A Non-Hyperarithmetical Gödel Logic*

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Abstract. Let $\mathsf{G}_\downarrow$ be the Gödel logic whose set of truth values is $V_\downarrow = \{0\} \cup \{1/n : n \in \mathbb{N} \setminus \{0\}\}$. Baaz-Leitsch-Zach have shown that $\mathsf{G}_\downarrow$ is not recursively axiomatizable and Hájek showed that it is not arithmetical. We find the optimal strengthening of their theorems and prove that the set of validities of $\mathsf{G}_\downarrow$ is Π^1_1 complete and the set of satisfiable formulas in $\mathsf{G}_\downarrow$ is Σ^1_1 complete.

Keywords: Gödel logic, Fuzzy logic, Hyperarithmetical set

1 Introduction

A family of finite-valued logics was introduced by Gödel in [6] to show there are propositional logics weaker than classical but stronger than intuitionistic propositional logic. A natural extension of those logics to many-valued logic followed in the paper of Dummet [5] who also showed that they can be axiomatised by adding the axiom $(p \to q) \vee (q \to p)$ into intuitionistic logic. Today we call those logics Gödel logics. In particular, Gödel logics are intermediate logics where propositions take truth values in $[0, 1]$. Different Gödel logics arise by choosing a subset $V \subseteq [0, 1]$ as truth values. In the case of propositional Gödel logic, any infinite subset of $[0, 1]$ will yield the same set of valid formulas, but this is not the case for first order Gödel logic. In this case we require that V be a closed set, as suprema and infima are used to evaluate the quantifiers.

In particular, we are interested in $\mathsf{G}_\downarrow$, the Gödel logic whose set of truth values is

$$V_\downarrow = \{0\} \cup \left\{1/n : n \in \mathbb{N} \setminus \{0\}\right\}.$$

$\mathsf{G}_\downarrow$ is the same as the logic defined by *linearly ordered Kripke structures on constant domains* [1] - a fundamental concept in the definition of *Temporal logic of programs* [9], an origin of the study of program verification.

Baaz-Leitsch-Zach [1] have shown that $\mathsf{G}_\downarrow$ is not recursively axiomatizable and Hájek [8] showed that the sets of validities and satisfiable formulas are not arithmetical. We will show that they are Σ^1_1-complete and Π^1_1-complete,

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respectively. As satisfiability is Σ^1_1 and validity is Π^1_1, this yields the optimal strengthening of their theorems. We remark that each of the results for Σ^1_1 and Π^1_1 is not immediate from the other, because satisfiability and validity are not dual in Gödel logic as they are in classical logic.

2 Preliminaries

First order Gödel logic uses the syntax of intuitionistic predicate logic, where the set of formulas is defined according to the following clauses:

$$\bot \quad | \quad P(\mathbf{x}) \quad | \quad \phi \wedge \psi \quad | \quad \phi \vee \psi \quad | \quad \phi \to \psi \quad | \quad \exists x \phi \quad | \quad \forall x \phi.$$

Here, P is a predicate symbol of arity n in some predetermined alphabet, $\mathbf{x}$ is a tuple of n variables, ϕ, ψ are formulas and x is a variable.

Definition 1. *Fix a closed set $V \subseteq [0,1]$ with $0 \in V$; such a set is a* set of truth values. *A V-valued model is a pair $\mathfrak{M} = (D, \|\cdot\|)$, where D is a set of elements and $\|\cdot\|$ assigns to each n-ary predicate symbol P a function $\|P(\cdot)\|\colon D^n \to V$. We then extend $\|\cdot\|$ to complex formulas according to the following clauses:*

- $\|\bot(\mathbf{a})\| = 0$
- $\|\phi \wedge \psi(\mathbf{a})\| = \min\{\|\phi(\mathbf{a})\|, \|\psi(\mathbf{a})\|\}$
- $\|\phi \vee \psi(\mathbf{a})\| = \max\{\|\phi(\mathbf{a})\|, \|\psi(\mathbf{a})\|\}$
- $\|\phi \to \psi(\mathbf{a})\| = \begin{cases} 1 & \textit{if } \|\phi(\mathbf{a})\| \leq \|\psi(\mathbf{a})\| \\ \|\psi(\mathbf{a})\| & \textit{otherwise} \end{cases}$
- $\|\exists x \phi(x, \mathbf{a})\| = \sup_{b \in D} \|\phi(b, \mathbf{a})\|$
- $\|\forall x \phi(x, \mathbf{a})\| = \inf_{b \in D} \|\phi(b, \mathbf{a})\|$.

