours similar to γ in asymptotical problems in combinatorics.

Theorem 3.2. *With the notation of the previous theorem the fdd's of Brownian bridge local time are given by*

$$\bar{\phi}_{\kappa_1\ldots\kappa_p}(t_1,\ldots,t_p)=\frac{1}{2i\sqrt{\pi}}\int_\gamma f_{2\kappa_1,\ldots,2\kappa_p}(x,2t_1,\ldots,2t_p)\frac{e^{-x}}{\sqrt{-x}}\,dx. \tag{3.4}$$

The density of rBB local time. As mentioned in the introduction Proskurin [19] calculated the limiting distribution of the number of nodes in the r-th stratum for $r/\sqrt{n}\to\rho>0$. His result implies that the one-dimensional density $f_\rho(x)$ of the total local time at level ρ has a representation of the form

$$f_\rho(x)=\frac{2}{\rho}\sum_{j=1}^{\infty}\frac{(-1)^j}{(j-1)!}\left[\frac{\partial^j}{\partial u^j}(u-j)^{j-1}e^{-2\rho^2u^2}\right]_{u=j+x/2\rho}. \tag{3.5}$$

Using a random walk approximation Takács [22] obtained a different representation, namely

$$f_\rho(x)=2\sum_{l=1}^{\infty}\sum_{j=1}^{l}\binom{l}{j}\frac{(-1)^{l+j}x^{j-1}}{(j-1)!}e^{-(2l\rho+x)^2/2}H_j(2l\rho+x) \tag{3.6}$$

where $H_j(x)$ are the Hermite polynomials defined by

$$H_j(x)=j!\sum_{i=0}^{\lfloor n/2\rfloor}\frac{(-1)^ix^{j-2i}}{2^ii!(j-2i)!}.$$

Using our approach the density can be determined by the appropriate coefficient in the generating function (cf. (2.1)) and this yields a third representation given by

$$f_\rho(x)=\frac{1}{i\sqrt{2\pi}}\int_{-1-\infty\cdot i}^{-1+\infty\cdot i}\frac{e^{-\rho\sqrt{-2u}-u}}{\cosh^2\left(\rho\sqrt{-2u}\right)}\exp\left(-\frac{x}{\sqrt{2}}\frac{\sqrt{-u}e^{\rho\sqrt{-2u}}}{\cosh\left(\rho\sqrt{-2u}\right)}\right)du.$$

This one is the analogous form of Cohen and Hooghiemstra's [7] represention for the Brownian excursion local time density (for a list of further representations, among them the analoga of (3.6), see [9]) and could be generalized to multi-dimensional densities by evaluating the corresponding coefficients in the multivariate generating functions. In case of Brownian excursion local time this has been done in [12]. However, it seems to be difficult to get multivariate extensions of (3.5) or (3.6) and the analogous problem for Brownian excursion is also unsolved up to now.

Proof of Theorem 3.1. In order to prove this theorem we have to calculate the right-hand side of (2.2). We will use Cauchy's integral formula with the integration contour $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$ defined as follows:

$$\begin{aligned}\Gamma_1&=\left\{z=\frac{1}{e}\left(1+\frac{x}{n}\right)\,\middle|\,\Re x\le 0\text{ and }|x|=1\right\}\\ \Gamma_2&=\left\{z=\frac{1}{e}\left(1+\frac{x}{n}\right)\,\middle|\,\Im x=1\text{ and }0\le\Re x\le\log^2 n\right\}\\ \Gamma_3&=\overline{\Gamma}_2\\ \Gamma_4&=\left\{z\,\middle|\,|z|=\frac{1}{e}\left|1+\frac{\log^2 n+i}{n}\right|\text{ and }\arg\left(1+\frac{\log^2 n+i}{n}\right)\le|\arg(z)|\le\pi\right\}.\end{aligned} \tag{3.7}$$

In order to see how the general scheme of the proof is running it suffices to consider the case $p=2$. Then the proof for $p=1$ is merely an obvious simplification of the presented proof and the remaining part is obtained by induction. Thus we have to calculate the integral

$$\frac{n!}{n^n}\frac{1}{2\pi i}\int_\Gamma\frac{1}{1-y_k(z,uy_h(z,va(z)))}\frac{dz}{z^{n+1}} \tag{3.8}$$

where $u = e^{is/\sqrt{n}}$, $v = e^{it/\sqrt{n}}$ and $k = \kappa\sqrt{n}$, $h = \eta\sqrt{n}$. Set $R_k(z,u) = y_k(z,u) - a(z)$. Using the well-known expansion

$$a(z) = 1 - \sqrt{2}\sqrt{1-ez} + \mathcal{O}\left(1-ez\right), \quad z \to \frac{1}{e}, \quad z \in \Delta \tag{3.9}$$

where

$$\Delta = \left\{ z : |z| < \frac{1}{e} + \eta,\ |\arg(z - z_0)| > \vartheta \right\},$$

