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Powers of Models in Weak Arithmetics

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Introduction

According to [HP93] the first use of what is called today an ultrapower construction for constructing models of PA was in [Sko34]. To the best knowledge of the author the first use of this technique to construct a model witnessing a consistency result for an arithmetical theory was in [KK82]. The technique was used by Saul Kripke and Simon Kochen to construct a model of PA in which a certain combinatorial principle true in the Natural numbers fail. Another example from this period could be the characterisation of countable models of the theory consisting from all true Π_2-sentences given by Joram Hirschfeld in [Hir75]. More recently in [Eny07] Ali Enyat used a technique of iterated ultrapower constructions to give a proof of the conjecture of James H.Schmerls, Jan Krajíček used an ultrapower construction in [Kra98] to construct extensions of models of theories related to the Complexity theory and Michal Garlík used an ultrapower constructions in [Gar15] to show that three pairs of theories relevant for Complexity theory are not logically equivalent if a certain complexity-theoretic assumption holds.

In the first chapter of this thesis we will give a definition of (ultra)power construction and show some basic properties of those constructions that will be necessary for the latter chapters. The Chapters 2&3, 4 and 5 are independent of each other and a reader familiar with the first chapter should be able to understand them without knowing the other. In the Chapters 2 we give a generalisation of the Construction B of [Gar15] augment by a sort of density arguments similar to the density arguments of forcing from the Set theory. In the Chapter 3 we will show that the Construction B is a special case of the construction developed in the Chapter 2. The Chapter 4 is a variation on the result from [Hir75] expanded by some corollaries for $\widetilde{\mathrm{PV}}$ - the universal theory of $\mathbb{N}$ in the language $\mathcal{L}_{PV}$. Finally the Chapter 5 uses an ultrapower construction over a countable Herbrand saturated model of $\widetilde{\mathrm{PV}}$ give a stronger version of the result from a recent paper [KO17].

Preliminaries

In this chapter we establish some basic definitions, notational conventions and folklore results that will be used throughout the thesis.

Notational conventions Suppose A, B are non-empty sets. We will follow the usual notation and use the symbol $\bar{\cdot}$ to denote tuples. Moreover if A is a set and $\bar{a} = (a_0, a_1, \ldots, a_{r-1})$ a tuple of objects from A then we will abuse the notation and write $\bar{a} \in A$ in place of $\bar{a} \in A^r$ where A^r denotes the set of r-tuples of objects from A. We will write ${}^A B$ to denote the class of functions from A to B. For $f \in {}^A B$ we denote by $dom(f)$ the domain of f (i.e. in this case $dom(f) = A$) and by $rng(f)$the range of f. Moreover for $C \subseteq A$ we denote by $f \restriction C$ the restriction of f on C i.e. the function with the domain C that maps $c \in C$ to $f(c)$. Further if $\bar{a} = (a_0, a_1, \ldots, a_{r-1}) \in A$ then we will abuse the notation and write $f(\bar{a})$ for $(f(a_0), \ldots, f(a_{r-1}))$. Similarly if $F : A^r \to B$ and $\bar{g} = (g_0, g_1, \ldots, g_{r-1}) \in {}^A B$ then $F \circ \bar{g}$ denotes the function mapping $a \in A$ to $F(g_0(a), \ldots, g_{r-1}(a))$ and $\bar{g}(a)$ denotes the tuple $(g_0(a), \ldots, g_{r-1}(a))$. Finally we say that f *is a function on* A if $f : A^r \to A$ for some natural number r.

Filter, Ultrafilter, Maximal filter, Prime filter Let A be a non-empty set. We will say that a family B of subsets of A *has a finite intersection property* if for every $b_0, b_1, \ldots, b_{r-1} \in B$, $\bigcap_{i<r} b_i \neq \emptyset$. Then a family $\mathcal{V}$ of subsets of A is said to be a *filter on* A *(over* A*)* if it is non-empty, has a finite intersection property and for any $a, b \subseteq A$, $b \in \mathcal{V}$ whenever $a \in \mathcal{V}$ and $b \supseteq a$. Moreover for a filter $\mathcal{V}$ over A the following are equivalent:

(i) $\mathcal{V}$ is an *ultrafilter* i.e. for every $a \in A$: $a \in \mathcal{V}$ or $A - a \in \mathcal{V}$

(ii) $\mathcal{V}$ is *maximal* i.e. for every filter $\mathcal{V}'$ over A: if $\mathcal{V}' \supseteq \mathcal{V}$ then $\mathcal{V}' = \mathcal{V}$

(iii) $\mathcal{V}$ is *prime* i.e. for every $a, b \subseteq A$: if $a \cup b \in \mathcal{V}$ then $a \in \mathcal{V}$ or $b \in \mathcal{V}$.

We will also say that a filter $\mathcal{V}$ over A is *non-principal* if $\bigcap \mathcal{V} = \emptyset$. It is not hard to see that every non-empty family of subsets with finite intersection property can be extended into a filter. Finally it is a folklore fact that every non-empty filter over a (non-empty) set can be extended into an ultrafilter.

Model theory

We will assume a basic knowledge of first-order logic covering [Bar99, Chapter A.1]. to fix our mode of speech we give basic definitions and facts about model theory following [ZT12, Chapter 1].

We will work exclusively in the first-order logic and so any structure and language that appears in this thesis will be first-order. We will use Greek letters $\varphi, \psi, \theta, \chi$... to denote the first order formulae and we will write $\varphi(\bar{x})$ to indicate that all free variables of φ are among $\bar{x}$ and similarly $t(\bar{x})$ to indicate that all free variables of the term t are between $\bar{x}$.

Suppose $\mathcal{L}$ is a language. Then for theories T, S in $\mathcal{L}$ we will write $T+S$ to denote $T \cup S$ and $\mathrm{Thm}(T)$ to denote the set of all $\mathcal{L}$-formulae provable from T. Finally if Γ is a set of $\mathcal{L}$-formulae then we let $\forall\Gamma = \{\forall\bar{x}\varphi(\bar{x}) \mid \varphi(\bar{x}) \in \Gamma\}$ and we will say that Γ *is closed under boolean combinations* if for any $\varphi, \psi \in \Gamma$: $\varphi \wedge \psi, \varphi \vee \psi, \neg\varphi \in \Gamma$.

If $\mathbb{M}$ is a model in language $\mathcal{L}$ (or equivalently an $\mathcal{L}$-structure) then we do not notationally distinguish between $\mathbb{M}$ and its domain. For $A \subseteq \mathbb{M}$ we will also not notationally distinguish between $\mathbb{M}$ and its expansion into a language $\mathcal{L}(A)$ consisting from $\mathcal{L}$ and a constant symbol for every element of A with the natural interpretation in $\mathbb{M}$. For a symbol s of $\mathcal{L}$ we denote by $s^{\mathbb{M}}$ the interpretation of s in $\mathbb{M}$. Moreover if $t(x_0, x_1, \ldots, x_{k-1})$ is an $\mathcal{L}$-term and $(a_0, a_1, \ldots, a_{k-1}) \in \mathbb{M}^k$ then we will denote by $t^{\mathbb{M}}(a_0, a_1, \ldots, a_{k-1})$ the value of t in $\mathbb{M}$ under the assignment mapping a_i to x_i for $i < k$. In particular, $t^{\mathbb{M}}$ is a function from $\mathbb{M}^k$ into $\mathbb{M}$.

Types Let $\mathbb{M}$ be an $\mathcal{L}$-structure and $A \subseteq \mathbb{M}$. Following the usual mode of speech we will call the set $p(x_0, x_1, \ldots, x_{r-1})$ of $\mathcal{L}(A)$ formulae with free variables $x_0, x_1, \ldots, x_{r-1}$ an *r-type over A* if $p(\bar{x})$ is maximal and consistent set of $\mathcal{L}$(A)-formulae. Further we say that such $p(\bar{x})$ is finitely satisfiable in $\mathbb{M}$ if for any $\varphi_0(\bar{x}), \ldots, \varphi_{k-1}(\bar{x}) \in p(\bar{x})$, $\mathbb{M} \models \exists\bar{x} \bigwedge_{i<k} \varphi(\bar{x})$ and that $p(\bar{x})$ *is realised in* $\mathbb{M}$ if there is $\bar{m} \in \mathbb{M}$ such that $\mathbb{M} \models \varphi(\bar{m})$ for every $\varphi(\bar{x}) \in \mathbb{M}$ Finally if $\bar{m} \in \mathbb{M}$ then we will denote by $tp^{\mathbb{M}}_A(\bar{m})$ the set of all $\mathcal{L}(\mathcal{A})$-formulae $\varphi(\bar{x})$ with $\mathbb{M} \models \varphi(\bar{m})$.

Theories $\mathrm{Th}(\mathbb{M}, \mathcal{L}')$**,** $\mathrm{Th}_\Gamma(\mathbb{M})$ **and** $\mathrm{Th}_\forall(\mathbb{M})$**:** Let $\mathbb{M}$ be an $\mathcal{L}$-structure, $\mathcal{L}' \subseteq \mathcal{L}$, and Γ a set of $\mathcal{L}$-sentences. Then we define the following sets:

$$\mathrm{Th}(\mathbb{M}, \mathcal{L}') = \{\sigma \mid \sigma \text{ is an } \mathcal{L}'\text{-sentence and } \mathbb{M} \models \sigma\}$$

$$\mathrm{Th}_\Gamma(\mathbb{M}) = \{\sigma \mid \sigma \in \Gamma \text{ and } \mathbb{M} \models \sigma\}$$

$$\mathrm{Th}_\forall(\mathbb{M}, \mathcal{L}') = \{\forall\bar{x}\varphi(\bar{x}) \mid \varphi(\bar{x}) \text{ is an open } \mathcal{L}'\text{-formula and } \mathbb{M} \models \forall\bar{x}\varphi(\bar{x})\}.$$

Moreover if $\mathcal{L}' = \mathcal{L}$ then we will write $\mathrm{Th}(\mathbb{M})$ for $\mathrm{Th}(\mathbb{M}, \mathcal{L}')$ and $\mathrm{Th}_\forall(\mathbb{M})$ for $\mathrm{Th}_\forall(\mathbb{M}, \mathcal{L}')$. Finally we call $\mathrm{Th}_\forall(\mathbb{M})$ *the universal theory of* $\mathbb{M}$ *(in language* $\mathcal{L}$*)* and $\mathrm{Th}(\mathbb{M})$ *the theory of* $\mathbb{M}$ *(in language* $\mathcal{L}$*).*

Definable functions Let T be a theory in language $\mathcal{L}$ and $\varphi(\bar{x}, y)$ an $\mathcal{L}$-formula. We will say that $\varphi(\bar{x}, y)$ *defines a function in* T or that T *proves that* $\varphi(\bar{x}, y)$ *defines a function* if $T \vdash \forall\bar{x}\exists y\varphi(\bar{x}, y)$ and $T \vdash \forall\bar{x}\forall y_1, y_2(\varphi(\bar{x}, y_1) \wedge \varphi(\bar{x}, y_2) \to y_1 = y_2)$.

Suppose $\mathbb{M}$ is an $\mathcal{L}$-structure, $f(\bar{x})$ is a function on $\mathbb{M}$ and $\varphi(\bar{x}, y)$ is an $\mathcal{L}(\mathbb{M})$ formula. Then we say that f *is definable by* $\varphi(\bar{x}, y)$ *(in* $\mathbb{M}$*)* if for any $\bar{a}, b \in \mathbb{M}$, $f(\bar{a}) = b$ if and only if $\mathbb{M} \models \varphi(\bar{a}, b)$. We say that a function $f(\bar{x})$ is $\mathbb{M}$-*definable* if there is an $\mathcal{L}(\mathbb{M})$-formula $\varphi(\bar{x}, y)$ such that $f(\bar{x})$ is definable by $\varphi(\bar{x}, y)$.

Finally if $\mathbb{M} \models T$ and $f(\bar{x})$ is a function on $\mathbb{M}$ and Γ a set of $\mathcal{L}(\mathbb{M})$-formulae then we say that $f(\bar{x})$ is Γ-*definable in* T if there is $\varphi(\bar{x}, y) \in \Gamma$ such that $\varphi(\bar{x}, y)$ defines a function in T and $f(\bar{x})$ is definable by $\varphi(\bar{x}, y)$.

Embedding, Isomorphism, Substructure, Absoluteness Let $\mathbb{M}$, $\mathbb{N}$ be $\mathcal{L}$-structures and $h : \mathbb{M} \to \mathbb{N}$. We say that h is an *embedding (of* $\mathbb{M}$ *into* $\mathbb{N}$*)* if it is injective and

(i) for any constant symbol $c \in \mathcal{L}$: $h(c^{\mathbb{M}}) = c^{\mathbb{N}}$

(ii) for any function symbol $F(\bar{x}) \in \mathcal{L}$ and $\bar{m} \in \mathbb{M}$: $h(F^{\mathbb{M}}(\bar{m})) = F^{\mathbb{N}}(h(\bar{m}))$

(iii) for any relation symbol $R(\bar{x}) \in \mathcal{L}$ and $\bar{m} \in \mathbb{M}$: $\bar{m} \in R^{\mathbb{M}}$ if and only if $h(\bar{m}) \in R^{\mathbb{N}}$.

Moreover we say that h is an *isomorphism* if it is surjective and we say that $\mathbb{M}$ *is isomorphic to* $\mathbb{N}$ if there is an isomorphism between $\mathbb{M}$ and $\mathbb{N}$.

Finally we say that $\mathbb{M}$ is a *substructure of* $\mathbb{N}$, in symbols $\mathbb{M} \leq \mathbb{N}$, if $\mathbb{M} \subseteq \mathbb{N}$ and the identity function on $\mathbb{M}$ is an embedding of $\mathbb{M}$ into $\mathbb{N}$. We will also say that $\mathbb{M}$ is *the substructure of* $\mathbb{N}$ *defined by* $A \subseteq \mathbb{N}$ if the domain of $\mathbb{M}$ is A and $\mathbb{M} \leq \mathbb{N}$.

Finally for $\mathcal{L}$-structures $\mathbb{M}_1, \mathbb{M}_2$ with $\mathbb{M}_1 \leq \mathbb{M}_2$ and an $\mathcal{L}$-formula $\varphi(\bar{x})$ we say that $\varphi(\bar{x})$ *is absolute between* $\mathbb{M}_1$ *and* $\mathbb{M}_2$ if for any $\bar{p} \in \mathbb{M}_1$:

$$\mathbb{M}_1 \models \varphi(\bar{p}) \text{ if and only if } \mathbb{M}_2 \models \varphi(\bar{p}).$$

Observation 1. *Suppose* $\mathbb{M}_1, \mathbb{M}_2$ *with* $|\mathbb{M}_1| = |\mathbb{M}_2|$ *are* $\mathcal{L}$*-structures and for* $i \in \{1, 2\}$*,* $G_i \subseteq \mathbb{M}_i$ *is such that* $\mathbb{M}_i = \{t^{\mathbb{M}_i}(\bar{a}) \mid t(\bar{x})$ *is an* $\mathcal{L}$*-term and* $\bar{a} \in G_i\}$*. Assume further* $h : G_1 \to G_2$ *is a bijection such that for any open* $\mathcal{L}$*-formula* $\varphi(\bar{x})$ *and* $\bar{a} \in G_1$*,* $\mathbb{M}_1 \models \varphi(\bar{a})$ *if and only if* $\mathbb{M}_2 \models \varphi(h(\bar{a}))$*. Then* $\mathbb{M}_1$ *is isomorphic to* $\mathbb{M}_2$*.*

Proof. Observe that if $t(\bar{x}), s(\bar{y})$ are $\mathcal{L}$-terms and $\bar{a}$, $\bar{b} \in G_1$ such that $\mathbb{M}_1 \models t(\bar{a}) = s(\bar{b})$ then $\mathbb{M}_2 \models t(h(\bar{a})) = s(h(\bar{b}))$. Hence we can extend h into $h^* : \mathbb{M}_1 \to \mathbb{M}_2$ by mapping $m \in \mathbb{M}_1$ to $t^{\mathbb{M}_2}(h(\bar{a}))$ for some $\mathcal{L}$-term $t(\bar{x})$ and $\bar{a} \in G_1$ with $t^{\mathbb{M}_1}(\bar{a}) = m$. To show that h^* is an isomorphism of $\mathbb{M}_1$ and $\mathbb{M}_2$ follows easily from the assumption on h. □

Arithmetical theories

In this section we give some basic definitions and references regarding theories TA, PA, $\mathrm{I}\Sigma_i$, S^i_2 and $\widetilde{\mathrm{PV}}$. We will often use an informal expression *arithmetical theory* meaning that the theory has language extending the language of PA (see bellow), is consistent with True arithmetic (see bellow) and is strong enough to capture some "meaningful" part of mathematics.

True arithmetic and $\mathcal{L}_{all}$ Let $\mathbb{N}$ be the structure of Natural Numbers. We denote by $\mathcal{L}_{all}$ the language containing a function (relation) symbol for every function from ${}^{\mathbb{N}}\mathbb{N}$ (relation on $\mathbb{N}$). We will call *True Arithmetic* or TA for short the countable theory $\mathrm{Th}(\mathbb{N}, \mathcal{L}_{all})$. The reason of this very generous definition is that we will have access to all function/relation symbols we will need. The fact that $\mathcal{L}_{all}$ is uncountable will play no role in this thesis.

If $\mathbb{M}$ is an $\mathcal{L}$-structure for $\mathcal{L}$ with $\mathcal{L}_{\mathrm{PA}} \subseteq \mathcal{L} \subseteq \mathcal{L}_{all}$ and $k \in \mathbb{N}$ then we will use natural expressions like x^k to denote the term $x \cdot x \cdot \ldots \cdot x$ (k-times) or k for the term $1 + 1 + \ldots + 1$ (k-times) and similarly for other usual terms if the meaning is clear.

Peano Arithmetic We denote by PA the theory of Peano Arithmetic in language $\mathcal{L}_{\mathrm{PA}} = \{0, 1, +, \cdot, \leq\}$. The axioms of PA consists from a finite set of open axioms PA^- that captures basic properties of $\mathcal{L}_{\mathrm{PA}}$ (see [Kay91, Chapter 2] for the axioms) and the induction scheme

$$\varphi(x, \bar{y}) - \mathrm{IND}: \qquad \varphi(0, \bar{y}) \wedge \forall x(\varphi(x, \bar{y}) \to \varphi(x + 1, \bar{y})) \to \forall x \varphi(x, \bar{y})$$

for every $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(x, \bar{y})$. Moreover for $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{PA}}$ we denote by $\mathrm{PA}(\mathcal{L})$ the theory in language $\mathcal{L}$ consisting from PA^- and the induction scheme for every $\mathcal{L}$-formula.

Bounded and strictly bounded quantifiers Suppose $\mathcal{L}$ contains a binary relation symbol $\leq$, $t(\bar{y})$ is an $\mathcal{L}$-term and $\varphi(x, \bar{y})$ is an $\mathcal{L}$-formula where x is not among $\bar{y}$. As usual we use the expression $\exists x \leq t(\bar{y})\varphi(x, \bar{y})$ as an abbreviation for $\exists x(x \leq t(\bar{y}) \wedge \varphi(x, \bar{y}))$ and $\forall x \leq t(\bar{y})\varphi(x, \bar{y})$ as an abbreviation for $\forall x(x \leq t(\bar{y}) \to \varphi(x, \bar{y}))$. We will call the quantifiers that appears in the form described above *bounded quantifiers* and we say that a quantifier Q is *bounded by* $t(\bar{y})$ if it is of the form $Qx \leq t(\bar{y})$. Further as usual $\exists x < t(\bar{y})\varphi(x, \bar{y})$ abbreviates $\exists x \leq (x \neq y \to \varphi(x, y, \bar{z}))$ and analogously for the universal quantifier. We call the quantifiers that appears in this form *strictly bounded quantifiers* and say that a quantifier Q is *strictly bounded by* $t(\bar{y})$ if it is of the form $Qx < t(\bar{y})$. Finally we say that an $\mathcal{L}$-formula is *(strictly) bounded* if all quantifiers occurring in it are (strictly) bounded.

Arithmetical Hierarchy Suppose a language $\mathcal{L}$ contains a binary relation symbol $\leq$. Let i be a natural number then we let $\Sigma_0(\mathcal{L}) = \Pi_0(\mathcal{L}) = \Delta_0(\mathcal{L})$ be the set of bounded $\mathcal{L}$-formulae and

$$\Sigma_{i+1}(\mathcal{L}) = \{\exists \bar{x}\varphi(\bar{x}, \bar{z}) \mid \varphi(\bar{x}, \bar{z}) \in \Pi_i(\mathcal{L})\}$$

$$\Pi_{i+1}(\mathcal{L}) = \{\forall \bar{x}\varphi(\bar{x}, \bar{z}) \mid \varphi(\bar{x}, \bar{z}) \in \Sigma_i(\mathcal{L})\}.$$

We denote by $\Delta_{i+1}(\mathcal{L})$ the set of $\mathcal{L}$-formulae $\varphi(\bar{x})$ such that there is $\psi(\bar{x}) \in \Sigma_{i+1}(\mathcal{L})$ and $\theta(\bar{x}) \in \Pi_{i+1}(\mathcal{L})$ with $\mathrm{PA}(\mathcal{L}) \vdash \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$ and $\mathrm{PA}(\mathcal{L}) \vdash \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \theta(\bar{x}))$.

Moreover if $\mathcal{L} = \mathcal{L}_{\mathrm{PA}}$ then we will write Σ_i, Π_i and Δ_i instead of $\Sigma_i(\mathcal{L})$, $\Pi_i(\mathcal{L})$ and $\Delta_i(\mathcal{L})$ respectively.