On occasion we may write $\|\cdot\|_{\mathfrak{M}}$ instead of $\|\cdot\|$ when we want to specify the model we are referring to. We write $\mathfrak{M} = (D, P_1, \ldots, P_n)$ instead of $\mathfrak{M} = (D, \|\cdot\|)$ to indicate that the alphabet of $\mathfrak{M}$ is $P_1, \ldots, P_n$. Given a closed set $V \subseteq [0,1]$ containing 0 and 1, we say that a sentence ϕ is *V-satisfiable* if there is a model $\mathfrak{M}$ such that $\|\phi\|_{\mathfrak{M}} = 1$ (in which case we write $\mathfrak{M} \models \phi$), and *weakly V-satisfiable* if there is a model $\mathfrak{M}$ such that $\|\phi\|_{\mathfrak{M}} > 0$. The formula ϕ is *V-valid* if for every model $\mathfrak{M}$, $\|\phi\|_{\mathfrak{M}} = 1$. A model $\mathfrak{M}$ is *crisp* if $V = \{0,1\}$; clearly, crisp models are equivalent to classical models. In the remainder of the text we fix $V = V_\downarrow = \{0\} \cup \{1/n : n \in \mathbb{N} \setminus \{0\}\}$, and *satisfiability*, etc. refer to this set of truth values. We will explicitly write e.g. *classical satisfiability* when referring to $V = \{0,1\}$.

A formula is $V_\downarrow$-satisfiable iff it is weakly $V_\downarrow$-satisfiable. In fact, a more general claim holds. Recall that a linear order is *Noetherian* if it contains no infinite strictly increasing sequences; note that $V_\downarrow$ is Noetherian.

Lemma 1. *Let V be a Noetherian set of truth values. Given a sentence φ and any set of truth values V, if there exists a model $\mathfrak{M}$ such that $\|\varphi\|_{\mathfrak{M}} > 0$, then there exists a model $\mathfrak{M}'$ such that for all formulas ψ and tuples $\mathbf{a}$,*

$$\|\psi(\mathbf{a})\|_{\mathfrak{M}'} = \begin{cases} \|\psi(\mathbf{a})\|_{\mathfrak{M}} & \textit{if } \|\psi(\mathbf{a})\|_{\mathfrak{M}} < \|\varphi\|_{\mathfrak{M}} \\ 1 & \textit{otherwise.} \end{cases} \tag{1}$$

Proof. Let $\mathfrak{M} = (D, \|\cdot\|)$ and define $\mathfrak{M}' = (D, \|\cdot\|')$ so that $\|\cdot\|$ is defined according to (1) for atomic formulas. Then proceed by induction on formula complexity. The assumption that V is Noetherian is used for the case of an existential quantifier, so we focus on this one. We have that $\|\exists x\psi(x, \mathbf{a})\|' = \sup_{b\in D} \|\psi(b, \mathbf{a})\|'$. By the induction hypothesis, $\|\psi(b, \mathbf{a})\|'$ satisfies (1) for all $b \in D$. Since V is Noetherian, there is $b_* \in D$ such that $\|\psi(b_*, \mathbf{a})\|' = \sup_{b\in D} \|\psi(b, \mathbf{a})\|'$. If $\|\psi(b_*, \mathbf{a})\| < \|\varphi\|$ then it is readily checked that for all $b \in D$, $\|\psi(b, \mathbf{a})\|' = \|\psi(b, \mathbf{a})\| < \|\varphi\|$, so $\|\exists x\psi(x, \mathbf{a})\| = \|\psi(b_*, \mathbf{a})\| = \|\psi(b_*, \mathbf{a})\|' = \|\exists x\psi(x, \mathbf{a})\|'$. Otherwise, $\|\psi(b_*, \mathbf{a})\| \geq \|\varphi\|$, so that $\|\exists x\psi(x, \mathbf{a})\| \geq \|\varphi\|$ and we must check that $\|\exists x\psi(x, \mathbf{a})\|' = 1$. But from $\|\psi(b_*, \mathbf{a})\| \geq \|\varphi\|$ we obtain $\|\psi(b_*, \mathbf{a})\|' = 1$ and thus $\|\exists x\psi(x, \mathbf{a})\|' = 1$, as needed.