$\eta > 0$ and $0 < \vartheta < \pi/2$ arbitrary but fixed, we get the following asymptotic expansions on $\gamma' = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$:

$$a(z) = 1 - \sqrt{2}\sqrt{-\frac{x}{n}} + \mathcal{O}\left(\frac{|x|}{n}\right)$$
$$a(z)^k = \exp\left(-\kappa\sqrt{-2x}\right)\left(1 + \mathcal{O}\left(\frac{k|x|}{n}\right)\right)$$

Applying Lemma 2.1 and these formulae yields

$$\begin{aligned}\frac{1}{1 - y_k(z, uy_h(z, va(z)))} &= \sqrt{-\frac{n}{2x}}\,\frac{\sqrt{-x}\exp\left(\kappa\sqrt{-x/2}\right) - (is/\sqrt{2} + \sqrt{n/2}R_h)\sinh\left(\kappa\sqrt{-x/2}\right)}{\sqrt{-x}\exp\left(\kappa\sqrt{-x/2}\right) - (is/\sqrt{2} + \sqrt{n/2}R_h)\cosh\left(\kappa\sqrt{-x/2}\right)} \\ &\quad \times \left(1 + \mathcal{O}\left(\frac{\log^2 n}{\sqrt{n}}\right)\right)\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}R_h = R_h(z, va(z)) &= \sqrt{-\frac{x}{n}}\,\frac{it\exp\left(-\eta\sqrt{-x/2}\right)}{\sqrt{-x}\exp\left(-\eta\sqrt{-x/2}\right) - \frac{it}{\sqrt{2}}\sinh\left(-\eta\sqrt{-x/2}\right)} \\ &\quad \times \left(1 + \mathcal{O}\left(\frac{\log^2 n}{\sqrt{n}}\right)\right)\end{aligned} \tag{3.11}$$

Now let us analyze the contribution of Γ_4. Observe that

$$\left[\frac{\partial}{\partial x_2} y_h(x_1, x_2)\right]_{x_1 = z, x_2 = a(z)} = a(z)^h.$$

Expanding the generating function in a Taylor series at $(x_1, x_2) = (z, a(z))$ gives

$$\begin{aligned}\frac{1}{1 - y_k(z, uy_h(z, va(z)))} &= \frac{1}{1 - a(z)} + \frac{a(z)^k}{(1 - a(z))^2}(uy_h(z, va(z)) - a(z)) \\ &\quad + \mathcal{O}\left((uy_h(z, va(z)) - a(z))^2\right) \\ &= \frac{1}{1 - a(z)} + \frac{a(z)^{k+1}}{(1 - a(z))^2}\left(u - 1 + u(v-1)a(z)^h + \mathcal{O}\left((v-1)^2\right)\right) \\ &\quad + \mathcal{O}\left(\left(u - 1 + u(v-1)a(z)^h + \mathcal{O}\left((v-1)^2\right)\right)\right)\end{aligned}$$

$a(z)$ and $1/(1 - a(z))$ have only positive coefficients and hence these functions attain their maximum on Γ_4 if and only if $z \in \Gamma_4 \cap \gamma'$, i.e. $z = e^{-1}(1 + (\log^2 n + i)/n)$. Thus we obtain

$$\max_{z \in \Gamma_4}\left|\frac{1}{1 - a(z)}\right| \sim \frac{1}{\sqrt{2}}\frac{\sqrt{n}}{\log n}$$
$$\max_{z \in \Gamma_4}|a(z)|^k \sim \exp\left(-\kappa\Re\sqrt{-\log^2 n - i}\right) = \mathcal{O}\left(1\right).$$

Furthermore we have $u-1=\mathcal{O}(1/\sqrt{n})$ and $|z|^{-n-1}\sim e^{-\log^2 n}$. Collecting all these estimates yields finally

$$\frac{n!}{n^n}\frac{1}{2\pi}\left|\int_{\Gamma_4}\frac{1}{1-y_k(z,uy_h(z,va(z)))}\frac{dz}{z^{n+1}}\right|=o(1).$$

and we are done. (3.8) – (3.11) give

$$\varphi_{\kappa,\kappa+\eta}(s,t)=\frac{1}{2i\sqrt{\pi}}\int_{\gamma'}f_{\kappa,\kappa+\eta}(x,s,t)\frac{e^{-x}}{\sqrt{-x}}\,dx.$$