Theories $\mathrm{I}\Sigma_i$: Suppose $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{PA}}$, $\varphi(x, \bar{y})$ is an $\mathcal{L}$-formula and Γ is a set of $\mathcal{L}$-formulae. Then we let

$$\varphi(x, \bar{y}) - \mathrm{LNP}: \qquad \exists x \varphi(x, \bar{y}) \to \exists x(\varphi(x, \bar{y}) \wedge \forall x' \leq x(x' \neq x \to \neg\varphi(x', \bar{y}))$$

and call it the *least number principle for* $\varphi(x, \bar{y})$. Moreover we let

$$\Gamma - \mathrm{IND} = \{\varphi(x, \bar{y}) - \mathrm{IND} \mid \varphi(x, \bar{y}) \in \Gamma\}$$

$$\Gamma - \mathrm{LNP} = \{\varphi(x,\bar{y}) - \mathrm{LNP} \mid \varphi(x,\bar{y}) \in \Gamma\}$$

and denote by $\mathrm{I}\Gamma$ the theory $\mathrm{PA}^- + \Gamma - \mathrm{IND}$ (in the language $\mathcal{L}$) and by $\mathrm{L}\Gamma$ the theory $\mathrm{PA}^- + \Gamma - \mathrm{LNP}$ (in the language $\mathcal{L}$). It is a folklore fact that for every natural number i, $\mathrm{Thm}(\mathrm{I}\Sigma_i) = \mathrm{Thm}(\mathrm{L}\Sigma_i)$ (see f.e. [HP93, Theorem 2.4]).

Suppose $\mathbb{M}$ is a model of PA^- in language $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{PA}}$ and $A \subseteq \mathbb{M}$. Then we say that A *is an initial segment of* $\mathbb{M}$ if $A = \{m \in \mathbb{M} \mid \mathbb{M} \models m \leq a$ for some $a \in A\}$.

Observation 2. *Suppose* $\mathbb{M}_1, \mathbb{M}_2$ *are models of* PA^- *with* $\mathbb{M}_1 \leq \mathbb{M}_2$ *and the domain of* $\mathbb{M}_1$ *is an initial segment of the domain of* $\mathbb{M}_2$*. Then* Δ_0*-formulae are absolute between* $\mathbb{M}_1$ *and* $\mathbb{M}_2$*. Moreover if in addition* $\mathbb{M}_2 \models \mathrm{I}\Delta_0$ *then* $\mathbb{M}_1 \models \mathrm{I}\Delta_0$*.*

Proof. For the absoluteness see [Kay91, Theorem 2.7] rest follows easily. □

Basic complexity-theory

We give some basic definitions and notational convention from complexity-theory to fix the mode of speech but we assume a reader have some knowledge of this field covering Sections 1-6 of [AB09, Part I].

For $w \in \mathbb{N}$ we let $|w| = \lceil \log_2(w+1) \rceil$ (="the length of the binary representation of w") and call it the *length of* w and for for a tuple $\bar{w} = (w_0, w_1, \ldots, w_{r-1}) \in \mathbb{N}^r$ we write $|\bar{w}|$ to denote the tuple $(|w_0|, |w_1|, \ldots, |w_{r-1}|)$. Moreover for $f, g : \mathbb{N} \to \mathbb{N}$ we write $f = O(g)$ and say f *is* $O(g)$ if there is a $c \in \mathbb{N}$ such that $f(n) \leq cg(n) + c$ for all $n \in \mathbb{N}$.

Turing machines We start with some notation regarding Turing machines (or algorithms).[1] The notion of Turing machine is mostly defined as a machine operating on binary strings (or strings of symbols from a finite alphabet). However, this can be equivalently seen as operating on natural numbers if one identifies natural numbers with their binary representation. Thus in this context we will not explicitly distinguish between natural numbers and its binary representation if there is no danger of confusion.

Suppose $t : \mathbb{N} \to \mathbb{N}$. Then a Turing machine $\mathbb{A}$ is $t(n)$*-time bounded* or *computes in time* $t(n)$ if $dom(\mathbb{A}) = \mathbb{N}$ and for every $w \in \mathbb{N}$, the computation of $\mathbb{A}$ with input w terminates in $\leq t(|w|)$ many steps. Moreover if $\mathbb{A}$ computes in time $c + n^c$ for some $c \in \mathbb{N}$ then we will say that $\mathbb{A}$ is *p-time bounded*.

Next we fix some coding of finite tuples from $\mathbb{N}$ such that there is some $c \in \mathbb{N}$ and a $c(c+n)$-time bounded Turing machine which on input x, i checks whether $x = code(\bar{w})$ for some $\bar{w}$ and if so then outputs the i-th element of $\bar{w}$, otherwise it outputs 0.

For our next purpose it will be convenient to work with the definition of Turing machines that uses an output tape - a tape which is write-only and serves as a storage for the output of a computation. Suppose $\mathbb{A}$ is a Turing machine. Then we let $dom(\mathbb{A}) = \{w \in \mathbb{N} \mid$ the computation of $\mathbb{A}$ with input w terminates$\}$ and for any $w \in dom(\mathbb{A})$ we denote by $\mathbb{A}(w)$ the content of the output-tape of $\mathbb{A}$ after the computation of $\mathbb{A}$ with the input w. Moreover if $\bar{w} \in \mathbb{N}$ is such that $code(\bar{w}) \in dom(\mathbb{A})$ then

[1] by Turing machine we will always mean deterministic Turing machine

we will denote by $\mathbb{A}(\bar{w})$ the content of the output-tape of $\mathbb{A}$ after the computation of $\mathbb{A}$ on $code(\bar{w})$. We say that $\mathbb{A}$ *accepts* $w \in \mathbb{N}$ if $w \in dom(\mathbb{A})$ and $\mathbb{A}(w) = 1$ and we say that $\mathbb{A}$ *rejects* $w \in \mathbb{N}$ if $w \in dom(\mathbb{A})$ and $\mathbb{A}(w) \neq 1$. Finally we say that $\mathbb{A}$ *decides* $w \in \mathbb{N}$ if $\mathbb{A}$ accepts w or $\mathbb{A}$ rejects w. We say that $\mathbb{A}$ accepts/rejects/decides $\bar{w} \in \mathbb{N}$ if $\mathbb{A}$ accepts/rejects/decides $code(\bar{w})$. Following the usual mode of speech we say that $\mathbb{A}$ *decides* $B \subseteq \mathbb{N}^k$ if $\mathbb{A}$ accepts any $\bar{w} \in B$ and rejects any $\bar{w} \notin B$. Moreover if $B \subseteq \mathbb{N}^k$ is such that it can be decided by a Turing machine then we will call B a *(k-ary) predicate* on $\mathbb{N}$ and write $B(x_0, x_1, \dots, x_{k-1})$ rather than $(x_0, x_1, \dots, x_{k-1}) \in B$.

The classes $\mathrm{DTIME}(t(n))$**,** P**,** FP We denote by $\mathrm{DTIME}(t(n))$ the class of predicates B such that there is $c \in \mathbb{N}$ and B can be decided by a Turing machine computing in time $ct(n) + c$. We further denote by P the set $\bigcup_{c\in\mathbb{N}} \mathrm{DTIME}(n^c)$.

Finally we define the class FP of functions on $\mathbb{N}$ by: $f(\bar{x}) \in \mathrm{FP}$ if and only if there is a p-time bounded Turing machine $\mathbb{A}$ such that for any $\bar{w} \in \mathbb{N}$, $f(\bar{w}) = \mathbb{A}(\bar{w})$. We will call the elements of FP *polynomialy-time computable* (or *p-time*) *functions* (on $\mathbb{N}$) and the elements of P *polynomial predicates (on* $\mathbb{N}$*)*.

Bounded arithmetic

Bounded arithmetic are sort of theories developed by Samuel Buss which first appeared in his PhD thesis [Bus86]. They were developed to study Complexity and Proof-complexity from the logical point of view. We define some of those theories which are relevant for this thesis and give some basic facts. For more details one can consult [Kra95],[CN10] or [Bus97].

The theory $\widetilde{\mathrm{PV}}$ Let $\mathbb{N}$ be the standard model of True arithmetic and denote by $\mathcal{L}_{PV} \subseteq \mathcal{L}_{all}$ the language containing a function and a predicate symbol for every polynomial-time computable function or polynomial relation on $\mathbb{N}$. We denote by $\widetilde{\mathrm{PV}}$ the theory $\mathrm{Th}_{\forall}(\mathbb{N}, \mathcal{L}_{PV})$.

Observation 3. *Any open* $\mathcal{L}_{PV}$*formula defines a polynomial predicate on* $\mathbb{N}$ *i.e. for any open* $\mathcal{L}_{PV}$*-formula* $\varphi(\bar{x})$ *there is a p-time bounded Turing machine* $\mathbb{A}$ *such that for any* $\bar{w} \in \mathbb{N}$*,* $\mathbb{A}$ *decides the set* $\{\bar{w} \in \mathbb{N} \mid \mathbb{N} \models \varphi(\bar{w})\}$*.*

Proof. Induction on the complexity of an open $\mathcal{L}_{PV}$formula. □

Observation 4. *Assume* $\varphi(\bar{x}, y)$ *is an* $\mathcal{L}_{\mathrm{PA}}$*-formula and* $t_0(\bar{x}), t_1(\bar{x}), \dots, t_{k-1}(\bar{x})$ *are* $\mathcal{L}_{PV}$*-terms with* $\widetilde{\mathrm{PV}} \vdash \forall \bar{x} \bigvee_{i<r} \varphi(\bar{x}, t_i(\bar{x}))$*. Then there is a function symbol* $f \in \mathcal{L}_{PV}$ *such that* $\widetilde{\mathrm{PV}} \vdash \forall \bar{x} \varphi(\bar{x}, f(\bar{x}))$*.*

Proof. Since FP is closed under composition we can by the definition of $\mathcal{L}_{PV}$wlog assume that $t_0, t_1, \dots, t_{r-1}$ are function symbols $f_0, f_1, \dots, f_{r-1}$ of $\mathcal{L}_{PV}$. For every $i < r$ let $\mathbb{A}_i$ be some p-time bounded Turing Machines witnessing that $f_i \in \mathcal{L}_{PV}$ and $B(\bar{x}, y)$ be the polynomial predicate on $\mathbb{N}$ defined by $\varphi(\bar{x}, y)$. Then we can let $f(\bar{x})$ be the function symbol such that $f^{\mathbb{N}}(\bar{x})$ is computed by the following algorithm:

Given input $\bar{x}$ find the smallest $i < r$ such that $B(\bar{x}, \mathbb{A}_i(\bar{x}))$ and output $\mathbb{A}_i(\bar{x})$.

By the assumption on such $i < r$ exists for every $\bar{w} \in \mathbb{N}$ and thus the computation is defined on every input. It is easy to see that the algorithm is p-time bounded and so $f(\bar{x}) \in \mathcal{L}_{PV}$. $\square$

Theorem 5 (Herbrand Theorem)**.** *Assume T is a universal theory in language $\mathcal{L}$ and $\varphi(\bar{x}, y)$ an $\mathcal{L}$-formula such that $T \vdash \forall\bar{x}\exists y\varphi(\bar{x}, y)$. Then there are $\mathcal{L}$ terms $t_0(\bar{x}), \ldots, t_{r-1}(\bar{x})$ such that $T \vdash \forall\bar{x} \bigvee_{i<r} \varphi(\bar{x}, t_i(\bar{x}))$*

Proof. See [Bus95, Theorem 1]. $\square$

Lemma 6. *Assume $\varphi(\bar{x}, y)$ is an open $\mathcal{L}_{PV}$-formula. Then $\widetilde{\mathrm{PV}} \vdash \forall\bar{x}\exists y\varphi(\bar{x}, y)$ if and only if $\mathbb{N} \models \forall\bar{x}\varphi(\bar{x}, f(\bar{x}))$ for a function symbol $f(\bar{x}) \in \mathcal{L}_{PV}$ i.e a polynomial-time computable function $f^{\mathbb{N}}(\bar{x})$.*

Proof. For the left-right implication we have by the Herbrand Theorem that there are finitely many $\mathcal{L}_{PV}$-terms $t_0(\bar{x}), \ldots, t_{r-1}(\bar{x})$ such that $\widetilde{\mathrm{PV}} \vdash \forall\bar{x} \bigvee_{i<r} \varphi(\bar{x}, t_i(\bar{x}))$. Then by the observation above there is a function symbol $f(\bar{x}) \in \mathcal{L}_{PV}$ i.e. $f^{\mathbb{N}}(\bar{x}) \in \mathrm{FP}$ with $\widetilde{\mathrm{PV}} \vdash \forall\bar{x}\varphi(\bar{x}, f(\bar{x}))$ and so $\mathbb{N} \models \forall\bar{x}\varphi(\bar{x}, f(\bar{x}))$.

For the right-left implication if $\mathbb{N} \models \forall\bar{x}\varphi(\bar{x}, f(\bar{x}))$ then $\forall\bar{x}\varphi(\bar{x}, f(\bar{x})) \in \widetilde{\mathrm{PV}}$ and so $\widetilde{\mathrm{PV}} \vdash \forall\bar{x}\exists y\varphi(\bar{x}, y)$. $\square$

Language $\mathcal{L}_{BUSS}$ and the theory BASIC Let $\mathcal{L}_{BUSS} = \mathcal{L}_{\mathrm{PA}} \cup \{\lfloor \frac{x}{2} \rfloor, |x|, x\#y\}$ be an extension of $\mathcal{L}_{\mathrm{PA}}$by two unary function symbols $\lfloor \frac{x}{2} \rfloor$ and $|x|$ and one binary function symbol $x\#y$. Then BASIC is a set of finitely many open axioms fixing the interpretation of function symbols from $\mathcal{L}_{BUSS}$ (for the list of axioms see for example [Kra95, Definition 5.2.1]) where the intended meaning of $|x|$ is "the length of the binary representation of x" and the intended meaning of $x\#y$ is $2^{|x||y|}$. As usual for any $k \in \mathbb{N}$ we will often use the expression $|x|^k$ to denote the term $|x| \cdot |x| \cdot \ldots \cdot |x|$ (k-times).

Suppose $\mathcal{L}$ contains a unary function symbol $|\cdot|$ and a binary relation symbol $\leq$. Recall the definition of bounded quantifiers. For $Q = \exists$ or $Q = \forall$ and any $\mathcal{L}$-term t we will say that the quantifier $Qx \leq t(\bar{y})$ is *sharply bounded* if $t(\bar{y}) = |g(\bar{y})|$ for some $\mathcal{L}$-term $g(\bar{y})$ and analogously for $Qx < t(\bar{y})$. Moreover we will call an $\mathcal{L}$-formula is *(strictly) sharply bounded* if all quantifiers occurring in it are (strictly) sharply bounded.

Bounded arithmetical hierarchy Suppose $\mathcal{L}$ contains a unary function symbol $|\cdot|$ and a binary relation symbol $\leq$.

Then $\Sigma_0^b(\mathcal{L}) = \Pi_0^b(\mathcal{L}) = \Delta_0^b(\mathcal{L})$ denotes the set of sharply bounded $\mathcal{L}$-formulae.

For $i \geq 0$ the classes $\Sigma_{i+1}^b(\mathcal{L})$ and $\Pi_{i+1}^b(\mathcal{L})$ are the smallest classes of $\mathcal{L}$-formulae such that:

(i) $\Sigma_i^b(\mathcal{L}) \cup \Pi_i^b(\mathcal{L}) \subseteq \Sigma_{i+1}^b(\mathcal{L}) \cup \Pi_{i+1}^b(\mathcal{L})$

(i) $\Sigma_i^b(\mathcal{L})$ and $\Pi_i^b(\mathcal{L})$ are closed under sharply bounded quantification, disjunction and conjunction,

(iii) $\Sigma_{i+1}^b(\mathcal{L})$ is closed under bounded existential quantification,

(iv) $\Pi_{i+1}^b(\mathcal{L})$ is closed under bounded universal quantification,

(v) the negation of $\Sigma_{i+1}^b(\mathcal{L})$-formula is $\Pi_{i+1}^b(\mathcal{L})$ and the negation of $\Pi_i^b(\mathcal{L})$-formula is $\Sigma_{i+1}^b(\mathcal{L})$-formula.

Observation 7. *For every $\Delta_0^b(\mathcal{L}_{PV})$-formula $\varphi(\bar{x})$ there is an open $\mathcal{L}_{PV}$-formula $\psi(\bar{x})$ such that $\widetilde{\mathrm{PV}} \vdash \forall\bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.*

Proof. By induction on complexity of a $\Delta_0^b(\mathcal{L}_{PV})$-formula. □

Suppose $\mathcal{L}$ contains constant symbol $0, 1$, a unary function symbol $|\cdot|$, a binary function symbol $+$ and a binary relation symbol $\leq$. Let Γ be a set of $\mathcal{L}$ formulae, then we let

$$\Gamma - \mathrm{LIND} = \{\varphi(0,\bar{y}) \wedge \forall x(\varphi(x,\bar{y}) \to \varphi(x+1,\bar{y})) \to \forall x\varphi(|x|,\bar{y}) \mid \varphi(x,\bar{y}) \in \Gamma\}$$
$$\Gamma - \mathrm{PIND} = \{\varphi(0,\bar{y}) \wedge \forall x(\varphi(\lfloor \tfrac{x}{2} \rfloor,\bar{y}) \to \varphi(x,\bar{y})) \to \forall x\varphi(x,\bar{y}) \mid \varphi(x,\bar{y}) \in \Gamma\}.$$

Theories $\mathrm{S}_2^i(\mathcal{L})$ Suppose $\mathcal{L} \supseteq \mathcal{L}_{BUSS}$. The theory $\mathrm{S}_2^i(\mathcal{L})$ is the theory in language $\mathcal{L}$ consisting from the axioms BASIC and $\Sigma_i^b(\mathcal{L}) - \mathrm{LIND}$. Moreover if $\mathcal{L} = \mathcal{L}_{BUSS}$ then we write S_2^i, in place of $\mathrm{S}_2^i(\mathcal{L})$ and if $\mathcal{L} = \mathcal{L}_{PV}$ then we will follow the usual notation and write $\mathrm{S}_2^i(PV)$ in place of $\mathrm{S}_2^i(\mathcal{L}_{PV})$.

Proof of the following lemma can be found in [Kra95, Lemma 5.2.5].

Lemma 8. *Let $i \geq 1$ be a natural number, then*

$$S_2^i \equiv \mathrm{BASIC} + \Pi_\mathrm{i}^\mathrm{b} - \mathrm{LIND} \equiv \mathrm{BASIC} + \Sigma_i^b - \mathrm{PIND} \equiv \mathrm{BASIC} + \Pi_i^b - \mathrm{PIND}.$$

Proofs of the following two theorems can be find in [Bus86, Chapter 3].

Theorem 9. *Let $f(\bar{x}) \in \mathrm{FP}$, $t(\bar{x})$ an $\mathcal{L}_{BUSS}$-term so that $\mathbb{N} \models \forall\bar{x}(f(\bar{x})) \leq t(\bar{x})$. Then there is a Σ_1^b-formula $\varphi(\bar{x},y)$ such that*
(i) $S_2^1 \vdash \forall\bar{x}\exists y \leq t(\bar{x})\varphi(\bar{x},y)$
(ii) $S_2^1 \vdash \forall\bar{x},y,z(\varphi(\bar{x},y) \wedge \varphi(\bar{x},z) \to y = z)$
(iii) $\mathbb{N} \models \forall\bar{x}\varphi(\bar{x},f(\bar{x}))$.

Theorem 10 (Buss witnessing theorem for S_2^1)**.** *Let $\varphi(\bar{x},y)$ be a Σ_1^b-formula with $S_2^1 \vdash \forall\bar{x}\exists y\varphi(\bar{x},y)$. Then there is a Σ_1^b-formula $\psi(\bar{x},y)$ and a function $f(\bar{x}) \in \mathrm{FP}$ such that:*
(i) $S_2^1 \vdash \forall\bar{x},y(\psi(\bar{x},y) \to \varphi(\bar{x},y))$
(ii) $S_2^1 \vdash \forall\bar{x}\exists!y\psi(\bar{x},y)$
(iii) $\mathbb{N} \models \forall\bar{x}\psi(\bar{x},f(\bar{x}))$.

Corollary 11. *A function f on $\mathbb{N}$ is Σ_1^b-definable in S_2^1 if and only if it is in* FP.

Finally in the Chapter 2 we will need the following class of formulae and an induction scheme:

The class $strict\Sigma_1^b(\mathcal{L})$ Let $\mathcal{L}$ be a language containing a unary function symbol $|\cdot|$ and a binary relation symbol $\leq$. Then by $strict\Sigma_1^b(\mathcal{L})$ we denote the set of $\mathcal{L}$-formulae of the form

$$\exists z_0 \leq t_0(\bar{x},\bar{y})\exists z_1 \leq t_1(z_0,\bar{x},\bar{y}) \ldots \exists z_{k-1} \leq t_{k-1}(z_0,\ldots z_{k-1},\bar{x},\bar{y})\theta(\bar{z},\bar{x},\bar{y})$$

where $t_0, t_1, \ldots, t_{k-1}$ are $\mathcal{L}$-terms and $\theta(\bar{z},\bar{x},\bar{y}) \in \Delta_0^b(\mathcal{L})$. Moreover if $\mathcal{L} = \mathcal{L}_{BUSS}$ then we will write $strict\Sigma_1^b$ in place of $strict\Sigma_1^b(\mathcal{L})$.

The induction scheme LLIND If $\mathcal{L}' \supseteq \mathcal{L}_{BUSS}$ and $\varphi(x, \bar{y})$ is an $\mathcal{L}'$-formula then by $\varphi(x, \bar{y}) - \mathrm{LLIND}$ we denote the formula

$$\varphi(0, \bar{y}) \wedge \forall x(\varphi(x, \bar{y}) \rightarrow \varphi(x + 1, \bar{y})) \rightarrow \forall x \varphi(||x||, \bar{y})$$

and if Γ is a set of $\mathcal{L}'$-formulae then $\Gamma - \mathrm{LLIND} = \{\varphi(x, \bar{y}) - \mathrm{LLIND} \mid \varphi(x, \bar{y}) \in \Gamma\}$.

Chapter 1

General Ultrapower Theory

Let $\mathbb{M}$ be an infinite $\mathcal{L}$-structure and Ω, D be infinite sets with $\Omega \subseteq D$. Let further $\mathcal{F} \subseteq {}^D\mathbb{M}$ and let $\mathcal{V}$ be a filter on Ω.

We define the relation $=_\mathcal{V} \subseteq \mathcal{F} \times \mathcal{F}$ by

$$f =_\mathcal{V} g \text{ if and only if } \{\omega \in \Omega \mid f(\omega) = g(\omega)\} \in \mathcal{V}$$

It is easy to see that $=_\mathcal{V}$ is an equivalence relation. We will denote the equivalence class of f wrt to $=_\mathcal{V}$ by $f^\mathcal{V}$. Moreover to make our text more readable we will often shorten a tuple $(f_0^\mathcal{V}, f_1^\mathcal{V}, \ldots, f_{n-1}^\mathcal{V})$ by $\bar{f}^\mathcal{V}$ and we will sometimes use $dom(\mathcal{F})$ to denote the set D

Throughout the following paragraphs, let $\mathbb{M}$, Ω, D and $\mathcal{F}$ be fixed of the form above.