We will make use of the abbreviation

$$\phi \prec \psi := (\psi \to \phi) \to \psi.$$

It can be checked by the semantics that $\phi \prec \psi$ evaluates to 1 if and only if either ϕ has a smaller value than ψ, or they both have value 1.

While first order logic in principle contains predicate symbols of all arities, in this paper it will suffice to work with a unary symbol N and binary symbols $=, \in$. The convention regarding equality is that it is interpreted as any fuzzy equivalence relation when $V \neq \{0, 1\}$, but is true equality when $V = \{0, 1\}$. The symbol $\in$ is meant to interpret Kripke-Platek set theory (KP), a weakening of ZFC which, in the version we consider, has the advantage of being finitely axiomatizable. In general, let $\mathcal{L}_{P_1 \dots P_n}$ be the language of first order Gödel logic whose predicate symbols are restricted to $P_1, \dots, P_n$. To define KP, first say that a Δ_0 formula is a $\mathcal{L}_\in$ formula such that all quantifiers are of the form $\exists x(x \in y \wedge \phi)$ or $\forall x(x \in y \to \phi)$ (often abbreviated as e.g. $\exists x \in y\phi$ and $\forall x(x \in y \to \phi)$, respectively). A Σ_1 formula is one of the form $\exists x\phi$, where ϕ is Δ_0, and a Π_1 formula is one of the form $\forall x\phi$ with the same restriction.

We will use the version of KP with infinity axiomatized by all axioms of ZFC except for powerset, but with foundation restricted to Π_1 classes, separation restricted to Δ_0 formulas and replacement restricted to Δ_0-collection. We will also assume that KP contains equality axioms asserting that $=$ is an equivalence relation, as well as the axioms

$$x = y \to x \in z \leftrightarrow y \in z$$
$$x = y \to z \in x \leftrightarrow z \in y.$$

The precise definitions are not needed to follow the text, but we will use some properties of this version of KP, including that it is finitely axiomatizable.

The axiom of infinity asserts that the set of natural numbers exist as a set, and there is a formula of $\mathcal{L}_\in$ (which we denote by $x \in \mathbb{N}$) defining the set of natural numbers as von Neumann ordinals, which have the property that $n < m$ iff $n \in m$. We will usually write $<$ instead of $\in$ when working with natural numbers within KP. This definition allows us to quantify over the set of natural numbers and define quantifiers $\forall x \in \mathbb{N}$, $\exists X \subseteq \mathbb{N}$, etc. as abbreviations.

We will also use the fact that addition (along with other standard arithmetical operations) is readily interpretable in KP. An *arithmetical formula* is one where all quantifiers are of the form $\forall x \in \mathbb{N}$ or $\exists x \in \mathbb{N}$, a Π^1_1*-formula* is one of the form $\forall X \subseteq \mathbb{N}\phi$ where ϕ is arithmetical, and a Σ^1_1*-formula* is one of the form $\exists X \subseteq \mathbb{N}\phi$, also with ϕ arithmetical. A model $\mathfrak{M}$ is an ω-model if the natural numbers in $\mathfrak{M}$ are isomorphic to the standard natural numbers.

As for the symbol N, its intended meaning is that $\|N(x)\| > 0$ iff $x \in \mathbb{N}$, with larger natural numbers receiving smaller truth values. This will be made precise later in the text.

Some familiarity with the class of ordinals, as well as the constructible hierarchy $\{\mathbb{L}_\alpha \mid \alpha \text{ is an ordinal}\}$ is assumed. An ordinal α is *admissible* if $\mathbb{L}_\alpha$ is a (classical) model of KP; because every recursive ordinal is provably well-ordered in KP, the smallest admissible ordinal is the Church-Kleene ordinal ω_1^{CK}. Note that ω_1^{CK} is countable.

We will use the following two results involving admissible sets.