It is easy to see that the numerator of $f_{\kappa,\kappa+\eta}$ is bounded by $\exp(C\sqrt{|x|})$ for a suitable constant C. Furthermore for $x=y+i$ the denominator $D(t,x)$ of $f_{\kappa,\kappa+\eta}$ satisfies

$$D(t,x)\sim\sqrt{-y}e^{\kappa\sqrt{-2(y+i)}+\eta\sqrt{-2(y+(1+t)i)}},\quad y\to\infty.$$

As the real part of the exponent converges to zero as $y\to\infty$, the denominator is bounded from below by a positive constant. Thus we may substitute the integration path γ' by γ due to the dominated convergence theorem. □

Proof of Theorem 3.2. For brevity, let us use the following notation in the sequel: If X is a random variable on a probability space $(\Omega,\mathcal{A},P)$ then set

$$P[X]:=\int_\Omega X\,dP.$$

Furthermore, let $W^+(s)$ denote reflecting BM and $\tau_s:=\min\{t:l_t(0)\ge s\}$ where $l_t(a)$ denotes the local time at level a of $(W^+(s),s\ge 0)$ up to time t. By P^{τ_s} we denote the distribution of $(W^+(t),t\ge 0)$ stopped at time τ_s and Q^u denotes the distribution of rBB of length u. Then the desired characteristic function is given by

$$\bar{\phi}_{\kappa_1\cdots\kappa_p}(t_1,\ldots,t_p)=Q^1\left[e^{i(t_1l_1(\kappa_1)+\cdots+t_pl_1(\kappa_p))}\right].$$

Moreover, note that (3.4) is an inverse Laplace transform and hence $\bar{\varphi}_{\kappa_1\cdots\kappa_p}$ can be obtained by transforming a proper function of Q^u. In fact we have

Proposition 3.1. *With the notations above, for $\Re x<0$ the following identity holds:*

$$\int_0^\infty Q^u\left[e^{i(t_1l_u(\kappa_1)+\cdots+t_pl_u(\kappa_p))}\right]\frac{e^{xu}\,du}{\sqrt{2\pi u}}=\frac{\sqrt{2}}{\sqrt{-x}}f_{2\kappa_1,\ldots,2\kappa_p}(2t_1,\ldots,2t_p)$$

This immediately implies Theorem 3.2.

Proof. By [21, Ch. VI, ex. 2.29 or Ch. XII, ex. 4.18] we have

$$\int_0^\infty Q^u\frac{du}{\sqrt{2\pi u}}=\int_0^\infty P^{\tau_s}\,ds.$$

This implies

$$\int_0^\infty Q^u\left[e^{i(t_1l_u(\kappa_1)+\cdots+t_pl_u(\kappa_p))}\right]\frac{e^{xu}\,du}{\sqrt{2\pi u}}=\int_0^\infty P^{\tau_s}\left[e^{i(t_1l_{\tau_s}(\kappa_1)+\cdots+t_pl_{\tau_s}(\kappa_p))}e^{x\tau_s}\right]\,ds.$$

Note that under P^{τ_s} the local time is identical in law to the square of a 0-dimensional Bessel process (see e.g. [5, p.78]) which we will denote by X_t in the sequel. Decomposing the duration of the rBM in the form

$$\tau_s=\int_0^\infty l_{\tau_s}(a)\,da=\int_0^{\kappa_1}l_{\tau_s}(a)\,da+\int_{\kappa_1}^{\kappa_2}l_{\tau_s}(a)\,da+\cdots+\int_{\kappa_{p-1}}^{\kappa_p}l_{\tau_s}(a)\,da+\int_{\kappa_p}^\infty l_{\tau_s}(a)\,da$$

gives

$$
\begin{aligned}
&P^{\tau_s}\left[e^{i(t_1 l_{\tau_s}(\kappa_1)+\cdots+t_p l_{\tau_s}(\kappa_p))}e^{x\tau_s}\right]\\
&\qquad=\mathbf{E}\left[e^{i(t_1X_{\kappa_1}+\cdots+t_pX_{\kappa_p})}\exp\left(x\int_0^{\kappa_1}X_u\,du+x\int_{\kappa_1}^{\infty}X_u\,du\right)\right].
\end{aligned}
$$