Definition 1.0.1. *Suppose $F : \mathbb{M}^r \to \mathbb{M}$ for some $r \in \mathbb{N}$. We say that $\mathcal{F}$ is F-closed if for any r-tuple of functions $\bar{f} = (f_0, f_1, \ldots, f_{r-1})$ from $\mathcal{F}$ we have $F \circ \bar{f} \in \mathcal{F}$. We say that $\mathcal{F}$ is $\mathcal{L}$-closed (wrt to $\mathbb{M}$) if it is $F^\mathbb{M}$-closed for any function symbol F from $\mathcal{L}$.*

The structure $\mathcal{F}/\mathcal{U}$ Assume $\mathcal{F}$ is $\mathcal{L}$-closed and $\mathcal{U}$ is a filter on Ω. We define a structure denoted by $\mathcal{F}/\mathcal{U}$ in language $\mathcal{L}$ as follows. Its universe is

$$\{f^\mathcal{U} \mid f \in \mathcal{F}\}.$$

The interpretation of symbols from language $\mathcal{L}$ is defined as follows:

For an r-ary function symbol $F \in \mathcal{L}$ and $f_0, f_1, \ldots, f_{r-1} \in \mathcal{F}$:

$$F^{\mathcal{F}/\mathcal{U}}(f_0^\mathcal{U}, f_1^\mathcal{U}, \ldots, f_{r-1}^\mathcal{U}) = [F^\mathbb{M} \circ (f_0, f_1, \ldots, f_{r-1})]^\mathcal{U}$$

For an r-ary relation symbol $P \in \mathcal{L}$ and $f_0, f_1, \ldots, f_{r-1} \in \mathcal{F}$:

$$(f_0^\mathcal{U}, f_1^\mathcal{U}, \ldots, f_{r-1}^\mathcal{U}) \in P^{\mathcal{F}/\mathcal{U}} \text{ if and only if } \{\omega \in \Omega \mid (f_1(\omega), \ldots, f_n(\omega)) \in P^\mathbb{M}\} \in \mathcal{U}$$

It is easy to check using the properties of the filter that the definition is correct. We call the structure $\mathcal{F}/\mathcal{U}$ *a power over* $\mathbb{M}$ *with domain* Ω *in language* $\mathcal{L}$. Moreover we say that $(\mathcal{F}, \Omega)$ *defines a power construction over* $\mathbb{M}$ *in language* $\mathcal{L}$ if $\mathbb{M}$,$\mathcal{L}$, Ω and $\mathcal{F}$

are as above and $\mathcal{F}$ is $\mathcal{L}$-closed.

Note that by the definition the domain D of functions from $\mathcal{F}$ may in general differ from the domain of a construction Ω. This is because in a typical situation we let $D = \mathbb{M}$ and use some prominent $\mathcal{L}$-closed family of functions $\mathcal{F} \subseteq {}^D\mathbb{M}$ for a power-construction over different domains $\Omega_1, \Omega_2 \subsetneq \mathbb{M}$. It would of course make no difference to restrict functions from $\mathcal{F}$ to the particular domains of power-construction and proceed with a definition where $D = \Omega$. However, our definition is more convenient for the purpose of this thesis.

Lemma 1.0.2. *Assume $t(x_0, x_1, \ldots, x)_{-1}$ is a term in $\mathcal{L}$, $\mathcal{F}/\mathcal{U}$ a power over* $\mathbb{M}$ *with domain Ω and $f_0^{\mathcal{U}}, f_1^{\mathcal{U}}, \ldots, f_{n-1}^{\mathcal{U}} \in \mathcal{F}/\mathcal{U}$. Then $t^{\mathcal{F}/\mathcal{U}}(f_0^{\mathcal{U}}, f_1^{\mathcal{U}}, \ldots, f_{n-1}^{\mathcal{U}}) = [t^{\mathbb{M}} \circ (f_0, f_1, \ldots, f_{n-1})]^{\mathcal{U}}$.*

Proof. By induction on complexity of term t. □

We will need to introduce two natural notational conventions which will be frequently used throughout this thesis. Let $F \in \mathcal{L}$ be a binary function symbol, $R \in \mathcal{L}$ a binary relation symbol and $\varphi(x, y)$ the $\mathcal{L}$-formula $\exists z(F(x, y) = z \vee R(x, z))$. Then for any $f, g \in \mathcal{F}$ and $\omega \in \Omega$ we would first like to give a meaning to the expressions like $\mathbb{M} \models \exists z(F(f(\omega), g(\omega)) = z \vee R(z, g(\omega)))$ which we would like to write as $\mathbb{M} \models \varphi(f(\omega), g(\omega))$. Second, we would like to work with sets of the form $\{\omega \in \Omega \mid \mathbb{M} \models \varphi(f(\omega), g(\omega))\}$ which we would like to denote as $\langle\langle\varphi(f, g)\rangle\rangle_\Omega$.

To make those intuitive concepts precise, let $\mathcal{L}(\mathcal{F})$ denote the language which consists from $\mathcal{L}$ augment by a constant symbol f for any function $f \in \mathcal{F}$. We have the following direct observation:

Observation 1.0.3. *Assume $\varphi(\bar{x})$ is an $\mathcal{L}(\mathcal{F})$-formula and $f_0, f_1, \ldots, f_{r-1}$ symbols from $\mathcal{L}(\mathcal{F}) - \mathcal{L}$ such that all symbols from $\mathcal{L}(\bar{F}) - \mathcal{L}$ which appears in $\varphi(\bar{x})$ are in between $f_0, f_1, \ldots, f_{r-1}$. Then there are variables $y_0, y_1, \ldots, y_{r-1}$ and an $\mathcal{L}$-formula $\varphi^*(\bar{x}, y_0, y_1, \ldots, y_{r-1})$ such that $y_0, y_1, \ldots, y_{r-1}$ do not appear in $\varphi(\bar{x})$ and $\varphi(\bar{x}) = \varphi^*(\bar{x}, \bar{y})[\bar{y}/\bar{f}]$ where $\varphi^*(\bar{x}, \bar{y})[\bar{y}/\bar{f}]$ denotes the formula which arises from $\varphi^*(\bar{x}, \bar{y})$ by simultaneous substitution of f_i for y_i for every $i < r$. Moreover if there are two such formulae $\varphi_1^*(\bar{x}, \bar{y})$ and $\varphi_2^*(\bar{x}, \bar{y})$ with $\varphi(\bar{x}) = \varphi_1^*(\bar{x}, \bar{y})[\bar{y}/\bar{f}] = \varphi_1^*(\bar{x}, \bar{y})[\bar{y}/\bar{f}]$ then for every $\omega \in \Omega$, $\bar{b} \in \mathbb{M}$ and $\bar{a} = \bar{f}(\omega)$: $\mathbb{M} \models \varphi_1^*(\bar{b}, \bar{a})$ if and only if $\mathbb{M} \models \varphi_1^*(\bar{b}, \bar{a})$.*

Proof. First we can wlog assume that $f_0, f_1, \ldots, f_{r-1}$ are pairwise distinct otherwise we can pick a subset of $f_0, f_1, \ldots, f_{r-1}$ which is pairwise distinct. Let $\bar{y}$ be variable which do not appear in $\varphi(\bar{x})$. Take for $\varphi^*(\bar{x}, \bar{y})$ the formula that arises from $\varphi(\bar{x})$ by simultaneous replacing f_i by x_i. It is easy to see induction on complexity of an $\mathcal{L}(\mathcal{F})$-formula that $\varphi^*(\bar{x}, \bar{y})$ is an $\mathcal{L}$-formula.

For the "moreover part" suppose $\varphi_1^*(\bar{x}, \bar{y})$, $\varphi_2^*(\bar{x}, \bar{y})$ are as stated and $\omega, \bar{b}, \bar{a}$ as above arbitrary given. Let s_1,s_2 be the first symbols (reading from left to right) where $\varphi_1^*(\bar{x}, \bar{y})$ differ from $\varphi_2^*(\bar{x}, \bar{y})$ respectively. Then the symbols s_1, s_2 can not be between $\bar{x}$ and can not be logical symbols or symbols of $\mathcal{L}$ as they would remain unchanged by the substitution ($\bar{x}$ is not among $\bar{y}$ by assumption on $\bar{y}$) and so $\varphi_1^*(\bar{x}, \bar{y})[\bar{y}/\bar{f}]$ would not be the same formula as $\varphi_2^*(\bar{x}, \bar{y})[\bar{y}/\bar{f}]$. Thus it must be the case that $s_1 = y_i$ and $s_2 = y_j$ for $y_i \neq y_j$ and some $i, j < r$. But then the assumption on substitution gives that $f_i = f_j$ and thus $a_i = a_j$. Repeating this argument for next symbols which differs until we consider all differences gives that $\varphi_1^*(\bar{b}, \bar{a})$ and $\varphi_2^*(\bar{b}, \bar{a})$ equals as $\mathcal{L}$-formulae with parameters from $\mathbb{M}$ and so the claim holds. □

Now assume φ is an $\mathcal{L}(\mathcal{F})$ sentence such that all symbols from $\mathcal{L}(\mathcal{F}) - \mathcal{L}$ which appears in φ are among $\bar{f} = (f_0, f_1, \ldots, f_{r-1}) \in \mathcal{F}$. Let $\varphi^*(\bar{x})$ be an $\mathcal{L}$-formula such that $\varphi = \varphi^*(\bar{x})[\bar{x}/\bar{f}]$. Then for any $\omega \in \Omega$ we will write $\mathbb{M} \models \varphi(f_0(\omega), f_1(\omega), \ldots, f_{r-1}(\omega))$ as a shorthand for $\mathbb{M} \models \varphi^*(a_0, a_1, \ldots, a_{r-1})$ where $a_i = f_i(\omega)$ for every $i < r$. Moreover if φ and $\bar{f}$ are as above then we define

$$\langle\langle \varphi \rangle\rangle_\Omega = \{\omega \in \Omega \mid \mathbb{M} \models \varphi(f_0(\omega), f_1(\omega), \ldots, f_{r-1}(\omega))\}.$$

Finally assume $\varphi(x_0, x_1, \ldots, x_{r-1}, \bar{y})$ is an $\mathcal{L}(\mathcal{F})$-formula and $f_0, f_1, \ldots, f_{r-1} \in \mathcal{F}$. Then we denote by $\varphi(f_0, f_1, \ldots, f_{r-1}, \bar{y})$ (or $\varphi(\bar{f}, \bar{y})$ for short) the formula $\varphi(\bar{x}, \bar{y})[\bar{x}/\bar{f}]$.

Note that if $\mathcal{U}$ is an filter on Ω then $\mathcal{F}/\mathcal{U}$ has a natural expansion into the language $\mathcal{L}(\mathcal{F})$ given by interpreting each symbol $f \in \mathcal{L}(\mathcal{F}) - \mathcal{L}$ by the corresponding element $f^{\mathcal{U}}$.

Observation 1.0.4. *Assume $(\mathcal{F}, \Omega)$ defines a power construction over $\mathbb{M}$ in language $\mathcal{L}$ and $\mathcal{V}$ is a filter on Ω. Suppose further that $f_0, f_1, \ldots, f_{k-1}, g_0, g_1, \ldots, g_{k-1} \in \mathcal{F}$ with $f_i^{\mathcal{V}} = g_i^{\mathcal{V}}$ for every $i < k$. Then for any $\mathcal{L}$-formula $\varphi(x_0, x_1, \ldots, x_{k-1})$:*

$$\langle\langle \varphi(f_0, f_1, \ldots, f_{k-1}) \rangle\rangle_\Omega \in \mathcal{V} \textit{ if and only if } \langle\langle \varphi(g_0, g_1, \ldots, g_{k-1}) \rangle\rangle_\Omega \in \mathcal{V}.$$

Proof. For every $i < k$ we have that $f_i^{\mathcal{V}} = g_i^{\mathcal{V}}$ i.e. $f_i =_{\mathcal{V}} g_i$ i.e. $\{\omega \in \Omega \mid f_i(\omega) = g_i(\omega)\} \in \mathcal{V}$ i.e $\langle\langle f_i = g_i \rangle\rangle_\Omega \in \mathcal{V}$ and so $A = \langle\langle f_0 = g_0 \rangle\rangle_\Omega \cap \ldots \cap \langle\langle f_{k-1} = g_{k-1} \rangle\rangle_\Omega \in \mathcal{V}$.then for any $\mathcal{L}$-formula $\varphi(\bar{x})$, $A \cap \langle\langle \varphi(f_0, f_1, \ldots, f_{k-1}) \rangle\rangle_\Omega = A \cap \langle\langle \varphi(g_0, g_1, \ldots, g_{k-1}) \rangle\rangle_\Omega$ and so for any $\mathcal{L}$-formula $\varphi(\bar{x})$, $\langle\langle \varphi(f_0, f_1, \ldots, f_{k-1}) \rangle\rangle_\Omega \in \mathcal{V}$ if and only if $A \cap \langle\langle \varphi(f_0, f_1, \ldots, f_{k-1}) \rangle\rangle_\Omega$ if and only if $A \cap \langle\langle \varphi(g_0, g_1, \ldots, g_{k-1}) \rangle\rangle_\Omega \in \mathcal{V}$ if and only if $\langle\langle \varphi(g_0, g_1, \ldots, g_{k-1}) \rangle\rangle_\Omega \in \mathcal{V}$. □

1.1 Łos property and Skolem functions

So far we only know that the construction defined above produces first order structures. Our aim is to use those constructions to produce models that will witness some consistency statements about arithmetical theories. In that case all of the constructions share the same idea which can be described as follows:

Assume T is a theory of our interest and σ is a sentence which we want to show is consistent with T. We choose a model $\mathbb{M}$ of the theory $T + \neg\sigma$ and make a (ultra)power construction over $\mathbb{M}$ to get a (ultra)power $\mathcal{F}/\mathcal{U} \models T + \sigma$. To do so, we have to ensure $\mathcal{F}/\mathcal{U} \models \varphi$ for every $\varphi \in T$ and $\mathcal{F}/\mathcal{U} \models \sigma$. To ensure the former we will mostly use that our construction preserves validity of certain class of formulae containing T and thus as $\mathbb{M} \models T$ we get $\mathcal{F}/\mathcal{U} \models T$. On the other hand, to show that $\mathcal{F}/\mathcal{U} \models \sigma$ is of the opposite manner since $\mathbb{M} \models \neg\sigma$ and thus in particular we will have to ensure that the construction does not preserves validity of $\neg\sigma$. The preservation of sentences from the groundmodel will be mostly ensured by a choice of a suitable family of functions $\mathcal{F}$ and the validity of $\neg\sigma$ in $\mathcal{F}/\mathcal{U}$ by a suitable choice of a (ultra)filter $\mathcal{U}$ and the domain Ω (however, the choice of Ω will be of course also depended on $\mathcal{F}$).

We will show in the Corollary 1.1.6 that $\mathcal{F}/\mathcal{U}$ models $\mathrm{Th}_\forall(\mathbb{M})$ whenever we choose $\mathcal{U}$ to be an ultrafilter independently on the set of functions $\mathcal{F}$ and Ω. Since arithmetical theories mostly consist from a set of universal sentences and an induction scheme for some class of formulae when working with ultrapowers one only have take care of an induction scheme for a given class of formulae. This seems to be a big advantage since the induction scheme is a set of formulae of a very specific and natural form. However, the techniques to ensure some induction scheme known are either trivial in the sense that the constructions preserves too big class of formulae and in particular are useless for proving consistency statements. Or they are very specific in the sense that they work only for some small class of induction schemes. In this chapter we will show some of the trivial techniques where the induction scheme is ensured by a right choice of the set of functions $\mathcal{F}$. Latter in the Chapter 2 we will show a more involved power construction producing models of a weak form of induction where $\mathcal{F}$ and $\mathcal{U}$ are constructed simultaneously.

Now we are ready to give the central definition of this thesis.

Definition 1.1.1. *Assume $(\mathcal{F},\Omega)$ defines a power construction over $\mathbb{M}$ in language $\mathcal{L}$ and $\mathcal{V}$ is a filter over Ω. We say that an $\mathcal{L}$-formula $\varphi(\bar{x})$ is Łos for $\mathcal{F}/\mathcal{V}$ if for any $\bar{f}\in\mathcal{F}$:*

$$\mathcal{F}/\mathcal{V}\models\varphi(\bar{f}^{\mathcal{V}}) \text{ if and only if } \langle\langle\varphi(\bar{f})\rangle\rangle_\Omega\in\mathcal{U}$$

Moreover we say that $\varphi(\bar{x})$ is *Łos* for $\mathcal{F}$ *if $\varphi(\bar{x})$ is Łos for $\mathcal{F}/\mathcal{U}$ for any ultrafilter $\mathcal{U}$ on Ω.*

The following series of statements shows basic properties of Łos formulae.

Lemma 1.1.2. *Assume $(\mathcal{F},\Omega)$ defines a power construction over $\mathbb{M}$ in language $\mathcal{L}$ and $\mathcal{V}$ is a filter over Ω. Suppose further that $\mathcal{L}$-formulae $\varphi(\bar{x}),\psi(\bar{y})$ are Łos for $\mathcal{F}/\mathcal{V}$. Then $\varphi(\bar{x})\wedge\psi(\bar{y})$ is Łos for $\mathcal{F}/\mathcal{V}$. If moreover for $\chi\in\{\varphi(\bar{x}),\psi(\bar{y})\}$ and any $\bar{f}\in\mathcal{F}$, $\langle\langle\chi(\bar{f})\rangle\rangle_\Omega\in\mathcal{V}$ or $\langle\langle\neg\chi(\bar{f})\rangle\rangle_\Omega\in\mathcal{V}$. Then $\neg\varphi(\bar{x}),\varphi(\bar{x})\vee\psi(\bar{y})$ are Łos for $\mathcal{F}/\mathcal{V}$.*

Proof. Assume $\varphi(\bar{x}),\psi(\bar{y})$ are Łos . To show that $\varphi(\bar{x})\wedge\psi(\bar{y})$ is Łos for $\mathcal{F}/\mathcal{V}$ is easy and we leave it to the reader. To show the second part of this lemma assume in addition that $(*)$: for $\chi\in\{\varphi(\bar{x}),\psi(\bar{y})\}$ and any $\bar{f}\in\mathcal{F}$, $\langle\langle\chi(\bar{f})\rangle\rangle_\Omega\in\mathcal{V}$ or $\langle\langle\neg\chi(\bar{f})\rangle\rangle_\Omega\in\mathcal{V}$.

To show that $\neg\varphi(\bar{x})$ is Łos let $\bar{f}\in\mathcal{F}$ be given. Then we have: $\mathcal{F}/\mathcal{V}\models\neg\varphi(\bar{f}^{\mathcal{V}})$ if and only if $\mathcal{F}/\mathcal{V}\not\models\varphi(\bar{f}^{\mathcal{V}})$. As $\varphi(\bar{x})$ is Łos for $\mathcal{F}/\mathcal{V}$ this is if and only if $\langle\langle\varphi(\bar{f})\rangle\rangle_\Omega\notin\mathcal{V}$ which is by $(*)$ if and only if $\langle\langle\neg\varphi(\bar{f})\rangle\rangle_\Omega\in\mathcal{V}$.

For $\varphi(\bar{x})\vee\psi(\bar{y})$ the argument is similar. Let $\bar{f},\bar{g}\in\mathcal{F}$ be given then: $\mathcal{F}/\mathcal{V}\models\varphi(\bar{f}^{\mathcal{V}})\vee\psi(\bar{g}^{\mathcal{V}})$ if and only if $\mathcal{F}/\mathcal{V}\models\varphi(\bar{f}^{\mathcal{V}})$ or $\mathcal{F}/\mathcal{V}\models\psi(\bar{g}^{\mathcal{V}})$. Since $\varphi(\bar{x}),\psi(\bar{y})$ are Łos for $\mathcal{F}/\mathcal{V}$ this is if and only if $\langle\langle\varphi(\bar{f})\rangle\rangle_\Omega\in\mathcal{V}$ or $\langle\langle\psi(\bar{g})\rangle\rangle_\Omega\in\mathcal{V}$. But by $(*)$ this is if and only if $\langle\langle\varphi(\bar{f})\vee\psi(\bar{g})\rangle\rangle_\Omega\in\mathcal{V}$. Indeed, the only problematic implication is the right-left implication. To show this implication assume for a contradiction $\langle\langle\varphi(\bar{f})\vee\psi(\bar{g})\rangle\rangle_\Omega\in\mathcal{V}$ and $\langle\langle\varphi(\bar{f})\rangle\rangle_\Omega,\langle\langle\psi(\bar{g})\rangle\rangle_\Omega\notin\mathcal{V}$. Then by (*)

we get $\langle\langle\neg\varphi(\bar f)\rangle\rangle_\Omega \in \mathcal{V}$ and $\langle\langle\neg\psi(\bar g)\rangle\rangle_\Omega \in \mathcal{V}$ thus $\mathcal{V} \ni \langle\langle\neg\varphi(\bar f)\rangle\rangle_\Omega \cap \langle\langle\neg\psi(\bar f)\rangle\rangle_\Omega = \langle\langle\neg\varphi(\bar f) \wedge \neg\psi(\bar f)\rangle\rangle_\Omega = \Omega - \langle\langle\varphi(\bar f) \vee \psi(\bar f)\rangle\rangle_\Omega$ contradicting $\langle\langle\varphi(\bar f) \vee \psi(\bar f)\rangle\rangle_\Omega \in \mathcal{V}$ and we are done. □

Corollary 1.1.3. *Assume $(\mathcal{F}, \Omega)$ defines a power construction over $\mathbb{M}$ in language $\mathcal{L}$ and $\mathcal{U}$ is an ultrafilter on Ω. Then the family of $\mathcal{L}$-formulae which are Łos for $\mathcal{F}/\mathcal{U}$ is closed under boolean combinations.*

Lemma 1.1.4. *Assume $(\mathcal{F}, \Omega)$ defines a power construction over $\mathbb{M}$ in language $\mathcal{L}$ and $\mathcal{V}$ is a filter over Ω. Assume further that for any open $\mathcal{L}$-formula φ and $\bar f \in \mathcal{F}$:*

$$\langle\langle\varphi(\bar f)\rangle\rangle_\Omega \in \mathcal{V} \text{ or } \langle\langle\neg\varphi(\bar f)\rangle\rangle_\Omega \in \mathcal{V}.$$

Then open formulae are Łos for $\mathcal{F}/\mathcal{V}$.