Theorem 1 (Ville [3]). *Let $\mathfrak{M}$ be any ω-model of KP. Then, the well-founded part of $\mathfrak{M}$ (with respect to $\in$) is admissible, and hence extends $\mathbb{L}_{\omega_1^{\mathrm{CK}}}$.*

Theorem 2 (Barwise-Gandy-Moschovakis [4]). *Given a Σ^1_1 formula ϕ, one can effectively and uniformly find a Π_1 $\mathcal{L}_\in$-formula $\psi(x)$ such that for every natural number n*

$$\mathbb{N} \models \phi(n) \leftrightarrow \mathbb{L}_{\omega_1^{\mathrm{CK}}} \models \psi(n).$$

3 Standard models via vagueness

Our proof of hardness follows by a variation of Hájek's proof. The high-level idea of Hájek's proof is to use the set of truth values to define an interpretation of the standard natural numbers. In our argument, we use the set of truth values to define an interpretation of the standard natural numbers in models of KP and then apply the theorem of Ville and of Barwise-Gandy-Moschovakis to them.

Recall that we are using a finitely axiomatizable presentation of KP, that $\mathbb{N}$ is definable in KP and that we write $<$ instead of $\in$ for natural numbers. Recall also that we are working with a monadic predicate N whose intended meaning is that $\|N(x)\| > 0$ iff $x \in \mathbb{N}$, and that we defined $\phi \prec \psi := (\psi \to \phi) \to \psi$. With this in mind, let Ψ be the sentence asserting the conjunction of the following statements:

(i) $=$ and $\in$ are crisp, i.e., they satisfy excluded middle;
(ii) KP holds of the predicate $\in$;
(iii) $\forall x, y\,(x = y \to (N(x) \leftrightarrow N(y)))$;
(iv) $\forall x, y \in \mathbb{N}\,(x < y \to (N(y) \prec N(x)))$;
(v) $\forall x\,\neg\neg N(x) \to x \in \mathbb{N}$;
(vi) $\neg\exists x \in \mathbb{N}\,\neg N(x)$;
(vii) $\neg\forall x \in \mathbb{N}\,N(x)$.

Let $\mathfrak{M} = (D, =, \in, N)$ be any model of Ψ. By (i), $\in$ is crisp, so for each $x \in D$, the formula $x \in \mathbb{N}$ has value 0 or 1. Formula (vi) asserts that whenever x is a natural number, $N(x)$ has a positive truth value. Conversely, formula (v) asserts that whenever $N(x)$ has positive truth value, then x is a natural number. Formula (vii) asserts that the infimum of the truth values of $N(x)$ is 0 as x ranges over the natural numbers. The intuition may be grasped easily through the following concrete construction.

Lemma 2. *Any model of Ψ satisfies the equality schema*

$$x = y \to \varphi(x) \leftrightarrow \varphi(y)$$

for all formulas φ in the vocabulary $\{=, \in, N\}$.

Proof. For atomic formulas involving the relations $\in$ and $=$ the result follows directly from the axioms of KP. For N, it follows from (iii). The general case follows by a straightforward induction.

Lemma 3. *Any ω-model of KP can be extended to a model of Ψ.*

Proof. Let $\mathfrak{M} = (D, =, \in)$ be an ω-model of KP. For $n \in \mathbb{N}$, define $\|N(n)\| = {}^1/_{n+1}$. For $x \notin \mathbb{N}$, define $\|N(x)\| = 0$. The model $\mathfrak{M} = (D, =, \in, N)$ thus defined satisfies (i) and (ii) since the interpretation of $=, \in$ did not change, and (iii)-(vii) are readily checked to hold using the definition of N.

The key property of Ψ is that *only* ω-models of KP can be extended to models of Ψ.

Lemma 4. *If $\mathfrak{M} = (D, =, \in, N)$ is such that $\mathfrak{M} \models \Psi$ and $=$ is identity in D, then $(D, =, \in)$ is an ω-model of KP.*

Proof. Fix a model $\mathfrak{M} = (D, \|\cdot\|)$ over the signature $\{=, \in, N\}$. Noting that $=, \in$ (and hence $<$) are crisp by (i), we may reason classically about these relations. First note that by (vi), for every $a \in \mathbb{N}$ we have that $N(a) > 0$.