In this form our term is amenable to an application of [20, formula (2.k)] which states that for $\kappa'<\kappa''$

$$
\begin{aligned}
&\mathbf{E}\left[\exp\left(itX_{\kappa''}+x\int_{\kappa'}^{\kappa''}X_u\,du\right)\middle|X_{\kappa'}\right]\\
&\qquad=\exp\left(-X_{\kappa'}\sqrt{-\frac{x}{2}}\,\frac{1-it\sqrt{-2/x}\coth\left((\kappa''-\kappa')\sqrt{-2x}\,\right)}{\coth\left((\kappa''-\kappa')\sqrt{-2x}\,\right)-it\sqrt{-2/x}}\right)
\end{aligned}
\tag{3.12}
$$

and

$$
\mathbf{E}\left[\exp\left(x\int_{\kappa''}^{\infty}X_u\,du\right)\middle|X_{\kappa''}\right]=\exp\left(-X_{\kappa''}\sqrt{-\frac{x}{2}}\right). \tag{3.13}
$$

Now proceed as in [9, pp. 440] (we will not detail this part): (3.12), (3.13) as well as the Markov property give

$$
\begin{aligned}
&\mathbf{E}\left[\exp\left(it_pX_{\kappa_p}+x\int_{\kappa_{p-1}}^{\infty}X_u\,du\right)\middle|X_{\kappa_{p-1}}\right]\\
&\quad=\mathbf{E}\left[\exp\left(it_pX_{\kappa_p}+x\int_{\kappa_{p-1}}^{\kappa_p}X_u\,du\right)\mathbf{E}\left[\exp\left(x\int_{\kappa_p}^{\infty}X_u\,du\right)\middle|X_{\kappa_p}\right]\middle|X_{\kappa_{p-1}}\right]\\
&\quad=\mathbf{E}\left[\exp\left(\left(it_p-\sqrt{-\frac{x}{2}}\right)X_{\kappa_p}+x\int_{\kappa_{p-1}}^{\kappa_p}X_u\,du\right)\middle|X_{\kappa_{p-1}}\right]\\
&\quad=\exp\left(-X_{\kappa_{p-1}}\left(\sqrt{-\frac{x}{2}}-\tilde\Psi_{2(\kappa_p-\kappa_{p-1})}\left(x,it_p\sqrt{2}\right)\right)\right)
\end{aligned}
$$

where $\tilde\Psi(x,t)$ is defined in (3.2). The next step gives

$$
\begin{aligned}
&\mathbf{E}\left[\exp\left(it_{p-1}X_{\kappa_{p-1}}+it_pX_{\kappa_p}+x\int_{\kappa_{p-2}}^{\infty}X_u\,du\right)\middle|X_{\kappa_{p-2}}\right]\\
&\quad=\exp\left(-X_{\kappa_{p-2}}\left(\sqrt{-\frac{x}{2}}-\tilde\Psi_{2(\kappa_{p-1}-\kappa_{p-2})}\left(x,it_{p-1}\sqrt{2}+\tilde\Psi_{2(\kappa_p-\kappa_{p-1})}\left(x,it_p\sqrt{2}\right)\right)\right)\right)
\end{aligned}
$$

and proceeding analogously we obtain after all

$$
\begin{aligned}
&\mathbf{E}\left[\exp\left(it_2X_{\kappa_2}+\cdots+it_pX_{\kappa_p}+x\int_{\kappa_1}^{\infty}X_u\,du\right)\middle|X_{\kappa_1}\right]\\
&\quad=\exp\left(-X_{\kappa_1}\left(\sqrt{-\frac{x}{2}}-\tilde f_{\kappa_2\cdots\kappa_p}(x,t_2,\ldots,t_p)\right)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
&\tilde f_{\kappa_2,\ldots,\kappa_p}(x,t_2,\ldots,t_p)\\
&\quad=\tilde\Psi_{2(\kappa_2-\kappa_1)}\left(\ldots\tilde\Psi_{2(\kappa_{p-1}-\kappa_{p-2})}\left(x,it_{p-1}\sqrt{2}+\tilde\Psi_{2(\kappa_p-\kappa_{p-1})}\left(x,it_p\sqrt{2}\right)\right)\cdots\right).
\end{aligned}
$$