Proof. We proceed by the induction on complexity of an open $\mathcal{L}$-formula. Assume $\varphi(\bar x)$ is an atomic formula of the form $P(t_0(\bar x), \ldots, t_{k-1}(\bar x))$ for some predicate symbol P and $\mathcal{L}$-terms $t_0(\bar x), \ldots, t_{k-1}(\bar x)$. Then for any tuple $\bar f^{\mathcal{V}} \in \mathcal{F}/\mathcal{V}$:

$$\begin{aligned}
&\mathcal{F}/\mathcal{V} \models P(t_0(\bar f^{\mathcal{V}}), \ldots, t_{k-1}(\bar f^{\mathcal{V}})) && \\
&\Leftrightarrow (t_0^{\mathcal{F}/\mathcal{V}}(\bar f^{\mathcal{V}}), \ldots, t_{k-1}^{\mathcal{F}/\mathcal{V}}(\bar f^{\mathcal{V}})) \in P^{\mathcal{F}/\mathcal{V}} && \text{Definition of a structure}\\
&\Leftrightarrow ([t_0^{\mathbb{M}} \circ \bar f]^{\mathcal{V}}, \ldots, [t_{k-1}^{\mathbb{M}} \circ \bar f]^{\mathcal{V}}) \in P^{\mathcal{F}/\mathcal{V}} && \text{Lemma 1.0.2}\\
&\Leftrightarrow \{\omega \in \Omega \mid (t_0^{\mathbb{M}}(\bar f(\omega)), \ldots, t_{k-1}^{\mathbb{M}}(\bar f(\omega))) \in P^{\mathbb{M}}\} \in \mathcal{V} && \text{Definition of power}\\
&\Leftrightarrow \langle\langle P(t_0(\bar f), \ldots, t_{k-1}(\bar f)\rangle\rangle_\Omega \in \mathcal{V} && \langle\langle\cdot\rangle\rangle_\Omega \text{ notation}
\end{aligned}$$

The case when $\varphi(\bar f)$ is of the form $t_1(\bar f) = t_2(\bar f)$ is similar and we leave it to the reader. Finally if $\varphi(\bar x) \in \{\neg\psi(\bar x), \psi(\bar x) \vee \theta(\bar x), \psi(\bar x) \wedge \theta(\bar x)\}$ for an open $\mathcal{L}$-formulae $\psi(\bar x), \theta(\bar x)$ which are Łos for $\mathcal{F}/\mathcal{V}$. Then $\varphi(\bar x)$ is Łos for $\mathcal{F}/\mathcal{V}$ by the assumption on $\mathcal{V}$ and the corollary above. This finishes the argument.

□

Corollary 1.1.5. *Suppose $(\mathcal{F}, \Omega)$ defines a power construction over $\mathbb{M}$ in language $\mathcal{L}$ and $\mathcal{U}$ is a ultrafilter on Ω. Then open $\mathcal{L}$-formulae are Łos for $\mathcal{F}/\mathcal{U}$.*

Theorem 1.1.6. *($\forall$-Preservation) Assume $(\mathcal{F}, \Omega)$ defines a power construction over $\mathbb{M}$ in language $\mathcal{L}$ and $\mathcal{V}$ is a filter on Ω. Then for any $\mathcal{L}$-formula $\varphi(\bar x)$*

$$\mathcal{F}/\mathcal{V} \models \forall\bar x\varphi(\bar x) \text{ whenever } \mathbb{M} \models \forall\bar x\varphi(\bar x) \text{ and } \varphi(\bar x) \text{ is Łos for } \mathcal{F}/\mathcal{V}.$$

Proof. Assume an $\mathcal{L}$-formula $\varphi(\bar x)$ is Łos for $\mathcal{F}/\mathcal{V}$ and $\mathbb{M} \models \forall\bar x\varphi(\bar x)$. We have to show that for any $\bar f \in \mathcal{F}$, $\mathcal{F}/\mathcal{V} \models \varphi(\bar f^{\mathcal{V}})$. Since $\varphi(\bar x)$ is Łos for $\mathcal{F}/\mathcal{V}$ this is equivalent to showing $\langle\langle\varphi(\bar f)\rangle\rangle_\Omega \in \mathcal{V}$ for any $\bar f \in \mathcal{F}$. But since $\mathbb{M} \models \forall\bar x\varphi(\bar x)$ we have $\langle\langle\varphi(\bar f))\rangle\rangle_\Omega = \Omega$ for any $\bar f \in \mathcal{F}$ and thus $\langle\langle\varphi(\bar f)\rangle\rangle_\Omega \in \mathcal{V}$ for any $\bar f \in \mathcal{F}$. □

Corollary 1.1.7. *Assume $(\mathcal{F}, \Omega)$ defines a power construction over $\mathbb{M}$ in language $\mathcal{L}$ and $\mathcal{U}$ is an ultrafilter on Ω. Then $\mathcal{F}/\mathcal{U} \models \mathrm{Th}_\forall(\mathbb{M})$.*

Proof. By the Corollary 1.1.5 open $\mathcal{L}$-formulae are Łos for $\mathcal{F}$ so the rest follows from the above lemma. $\square$

Corollary 1.1.8. *($\forall$-Preservation, parametrical version) Assume $(\mathcal{F}, \Omega)$ defines a power construction over $\mathbb{M}$ in language $\mathcal{L}$ and $\mathcal{V}$ is an filter on Ω. Let further $\bar{a} = (a_0, a_1, \ldots, a_{k-1}) \in \mathbb{M}$ and suppose there are functions $\bar{c}_{\bar{a}} = (c_{a_0}, \ldots, c_{a_{k-1}}) \in \mathcal{F}$ such that c_{a_i} is constant a_i on Ω. Then for any $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$*

$$\mathcal{F}/\mathcal{V} \models \forall \bar{x} \varphi(\bar{x}, \bar{c}_{\bar{a}}^{\mathcal{V}}) \textit{ whenever } \mathbb{M} \models \forall \bar{x} \varphi(\bar{x}, \bar{a}) \textit{ and } \varphi(\bar{x}, \bar{y}) \textit{ is Łos for } \mathcal{F}/\mathcal{V}.$$

Proof. Assume $\mathbb{M}'$ is an expansion of $\mathbb{M}$ into the language $\mathcal{L}' = \mathcal{L} \cup \{d_0, \ldots, d_{k-1}\}$ with $d_i^{\mathbb{M}'} = a_i$. Since $c_{a_0}, \ldots, c_{a_{k-1}} \in \mathcal{F}$ and $\mathcal{F}$ is $\mathcal{L}$-closed we get that $\mathcal{F}$ is $\mathcal{L}'$-closed. Thus $(\mathcal{F}, \Omega)$ defines a power construction over $\mathbb{M}'$ (in language $\mathcal{L}'$). Now let a filter $\mathcal{V}$ on Ω and an $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ which is Łos for $\mathcal{F}/\mathcal{V}$ be given and assume $\mathbb{M} \models \forall \bar{x} \varphi(\bar{x}, \bar{a})$. Then $\mathbb{M}' \models \forall \bar{x} \varphi(\bar{x}, d_0, d_1, \ldots, d_{k-1})$ thus by the previous theorem $\mathcal{F}/\mathcal{V} \models \forall \bar{x} \varphi(\bar{x}, d_0, d_1, \ldots, d_{k-1})$. But since $\langle\langle c_{a_i} = d_i \rangle\rangle_\Omega = \Omega$ for any $i < k$, we have $\mathcal{F}/\mathcal{V} \models \forall \bar{x} \varphi(\bar{x}, \bar{c}_{\bar{a}}^{\mathcal{V}})$ and thus the same holds for $\mathcal{F}/\mathcal{U}$ restricted to $\mathcal{L}$. $\square$

As described in the beginning of this section one very often needs to ensure Łos property for some family of formulae. One way to ensure this is to choose rich enough set of functions as described in the next paragraphs.

Definition 1.1.9. *Let $\mathbb{M}$ be an $\mathcal{L}$-structure and $\varphi(y, x_0, x_1, \ldots, x_{n-1})$ an $\mathcal{L}$-formula. We say that a function $\mathrm{W}_{\exists y \varphi} : \mathbb{M}^n \to \mathbb{M}$ is a Skolem function (or a witnessing function) for $\exists y \varphi(y, \bar{x})$ (on $\mathbb{M}$) if $\mathbb{M} \models \forall \bar{x}(\exists y \varphi(y, \bar{x}) \to \varphi(\mathrm{W}_{\exists x \varphi}(\bar{x}), \bar{x}))$.*

Lemma 1.1.10. *Suppose $(\mathcal{F}, \Omega)$ defines a power construction over $\mathbb{M}$ in language $\mathcal{L}$, $\mathcal{V}$ is a filter on Ω and $\varphi(y, \bar{x})$ is an $\mathcal{L}$-formula. Assume that $\mathcal{F}$ is closed under some Skolem function for $\exists y \varphi(y, \bar{x})$ and $\varphi(y, \bar{x})$ is Łos for $\mathcal{F}/\mathcal{V}$. Then $\exists y \varphi(y, \bar{x})$ is Łos for $\mathcal{F}/\mathcal{U}$.*

Proof. We have to show that for any $\bar{f} \in \mathcal{F}$, $\mathcal{F}/\mathcal{V} \models \exists y \varphi(y, \bar{f}^{\mathcal{V}})$ if and only if $\langle\langle \exists y \varphi(y, \bar{f}) \rangle\rangle_\Omega \in \mathcal{V}$.

Let $\bar{f} \in \mathcal{F}$ be given. To show the right-left implication assume $\langle\langle \exists y \varphi(y, \bar{f}) \rangle\rangle_\Omega \in \mathcal{V}$. Let $W_{\exists y \varphi}$ be a Skolem function for $\exists y \varphi(y, \bar{x})$ on $\mathbb{M}$ such that $\mathcal{F}$ is closed under $W_{\exists y \varphi}$. Then for any $\omega \in \Omega$, $\mathbb{M} \models \exists y \varphi(y, \bar{f}(\omega)) \to \varphi(\mathrm{W}_{\exists y \varphi}(\bar{f}(\omega)), \bar{f}(\omega)))$ and so $\langle\langle \exists y \varphi(y, \bar{f}) \rangle\rangle_\Omega \subseteq \langle\langle \varphi(\mathrm{W}_{\exists y \varphi} \circ \bar{f}, \bar{f}) \rangle\rangle_\Omega \in \mathcal{V}$. But then as $\mathrm{W}_{\exists y \varphi} \circ \bar{f} \in \mathcal{F}$ and $\varphi(y, \bar{x})$ is Łos for $\mathcal{F}/\mathcal{V}$ we get that $\mathcal{F}/\mathcal{V} \models \varphi([\mathrm{W}_{\exists y \varphi} \circ \bar{f}]^{\mathcal{V}}, \bar{f}^{\mathcal{V}})$ i.e. $\mathcal{F}/\mathcal{V} \models \exists y \varphi(x, \bar{f}^{\mathcal{V}})$.

To show the left-right implication suppose $\mathcal{F}/\mathcal{V} \models \exists y \varphi(y, f^{\mathcal{V}})$. Then $\mathcal{F}/\mathcal{V} \models \varphi(g^{\mathcal{V}}, \bar{f}^{\mathcal{V}})$ for some $g \in \mathcal{F}$ but since $\varphi(y, \bar{x})$ is Łos for $\mathcal{F}/\mathcal{V}$ this gives $\langle\langle \varphi(g, \bar{f}) \rangle\rangle_\Omega \in \mathcal{V}$ and we are done as $\langle\langle \varphi(g, \bar{f}) \rangle\rangle_\Omega \subseteq \langle\langle \exists y \varphi(y, \bar{f}) \rangle\rangle_\Omega$.

$\square$

Corollary 1.1.11. *Assume $\mathbb{M}$ is an infinite $\mathcal{L}$-structure, Ω an infinite set and $\mathcal{U}$ an ultrafilter on Ω. Then all $\mathcal{L}$-formulae are Łos for ${}^{\Omega}\mathbb{M}/\mathcal{U}$ and in particular ${}^{\Omega}\mathbb{M}/\mathcal{U} \models \mathrm{Th}(\mathbb{M})$.*

Proof. We will proceed by induction on the complexity of an open $\mathcal{L}$-formula $\varphi(y, \bar{x})$. Note that by the Corollary 1.1.5 open $\mathcal{L}$-formulae are Łos for ${}^{\Omega}\mathbb{M}/\mathcal{U}$ and so base of the induction holds. Moreover by the Corollary 1.1.3 any boolean combination of $\mathcal{L}$-formulae which are Łos for ${}^{\Omega}\mathbb{M}/\mathcal{U}$ is Łos for ${}^{\Omega}\mathbb{M}/\mathcal{U}$. Thus the only step of the induction we have to consider is that when $\varphi(\bar{x}) = \exists y\psi(y,\bar{x})$ for an $\mathcal{L}$-formula $\psi(y,\bar{x})$ which is Łos for ${}^{\Omega}\mathbb{M}/\mathcal{U}$.

To do so, fix some well-ordering $<$ of $\mathbb{M}$[1] and $a \in \mathbb{M}$. Then the function $\mathrm{W}_{\exists y\psi}$ with domain $\mathbb{M}^r$ where r is the length of the tuple $\bar{x}$ defined by

$$\mathrm{W}_{\exists y\psi}(\bar{m}) = \begin{cases} \min_{<}\{w \in \mathbb{M} \mid \mathbb{M} \models \psi(w,\bar{m})\} & \text{if } \mathbb{M} \models \exists y\psi(y,\bar{m}) \\ a & \text{otherwise} \end{cases}$$

is clearly a Skolem function for $\exists y\psi(y,\bar{x})$ in $\mathbb{M}$. Moreover ${}^{\Omega}\mathbb{M}$ is closed under $\mathrm{W}_{\exists y\varphi}$ since $rng(\mathrm{W}_{\exists y\psi}) \subseteq \mathbb{M}$. But then $\varphi(\bar{x}) = \exists y\psi(y,\bar{x})$ is Łos for ${}^{\Omega}\mathbb{M}/\mathcal{U}$ by the previous lemma and we are done.

The in particular part follows easily from the Theorem 1.1.6 since by the previous paragraph all $\mathcal{L}$-formulae are Łos for ${}^{\Omega}\mathbb{M}/\mathcal{U}$.

□

Example 1.1.12. Recall that by Δ_n we mean Δ_n wrt to PA (see Section). Assume $\mathbb{M}$ is a model of PA and $\exists z\psi(z,\bar{x},\bar{w})$ an $\mathcal{L}_{\mathrm{PA}}$-formula and $\bar{m} \in \mathbb{M}$. Let the function $\mathrm{W}_{\exists z\psi} : \mathbb{M}^r \to \mathbb{M}$ where r denotes the length of the tuple $\bar{x}$ be defined in $\mathbb{M}$ by

$$\mathrm{W}_{\exists z\psi}(\bar{x}) = \begin{cases} \min\{y \mid \psi(y,\bar{x})\} & \text{if } \exists z\psi(z,\bar{x},\bar{m}) \\ 0 & \text{otherwise.} \end{cases}$$

Since PA proves least number principle for all $\mathcal{L}_{\mathrm{PA}}$-formulae, we have that $\mathbb{M} \models \forall\bar{x}\exists! y \mathrm{W}_{\exists z\psi}(\bar{x}) = y$ and so $\mathrm{W}_{\exists z\psi}(\bar{x})$ is well-defined function. But we also clearly have $\mathbb{M} \models \forall\bar{x}(\exists z\psi(z,\bar{x}) \to \psi(\mathrm{W}_{\exists z\psi}(\bar{x}),\bar{x}))$ and so $\mathrm{W}_{\exists z\psi}(\bar{x})$ is a Skolem function for $\exists z\psi(z,\bar{x},\bar{m})$. Moreover if $\exists z\psi(z,\bar{x},\bar{w}) \in \Sigma_n$ then it is not hard to see that the formula defining $\mathrm{W}_{\exists z\varphi}$ is a Δ_{n+1}-formula with parameter from $\mathbb{M}$.

Now fix some $n \in \mathbb{N}$, let $\Omega = \mathbb{M}$ and let $\mathcal{F}$ be the set of all unary functions definable in $\mathbb{M}$ by a Δ_{n+1} formula with parameters from $\mathbb{M}$. Again it is not hard to see that $\mathcal{F}$ is closed under any function which is definable by a Δ_{n+1}-formula with a parameter from $\mathbb{M}$. Thus in particular $\mathcal{F}$ is $\mathcal{L}_{\mathrm{PA}}$-closed and by the previous paragraph closed under some Skolem function for every Σ_n-formula with parameters from $\mathbb{M}$.[2]

We claim that Σ_n-formulae are Łos for $\mathcal{F}$. Let $\mathcal{U}$ be arbitrary given ultrafilter on Ω. To show that Σ_n-formulae are Łos for $\mathcal{F}/\mathcal{U}$ we proceed by induction on complexity of Σ_n formula $\varphi(\bar{x})$. Similarly as in the proof of the corollary above the only step we need to consider in the induction is that when $\varphi(\bar{x}) = \exists y\psi(y,\bar{x})$ for $\psi(y,\bar{x}) \in \Sigma_n$ which is Łos for $\mathcal{F}/\mathcal{U}$. But by the previous paragraph and the Lemma 1.1.10, $\varphi(\bar{x})$ of this form is Łos for $\mathcal{F}/\mathcal{U}$. Thus all Σ_n formulae are Łos for $\mathcal{F}$. Finally by the Theorem 1.1.6 we get that $\mathcal{F}/\mathcal{U} \models \mathrm{Th}_{\Pi_{n+1}}(\mathbb{M})$. In particular, if $\mathcal{D}$ is the set of all $\mathbb{M}$-definable functions then $\mathcal{L}_{\mathrm{PA}}$-formulae are Łos for $\mathcal{D}/\mathcal{U}$ for any ultrafilter $\mathcal{U}$ on

[1] A set A is well-ordered by $<$ if for every $B \subseteq A$ there exists an $<$-minimal element of B.

[2] Assume $\psi_f(y_0,y_1,\ldots,y_{r-1},z)$, $\psi_{g_0}(x_0,y_0),\ldots\psi_{g_{r-1}}(x_{r-1},y_{r-1})$ are Δ_{n+1} formulae with parameters from $\mathbb{M}$ defining functions $f,g_0,\ldots,g_{r-1}$ on $\mathbb{M}$ respectively. Then $\exists\bar{y}(\psi_f(\bar{y},z) \wedge \bigwedge_{i<r}\psi_{g_i}(x_i,y_i))$ is (equivalent to) a Δ_{n+1}-formula with parameters from $\mathbb{M}$ and defines the function $f \circ (g_0,g_1,\ldots,g_{r-1})$.

$\mathbb{M}$. Indeed, $\mathcal{D}$ is closed under any $\mathbb{M}$-definable function and so for any $n \in \mathbb{N}$, $\mathcal{D}$ is closed under some Skolem function for any Σ_n-formula. The rest follows by the same induction argument as above.

The previous paragraphs gave us a tool how to ensure Łos property by defining the set of functions $\mathcal{F}$. Latter in the Lemma 1.2.6 we show that Łos property can equivalently be viewed as a statement about absoluteness between powers.

Note that by the Corollary 1.1.7 any ultrapower construction will always produce models of the universal theory of the groundmodel. We will show in the Section 1.3 that if $\mathbb{M}$ and $\mathbb{K}$ are countable $\mathcal{L}$-structures and $\mathbb{K} \models \mathrm{Th}_\forall(\mathbb{M})$ then $\mathbb{K}$ is isomorphic to an ultrapower $\mathcal{F}/\mathcal{U}$ over $\mathbb{M}$ with ($\mathcal{L}$-closed) $\mathcal{F} \subseteq {}^{\mathbb{M}}\mathbb{M}$ and any ultrafilter $\mathcal{U}$ over $\mathbb{M}$. In particular, this will show that countable ultrapowers over a model $\mathbb{M}$ with $\Omega = \mathbb{M}$ and $\mathcal{F} \subseteq {}^{\mathbb{M}}\mathbb{M}$ are (up to an isomorphism) exactly all countable models of $\mathrm{Th}_\forall(\mathbb{M})$. Finally latter in the Section 4.1 we will show that if a certain assumption on $\mathrm{Th}_\forall(\mathbb{M})$ and the language $\mathcal{L}$ holds then the "In particular" above holds with $\mathcal{F} \subseteq \{t^{\mathbb{M}} \mid t(x) \text{ is a term of } \mathcal{L}\}$.

Example 1.1.13 (weak Herbrand theorem)**.** [3] Let T be a universal theory in language $\mathcal{L}$ and φ an open $\mathcal{L}$-formula with $T \vdash \forall x \exists y \varphi(x, y)$. Then there are $\mathcal{L}$-terms $t_0, t_1, \dots, t_{k-1}$ such that $T \vdash \forall x \bigvee_{i<k} \varphi(x, t_i(x))$.

Proof. Suppose φ is an open $\mathcal{L}$-formula with $T \vdash \forall x \exists y \varphi(x, y)$ and assume for a contradiction that for any $\mathcal{L}$-terms $t_0, t_1, \dots, t_{k-1}$, $T \nvdash \forall x \bigvee_{i<k} \varphi(x, t_i(x))$. Then by the Compactness there is a model $\mathbb{M}$ of T such that for any $\mathcal{L}$-terms $t_0(x), \dots, t_{k-1}(x)$, $\mathbb{M} \not\models \forall x \bigvee_{i<k} \varphi(x, t_i(x))$. Thus the set

$$\{\langle\langle \neg\varphi(id, t^{\mathbb{M}})\rangle\rangle_\Omega \mid t(x) \text{ is an } \mathcal{L}\text{-term}\}$$

where id denote the identity function on Ω has finite intersection property and we can extend it into an ultrafilter $\mathcal{U}$ on $\Omega = \mathbb{M}$. Now we let

$$\mathcal{F} = \{t^{\mathbb{M}} \mid t(x) \text{ is an } \mathcal{L}\text{-term}\}$$

which is clearly $\mathcal{L}$-closed and $id \in \mathcal{F}$. But also $\mathcal{F}/\mathcal{U} \models \forall y \neg\varphi(id^{\mathcal{U}}, y)$ as otherwise there is an $\mathcal{L}$-term t such that $\mathcal{F}/\mathcal{U} \models \varphi(id^{\mathcal{U}}, [t^{\mathbb{M}}]^{\mathcal{U}})$. But since open formulae are Łos for $\mathcal{F}/\mathcal{U}$ by the Corollary 1.1.5 this implies $\langle\langle \varphi(id, t^{\mathbb{M}})\rangle\rangle_\Omega \in \mathcal{U}$ contradicting the definition of $\mathcal{U}$. Finally by the Corollary 1.1.7 we have that $\mathcal{F}/\mathcal{U} \models T$ since T is a universal theory. But this contradicts $T \vdash \forall x \exists y \varphi(x, y)$ and we are done.