Claim. If $a < b$ are such that $\|N(a)\| < 1$, then $\|N(b)\| < \|N(a)\|$.

Proof of the Claim: By (iv) we have that $\|N(b) \prec N(a)\| = 1$. By the truth conditions of $\prec$ we have that either $\|N(b)\| = \|N(a)\| = 1$ or else $\|N(b)\| < \|N(a)\|$. As we do not have that $\|N(a)\| = 1$ by assumption, we conclude that $\|N(b)\| < \|N(a)\|$ as needed. This establishes the Claim.

By (vii), there is an $a_0 \in D$ with $\|N(a_0)\| < 1$. We claim that if $a \in D$ is such that $a_0 < a \wedge a \in \mathbb{N}$, then a is standard, in the sense that $\{b \in D : b < a\}$ is finite. This will conclude the proof, since $\mathfrak{M} \models \forall x \in \mathbb{N}(x \leq a_0 \vee a_0 < x)$, so then every natural number is standard.

So fix $a > a_0$. By the Claim, $\|N(a)\| < \|N(a_0)\|$. Now, let $c < d \in \mathbb{N}$. By the Claim once again, $\|N(a + d)\| < \|N(a + c)\|$. It follows that the sequence

$(\|N(a+n)\|)_{n\in\mathbb{N}}$ is strictly decreasing, hence its infimum is zero (as this is the case for any strictly decreasing sequence in $V_\downarrow$). Now, if a were non-standard, we would have that $\|N(a+a)\| < \|N(a+n)\|$ for all $n \in \mathbb{N}$, hence $\|N(a+a)\| = 0$, which contradicts (vi). We conclude that a is indeed standard, and hence $\mathfrak{M}$ is an ω-model of KP.

4 Satisfiability in $\mathsf{G}_\downarrow$

Lemma 4 suffices to establish our hardness results. We begin with satisfiability; recall that by this we mean *$V_\downarrow$-satisfiability.* In view of Lemma 1, satisfiability can be replaced by weak satisfiability in the theorem below.

Theorem 3. *The set of all (weakly) $V_\downarrow$-satisfiable formulas is Σ^1_1-complete.*

Proof. First, a formula ϕ is $V_\downarrow$-satisfiable if and only if it has a model. By downwards Löwenheim-Skolem (see e.g., Baaz et al. [2]), this is equivalent to it having a countable model. Hence, ϕ is satisfiable if and only if there is a subset of $\mathbb{N}$ coding a model of ϕ. This is clearly Σ^1_1.

Now, fix a Σ^1_1 formula $\phi(x)$. We find a many-one reduction of $\{n : \mathbb{N} \models \phi(n)\}$ to the set of satisfiable formulas of $\mathsf{G}_\downarrow$. By Lemma 2, one can effectively and uniformly find a Π_1 $\mathcal{L}_\in$-formula $\psi(x)$ such that for every natural number n

$$\mathbb{N} \models \phi(n) \leftrightarrow \mathbb{L}_{\omega_1^{\text{CK}}} \models \psi(n).$$

We will show that for every standard natural number n, $\mathbb{N} \models \phi(n)$ if and only if $\Psi \wedge \psi(n)$ is $\mathsf{G}_\downarrow$-satisfiable. First, suppose that $\mathbb{N} \models \phi(n)$. Then, $\mathbb{L}_{\omega_1^{\text{CK}}} \models \psi(n)$. By Lemma 3, $\mathbb{L}_{\omega_1^{\text{CK}}}$ can be extended to a model $\mathfrak{M}$ of Ψ, and since $\psi(n)$ does not contain the symbol N, we have that $\mathfrak{M} \models \psi(n)$ as well.

Conversely, let $\mathfrak{M} = (D, \in, N)$ be a model of $\Psi \wedge \psi(n)$. By Lemma 3, $(D, =, \in)$ is an ω-model. By Theorem 1, the well-founded part of $\mathfrak{M}$ extends $\mathbb{L}_{\omega_1^{\text{CK}}}$. Since $\mathfrak{M}$ is a model of $\Psi \wedge \psi(n)$, we have that $\mathfrak{M} \models \psi(n)$. Since ψ is Π_1, we have $\mathbb{L}_{\omega_1^{\text{CK}}} \models \psi(n)$, as desired. This completes the proof.