Finally, observe that since we stop the BM at τ_s we have $X_0 = s$. Hence we may apply again [20, formula (2.k)] with $d = 0$ and $x = s$ and get

$$\begin{aligned}
&\int_0^\infty \mathbf{E}\left[\exp\left(it_1 X_{\kappa_1} + \cdots + it_p X_{\kappa_p} + x\tau_s\right)\right] ds \\
&\quad = \int_0^\infty \mathbf{E}\left[\exp\left(itX_{\kappa_1} + x\int_0^{\kappa_1} X_u\,du\right)\right. \\
&\qquad \times \mathbf{E}\left[\exp\left(it_2 X_{\kappa_2} + \cdots + it_p X_{\kappa_p} + x\int_{\kappa_1}^\infty X_u\,du\right)\Bigg|\, X_{\kappa_1}\right]\Bigg|\, X_0 = s\Bigg]\, ds \\
&\quad = \int_0^\infty \mathbf{E}\left[\exp\left(\left(it - \sqrt{-\frac{x}{2}} + \tilde{f}_{\kappa_2\cdots\kappa_p}(x, t_2, \ldots, t_p)\right) X_{\kappa_1} + x\int_0^{\kappa_1} X_u\,du\right)\Bigg|\, X_0 = s\right] ds \\
&\quad = \sqrt{-\frac{2}{x}}\, f_{2\kappa_1,\ldots,2\kappa_p}(2t_1, \ldots, 2t_p)
\end{aligned}$$

as desired. □

Remark . We would like to mention that it is also possible to use the generating function approach for proving Theorem 3.2. But this requires a technically complicated detour via occupation times, so we will only sketch how this can be done: Let $L([a,b]) = \int_0^1 I_{[a,b]}(B(s))\,ds$ denote rBB occupation time of the intervall $[a,b]$. [4, Theorem 8] immediately implies

$$h_n(t) = \frac{1}{n}\sum_{k\le\lfloor t\sqrt{n}\rfloor} L_n(k) \xrightarrow{w} L\left(\left[0, \frac{t}{2}\right]\right). \tag{3.14}$$

Hence the problem of determining the local time distributions can be managed by computing the characteristic function $\Phi_{\kappa_1\cdots\kappa_p\eta}(t_1,\ldots,t_p)$ of the joint distribution of $L([\kappa_1,\kappa_1+\eta]),\ldots,L([\kappa_p,\kappa_p+\eta])$ and applying the relation

$$\bar{\phi}_{\kappa_1\cdots\kappa_p}(t_1,\ldots,t_p) = \lim_{\eta\to 0}\Phi_{\kappa_1\cdots\kappa_p\eta}\left(\frac{t_1}{\eta},\ldots,\frac{t_p}{\eta}\right).$$

Let $c_{k_1m_1k_2m_2\ldots k_pm_pn}$ denote the number of all random mappings in $\mathcal{F}_n$ with m_i nodes between the k_i-th and the (k_i+h)-th stratum. Then the corresponding generating function is given by

$$\begin{aligned}
C_{k_1\cdots k_p}(z, u_1, \ldots, u_p) &= \sum_{m_1,\ldots,m_p,n\ge 0} c_{k_1m_1k_2m_2\ldots k_pm_pn} u_1^{m_1}\cdots u_p^{m_p}\frac{z^n}{n!} \\
&= [1 - y_{k_1}(z, y_h(u_1z, u_1y_{k_2-k_1}(\ldots, \\
&\qquad u_{p-1}y_{k_p-k_{p-1}}(z, y_h(u_pz, u_pa(z)))\cdots)]^{-1}.
\end{aligned}$$

Setting $k_j = \lfloor\kappa_j\sqrt{n}\rfloor$ and $h = \lfloor\eta\sqrt{n}\rfloor$ we obtain by (3.14)

$$\Phi_{\kappa_1/2,\ldots,\kappa_p/2,\eta/2}(t_1,\ldots,t_p) = \lim_{n\to\infty}\frac{n!}{n^n}[z^n]C_{k_1\cdots k_p}(z, e^{it_1/n},\ldots,e^{it_p/n}). \tag{3.15}$$

Using the techniques in the proof of Theorem 3.1 we can prove

Theorem 3.3. *The characteristic function of the joint distribution of* $L([\kappa_1,\kappa_1+\eta]),\ldots,L([\kappa_p,\kappa_p+\eta])$ *satisfies*

$$\Phi_{\kappa_1\ldots\kappa_p\eta}(t_1,\ldots,t_p) = \frac{1}{2i\sqrt{\pi}}\int_\gamma F_{\kappa_1,\ldots,\kappa_p,\eta}(x, t_1,\ldots,t_p)\frac{e^{-x}}{\sqrt{-x}}\,dx,$$

where

$$\begin{aligned}
&F_{\kappa_1,\ldots,\kappa_p,\eta}(x, t_1,\ldots,t_p) = \\
&\qquad \Xi_{\kappa_1,\eta}(x, t_1, \Xi_{\kappa_2-\kappa_1,\eta}(\ldots\Xi_{\kappa_{p-1}-\kappa_{p-2},\eta}(x, t_{p-1}, \Xi_{\kappa_p-\kappa_{p-1},\eta}(x, t_p, 0))\cdots)
\end{aligned}$$