□

Example 1.1.14 ($\forall\exists$-completeness for ultrapowers)**.** Let $\mathbb{M}$ be an $\mathcal{L}$-structure, $T = \mathrm{Th}_\forall(\mathbb{M})$,

$$\mathcal{F} = \{t^{\mathbb{M}} \mid t(x) \text{ is an } \mathcal{L}\text{-term}\}$$

and $\Omega = \mathbb{M}$. It is easy to see that $\mathcal{F}$ is an $\mathcal{L}$-closed family of functions. We have the following variant of $\forall\exists$-completeness:

For any open $\mathcal{L}$-formula $\varphi(x, y)$ the following is equivalent:

[3] We use the word "weak" because we state the theorem for $\varphi(x, y)$ in place of $\varphi(\bar{x}, y)$. It could be possible to proceed with a similar proof if f.e. one assume that T admits coding of standard length tuples (see Definition 4.0.3) but to keep the example simple we will not expand on this here.

(i) $T \vdash \forall x \exists y \varphi(x,y)$

(ii) for any ultrafilter $\mathcal{U}$ on $\mathbb{M}$: $\mathcal{F}/\mathcal{U} \models \forall x \exists y \varphi(x,y)$.

By the Corollary 1.1.7 we have that for any ultrafilter $\mathcal{U}$, $\mathcal{F}/\mathcal{U} \models T$ and thus the direction from (i) to (ii) is clear. The direction from (ii) to (i) is of similar manner as the previous example. Assume $T \nvdash \forall x \exists y \varphi(x,y)$ i.e. for any n and an n-tuple of terms $t_0, t_1, \dots, t_{n-1}$ from $\mathcal{L}$ we have that $T \nvdash \forall x \bigvee_{i\leq n} \varphi(x, t_i(x))$ and so $\mathbb{M} \models \exists x \bigwedge_{i\leq n} \neg\varphi(x, t_i(x))$. Let $id \in \mathcal{F}$ denote the identity function on Ω. Then the set $B = \{\langle\langle \neg\varphi(id, f)\rangle\rangle_\Omega \mid f \in \mathcal{F}\}$ has a finite intersection property and so there is an ultrafilter $\mathcal{U}$ (on Ω) extending B. Since open $\mathcal{L}$-formulae are Łos for $\mathcal{F}/\mathcal{U}$ by the Corollary 1.1.5 it is easy to see that $\mathcal{F}/\mathcal{U} \models \forall x \neg\varphi(id^{\mathcal{U}}, x)$ and so we are done.

Note that if we let $\mathcal{P} = \{f \mid f \text{ is a unary p-time computable function on } \mathbb{N}\}$[4] then we directly get: For any open $\mathcal{L}_{PV}$-formula $\varphi(x,y)$: $\widetilde{\mathrm{PV}} \nvdash \forall x \exists y \varphi(x,y)$ if and only if there is an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $\mathcal{P}/\mathcal{U} \models \exists x \forall y \neg\varphi(x,y)$

1.2 Model-theoretic properties

In this section we discuss some basic model-theoretic properties of powers which we will extensively use in latter sections. We start with a brief observation and leave comments after the proof.

Observation 1.2.1. *Let $\mathbb{M}$ be an infinite $\mathcal{L}$-structure. Suppose $(\mathcal{F}, \Omega)$ defines a power construction over $\mathbb{M}$ in language $\mathcal{L}$ and $\mathcal{U}, \mathcal{V}$ are filters over Ω such that for any open $\mathcal{L}(\mathcal{F})$-sentence φ: $\langle\langle\varphi\rangle\rangle_\Omega \in \mathcal{U}$ if and only if $\langle\langle\varphi\rangle\rangle_\Omega \in \mathcal{V}$. Then $\mathcal{F}/\mathcal{U}$ and $\mathcal{F}/\mathcal{V}$ are one structure. Whence the structures equal on domains and interpretation of the language $\mathcal{L}$ and thus are identical.*

Proof. First observe that $g^{\mathcal{U}} = g^{\mathcal{V}}$ for every $g \in \mathcal{F}$ since for every $g, h \in \mathcal{F}$, $\langle\langle g = h\rangle\rangle_\Omega \in \mathcal{U}$ if and only if $\langle\langle g = h\rangle\rangle_\Omega \in \mathcal{V}$ and so $\mathcal{F}/\mathcal{U} = \mathcal{F}/\mathcal{V}$. Then for any function symbol $F(\bar{x}) \in \mathcal{L}$ and $\bar{f}^{\mathcal{U}} \in \mathcal{F}$, $F^{\mathcal{F}/\mathcal{U}}(\bar{f}^{\mathcal{U}}) = [F^{\mathbb{M}} \circ \bar{f}]^{\mathcal{U}} = [F^{\mathbb{M}} \circ \bar{f}]^{\mathcal{V}} = F^{\mathcal{F}/\mathcal{V}}(\bar{f}^{\mathcal{V}})$. But also for any relation symbol $R \in \mathcal{L}$ and $\bar{g} \in \mathcal{F}$, $\bar{g}^{\mathcal{U}} \in R^{\mathcal{F}/\mathcal{U}}$ if and only if $\{\omega \in \Omega \mid \bar{g}(\omega) \in R^{\mathbb{M}}\} = \langle\langle R(\bar{g})\rangle\rangle_\Omega \in \mathcal{U}$ if and only if $\langle\langle R(\bar{g})\rangle\rangle_\Omega = \{\omega \in \Omega \mid \bar{g}(\omega) \in R^{\mathbb{M}}\} \in \mathcal{V}$ if and only if $\bar{g}^{\mathcal{V}} \in R^{\mathcal{F}/\mathcal{V}}$. Whence □

Now consider the following situation: Assume $\mathbb{M}$ is an infinite $\mathcal{L}$-structure, $\Omega \subseteq \mathbb{M}$ an $\mathbb{M}$-definable infinite set and $\mathcal{F} \subseteq {}^{\mathbb{M}}\mathbb{M}$ an infinite $\mathcal{L}$-closed family of $\mathbb{M}$-definable functions. Then for any $\mathcal{L}$-formula $\varphi(\bar{x})$ and $\bar{f} \in \mathcal{F}$, the set $\langle\langle\varphi(\bar{f})\rangle\rangle_\Omega$ is $\mathbb{M}$-definable. Thus if $\mathcal{U}$ is a filter on Ω then by the observation above the properties of $\mathcal{F}/\mathcal{U}$ depends only on definable sets contained in $\mathcal{U}$. Hence when working in this setting working with ultrafilters on the algebra of all subsets of Ω does not give any possibility to construct more models.

We will continue with model theoretic properties of powers.

Lemma 1.2.2. *Assume $(\mathcal{F}, \Omega)$ defines a power construction over $\mathbb{M}$ in language $\mathcal{L}$ and $\mathcal{U}$ is an ultrafilter on Ω. Suppose that for every $m \in \mathbb{M}$ there is a function $c_m \in \mathcal{F}$ which is constant m on Ω. Then the function $\pi : \mathbb{M} \to \mathcal{F}/\mathcal{U}$ defined by $\pi(m) = c_m^{\mathcal{U}}$ is an embedding of $\mathbb{M}$ into $\mathcal{F}/\mathcal{U}$.*

[4] i.e. $\mathcal{P} = \{t^{\mathbb{N}} \mid t(x) \text{ is a term of } \mathcal{L}_{PV}\}$

Proof. Observe that π is injective as for any m_1, m_2, $\langle\langle c_{m_1} = c_{m_2}\rangle\rangle_\Omega = \emptyset$ and thus $c^{\mathcal{U}}_{m_1} \neq c^{\mathcal{U}}_{m_2}$. We will show that for any $m_0, m_1, \ldots, m_{k-1} \in \mathbb{M}$ and an open $\mathcal{L}$-formula $\varphi(\bar{x})$:

$$\mathbb{M} \models \varphi(m_0, m_1, \ldots, m_{k-1}) \text{ if and only if } \mathcal{F}/\mathcal{U} \models \varphi(\pi(m_0), \ldots, \pi(m_{k-1})).$$

Since by Corollary 1.1.5 open formulae are Łos for $\mathcal{F}$, we have that $\mathcal{F}/\mathcal{U} \models \varphi(\pi(m_0), \ldots, \pi(m_{k-1}))$ if and only if $\langle\langle \varphi(c_{m_0}, \ldots, c_{m_{k-1}})\rangle\rangle_\Omega \in \mathcal{U}$. But $A = \langle\langle \varphi(c_{m_0}, \ldots, c_{m_{k-1}})\rangle\rangle_\Omega = \{\omega \in \Omega \mid \mathbb{M} \models \varphi(m_0, \ldots, m_{k-1})\}$ and so A is either Ω in case that $\mathbb{M} \models \varphi(m_0, m_1, \ldots, m_{k-1})$ or $\emptyset$ otherwise. Thus $\mathbb{M} \models \varphi(m_0, m_1, \ldots, m_{k-1})$ if and only if $\langle\langle \varphi(\pi(m_0), \ldots, \pi(m_{k-1}))\rangle\rangle_\Omega \in \mathcal{U}$ and we are done. $\square$

Recall that an uncountable model $\mathbb{M}$ in language $\mathcal{L}$ is $\aleph_1$-saturated if for every countable $A \subseteq \mathbb{M}$, any natural number r and an r-type p over A, p is realised in $\mathbb{M}$ whenever it is finitely satisfiable in $\mathbb{M}$.

Lemma 1.2.3. *Let $\mathbb{M}$ be a countable infinite structure in countable language $\mathcal{L}$, $\Omega \subseteq \mathbb{M}$ an infinite countable set and $\mathcal{U}$ a non-principal ultrafilter on Ω. Then ${}^\Omega\mathbb{M}/\mathcal{U}$ is $\aleph_1$-saturated (of size continuum) and ${}^\Omega\mathbb{M}/\mathcal{U} \models \mathrm{Th}(\mathbb{M})$.*

Proof. Let a non-principal ultrafilter $\mathcal{U}$ over Ω be given. To show that ${}^\Omega\mathbb{M}/\mathcal{U}$ is $\aleph_1$-saturated let $A \subseteq {}^\Omega\mathbb{M}/\mathcal{U}$ be a countable set and p be an $\mathcal{L}$-type over A which is finitely satisfiable in $\mathbb{M}$. For notational simplicity we assume that p is a 1-type. Let $\{\varphi_i(x, \bar{f}^{\mathcal{U}}_i)\}_{i\in\mathbb{N}}$ be some enumeration of p where $\bar{f}^{\mathcal{U}}_i$ denotes the tuple of parameters from A that appears in φ_i. Now for every $n \in \mathbb{N}$ let $g_n \in {}^\Omega\mathbb{M}$ and $B_n \subseteq \Omega$ be such that ${}^\Omega\mathbb{M}/\mathcal{U} \models \wedge_{i\leq n}\varphi_i(g^{\mathcal{U}}_i, \bar{f}^{\mathcal{U}}_i)$ and $B_n = \langle\langle \wedge_{i\leq n}\varphi(g_i, \bar{f}_i)\rangle\rangle_\Omega$. Such g_n exists for every $n \in \mathbb{N}$ since p is finitely satisfiable in ${}^\Omega\mathbb{M}/\mathcal{U}$ and all $\mathcal{L}$-formulae are Łos by the Corollary 1.1.11. We clearly have that for any $n \in \mathbb{N}, B_n \supseteq B_{n+1}$. Moreover for every $\omega \in \Omega$ we define

$$r(\omega) = \begin{cases} \max\{n \mid \omega \in B_n\} & \text{if } \omega \in \bigcup_{n\in\mathbb{N}} B_n \text{ and } \max\{n \mid \omega \in B_n\} \text{ exists} \\ \infty & \text{otherwise.} \end{cases}$$

Now we define a function $\delta \in {}^\Omega\mathbb{M}$ such that $\delta^{\mathcal{U}}$ realises p in ${}^\Omega\mathbb{M}/\mathcal{U}$. Let $\Omega = \{\omega_i\}_{i\in\mathbb{N}}$ be some enumeration of Ω and for ω_i define

$$\delta(\omega_i) = \begin{cases} g_{r(\omega_i)}(\omega_i) & \text{if } r(\omega_i) < \infty \\ g_i(\omega_i) & \text{otherwise.} \end{cases}$$

Now assume $B = \bigcap_{i\in\omega} B_i \in \mathcal{U}$. We claim that for any $n \in \mathbb{N}$, $B - \{\omega_0, \ldots \omega_{n-1}\} \in \mathcal{U}$ and thus for any $n \in \mathbb{N}$, ${}^\Omega\mathbb{M}/\mathcal{U} \models \wedge_{i\leq n}\varphi_i(\delta^{\mathcal{U}}, \bar{f}^{\mathcal{U}}_i)$. To see this note that B is infinite since $\mathcal{U}$ is non-principle and $B - \{\omega_0, \ldots \omega_{n-1}\} \in \mathcal{U}$ for any n since $\mathcal{U}$ is an ultrafilter. Moreover if $n \in \mathbb{N}$ is given and $\omega \in B - \{\omega_0, \ldots \omega_{n-1}\}$ then $\omega = \omega_{n'}$ for some $n' \in \mathbb{N}$ with $n' \geq n$. But since $\omega \in B$ we get $r(\omega) = \infty$ and thus $\delta(\omega) = g_{n'}(\omega)$. As $\omega \in B_{n'}$ the definition of $g_{n'}$ and $B_{n'}$ gives $\mathbb{M} \models \wedge_{i\leq n'}\varphi_i(\delta(\omega), \bar{f}_i(\omega))$. But since $n' \geq n$ we have that for any $\omega \in B - \{\omega_0, \ldots, \omega_{n-1}\}$, $\mathbb{M} \models \wedge_{i\leq n}\varphi_i(\delta(\omega), \bar{f}_i(\omega))$. Finally by the Corollary 1.1.11 all $\mathcal{L}$-formulae are Łos for ${}^\Omega\mathbb{M}/\mathcal{U}$ and that $B - \{\omega_0, \ldots, \omega_{n-1}\} \in \mathcal{U}$ gives ${}^\Omega\mathbb{M}/\mathcal{U} \models \wedge_{i\leq n}\varphi_i(\delta^{\mathcal{U}}, \bar{f}^{\mathcal{U}}_i)$.

If on the other hand $B \notin \mathcal{U}$ then $B_n - B \in \mathcal{U}$ for every $n \in \mathbb{N}$. But if $\omega \in B_n - B$ then $n \leq r(\omega) < \infty$ (by $B_k \supseteq B_{k+1}$ for all $k \in \mathbb{N}$) and so $\delta(\omega) = g_{r(\omega)}(\omega)$. Thus by the definition of $B_{r(\omega)}$ and $g_{r(\omega)}$, $\mathbb{M} \models \bigwedge_{i \leq r(\omega)} \varphi_i(\delta(\omega), \bar{f}_i(\omega))$ and so since all $\mathcal{L}$-formulae are Łos for ${}^{\Omega}\mathbb{M}/\mathcal{U}$ and $B_n - B \in \mathcal{U}$ we get that ${}^{\Omega}\mathbb{M}/\mathcal{U} \models \wedge_{i \leq n} \varphi_i(\delta^{\mathcal{U}}, \bar{f}_i^{\mathcal{U}})$. Since for every $n \in \mathbb{N}$, $B_n - B \in \mathcal{U}$ we get that $\delta^{\mathcal{U}}$ realises p in ${}^{\Omega}\mathbb{M}/\mathcal{U}$.

The statement ${}^{\Omega}\mathbb{M}/\mathcal{U} \models \mathrm{Th}(\mathbb{M})$ is a part of the Corollary 1.1.11.

□

Lemma 1.2.4. *Assume $(\mathcal{F}, \Omega)$ defines a power construction over $\mathbb{M}$ in language $\mathcal{L}$ and $\mathcal{U}$ is an ultrafilter on Ω. Suppose further that $\mathbb{K}$ is a substructure of $\mathcal{F}/\mathcal{U}$. Then $\mathcal{G} = \{f \in \mathcal{F} \mid f^{\mathcal{U}} \in \mathbb{K}\}$ is $\mathcal{L}$-closed and in particular $\mathcal{G}/\mathcal{U} = \mathbb{K}$.*

Proof. Let $\mathcal{G}$ be as stated. We first show that $\mathcal{G}$ is $\mathcal{L}$-closed. To do so, let a functional symbol $F(\bar{x}) \in \mathcal{L}$ and $\bar{g} \in \mathcal{G}$ be given. Since $\mathcal{F}$ is $\mathcal{L}$-closed, there is a function $h \in \mathcal{F}$ such that $F^{\mathbb{M}} \circ \bar{g} = h$. By the Lemma 1.0.2 we have $h^{\mathcal{U}} = [F^{\mathbb{M}} \circ \bar{g}]^{\mathcal{U}} = F^{\mathcal{F}/\mathcal{U}}(\bar{g}^{\mathcal{U}})$. By the definition of $\mathcal{G}$ we have that $\bar{g}^{\mathcal{U}} \in \mathbb{K}$ and thus $F^{\mathbb{K}}(\bar{g}^{\mathcal{U}}) = F^{\mathcal{F}/\mathcal{U}}(\bar{g}^{\mathcal{U}}) = h^{\mathcal{U}} \in \mathbb{K}$ i.e. $h \in \mathcal{G}$ and we are done.

To show that $\mathcal{G}/\mathcal{U} \cong \mathbb{K}$ via the identity function let an open $\mathcal{L}$-formula $\varphi(\bar{x})$ and $\bar{g}^{\mathcal{U}} \in \mathcal{G}/\mathcal{U}$ be given. Recall that by the Corollary 1.1.5 $\varphi(\bar{x})$ is Łos for $\mathcal{G}/\mathcal{U}$ and $\mathcal{F}/\mathcal{U}$ thus we have that $\mathcal{G}/\mathcal{U} \models \varphi(\bar{g}^{\mathcal{U}})$ if and only if $\langle\langle \varphi(\bar{g}) \rangle\rangle_{\Omega} \in \mathcal{U}$ if and only if $\mathcal{F}/\mathcal{U} \models \varphi(\bar{g}^{\mathcal{U}})$ if and only if $\mathbb{K} \models \varphi(\bar{g}^{\mathcal{U}})$ and we are done.

□

Lemma 1.2.5. *Suppose $(\mathcal{F}, \Omega), (\mathcal{G}, \Omega)$ define a power construction over $\mathbb{M}$ in language $\mathcal{L}$, $\mathcal{F} \subseteq \mathcal{G}$. Then $\mathcal{F}/\mathcal{V} \leq \mathcal{G}/\mathcal{V}$ for every ultrafilter $\mathcal{V}$ over Ω.*

Proof. Let a filter $\mathcal{V}$ over Ω be given. Then $\mathcal{F} \subseteq \mathcal{G}$ gives $\mathcal{F}/\mathcal{V} \subseteq \mathcal{G}/\mathcal{V}$ and the rest is the definition of a power. □

The following Lemma gives a different perspective on Łos property.

Lemma 1.2.6. *Suppose $(\mathcal{F}, \Omega), (\mathcal{G}, \Omega)$ define a power construction over $\mathbb{M}$ in language $\mathcal{L}$ an $\mathcal{F} \subseteq \mathcal{G}$. Suppose further $\mathcal{V}$ is a filter on Ω and $\varphi(\bar{x})$ an $\mathcal{L}$-formula which is Łos for $\mathcal{G}/\mathcal{V}$.*

Then $\varphi(\bar{x})$ is Łos for $\mathcal{F}/\mathcal{V}$ if and only if it is absolute between $\mathcal{F}/\mathcal{V}$ and $\mathcal{G}/\mathcal{V}$.

Proof. First note that by the previous lemma $\mathcal{F}/\mathcal{V}$ is a substructure of $\mathcal{G}/\mathcal{V}$ and so the notion of absoluteness between $\mathcal{F}/\mathcal{V}$ and $\mathcal{G}/\mathcal{V}$ is well-defined. To shows the left-right implication let $\bar{f}^{\mathcal{V}} \in \mathcal{F}/\mathcal{V}$ be given and assume $\varphi(\bar{x})$ is Łos for $\mathcal{F}/\mathcal{V}$. Since $\varphi(\bar{x})$ is also Łos for $\mathcal{G}/\mathcal{V}$ we have that $\mathcal{F}/\mathcal{V} \models \varphi(\bar{f}^{\mathcal{V}})$ if and only if $\langle\langle \varphi(\bar{f}) \rangle\rangle_{\Omega} \in \mathcal{V}$ if and only if $\mathcal{G}/\mathcal{V} \models \varphi(\bar{f}^{\mathcal{V}})$ and we are done.