5 Validity in $\mathsf{G}_\downarrow$

Finally, we show that validity is Π^1_1-complete.

Theorem 4. *The set of all $V_\downarrow$-valid formulas is Π^1_1-complete.*

Proof. Note that a formula is valid in $\mathsf{G}_\downarrow$ if and only if it holds in every model and this is equivalent to holding in every countable model, which is clearly Π^1_1. Thus the only problem will be to show the hardness. For this we use the complementary statement of Theorem 2, that is for any Π^1_1-formula $\phi(x)$ there is a Σ_1-formula $\psi(x)$ in the language of set theory such that for every natural number n

$$\mathbb{N} \models \phi(n) \leftrightarrow \mathbb{L}_{\omega_1^{\text{CK}}} \models \psi(n).$$

We claim that $\mathbb{N} \models \phi(n)$ iff $\Psi \to \psi(n)$ is $V_\downarrow$-valid. For the easy direction assume $\Psi \to \psi(n)$ is $V_\downarrow$-valid. In that case extend $\mathbb{L}_{\omega_1^{\mathrm{CK}}}$ to a model of Ψ using Lemma 3 and call the resulting $V_\downarrow$-model $\mathfrak{M}$. By the assumption $\|\psi(n)\|_{\mathfrak{M}} = 1$, but since $\psi(n)$ does not contain the symbol N, it follows that $\mathbb{L}_{\omega_1^{\mathrm{CK}}} \models \psi(n)$, which gives $\mathbb{N} \models \phi(n)$.

For the other direction, assume that $\mathbb{N} \models \phi(n)$; we claim that $\Psi \to \psi(n)$ is $V_\downarrow$-valid. If it were not, by Lemma 1 there would be a model $\mathfrak{M} = (D, \|\cdot\|)$ with $\|\Psi\| = 1$ and $\|\psi(n)\| < \|\Psi\|$. We construct the model $\mathfrak{M}/E = (D/E, \|\cdot\|_E)$ by factorising D modulo the relation $E = \{(a,b) : \|a = b\| = 1\}$. In more details, let the universe of the new model be the set of all equivalence classes of E. We denote by $[a]$ the E-equivalence class of a and for any a, b we set $\|[a] \in [b]\|_E := \|a \in b\|$. To interpret N we fix for every $[a]$ a unique value from $\{\|N(b)\| : b \in [a]\}$ and let $\|N([a])\|_E$ be that value. Note that these values do not depend on the choice of representatives in view of (iii) and Lemma 2.

Claim. $\mathfrak{M}/E \models \Psi$

Proof of the Claim: By (i) E is a congruence for the interpretation of $\in$, so the definition of the model is correct and the model is a model of KP. For the other clauses, we use the fact that the value of $N([a])$ does not depend on the choice of representatives, as well as the fact that $\mathfrak{M}$ is a model of Ψ. This proves the Claim.

Since $\mathfrak{M}/E \models \Psi$, the natural numbers in $\mathfrak{M}/E$ are standard by Lemma 4. Then by Theorem 1, the well-founded part of $\mathfrak{M}/E$ is admissible and hence contains $\mathbb{L}_{\omega_1^{\mathrm{CK}}}$. Moreover if $\mathbb{N} \models \phi(n)$ then $\mathbb{L}_{\omega_1^{\mathrm{CK}}} \models \psi(n)$ and so $\mathfrak{M}/E \models \psi(n)$ as $\psi(n)$ is Σ_1. Since $\psi(n)$ does not contain the symbol N we get $\mathfrak{M} \models \psi(n)$, which was to be shown. This completes the proof of the Theorem.

6 Concluding remarks

We have provided precise complexity bounds for $\mathsf{G}_\downarrow$, previously only known to be non-arithmetical. It is possible that similar results hold for other fuzzy logics. Hájek [7] showed that $\Pi\forall$SAT is non-arithmetical and Montagna [10] that $\Pi\forall$TAUT, BL$\forall$TAUT, and BL$\forall$SAT are non-arithmetical. We leave the question of whether these logics are Π^1_1/Σ^1_1 complete open.

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