with

$$
\begin{aligned}
\Xi_\kappa(x,t,y) = \Big(&\sqrt{-x(-x-it)}e^{\kappa\sqrt{-2x}}\cosh\left(\eta\sqrt{-2(x+it)}\right) - xe^{\kappa\sqrt{-2x}}\sinh\left(\eta\sqrt{-2(x+it)}\right)\\
&- it\sinh\left(\kappa\sqrt{-2x}\right)\sinh\left(\eta\sqrt{-2(x+it)}\right)\\
&- y\left(\sqrt{-x}\cosh\left(\kappa\sqrt{-2x}\right)\sinh\left(\eta\sqrt{-2(x+it)}\right)\right.\\
&\left.+\sqrt{-x-it}\sinh\left(\kappa\sqrt{-2x}\right)\cosh\left(\eta\sqrt{-2(x+it)}\right)\right)\Big)\\
\times\Big[&\sqrt{-x(-x-it)}e^{\kappa\sqrt{-2x}}\cosh\left(\eta\sqrt{-2(x+it)}\right) - xe^{\kappa\sqrt{-2x}}\sinh\left(\eta\sqrt{-2(x+it)}\right)\\
&- it\cosh\left(\kappa\sqrt{-2x}\right)\sinh\left(\eta\sqrt{-2(x+it)}\right)\\
&- y\left(\sqrt{-x}\sinh\left(\kappa\sqrt{-2x}\right)\sinh\left(\eta\sqrt{-2(x+it)}\right)\right.\\
&\left.+\sqrt{-x-it}\cosh\left(\kappa\sqrt{-2x}\right)\cosh\left(\eta\sqrt{-2(x+it)}\right)\right)\Big]^{-1}
\end{aligned}
$$

This theorem in conjunction with (3.15) and the relations

$$\sinh\left(\eta\sqrt{2}\sqrt{-x-\frac{it}{\eta}}\right) \sim \eta\sqrt{2}\sqrt{-x-\frac{it}{\eta}}, \quad \eta\to 0,$$

and $$\cosh\left(\eta\sqrt{2}\sqrt{-x-\frac{it}{\eta}}\right) \sim 1, \quad \eta\to 0.$$

immediately yields (2.3) after performing the substitutions $t \to t/2$ and $\kappa \to \kappa/2$.

4. Tightness

In this section we will show that the sequence of random variables $l_n(t) = n^{-1/2}L_n(t\sqrt{n})$, $t \ge 0$, is tight in $C[0,\infty)$. Since a sequence of stochastic processes $X_n(t)$, $t \ge 0$, is tight in $C[0,\infty)$ if and only if $X_n(t)$, $0 \le t \le T$, is tight in $C[0,T]$ for all $T > 0$ (see [16, p. 63]) we may restrict ourselves to finite intervals, i.e. it suffices to consider $L_n(t)$, $0 \le t \le A\sqrt{n}$, where $A > 0$ is an arbitrary real constant.

By [6, Theorem 12.3] tightness of $l_n(t)$, $0 \le t \le A$, follows from tightness of $l_n(0)$ (which is obvioulsy satisfied) and from an estimate of the form

$$\mathbf{P}\left\{|L_n(\rho\sqrt{n}) - L_n((\rho+\eta)\sqrt{n})| \ge \varepsilon\sqrt{n}\right\} \le C\frac{\eta^\alpha}{\varepsilon^\beta} \tag{4.1}$$

for some $\alpha > 1$, $\beta \ge 0$, and $C > 0$ uniformly for $0 \le \rho \le \rho+\eta \le A$. We will derive (4.1) from the following property:

Theorem 4.1. *There exists a constant $C > 0$ such that*

$$\mathbf{E}\left(L_n(r) - L_n(r+h)\right)^4 \le C\,h^2 n \tag{4.2}$$

holds for all non-negative integers n, r, h.

Obviously Theorem 4.1 proves (4.1) for $\alpha = 2$ and $\beta = 4$ if $\rho\sqrt{n}$ and $\eta\sqrt{n}$ are non-negative integers. However, in the case of linear interpolation it is an easy exercise (see [13] or [11]) to extend (4.1) to arbitrary $\rho, \eta \ge 0$ (probably with a different constant C).