For the right-left implication assume for a contradiction $\varphi(\bar{x})$ is absolute between $\mathcal{G}/\mathcal{V}$ and $\mathcal{F}/\mathcal{V}$ but not Łos for $\mathcal{F}/\mathcal{V}$. Then there is $\bar{f}^{\mathcal{V}} \in \mathcal{F}/\mathcal{V}$ with either $\mathcal{F}/\mathcal{V} \models \varphi(\bar{f}^{\mathcal{V}})$ and $\langle\langle \varphi(\bar{f}) \rangle\rangle_{\Omega} \notin \mathcal{V}$ or $\mathcal{F}/\mathcal{V} \models \neg\varphi(\bar{f}^{\mathcal{V}})$ and $\langle\langle \varphi(\bar{f}) \rangle\rangle_{\Omega} \in \mathcal{V}$. In the former case we get by the absoluteness of $\varphi(\bar{x})$ that $\mathcal{G}/\mathcal{V} \models \varphi(\bar{f}^{\mathcal{V}})$. But then $\langle\langle \varphi(\bar{f}) \rangle\rangle_{\Omega} \in \mathcal{V}$ since $\varphi(\bar{x})$ is Łos for $\mathcal{G}/\mathcal{V}$ contradicting the assumptions of the former case. On the other hand the latter case gives $\mathcal{G}/\mathcal{V} \not\models \varphi(\bar{f}^{\mathcal{V}})$ thus $\langle\langle \varphi(\bar{f}) \rangle\rangle_{\Omega} \notin \mathcal{V}$ contradicting $\langle\langle \varphi(\bar{f}) \rangle\rangle_{\Omega} \in \mathcal{V}$ which finishes the argument. □

Corollary 1.2.7. *(1)Let $\mathbb{M}$ be an $\mathcal{L}$-structure. Then for any $\mathcal{L}$-closed set of functions $\mathcal{F} \subseteq {}^{\mathbb{M}}\mathbb{M}$ and an ultrafilter $\mathcal{U}$ on $\mathbb{M}$ we have that for any $\mathcal{L}$-formula $\varphi(\bar{x})$: $\varphi(\bar{x})$ is Łos for $\mathcal{F}/\mathcal{U}$ if and only if $\varphi(\bar{x})$ is absolute between $\mathcal{F}/\mathcal{U}$ and ${}^{\mathbb{M}}\mathbb{M}/\mathcal{U}$.*

(2) Assume $\mathcal{L} = \mathcal{L}_{\mathrm{PA}}$ and $\mathbb{M} \models \mathrm{PA}$. Let $\mathcal{D}$ be the set of all unary $\mathbb{M}$-definable functions. Then for any $\mathcal{L}_{\mathrm{PA}}$-closed set of functions $\mathcal{F} \subseteq \mathcal{D}$ and an ultrafilter $\mathcal{U}$ over $\mathbb{M}$ we have for any $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(\bar{x})$: $\varphi(\bar{x})$ is Łos for $\mathcal{F}/\mathcal{U}$ if and only if $\varphi(\bar{x})$ is absolute between $\mathcal{F}/\mathcal{U}$ and $\mathcal{D}/\mathcal{U}$.

Proof. By the Corollary 1.1.11, $\mathcal{L}_{\mathrm{PA}}$-formulae are Łos for ${}^{\mathbb{M}}\mathbb{M}/\mathcal{U}$ and by the Example 1.1.12 all $\mathcal{L}_{\mathrm{PA}}$-formulae are Łos for $\mathcal{D}$. Thus (1) and (2) follows by the previous lemma. □

We will give two examples of a use of the previous lemma. Recall that for any models $\mathbb{M}_1, \mathbb{M}_2$ with $\mathbb{M}_2 \models \mathrm{I}\Delta_0$ we have that $\mathbb{M}_1 \models \mathrm{I}\Delta_0$ and Δ_0 formulae are absolute between $\mathbb{M}_1$ and $\mathbb{M}_2$ whenever $\mathbb{M}_1$ is an initial segment of $\mathbb{M}_2$.[5].

Example 1.2.8. We first show a general way how to construct models of $\mathrm{I}\Delta_0$ via ultrapower constructions for which Δ_0-formulae are Łos .

Let $\mathbb{M}$ be a countable model of PA and let $\mathcal{D}$ be the set of unary $\mathbb{M}$-definable functions. Define the ordering $<^*$ on $\mathcal{D}$ by $f <^* g$ if and only if $f(n) < g(n)$ for all but finitely many $n \in \mathbb{M}$.

Now assume $\{f_i\}_{i\in\mathbb{N}} \subseteq \mathcal{D}$ is such that f_0 is constant 0, f_1 is constant 1 and for any $i \in \mathbb{M}$ there is $j \in \mathbb{M}$ with $f_i^2 <^* f_j$. The assumption on $\{f_i\}_{i\in\mathbb{N}}$ is strong enough to ensure that the set

$$\mathcal{F} = \{g \in \mathcal{D} \mid \text{there is } i \in \mathbb{N} \text{ such that } g <^* f_i\}$$

is $\mathcal{L}_{\mathrm{PA}}$-closed. Indeed, let $\diamond \in \{+, \cdot\} \subseteq \mathcal{L}_{\mathrm{PA}}$ and $g_1, g_2 \in \mathcal{F}$ be given. Then $g_1, g_2 <^* f_i$ for some $i \in \mathbb{N}$ and so there is $j \in \mathbb{N}$ such that $g_1 \diamond g_2 <^* f_i^2 <^* f_j$ i.e. $g_1 \diamond g_2 \in \mathcal{F}$. On the other hand the assumption on $\{f_i\}_{i\in\mathbb{N}}$ does not imply $\mathcal{F} = \mathcal{D}(\mathbb{M})$ as the assumption can be true even if all functions from $\{f_i\}_{i\in\mathbb{N}}$ are $<^*$-bounded by the function 2^x.

If we now let $\mathcal{U}$ be an ultrafilter on $\mathbb{M}$ then $\mathcal{F}/\mathcal{U}$ is an initial segment of $\mathcal{D}/\mathcal{U}$ by the definition of $\mathcal{F}$. But by the Example 1.1.12 we have $\mathcal{D}/\mathcal{U} \models \mathrm{PA}$ and so $\mathcal{F}/\mathcal{U} \models \mathrm{I}\Delta_0$. Moreover Δ_0-formulae are absolute between $\mathcal{F}/\mathcal{U}$ and $\mathcal{D}/\mathcal{U}$ and thus by the previous corollary we also get that Δ_0-formulae are Łos for $\mathcal{F}/\mathcal{U}$.

Example 1.2.9. (Parikh's theorem) Assume $\varphi(x, y) \in \Delta_0$ and $\mathrm{I}\Delta_0 \vdash \forall x \exists y \varphi(x, y)$. Then there is an $\mathcal{L}_{\mathrm{PA}}$-term $t(x)$ such that $\mathrm{I}\Delta_0 \vdash \forall x \exists y < t(x) \varphi(x, y)$.

Proof. Let $\varphi(x, y) \in \Delta_0$ with $\mathrm{I}\Delta_0 \vdash \forall \bar{x} \exists y \varphi(x, y)$ be given and assume for a contradiction that there is no $\mathcal{L}_{\mathrm{PA}}$-term $t(x)$ such that $\mathrm{I}\Delta_0 \vdash \forall x \exists y < t(x) \varphi(x, y)$. Then by the Compactness there is a countable model $\mathbb{M}$ of $\mathrm{I}\Delta_0$ such that for any $\mathcal{L}_{\mathrm{PA}}$-term $t(x)$ we have $\mathbb{M} \models \exists x \forall y < t(x) \neg\varphi(x, y)$.[6]

[5] i.e. whenever $\mathbb{M}_1 \le \mathbb{M}_2$ and $\mathbb{M}_1 = \{m \in \mathbb{M}_2 \mid \mathbb{M}_2 \models m \le m' \text{ for some } m' \in \mathbb{M}_1\}$

[6] The set $\mathrm{I}\Delta_0 \cup \{\exists x \forall y < t(x) \varphi(x, y) \mid t(x) \text{ is an } \mathcal{L}_{\mathrm{PA}} \text{ term}\}$ is consistent by the Compactness since $\mathrm{I}\Delta_0 \vdash \bigvee_{i \le n} \forall x \exists y < t_i(x) \varphi(x, y)$ for some $\mathcal{L}_{\mathrm{PA}}$terms $t_i(x)$ implies $\mathrm{I}\Delta_0 \vdash \forall x \exists y < t_0(x) + \ldots + t_n(x) \varphi(x, y)$.

Now we let for $f, g \in {}^{\mathbb{M}}\mathbb{M}$ the relation $\leq^*$ be defined as $f \leq^* g$ if and only if $f(m) \leq g(m)$ for all but finitely many $m \in \mathbb{M}$ and let

$$\mathcal{F} = \{f : \mathbb{M} \to \mathbb{M} \mid f \leq^* t^{\mathbb{M}} \text{ for some } \mathcal{L}_{\mathrm{PA}}\text{-term } t(x)\}.$$

which is clearly $\mathcal{L}_{\mathrm{PA}}$-closed[7]. We will first show that for any ultrafilter $\mathcal{U}$ on $\mathbb{M}$, $\mathcal{F}/\mathcal{U} \models \mathrm{I}\Delta_0$ and Δ_0-formulae are Łos for $\mathcal{F}/\mathcal{U}$. Then we define an ultrafilter $\mathcal{U}$ on $\mathbb{M}$ such that $\mathcal{F}/\mathcal{U} \models \forall y \neg\varphi(id^{\mathcal{U}}, y)$. This will contradict the assumption $\mathrm{I}\Delta_0 \vdash \forall x \exists y \varphi(x, y)$ and finish the argument.

Let an ultrafilter $\mathcal{U}$ on $\mathbb{M}$ be arbitrary. By the Lemma 1.2.3 we have that ${}^{\mathbb{M}}\mathbb{M}/\mathcal{U} \models \mathrm{I}\Delta_0$ as $\mathbb{M} \models \mathrm{I}\Delta_0$. But by the definition of $\mathcal{F}$, $\mathcal{F}/\mathcal{U}$ is an initial segment of ${}^{\mathbb{M}}\mathbb{M}/\mathcal{U}$ and so $\mathcal{F}/\mathcal{U} \models \mathrm{I}\Delta_0$. Moreover Δ_0-formulae are absolute between $\mathcal{F}/\mathcal{U}$ and ${}^{\mathbb{M}}\mathbb{M}/\mathcal{U}$ and so by the corollary above Δ_0-formulae are Łos for $\mathcal{F}/\mathcal{U}$.

To find the promised ultrafilter $\mathcal{U}$ over $\mathbb{M}$ let $id \in \mathcal{F}$ denote the identity function on $\mathbb{M}$ and let

$$A = \{\langle\langle \neg\varphi(id, f)\rangle\rangle_\Omega \mid f \in \mathcal{F}\}.$$

We claim that A has a finite intersection property. To show this, let $f_0, f_1, \ldots, f_{k-1} \in \mathcal{F}$ be given and assume for a contradiction that $\bigcap_{i<k} \langle\langle \neg\varphi(id, f_i)\rangle\rangle_\Omega = \emptyset$. Let $t_0, t_1, \ldots, t_{k-1}$ be unary $\mathcal{L}_{\mathrm{PA}}$-terms such that $f_i \leq^* t_i^{\mathbb{M}}$ for every $i < k$. Then $\mathbb{M} \models \forall x \bigvee_{i<k} \exists y < t_i(x) + 1 \varphi(x, y)$. But then also $\mathbb{M} \models \forall x \exists y < t_1(x) +, \ldots + t_n(x) + k \varphi(x, y)$ thus there is a single $\mathcal{L}_{\mathrm{PA}}$-term $t_1(x) +, \ldots + t_n(x) + k$ contradicting the assumption on $\mathbb{M}$.

Now let $\mathcal{U}$ be an ultrafilter on $\mathbb{M}$ extending A and assume $\mathcal{F}/\mathcal{U} \models \exists y \varphi(id^{\mathcal{U}}, y)$. Then there is $f \in \mathcal{F}$ with $\mathcal{F}/\mathcal{U} \models \varphi(id^{\mathcal{U}}, f^{\mathcal{U}})$ but since Δ_0-formulae are Łos for $\mathcal{F}/\mathcal{U}$ this implies $\langle\langle \varphi(id, f)\rangle\rangle_\Omega \in \mathcal{U}$ contradicting the definition of $\mathcal{U} \supseteq A$.

□

[7]Clearly constant 0 and constant 1 functions are in $\mathcal{F}$. Thus it suffices to observe that if $f, g \in \mathcal{F}$ are such that for some $\mathcal{L}_{\mathrm{PA}}$-terms s, t, $f \leq^* s^{\mathbb{M}}$ and $g \leq^* t^{\mathbb{M}}$ then $f \cdot g, f + g \leq^* s^{\mathbb{M}} \cdot t^{\mathbb{M}}$.

1.3 Strong ultrapower

We prove the theorem we noted in the section discussing Łos property and Skolem functions. Recall that by the Corollary 1.1.7 any ultrapower over a model $\mathbb{N}$ is a model of the universal theory of $\mathbb{N}$. The following theorem shows that the other direction is true as well. Namely that any countable model $\mathbb{M}$ of the universal theory of (a countable model) $\mathbb{N}$ is isomorphic to an ultrapower over $\mathbb{N}$. We will start with definition of a special kind of ultrapower:

Definition 1.3.1. *Let $\mathbb{M}$ be an $\mathcal{L}$-structure, Ω infinite set and $\mathcal{F} \subseteq {}^{\Omega}\mathbb{M}$ infinite. We say that $\mathcal{F}$ is strong wrt to $\mathcal{L}$ if:*

(i) for any $f, g \in \mathcal{F}$ with $f \neq g$ the set $\langle\langle f = g\rangle\rangle_\Omega$ is finite,

(ii) for any $k \in \mathbb{N}$, k-ary function symbol $F \in \mathcal{L}$ and $g_0, g_1, \ldots, g_{k-1} \in \mathcal{F}$ there is $g \in \mathcal{F}$ such that $\langle\langle F(g_0, g_1, \ldots, g_{k-1}) = g\rangle\rangle_\Omega$ is co-finite,

(iii) for any $k \in \mathbb{N}$, k-ary relation symbol $R \in \mathcal{L}$ and $g_0, g_1, \ldots, g_{k-1} \in \mathcal{F}$, $\langle\langle R(g_0, g_1, \ldots, g_{k-1})\rangle\rangle_\Omega$ is finite or co-finite.

Observe that in the definition above there is exactly one such g in (ii) as by the condition (i) there is at most one such g.

The structure $\mathcal{F}_{Fin}$ Let $\mathbb{M}$ be an $\mathcal{L}$-structure, Ω infinite set and $\mathcal{F} \subseteq {}^{\Omega}\mathbb{M}$ strong wrt to $\mathcal{L}$. Then we denoted by $\mathcal{F}_{Fin}$ an $\mathcal{L}$-structure with the universe $\mathcal{F}$ and the interpretation of symblos from $\mathcal{L}$ defined as follows:

- For a k-ary function symbol $F \in \mathcal{L}$ and $g_0, g_1, \ldots, g_{k-1}$,

$$F^{\mathcal{F}_{Fin}}(g_0, g_1, \ldots, g_{k-1}) = g$$

for the $g \in \mathcal{F}$ such that $\langle\langle F(g_0, g_1, \ldots, g_{k-1}) = g\rangle\rangle_\Omega$ is co-finite.

- For a k-ary relation symbol $R \in \mathcal{L}$ we let

$$R^{\mathcal{F}_{Fin}} = \{(f_0, f_1, \ldots, f_{k-1}) \in \mathcal{F}^k \mid \langle\langle R(f_0, f_1, \ldots, f_{k-1})\rangle\rangle_\Omega \text{ is co-finite } \}.$$

We will call the structure $\mathcal{F}_{Fin}$ of this form a *strong ultrapower over a model* $\mathbb{M}$ *with domain* Ω (in language $\mathcal{L}$). and leave out the "with domain Ω" if $\Omega = \mathbb{M}$. The following observation shows that strong ultrapowers are indeed a special kind of ultrapowers.

Observation 1.3.2. *Let $\mathbb{M}$ be an $\mathcal{L}$-structure, Ω an infinite set and $\mathcal{F}_{Fin}$ a strong ultrapower over $\mathbb{M}$ with domain Ω. Let $\mathcal{F}^* = \{f \in {}^{\Omega}\mathbb{M} \mid$ there is $g \in \mathcal{F}$ and $\langle\langle f = g\rangle\rangle_\Omega$ is co-finite$\}$. Then*

(i) for any open $\mathcal{L}$-formula $\varphi(\bar{x})$ and $\bar{f} \in \mathcal{F}$, $\langle\langle \varphi(\bar{f})\rangle\rangle_\Omega$ is finite or co-finite,

(ii) for any open $\mathcal{L}$-formula $\varphi(\bar{x})$ and $\bar{f} \in \mathcal{F}$, $\mathcal{F}_{Fin} \models \varphi(\bar{f})$ if and only if $\langle\langle \varphi(\bar{f})\rangle\rangle_\Omega$ is co-finite,

(iii) $\mathcal{F}^*$ is $\mathcal{L}$-closed and $\mathcal{F}_{Fin} \cong \mathcal{F}^*/\mathcal{U}$ for any non-principal ultrafilter $\mathcal{U}$ on Ω.

Proof. To show (i) we proceed by induction on complexity of an open $\mathcal{L}$-formula $\varphi(\bar{x})$.

a The case when $\varphi(\bar{x})$ is of the form $t(\bar{x}) = s(\bar{x})$ for $\mathcal{L}$-terms $t(\bar{x}), s(\bar{x})$ or of the form $R(t_0(\bar{x}), \ldots, t_k(\bar{x}))$ for a relation symbol $R \in \mathcal{L}$ and $\mathcal{L}$-terms $t_0(\bar{x}), \ldots, t_k(\bar{x})$ follows easily from the definition of $\mathcal{F}$ being strong.

b The case when $\varphi(\bar{x})$ is a boolean combination of formulae for which the induction assumption holds follows by an observation that a boolean combination of finite and co-finite sets is a finite or co-finite set.

To show (ii) we will proceed by the induction on complexity of an open $\mathcal{L}$-formula $\varphi(\bar{x})$.

a The case when $\varphi(\bar{x})$ is of the form $t(\bar{x}) = s(\bar{x})$ for $\mathcal{L}$-terms $t(\bar{x}), s(\bar{x})$ or of the form $R(t_0(\bar{x}), \ldots, t_k(\bar{x}))$ for a relation symbol $R \in \mathcal{L}$ and $\mathcal{L}$-terms $t_0(\bar{x}), \ldots, t_k(\bar{x})$ follows easily from the definition of $\mathcal{F}$ being strong.

b Assume $\varphi(\bar{x}) = \neg\psi(\bar{x})$ for some $\psi(\bar{x})$ for which the induction assumption holds and $\bar{f} \in \mathcal{F}$ is given. Then $\mathcal{F}_{Fin} \models \neg\psi(\bar{f})$ if and only if $\mathcal{F}_{Fin} \not\models \psi(\bar{f})$ which is by induction assumption if and only if $\langle\langle\psi(\bar{f})\rangle\rangle_\Omega$ is not co-finite which is by (i) if and only if $\langle\langle\psi(\bar{f})\rangle\rangle_\Omega$ is finite. But the last is if and only if $\langle\langle\varphi(\bar{f})\rangle\rangle_\Omega$ is co-finite.

c The case where $\varphi(\bar{x}) = \psi(\bar{x}) \circ \theta(\bar{x})$ for $\circ \in \{\vee, \wedge\}$ is similar and we leave it to the reader.

To show (iii) fix a non-principal ultrafilter $\mathcal{U}$ on Ω. We claim that the function $i : \mathcal{F}_{Fin} \to \mathcal{F}^*/\mathcal{U}$ defined by $i : f \mapsto f^{\mathcal{U}}$ is an isomorphism. To show it is onto let $g^{\mathcal{U}} \in \mathcal{F}^*/\mathcal{U}$ be given. Then there is $f \in \mathcal{F}$ such that $\langle\langle f = g\rangle\rangle_\Omega$ is co-finite. But as $\mathcal{U}$ is non-principal and maximal, this gives $\langle\langle f = g\rangle\rangle_\Omega \in \mathcal{U}$ and so $f^{\mathcal{U}} = g^{\mathcal{U}}$ i.e. $i(f) = g^{\mathcal{U}}$. To show i is an injective embedding let $\varphi(\bar{x})$ be an open $\mathcal{L}$-formula and $\bar{f} \in \mathcal{F}$. Then by (ii) we have $\mathcal{F}_{Fin} \models \varphi(\bar{f})$ if and only if $\langle\langle\varphi(\bar{f})\rangle\rangle_\Omega$ is co-finite which is if and only if $\langle\langle\varphi(\bar{f})\rangle\rangle_\Omega \in \mathcal{U}$ since $\mathcal{U}$ is non-principal and maximal. But the last holds if and only if $\mathcal{F}^*/\mathcal{U} \models \varphi(\bar{f})$ since open formulae are Łos for $\mathcal{F}^*/\mathcal{U}$ and we are done. □

Now we can state the theorem of this section:

Theorem 1.3.3. *Let* $\mathbb{N}$, $\mathbb{M}$ *be countable models in a countable language* $\mathcal{L}$ *and assume that* $\mathbb{M}$ *models the universal theory of model* $\mathbb{N}$. *Then* $\mathbb{M}$ *is isomorphic to a strong ultrapower over* $\mathbb{N}$.

Proof. Let $\{m_i\}_{i\in\omega}$ be an injective enumeration of $\mathbb{M}$ and wlog assume $\mathbb{N} = \omega$. The idea of the proof goes as follows: we are aiming to define a set of functions $\mathcal{F} = \{f_i\}_{i\in\omega} \subseteq {}^{\mathbb{N}}\mathbb{N}$ such that for any open $\mathcal{L}$-formula $\varphi(x_1, ..., x_k)$ and $i_1, ..., i_k$ from ω:

$$\text{(1)} \qquad \mathbb{M} \models \varphi(m_{i_1}, ..., m_{i_k}) \text{ if and only if } \mathbb{N} \models \varphi(f_{i_1}(n), ..., f_{i_k}(n)) \text{ for all but finitely many } n \in \mathbb{N}$$

Using (1) we show that $\mathcal{F}$ is strong wrt to $\mathcal{L}$ and that the function from $\mathcal{F}_{Fin}$ to $\mathbb{M}$ mapping m_i to f_i is an isomorphism.