It remains to prove Theorem 4.1. Since the coefficient

$$b_{nkl,rh} = n![z^n u^k v^l]\frac{1}{1 - y_r(z, uy_h(z, va(z)))}$$

is the number of random mappings of size n with k nodes in layer r and l nodes in layer $r+h$, i.e.

$$\mathbf{P}\left\{L_n(r) = k, L_n(r+h) = l\right\} = \frac{b_{nkl,rh}}{n^n},$$

we obtain

$$\mathbf{P}\left\{L_n(r) - L_n(r+h) = m\right\} = \frac{n!}{n^n}[z^n u^m]\frac{1}{1 - y_r(z, uy_h(z, u^{-1}a(z)))}$$

and consequently

$$\mathbf{E}\left(L_n(r) - L_n(r+h)\right)^4 = \frac{n!}{n^n}[z^n]H_{rh}(z), \tag{4.3}$$

in which

$$\begin{aligned}
H_{rh}(z) &= \left(\frac{\partial}{\partial u} + 7\frac{\partial^2}{\partial u^2} + 6\frac{\partial^3}{\partial u^3} + \frac{\partial^4}{\partial u^4}\right)\frac{1}{1 - y_r(z, uy_h(z, u^{-1}a(z)))}\bigg|_{u=1} \\
&= \frac{1}{(1-a(z))^2}\left(h_{1,rh}(z) + 7h_{2,rh}(z) + 6h_{3,rh}(z) + h_{4,rh}(z)\right) \\
&+ \frac{2}{(1-a(z))^3}\left(7h_{1,rh}(z)^2 + 18h_{1,rh}(z)h_{2,rh}(z) + 3h_{2,rh}(z)^2 + 4h_{1,rh}(z)h_{3,rh}(z)\right) \\
&+ \frac{36}{(1-a(z))^4}\left(h_{1,rh}(z)^3 + h_{1,rh}(z)^2h_{2,rh}(z)\right) \\
&+ \frac{24}{(1-a(z))^5}h_{1,rh}(z)^4,
\end{aligned}$$

and

$$h_{j,rh}(z) = \frac{\partial^j}{\partial u^j}y_r(z, uy_h(z, u^{-1}a(z)))\bigg|_{u=1}, \qquad (1 \le j \le 4).$$

In [9] these functions have been calculated (in a little bit more general setting) in terms of $a(z)$.

Lemma 4.1. *Set $a = a(z)$. Then we have*

$$\begin{aligned}
h_{1,rh}(z) &= a^{r+1}(1-a^h), \\
h_{2,rh}(z) &= a^{r+2}\frac{1-a^r}{1-a}(1-a^h)^2 + a^{r+h+2}\frac{1-a^h}{1-a}, \\
h_{3,rh}(z) &= a^{r+3}\frac{1-a^{2r}}{1-a^2} + 3a^{r+4}\frac{(1-a^r)(1-a^{r-1})}{(1-a)(1-a^2)}(1-a^h)^3 \\
&\quad + 3a^{r+h+3}\frac{(1-a^r)(1-a^h)^2}{(1-a)^2} - 3a^{r+h+2}\frac{1-a^h}{1-a} \\
&\quad - a^{r+3}\left(a^h\frac{1-a^{2h}}{1-a^2} + 3a^{h+1}\frac{(1-a^h)(1-a^{h-1})}{(1-a)(1-a^2)}\right),
\end{aligned}$$

and

$$
\begin{aligned}
h_{4,rh}(z) = a^{r+4}\Bigg(&\frac{1-a^{3r}}{1-a^3} + a^2(7+10a+10^r+6a^{r+1})\frac{(1-a^r)(1-a^{r-1})}{(1-a^2)(1-a^3)} \\
&+ 3(1+5a)a^3\frac{(1-a^r)(1-a^{r-1})(1-a^{r-2})}{(1-a)(1-a^2)(1-a^3)}\Bigg)(1-a^h)^4 \\
&+ 7a^{r+4}\left(\frac{1-a^{2r}}{1-a^2} + 3a\frac{(1-a^r)(1-a^{r-1})}{(1-a)(1-a^2)}\right)a^h\frac{(1-a^h)^3}{1-a} \\
&- 12a^{r+h+4}\frac{(1-a^r)(1-a^h)^2}{(1-a)^2} + 3a^{r+2h+4}\frac{(1-a^r)(1-a^h)^2}{(1-a)^3} \\
&- 4a^{r+4}\frac{1-a^r}{1-a}\left(a^h\frac{(1-a^h)(1-a^{2h})}{1-a^2} + 3a^{h+1}\frac{(1-a^h)^2(1-a^{h-1})}{(1-a)(1-a^2)}\right) \\
&+ 12a^{r+h+2}\frac{1-a^h}{1-a} + 8a^{r+h+3}\left(\frac{1-a^{2h}}{1-a^2} + 3a\frac{(1-a^h)(1-a^{h-1})}{(1-a)(1-a^2)}\right) \\
&+ a^{r+h+4}\Bigg(\frac{1-a^{3h}}{1-a^3} + a^2(7+10a+10a^h+6a^{h+1})\frac{(1-a^h)(1-a^{h-1})}{(1-a^2)(1-a^3)} \\
&\qquad + 3a^3(1+5a)\frac{(1-a^h)(1-a^{h-1})(1-a^{h-2})}{(1-a)(1-a^2)(1-a^3)}\Bigg),
\end{aligned}
$$