We will start with some notations. For every $i \in \omega$ we let

$$p_i = \{\varphi(x_0, x_1, \dots, x_i) \mid \mathbb{M} \models \varphi(m_0, m_1, \dots, m_i) \text{ and } \varphi \text{ is an open } \mathcal{L}\text{-formula}\}.$$

For every $i \in \omega$ let $\{\chi^i_n\}_{n\in\omega} \subseteq p_i$ be such that for every $n \in \omega$, $\chi^i_{n+1} \to \chi^i_n$ and for any $\varphi \in p_i$ there is n such that $\chi^i_n \to \varphi$. Such a set exists for every $i \in \omega$ as for every $i \in \omega$, p_i is countable and closed under conjunction. Finally for $n < i$ we define:

$$\psi^i_n(y_0, y_1, \dots, y_i) = \chi^i_n(y_0, y_1, \dots, y_i)$$

and for $n \geq i$:

$$\psi^i_n(y_0, y_1, \dots, y_i) = \exists y_{i+1}, \dots, y_{n+1} \bigwedge_{j \leq n+1} \chi^j_n(y_0, y_1, \dots, y_j)$$

Where $y_0, y_1, \dots, y_i$ are not among $y_{i+1}, \dots, y_{n+1}$. Observe that for any $n \in \omega$, $\mathbb{M} \models \bigwedge_{j\leq n+1} \chi^j_n(m_0, m_1, \dots, m_j)$ and so $\mathbb{M} \models \psi^0_n(m_0)$.

Now we first inductively define the functions f_i and then show the construction is correct:

Let $f_0 \in {}^{\mathbb{N}}\mathbb{N}$ be arbitrary such that

$$\forall n \in \mathbb{N} : f_0(n) \in \psi^0_n(\mathbb{N})$$

where $\psi^0_n(\mathbb{N}) = \{a \in \mathbb{N} \mid \mathbb{N} \models \psi^0_n(a)\}$ and for $i > 0$ let $f_i \in {}^{\mathbb{N}}\mathbb{N}$ be arbitrary such that :

$$(2) \qquad \forall n \geq i : f_i(n) \in \psi^i_n(f_0(n), ..., f_{i-1}(n), \mathbb{N})$$

where $\psi^i_n(f_0(n), ..., f_{i-1}(n), \mathbb{N}) = \{a \in \mathbb{N} \mid \mathbb{N} \models \psi^i_n(f_0(n), ..., f_{i-1}(n), a)\}$.

To show that such functions exist, we have to show that for every i and every $n \geq i$ the set $\psi^i_n(f_0(n), ..., f_{i-1}(n), \mathbb{N})$ is non-empty To do so, we proceed by induction on i:

For $i = 0$ and an arbitrary n we have that $\mathbb{M} \models \psi^0_n(m_0)$ i.e. $\mathbb{M} \models \exists x \psi^0_n(x)$ and as ψ^0_n is existential formula and $\mathbb{M}$ is an models the universal theory of $\mathbb{N}$, we get that $\mathbb{N} \models \exists x \psi^0_n(x)$ and thus the set $\psi^0_n(\mathbb{N})$ is non-empty

Now assume that the induction hypothesis holds for i, i.e. there are $f_0, ..., f_i \in {}^{\mathbb{N}}\mathbb{N}$ satisfying (2). To show that $\psi^{i+1}_n(f_0(n), ..., f_i(n), \mathbb{N})$ is non-empty for every $n \geq i+1$ take an arbitrary n such. Then we have that

$$\mathbb{N} \models \psi^i_n(f_0(n), \dots, f_{i-1}(n), f_i(n))$$

and

$$\mathbb{N} \models \psi^i_n(f_0(n), ..., f_{i-1}(n), f_i(n)) \to \exists x \psi^{i+1}_n(f_0(n), ..., f_i(n), x)$$

since it is a tautology by the definition of ψ^i_n. Hence $\mathbb{N} \models \exists x \psi^{i+1}_n(f_0(n), ..., f_i(n), x)$ and thus $\psi^{i+1}_n(f_0(n), ..., f_i(n), \mathbb{N})$ is non-empty

Now we are ready to show that the property (1) holds i.e. that for any open formula φ and $i_1, ..., i_k \in \omega$:

$$\mathbb{M} \models \varphi(m_{i_1}, ..., m_{i_k}) \text{ if and only if}$$
$$\mathbb{N} \models \varphi(f_{i_1}(n), ..., f_{i_k}(n)) \text{ for all but finitely many } n \in \mathbb{N}$$

Assume $\mathbb{M} \models \varphi(m_{i_1}, ..., m_{i_k})$ and i_j is the maximal index, then there is $n_j \geq i_j$ with $\chi^{i_j}_{n_j} \to \varphi$ as $\varphi \in p_{i_j}$ and therefore $\psi^{i_j}_{n_j} \to \varphi$ as $\psi^{i_j}_{n_j} \to \chi^{i_j}_{n_j}$ by the definition of $\psi^{i_j}_{n_j}$. Moreover for any $n \geq n_j$ we have $\chi^{i_j}_{n} \to \chi^{i_j}_{n_j}$ and whence $\psi^{i_j}_{n} \to \varphi$ as $\psi^{i_j}_{n} \to \chi^{i_j}_{n}$. But by the definition of $f_{i_1}, ..., f_{i_k}$, for any $n \geq i_j : \mathbb{N} \models \psi^{i_j}_{n}(f_{i_0}(n)..., f_{i_j-1}(n), f_{i_j}(n))$ whence for any $n \geq n_j : \mathbb{N} \models \varphi(f_{i_1}(n), ..., f_{i_k}(n))$ and we are done.

On the other hand if $\mathbb{M} \not\models \varphi(m_{i_1}, ..., m_{i_k})$ then $\mathbb{M} \models \neg\varphi(m_{i_1}, ..., m_{i_k})$ hence $\mathbb{N} \models \neg\varphi(f_{i_1}(n), ..., f_{i_k}(n))$ for all but finitely many n's by the previous paragraph and whence $\mathbb{N} \not\models \neg\varphi(f_{i_1}(n), ..., f_{i_k}(n))$ for at most finitely many n's.

Armed with property (1) we can now show that $\mathcal{F}$ is strong wrt to $\mathcal{L}$. To check (i) of the definition assume $f_i, f_j \in \mathcal{F}$ and $i \neq j$ then since $\mathbb{M} \models m_i \neq m_j$ we get by (1) that $\langle\langle f_i = f_j \rangle\rangle_{\mathbb{N}}$ is finite. Similarly for (ii), let a k-ary function symbol $F \in \mathcal{L}$ and $f_{i_1}, ..., f_{i_k} \in \mathcal{F}$ be given. Then there is $m_i \in \mathbb{M}$ such that $\mathbb{M} \models F(m_{i_1}, ..., m_{i_k}) = m_i$ and so $\langle\langle F(f_{i_1}, ..., f_{i_k}) = f_i \rangle\rangle_{\mathbb{N}}$ is co-finite by (1). The condition (iii) follows in the same way and we leave it to the reader. Thus $\mathcal{F}$ is strong wrt to $\mathcal{L}$ and so defines a strong ultrapower $\mathcal{F}_{Fin}$ in language $\mathcal{L}$ over $\mathbb{N}$.

Finally we can show that the map $\xi : \mathcal{F}_{Fin} \to \mathbb{M}$ defined by $\xi(f_i) = m_i$ is an isomorphism. But this is clear since it is onto by definition and the rest follows by (1). Indeed, by (ii) of the Observation 1.3.2 we have that for any open $\mathcal{L}$-formula $\varphi(\bar{x})$ and $\bar{f} \in \mathcal{F}$:

$$\mathcal{F}_{Fin} \models \varphi(\bar{f}) \text{ if and only if } \langle\langle \varphi(\bar{f}) \rangle\rangle_{\mathbb{N}} \text{ is co-finite}$$

and so by (1) we get that that for any open $\mathcal{L}$-formula $\varphi(\bar{x})$ and $f_{i_1}, \ldots, f_{i_k} \in \mathcal{F}$:

$$\mathcal{F}_{Fin} \models \varphi(f_{i_1}, \ldots, f_{i_k}) \text{ if and only if } \mathbb{M} \models \varphi(m_{i_1}, \ldots, m_{i_k})$$

which finishes the proof.

$\square$

Chapter 2

Density arguments and weak inductions

Let $\mathbb{M}$ be a countable model of True Aritmetic in language $\mathcal{L}_{all}$. In this chapter we investigate some properties of power constructions with domain $\Omega \subseteq \mathbb{M}$ which is coded in $\mathbb{M}$ and a set of $\mathbb{M}$-definable functions $\mathcal{F}$.

In the first part of this chapter we show that there is some sort of density arguments which can be described as follows. Denote by $\mathcal{B}$ the algebra of $\mathbb{M}$-definable subsets of Ω and assume P is some model-theoretic property. Then one can sometimes find a countable family $\mathcal{D}_{\mathrm{P}}$ of subsets of $\mathcal{B}$ such that if a filter $\mathcal{V}$ on $\mathcal{B}$ intersects all sets from $\mathcal{D}_{\mathrm{P}}$ and $\mathcal{F}$ satisfies some additional properties corresponding to the property P then $\mathcal{F}/\mathcal{U}$ posses the property P.

In the next part of this chapter we combine this method with an idea from the Construction B of [Gar15] to give a general construction theorem that produces powers which are models of a weak form of induction. This theorem will be used latter in the Chapter 3 to derive the Construction B of Garlík given in [Gar15, Theorem 3.4]. Before we start, some preparations are in order:

Let us fix a countable non-standard model $\mathbb{M}$ of True arithmetic in the language $\mathcal{L}_{all}$. We will not distinguish between $\mathbb{M}$ and its expansion into a language augment by a function/realtion symbol for each $\mathbb{M}$-definable function/relation. Recall that a set definable in $\mathbb{M}$ is coded in $\mathbb{M}$ if and only if it is bounded. Suppose $A \subseteq \mathbb{M}$ is coded in $\mathbb{M}$ then we will not distinguish between A and its code in $\mathbb{M}$. We will stick to the common notation and write $m > \mathbb{N}$ if $m \in \mathbb{M} - \mathbb{N}$ or equivalently if $m \in \mathbb{M}$ is non-standard.

We will use basic concepts from the non-standard analysis and denote by $\mathbb{Q}_{\mathbb{M}}$ the set of codes of tuples (a, b) with $a, b \in \mathbb{M}$ and $b \geq 1$. We will call the elements of $\mathbb{Q}_{\mathbb{M}}$ as *(positive)* $\mathbb{M}$*-rationals* and write $\frac{a}{b}$ or a/b rather then (a, b) or simply q if the particular form is not important. The operations are defined on $\mathbb{Q}_{\mathbb{M}}$ as usual i.e. for example $a/b + c/d = (ad + bc)/(bd)$, $(a/b) \cdot (c/d) = ac/bd$, $(a/b)^c = a^c/b^c$ or $(a/b)/c$ is a shorthand for $(a/b) \cdot (1/c)$ and similarly for other common operations.[1] Moreover the order relation $\leq$ is extended to $\mathbb{M}$-rationals by $a/b \leq c/d$ if and only if $ad \leq bc$

[1] the last is an abuse of notation as $(a/b)/c$ should denotes the tuple $(\mathit{code}((a, b)), c)$ but this will never be the case in this thesis

and an $\mathbb{M}$-rational q is considered to be non-standard if $q > k/1$ for every $k \in \mathbb{N}$. Strictly speaking we should define another symbol for the order relation on $\mathbb{Q}_{\mathbb{M}}$ since one could confuse $a/b \leq c/d$ with comparing the codes of a/b and c/d. However, in the context of what will come the use of the relation $\leq$ will always be entirely clear and in particular it will always mean comparing two $\mathbb{M}$-rationals not its codes. For $a, b \in \mathbb{M}$ we will denote by $[a, b]_{\mathbb{Q}}$ the set $\{p/q \in \mathbb{Q}_{\mathbb{M}} \mid \mathbb{M} \models a/1 \leq p/q \leq b/1\}$. Moreover if $A \subseteq \mathbb{Q}_{\mathbb{M}}$ and $f : A \to \mathbb{Q}_{\mathbb{M}}$ then for $a, b, m \in \mathbb{M}$ we will often write $f(m)$ in place of $f(m/1)$ or $a/b \leq m$ in place of $a/b \leq m/1$ but this will always be clear from the context. Finally we will use the binary function symbol $-$ such that $\mathbb{M} \models \forall x, y(x - y =$ "the w such that $w + y = x$ if it exists and 0 otherwise").

Now we can finally fix

- a unary function symbol $\#(\cdot) \in \mathcal{L}_{all}$ with $rng(\#(\cdot)^{\mathbb{M}} \subseteq \mathbb{Q}_{\mathbb{M}}$ such that $\#(\cdot)^{\mathbb{M}}$ assign to every $A \subseteq \mathbb{M}$ which is coded in $\mathbb{M}$ its cardinality (or size) in $\mathbb{M}$,
- a set $\Omega \subseteq \mathbb{M}$ coded in $\mathbb{M}$ with $\#(\Omega) > \mathbb{N}$,
- the set $\mathcal{H}$ of all $\mathbb{M}$-definable functions from Ω to $\mathbb{M}$ and
- the algebra $\mathcal{B}$ of $\mathbb{M}$-definable subsets of Ω.

Note that since Ω is coded in $\mathbb{M}$ every function from $\mathcal{H}$ is coded in $\mathbb{M}$ and so $\mathcal{H}$ is countable as $\mathbb{M}$ is countable. Since we will work exclusively with filters on $\mathcal{B}$ we will not mention the algebra $\mathcal{B}$ i.e. from now on any filter considered in this section is a subset of $\mathcal{B}$.

Let further

- a non-standard $n \in \mathbb{M}$ be fixed,
- $\tilde{n}$ be a new constant symbol such that $\tilde{n}^{\mathbb{M}} = n$ and
- let $\mathcal{L}$ range over countable languages with $\tilde{n}, \leq \in \mathcal{L}$ such that the interpretation of function/relation symbols from $\mathcal{L}$ is $\mathbb{M}$-definable function/relation.

We denote by $\Delta_0^{<\tilde{n}}(\mathcal{L})$ the set of $\mathcal{L}$-formulae with all quantifiers strictly bounded by the term $\tilde{n}$ i.e. of the form $\exists x < \tilde{n}$ or $\forall x < \tilde{n}$. Moreover we let $\exists\Delta_0^{<\tilde{n}}(\mathcal{L}) = \{\exists \bar{y} \varphi(\bar{x}, \bar{y}) \mid \varphi(\bar{x}, \bar{y}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})\}$.

If $v \in \mathbb{M}$ then by c_v we denote the function from $\mathcal{H}$ which is constant v on Ω. If t, s are terms of $\mathcal{L}_{all}$ and $\bar{m} \in \mathbb{M}$ then we will often abuse the notation and write $t(\bar{m}) = s(\bar{m})$ or $t(\bar{m}) > s(\bar{m})$ instead of $t^{\mathbb{M}}(\bar{m}) = s^{\mathbb{M}}(\bar{m})$ or $t^{\mathbb{M}}(\bar{m}) >^{\mathbb{M}} s^{\mathbb{M}}(\bar{m})$, this is to make the arguments more readable. If f is a unary function (not necessarily $\mathbb{M}$-definable) such that $rng(f) \subseteq dom(f)$ then we denote by $f^{(0)}$ the identity function on $dom(f)$ and for any $k \in \mathbb{N}$ we denote by $f^{(k+1)}$ the function $f \circ f^{(k)}$. Moreover if $a, b \in \mathbb{M}$ then $[a, b]$ denotes as usual the set $\{m \in \mathbb{M} \mid \mathbb{M} \models a \leq m \leq b\}$.

2.1 Density arguments

Definition 2.1.1. *Let* $\beth : [0, \#(\Omega)]_{\mathbb{Q}} \to [0, \#(\Omega)]_{\mathbb{Q}}$, *we say that* $\beth$ *is large (wrt to* Ω*) if it is non-decreasing and for any* $k \in \mathbb{N}$, $\beth^{(k)}(\#(\Omega)) > \mathbb{N}$.

Definition 2.1.2. *Let* $D \subseteq \mathcal{B}$ *and* $\beth : [0, \#(\Omega)]_{\mathbb{Q}} \to [0, \#(\Omega)]_{\mathbb{Q}}$. *We say that* D *is* $\beth$*-dense (subset of* $\mathcal{B}$*) if for every* $A \in \mathcal{B}$ *there is a set* $B \in D$ *such that* $B \subseteq A$ *and* $\#(B) \geq \beth(\#(A))$.

Note that we do not demand $\gimel$ to be $\mathbb{M}$-definable and that in the context of the last definition $\#(A) \geq \#(B) \geq \gimel(\#(A))$ and thus it follows that $x \geq \gimel(x)$ for any $x \in [0, \#(\Omega)]_{\mathbb{Q}}$.

We will often write that D is a x/n-dense set or $(x-n)$-dense set etc. meaning that D is $\gimel$-dense set for $\gimel(x) = x/n$ or for $\gimel(x) = x - n$ both restricted to $[0, \#(\Omega)]_{\mathbb{Q}}$ etc.

Observation 2.1.3. *Let $\gimel, \daleth : [0, \#(\Omega)]_{\mathbb{Q}} \to [0, \#(\Omega)]_{\mathbb{Q}}$ be non-decreasing functions and $\mathcal{D}_{\gimel}$, $\mathcal{D}_{\daleth}$ countable non-empty families of $\gimel$-, $\daleth$-dense sets respectively. Then $\mathcal{D} = \mathcal{D}_{\gimel} \cup \mathcal{D}_{\daleth}$ is a countable family of $\gimel \circ \daleth$-dense sets. Moreover if $\gimel(x) \geq \daleth(x)$ for any $x \in [0, \#(\Omega)]_{\mathbb{Q}}$ then $\mathcal{D}_{\gimel}$ is a countable family of $\daleth$-dense sets.*

Proof. Let $D \in \mathcal{D}$ and $A \in \mathcal{B}$ be given. We have to show that there is $B \in D$, $B \subseteq A$ such that $\#(B) \geq \gimel \circ \daleth(\#(A))$. First assume $D \in \mathcal{D}_{\daleth}$, then there is $B \in D$ with $B \subseteq A$ such that $\#(B) \geq \daleth(\#(A)) \geq \gimel \circ \daleth(\#(A))$ where the last inequality holds as $x \geq \gimel(x)$ for any $x \in [0, \#(\Omega)]_{\mathbb{Q}}$. If $D \in \mathcal{D}_{\gimel}$ then there is $B \in D$ with $B \subseteq A$ such that $\#(B) \geq \gimel(\#(A)) \geq \gimel \circ \daleth(\#(A))$ where the last inequality holds as $x \geq \daleth(x)$ for any $x \in [0, \#(\Omega)]_{\mathbb{Q}}$ and $\gimel$ in non-decreasing. The moreover part follows easily from the definition of $\daleth$-dense set. $\square$

Lemma 2.1.4. *(Generic lemma) Let $\gimel : [0, \#(\Omega)]_{\mathbb{Q}} \to [0, \#(\Omega)]_{\mathbb{Q}}$ be a large function and let $\mathcal{D}$ be a countable family of $\gimel$-dense sets on $\mathcal{B}$. Then there is a filter $\mathcal{V}$ on $\mathcal{B}$ intersecting all sets from $\mathcal{D}$. Moreover $\mathcal{V}$ can be chosen such that for any $B \in \mathcal{V}$, $\#(B) > \mathbb{N}$.*

Proof. Let $\{D_k\}_{k \in \mathbb{N}}$ be some enumeration of $\mathcal{D}$. We construct a decreasing sequence $(A_k)_{k \in \mathbb{N}}$ of elements from $\mathcal{B}$ such that for any $k \in \mathbb{N}$, $A_{k+1} \in D_k$ and $\#(A_k) \geq \gimel^{(k)}(\#(\Omega))$. Then we let $\mathcal{V} = \{B \in \mathcal{B} \mid \text{there is } k \in \mathbb{N} \text{ with } B \supseteq A_k\}$. Such $\mathcal{V}$ will clearly intersects all dense sets from $\mathcal{D}$ and will also satisfy that for any $B \in \mathcal{V}$, $\#(B) > \mathbb{N}$. We will construct $(A_k)_{k \in \mathbb{N}}$ by induction on $\mathbb{N}$:

Let $A_0 = \Omega$. Assume A_k with $\#(A_k) \geq \gimel^{(k)}(\#(\Omega))$ is constructed. Since D_k is a $\gimel$-dense set, there is $A_{k+1} \in D_k$ such that $A_{k+1} \subseteq A_k$ and $\#(A_{k+1}) \geq \gimel(\#(A_k)) \geq \gimel(\gimel^{(k)}(\#(\Omega))) = \gimel^{(k+1)}(\#(\Omega))$ where we used that $\gimel$ is non-decreasing and the induction assumption on $\#(A_k)$. It is easy to see that the sequence constructed in this way posses the properties described above. $\square$

Corollary 2.1.5. *Let k be a natural number and $\gimel_0, \ldots, \gimel_{k-1} : [0, \#(\Omega)]_{\mathbb{Q}} \to [0, \#(\Omega)]_{\mathbb{Q}}$ be large functions. Assume that for every $i < k$, $\mathcal{D}_i$ is a countable family of $\gimel_i$-dense sets and that $\gimel_0 \circ \ldots \circ \gimel_{k-1}$ is large. Then there is a filter $\mathcal{V}$ on $\mathcal{B}$ intersecting all sets from $\mathcal{D} = \mathcal{D}_0 \cup \ldots \cup \mathcal{D}_{k-1}$ and for any $B \in \mathcal{V}$, $\#(B) > \mathbb{N}$.*

Proof. Using the Observation 2.1.3 ($k-1$ many times) $\mathcal{D}$ is a family of $\gimel_0 \circ \ldots \circ \gimel_{k-1}$-dense sets thus by the assumption on $\gimel_0 \circ \ldots \circ \gimel_{k-1}$ we can apply the *Generic Lemma* for $\mathcal{D}$. $\square$

Moving toward definition of some prominent countable families of dense sets we start with a notational convention and a simple observation. Let C be a covering of Ω. Then we define

$$D_C = \{B \in \mathcal{B} \mid B \subseteq c \text{ for some } c \in C\}.$$

Observation 2.1.6. *Let $C \in \mathbb{M}$ be a covering of Ω by $p \in \mathbb{M}$ many sets. Then D_C is x/p-dense set on $\mathcal{B}$.*

Proof. Assume $C = \{c_1, \ldots, c_p\}$ and let $A \in \mathcal{B}$ be given. Since $\{A \cap c_1, \ldots, A \cap c_p\} \in \mathbb{M}$ is a covering of A by p-many sets there is a $c \in C$ with $\#(A \cap c) \geq \#(A)/p$. But then also $A \cap c \in D_C$ by the definition of D_C and we are done. $\square$

The assumption that $C \in \mathbb{M}$ is crucial and in general we can not avoid it.