In a final step we will estimate the coefficients of $H_{rh}(z)$. Since $n!n^{-n} \sim e^{-n}\sqrt{2\pi n}$ Theorem 4.1 is equivalent to

$$[z^n]H_{rh}(z) = \mathcal{O}\left(e^n h^2\sqrt{n}\right) \qquad \text{uniformly for all } r,h \geq 0. \tag{4.4}$$

Essentially (4.4) follows from a lemma from singularity analysis [10]:

Lemma 4.2. *Let $F(z)$ be analytic in a region*

$$\Delta = \{z : |z| < z_0 + \eta,\ |\arg(z - z_0)| > \vartheta\},$$

in which z_0 and η are positive real numbers and $0 < \vartheta < \pi/2$. Furthermore suppose that there exists a real number $\alpha \notin \{0, -1, -2, \dots\}$ such that

$$F(z) = \mathcal{O}\left((1 - z/z_0)^{-\alpha}\right) \qquad (z \in \Delta).$$

Then

$$[z^n]F(z) = \mathcal{O}\left(z_0^{-n} n^{\alpha-1}\right).$$

Corollary. *Suppose that $G(z)$ is a bounded analytic function in Δ. Then*

$$[z^n]G(z)\frac{a(z)^k}{(1-a(z))^3} = \mathcal{O}\left(e^n n^{1/2}\right)$$

uniformly for all $k \geq 0$.

Proof. (3.9) implies

$$\sup_{z\in\Delta} |a(z)| = 1 \tag{4.5}$$

and

$$\frac{1}{(1-a(z))^3} = \mathcal{O}\left((1-ez)^{-3/2}\right).$$

Hence we can apply Lemma 4.2 with $\alpha = 3/2$. ∎

By Lemma 4.1 and (4.5) it follows that $H_{rh}(z)$ can be represented as

$$H_{rh}(z) = G_{1,rh}(z)\frac{(1-a(z)^h)^2}{(1-a(z))^5} + G_{2,rh}(z)\frac{1-a(z)^h}{(1-a(z))^4} + G_{3,rh}(z)\frac{1}{(1-a(z))^3},$$

in which $G_{j,rh}(z)$, $1 \le j \le 3$, are uniformly bounded in Δ. Note that $H_{r0}(z) \equiv 0$. So we may assume that $h \ge 1$. The coefficient of the first term of $H_{rh}(z)$ can be estimated by

$$\begin{aligned}[z^n]G_{1,rh}(z)\frac{(1-a(z)^h)^2}{(1-a(z))^5} &= [z^n]G_{1,rh}(z)\frac{1}{(1-a(z))^3}\sum_{i=0}^{h-1}a(z)^i\sum_{j=0}^{h-1}a(z)^j \\ &= \sum_{i,j=0}^{h-1}G_{1,rh}(z)\frac{a(z)^{i+j}}{(1-a(z))^3} \\ &= \mathcal{O}\left(e^n h^2 n^{1/2}\right).\end{aligned}$$

The coefficient of the second term is even smaller:

$$\begin{aligned}[z^n]G_{2,rh}(z)\frac{(1-a(z)^h)}{(1-a(z))^4} &= [z^n]G_{2,rh}(z)\frac{1}{(1-a(z))^3}\sum_{i=0}^{h-1}a(z)^i \\ &= \sum_{i=0}^{h-1}G_{2,rh}(z)\frac{a(z)^i}{(1-a(z))^3} \\ &= \mathcal{O}\left(e^n h n^{1/2}\right) = \mathcal{O}\left(e^n h^2 n^{1/2}\right).\end{aligned}$$

Similarly we can treat the remaining term

$$[z^n]G_{3,rh}(z)\frac{1}{(1-a(z))^3} = \mathcal{O}\left(e^n n^{1/2}\right) = \mathcal{O}\left(e^n h^2 n^{1/2}\right).$$

Thus we have proved (4.4) which is equivalent to (4.2). This completes the proof of tightness of the sequence $l_n(t)$ and consequently the proof of Theorem 1.1.

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