Suppose $\varphi(x, \bar{y}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})$ and $\bar{f} \in \mathcal{H}$. Then we denote by $\langle \exists x < \tilde{n}\varphi(x, \bar{f})\rangle$ the covering of Ω by the sets $\langle\langle \varphi(c_0, \bar{f})\rangle\rangle_\Omega, \langle\langle \varphi(c_1, \bar{f})\rangle\rangle_\Omega, \ldots, \langle\langle \varphi(c_{n-1}, \bar{f})\rangle\rangle_\Omega, \langle\langle \forall x < n\neg\varphi(x, \bar{f})\rangle\rangle_\Omega$ and define the set

$$\mathcal{D}_{\Delta_0^{<\tilde{n}}(\mathcal{L})-\text{Ł}os} = \{D_{\langle \exists x < \tilde{n}\varphi(x, \bar{f})\rangle} \mid \varphi(x, \bar{y}) \in \Delta_0^{<\tilde{n}}(\mathcal{L}) \text{ and } \bar{f} \in \mathcal{H}\}.$$

Since the functions from $\mathcal{H}$ are $\mathbb{M}$-definable and $\Omega \in \mathbb{M}$ we have that for any $\varphi(x, \bar{y})$ and $\bar{f}$ as above $\langle \exists x < \tilde{n}\varphi(x, \bar{f})\rangle \in \mathbb{M}$. As $\langle \exists x < \tilde{n}\varphi(x, \bar{f})\rangle$ is a partition of Ω into $n+1$ many sets, we get by the above observation that $\mathcal{D}_{\Delta_0^{<\tilde{n}}-\text{Ł}os}$ is a countable family of $(x/(n+1))$-dense sets.

Lemma 2.1.7. *Suppose $\mathcal{F} \subseteq \mathcal{H}$ is an $\mathcal{L}$-closed family of functions and $c_v \in \mathcal{F}$ for every $v \leq n$. Assume further $\mathcal{V}$ is a filter intersecting all sets from $\mathcal{D}_{\Delta_0^{<\tilde{n}}-\text{Ł}os}$. Then $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae are Łos for $\mathcal{F}/\mathcal{V}$ and for any $\alpha \in \mathcal{F}/\mathcal{U}$ such that $\mathcal{F}/\mathcal{V} \models \alpha \leq c_n^{\mathcal{V}}$ there is $v \leq n$ such that $\mathcal{F}/\mathcal{V} \models \alpha = c_v^{\mathcal{V}}$.*

In particular such filer $\mathcal{V}$ exists whenever $\#(\Omega)/(n+1)^k > \mathbb{N}$ for any $k \in \mathbb{N}$.

Proof. Fist observe that for any $\exists y\psi(y, \bar{x}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})$ and a tuple $\bar{f} \in \mathcal{H}$ either there is $w < n$ such that $\langle\langle c_w < n \wedge \psi(c_w, \bar{f})\rangle\rangle_\Omega \in \mathcal{V}$ or $\langle\langle \forall y < \tilde{n}\neg\psi(y, \bar{f})\rangle\rangle_\Omega \in \mathcal{V}$. Indeed, since $\mathcal{V}$ intersects the dense set $D_{\langle \exists y < \tilde{n}\psi(y, \bar{f})\rangle}$, there is $A \in \mathcal{V}$ such that $A \subseteq \langle\langle c_w < \tilde{n} \wedge \psi(c_w, \bar{f})\rangle\rangle_\Omega \subseteq \langle\langle \varphi(\bar{f})\rangle\rangle_\Omega$ for some $w < n$ or $A \subseteq \langle\langle \neg\varphi(\bar{f})\rangle\rangle_\Omega$ and we are done.

By the previous paragraph we immediately get that for any $\varphi(\bar{x}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})$ and $\bar{f} \in \mathcal{F}$: $\langle\langle \varphi(\bar{f})\rangle\rangle_\Omega \in \mathcal{V}$ or $\langle\langle \neg\varphi(\bar{f})\rangle\rangle_\Omega \in \mathcal{V}$. To show this, we can wlog assume $\varphi(\bar{x}) = \exists y < \tilde{n}\psi(y, \bar{x})$ for some $\psi(y, \bar{x}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})$ (otherwise go to an equivalent formula). But then by the previous paragraph for any $\bar{f} \in \mathcal{F}$ either there is $w < n$ with $\langle\langle c_w < \tilde{n} \wedge \psi(c_w, \bar{f})\rangle\rangle_\Omega \in \mathcal{V}$ and we are done as $\langle\langle c_w < \tilde{n} \wedge \psi(c_w, \bar{f})\rangle\rangle_\Omega \subseteq \langle\langle \varphi(\bar{f})\rangle\rangle_\Omega$. Or $\langle\langle \forall y\neg\psi(y, \bar{f})\rangle\rangle_\Omega \in \mathcal{V}$ and we are done as well.

Now we can proceed by induction on complexity of $\varphi(\bar{x}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})$. By the previous paragraph the Lemma 1.1.4 gives that open formulae are Łos for $\mathcal{F}/\mathcal{V}$ and so the base of the induction holds. Moreover by the Lemma 1.1.2 and the previous paragraph any boolean combination of formulae from $\Delta_0^{<\tilde{n}}(\mathcal{L})$ which are Łos for $\mathcal{F}/\mathcal{V}$ is Łos for $\mathcal{F}/\mathcal{V}$. Thus the only induction step which is to be considered is that if $\varphi(\bar{x}) = \exists x < \tilde{n}\psi(y, \bar{x})$ for some $\psi(y, \bar{x}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})$ which is Łos for $\mathcal{F}/\mathcal{V}$.

To do so, assume $\varphi(\bar{x})$ is of the form described above and let $\bar{f} \in \mathcal{F}$ be given. Suppose $\langle\langle \exists y < \tilde{n}\psi(y, \bar{f})\rangle\rangle_\Omega \in \mathcal{V}$. We have to show that $\mathcal{F}/\mathcal{V} \models \exists y < \tilde{n}\psi(y, f^{\mathcal{V}})$. By the first paragraph there is $w < n$ with $\langle\langle c_w < \tilde{n} \wedge \psi(c_w, \bar{f})\rangle\rangle_\Omega \in \mathcal{V}$ and so as $y < \tilde{n}$ and $\psi(y, \bar{x})$ are Łos for $\mathcal{F}/\mathcal{V}$ and so are its boolean combinations we get that $\mathcal{F}/\mathcal{V} \models c_w < \tilde{n} \wedge \psi(c_w^{\mathcal{V}}, \bar{f})$ and we are done.

If on the other hand $\mathcal{F}/\mathcal{V} \models \exists y < \tilde{n}\psi(y, \bar{f}^{\mathcal{V}})$ then $\langle\langle g < \tilde{n} \wedge \psi(g, \bar{f})\rangle\rangle_\Omega \in \mathcal{V}$ for some $g \in \mathcal{F}$ as $\psi(y, \bar{x})$ and $y < \tilde{n}$ are Łos for $\mathcal{F}/\mathcal{V}$ and so is its boolean combination. But then $\langle\langle g < \tilde{n} \wedge \psi(g, \bar{f})\rangle\rangle_\Omega \subseteq \langle\langle \exists y < \tilde{n}\psi(y, \bar{f})\rangle\rangle_\Omega \in \mathcal{V}$ and we are done.

Finally if $\mathcal{F}/\mathcal{V} \models f^{\mathcal{V}} \leq c_n^{\mathcal{V}}$ for some $f^{\mathcal{V}} \in \mathcal{F}/\mathcal{V}$ then either $f^{\mathcal{V}} = c_n^{\mathcal{V}}$ and we are done or $\mathcal{F}/\mathcal{V} \models \exists x < \tilde{n}(f^{\mathcal{V}} = x)$. But then $\langle\langle \exists x < \tilde{n}(f = x)\rangle\rangle_\Omega \in \mathcal{V}$ as $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae are Łos for $\mathcal{F}/\mathcal{V}$. Hence there is $w < n$ with $\langle\langle c_w < \tilde{n} \wedge f = c_w\rangle\rangle_\Omega \in \mathcal{V}$ as $\mathcal{V}$ intersects $D_{\langle \exists x < \tilde{n}(f=x)\rangle}$. But this means $\mathcal{F}/\mathcal{V} \models f^{\mathcal{V}} = c_v^{\mathcal{V}}$ which finishes the argument.

The "moreover" part follows from the Generic Lemma since the assumption gives that the function $x/(n+1)$ (restricted on $[0, \#(\Omega)]_{\mathbb{Q}}$) is large. $\square$

Now let $\mathcal{F} \subseteq \mathcal{H}$, $\bar{f} \in \mathcal{H}$ and $\varphi(x, \bar{y}, \bar{z})$ be an $\mathcal{L}$-formula. We define

$$\mathcal{D}_{Err_\Omega^{\mathcal{F}}(\varphi(x,\bar{y},\bar{f}))} = \{\{B \in \mathcal{B} \mid B \cap \langle\langle \varphi(id, \bar{g}, \bar{f})\rangle\rangle_\Omega = \emptyset\} \mid \bar{g} \in \mathcal{F}\}.$$

Lemma 2.1.8. *Assume $\mathcal{F} \subseteq \mathcal{H}$ is $\mathcal{L}$-closed, $\varphi(x, \bar{y}, \bar{z})$ an $\mathcal{L}$-formula and $id, \bar{f} \in \mathcal{F}$. Then*

(i) $\mathcal{D}_{Err_\Omega^{\mathcal{F}}(\varphi(x,\bar{y},\bar{f}))}$ is a countable family of $(x - \#(\Omega)q)$-dense sets whenever there is an $\mathbb{M}$-rational q such that for any $\bar{g} \in \mathcal{F}$, $\#(\langle\langle \varphi(id, \bar{g}, \bar{f})\rangle\rangle_\Omega)/\#(\Omega) \leq q$ and

(ii) $\mathcal{F}/\mathcal{V} \models \forall y \neg\varphi(id^{\mathcal{V}}, y, \bar{f}^{\mathcal{V}})$ whenever $\mathcal{V}$ is a filter intersecting all sets from $\mathcal{D}_{Err_\Omega^{\mathcal{F}}(\varphi(x,\bar{y},\bar{f}))}$ and $\varphi(x, \bar{y}, \bar{z})$ is Łos for $\mathcal{F}/\mathcal{V}$

Proof. For (i) we are obligated to show that for any $\bar{g} \in \mathcal{F}$, $D_{\neg\varphi(id,\bar{g},\bar{f})} = \{B \in \mathcal{B} \mid B \cap \langle\langle \varphi(id, \bar{g}, \bar{f})\rangle\rangle_\Omega = \emptyset\}$ is $(x - \#(\Omega)q)$-dense set. To do so, let $A \in \mathcal{B}$ be given and let $B = A - \langle\langle \varphi(id, \bar{g}, \bar{f})\rangle\rangle_\Omega$. Then $\#(B) \geq \#(A) - \#(\langle\langle \varphi(id, \bar{g}, \bar{f})\rangle\rangle_\Omega) \geq \#(A) - \#(\Omega)q$ where the last inequality is by the assumption $\langle\langle \varphi(id, \bar{g}, \bar{f})\rangle\rangle_\Omega/\#(\Omega) \leq q$. But we also clearly have $B \in D_{\neg\varphi(id,\bar{g},\bar{f})}$.

To show (ii) let a filter $\mathcal{V}$ intersecting all sets from $\mathcal{D}_{Err_\Omega^{\mathcal{F}}(\varphi(x,\bar{y},\bar{f}))}$ be given. Assume $\varphi(x, \bar{y}, \bar{z})$ is Łos for $\mathcal{F}/\mathcal{V}$ and assume for a contradiction that $\mathcal{F}/\mathcal{V} \models \varphi(id^{\mathcal{V}}, \bar{g}^{\mathcal{V}}, \bar{f}^{\mathcal{V}})$ for some $\bar{g} \in \mathcal{F}$. But there is $B \in \mathcal{V} \cap D_{\neg\varphi(id,\bar{g},\bar{f})}$ such that $B \cap \langle\langle \varphi(id, \bar{g}, \bar{f})\rangle\rangle_\Omega = \emptyset$ contradicting that $\varphi(x, \bar{y}, \bar{z})$ is Łos for $\mathcal{F}/\mathcal{V}$. $\square$

2.2 Powers of weak forms of induction

Suppose $\mathcal{L}_0$ is a language with binary predicate symbol $<$, a binary function symbol $+$ and constant symbols $0, 1$. Let $\varphi(x, \bar{y})$ be an $\mathcal{L}_0$-formula and Γ a set of $\mathcal{L}_0$-formulae. Then we denote by $\varphi(x, \bar{y}) - \mathrm{IND}^{\leq a}$ the induction formula

$$\forall \bar{y}[\varphi(0, \bar{y}) \to \forall x < a(\varphi(x, \bar{y}) \to \varphi(x + 1, \bar{y})) \to \varphi(a, \bar{y})]$$

which has a free variable a and by $\Gamma - \mathrm{IND}^{\leq a}$ the set $\{\varphi(x, \bar{y}) - \mathrm{IND}^{\leq a} \mid \varphi(x, \bar{y}) \in \Gamma\}$. Moreover if $\mathbb{K}$ is an $\mathcal{L}_0$-structure with $m \in \mathbb{K}$ and $\mathbb{K} \models \Gamma - \mathrm{IND}^{\leq m}$ then we say that $\mathbb{K}$ satisfies Γ-*induction up to m* and similarly for any $\varphi(x, \bar{y}) \in \Gamma$.

The following lemma is a motivation for models of such form of a weak induction and in particular is a motivation for the construction presented in this chapter. The proof of this lemma is not hard but long and tedious. Since we only show the statement as a motivation for our construction we will skip the proof for this moment. An interested reader can find the proof in the last section of this chapter.

Lemma 2.2.1. *Suppose $\mathcal{L}'$ is a language with $\mathcal{L}_{all} \supseteq \mathcal{L}' \supseteq \mathcal{L}_{\mathrm{BUSS}} \cup \{\tilde{n}\}$ and $\mathbb{K}$ an infinite $\mathcal{L}'$-structure with $m \in \mathbb{K}$ such that*

$$\mathbb{K} \models \mathrm{Th}_{\forall\Delta_0^{<\tilde{n}}(\mathcal{L}')}(\mathbb{M}) \text{ and } \mathbb{K} \models \exists\Delta_0^{<\tilde{n}}(\mathcal{L}') - \mathrm{IND}^{\leq m}.$$

Suppose further that (L)*:*
For any r-ary function symbol $F \in \mathcal{L}'$ there are $c, k \in \mathbb{N}$ with

$$\mathbb{M} \models \forall y_0, y_1, \ldots, y_{r-1}(|F(y_0, y_1, \ldots, y_{r-1})| \leq c(c + |y_1| + \ldots + |y_r|)^k).$$

Then $K' = \{b \in \mathbb{K} \mid \mathbb{K} \models |b| < \tilde{n}^k$ for some $k \in \mathbb{N}\}$ is a domain of a structure $\mathbb{K}' \leq \mathbb{K}$, $\mathbb{K}' \models \mathrm{BASIC}$ and
(i) if $\mathbb{K} \models \tilde{n} = m$ then $\mathbb{K}' \models \mathrm{strict}\Sigma_1^b(\mathcal{L}') - \mathrm{LIND}$ and
(ii) if $\mathbb{K} \models |\tilde{n}| = m$ then $\mathbb{K}' \models \mathrm{strict}\Sigma_1^b(\mathcal{L}') - \mathrm{LLIND}$.

Thus our aim of this section is to develop a construction producing models like $\mathbb{K}$ from the previous lemma that satisfies the assumption of the case (*i*) or (*ii*) to get a suitable model $\mathbb{K}'$ of $\mathrm{strict}\Sigma_1^b(\mathcal{L}') - \mathrm{LIND}$ or $\mathrm{strict}\Sigma_1^b(\mathcal{L}') - \mathrm{LLIND}$ which could possibly serve as witness of some consistency result. We will show how to use our construction for independence results in the Corollary 2.2.8 and the Corollary 2.2.13 of the construction theorem.

Before we present the idea of the construction some preparations are in order: Let $\ell \in \mathbb{M}$ and $A \subseteq \mathbb{M}$ be $\mathbb{M}$-definable. Then the we can define the following in $\mathbb{M}$: Denote by A^{ℓ} the set of all sequences of elements from A of length ℓ, by $A^{<\ell}$ the set of sequences of elements from A of length $< \ell$ and by $A^{\leq \ell}$ the set $A^{<\ell} \cup A^{\ell}$. For a sequence s we will write $len(s)$ to denote the length of the sequence s. Moreover for sequences s, t we will write $s \sqsubseteq t$ if s is the initial segment of t or equivalently t extends s. We use the symbol $\emptyset$ to denote an empty sequence which by definition initial segment of any sequence. If s, s' are sequences and a some element (of $\mathbb{M}$) then we denote by $s^\frown s'$ the concatenation of s and s' (in the order shown) and by $s^\frown a$ the sequence consisting from s followed by the element a. Finally we will abuse the notation and write $a \in s$ to denote that a appears in the sequence s and if $u \in \mathcal{H}^{\leq \ell}$ then then we let $Fct(u) = \{h \mid h \in u\}$.

Definition 2.2.2. *Assume $\mathcal{F} \subseteq \mathcal{H}$, $\ell \in \mathbb{M}$ and $\mathcal{X} \subseteq \mathcal{F}$. We say that $\mathcal{T}^{\mathcal{F}} \subseteq \mathcal{F}^{\leq \ell}$ is an $\mathcal{L}$-tree of height ℓ over $\mathcal{X}$ in $\mathcal{F}$ if:*
(i) $\emptyset \neq \mathcal{T}^{\mathcal{F}} \in \mathbb{M}$,
(ii) for every $u, s \in \mathcal{F}^{\leq \ell}$: if $s \in \mathcal{T}^{\mathcal{F}}$ and $u \sqsubseteq s$ then $u \in \mathcal{T}^{\mathcal{F}}$ and finally
(iii) for every $u \in \mathcal{T}_{\mathcal{F}}$, for every function symbol $F \in \mathcal{L}$ and any $\bar{f} \in Fct(u) \cup \mathcal{X}$: if $len(u) < \ell$ then $u^\frown F^{\mathbb{M}} \circ \bar{f} \in \mathcal{T}_{\mathcal{F}}$.
Moreover if $\mathcal{T}^{\mathcal{F}}$ is such a tree, $i \leq \ell$ and $u \in \mathcal{T}^{\mathcal{F}}$ then we denote by $\mathcal{T}_i^{\mathcal{F}}[u]$ the set $\{s \in \mathcal{T}^{\mathcal{F}} \cap \mathcal{F}^{\leq i} \mid s \sqsubseteq u$ or $u \sqsubseteq s\}$ and by $\mathcal{T}^{\mathcal{F}}[u]$ the set $\mathcal{T}_{\ell}^{\mathcal{F}}[u]$.

Observe that if $\mathcal{T}^{\mathcal{F}}$ is as above, then any $u \in \mathcal{T}^{\mathcal{F}}$ has an extension in $\mathcal{T}^{\mathcal{F}}$ which is of length i for any $i \in [len(u), \ell]$. The following example illustrates the definition above and is used in Construction B of [Gar15] which we will derive in the next chapter.

Example 2.2.3. (SLP) Suppose $\ell \in \mathbb{M}$, $\mathcal{X} \subseteq \mathcal{H}$ with $\emptyset \neq \mathcal{X} \in \mathbb{M}$ and $\mathcal{L}$ contains finitely many function symbols. We say that a straight-line program (SLP for short) over $\mathcal{L}$ and $\mathcal{X}$ of size ℓ is a sequence of functions $y_0, y_1, \ldots, y_{\ell-1}$ of the following form: for $i < \ell$ the $i-th$ function y_i equals to $F^{\mathbb{M}} \circ (y_{i_0}, \ldots, y_{i_{r-1}}, f_0, \ldots, f_{k-1})$ where $F \in \mathcal{L}$ is some $(r+k)$-ary function symbol, $i_j < i$ for all $j < r$ and $f_0, f_1, \ldots, f_{k-1} \in \mathcal{X}$. We denote by $\mathrm{SLP}_\ell(\mathcal{X})$ the set of all SLP programs over $\mathcal{L}$ and $\mathcal{X}$ of size ℓ and by $\mathrm{SLP}_{\leq \ell}(\mathcal{X})$ the set of all SLP programs over $\mathcal{L}$ and $\mathcal{X}$ of size $\leq \ell$. Finally we let $FCT_\ell(\mathcal{X}) = \mathcal{X} \cup \bigcup_{P \in \mathrm{SLP}_\ell} Fct(P)$.

Then $\mathrm{SLP}_{\leq \ell}(\mathcal{X})$ is an $\mathcal{L}$-tree of height ℓ over $\mathcal{X}$ in $FCT_\ell(\mathcal{X})$. Indeed, since $\emptyset \neq \mathcal{X} \in \mathbb{M}$, $\ell \in \mathbb{M}$ and $\mathcal{L}$ contains finitely many function symbols we have that $\emptyset \neq \mathrm{SLP}_{\leq \ell}(\mathcal{X}) \in \mathbb{M}$. Moreover if $P \in \mathrm{SLP}_{\leq \ell}(\mathcal{X})$ then $P' \in \mathrm{SLP}_{\leq \ell}(\mathcal{X})$ for any $P' \sqsubseteq P$ and finally if $P \in \mathrm{SLP}_{\leq \ell}(\mathcal{X})$, $\bar{h} \in Fct(P) \cup \mathcal{X}$, $F \in \mathcal{L}$ is a function symbol and P is of size $< \ell$ (i.e. $len(P) < \ell$) then $P^\frown(F^{\mathbb{M}} \circ \bar{h}) \in \mathrm{SLP}_{\leq \ell}(\mathcal{X})$.

Definition 2.2.4. *Let $m \in \mathbb{M}$, $A \subseteq \mathcal{H}$ and $\varphi(x, \bar{z})$ be an $\mathcal{L}(\mathcal{H})$-formula. We say that $\varphi(x, \bar{z})$ is m-*good *for A if there is $\bar{g} \in A$ with $\langle\langle \varphi(c_0, \bar{g}) \rangle\rangle_\Omega = \Omega$ and for any $\bar{h} \in \mathcal{H}$, $\langle\langle \varphi(c_m, \bar{h}) \rangle\rangle_\Omega = \emptyset$.*

The following observation shows that we will only need to consider m-good formulae in our construction.

Observation 2.2.5. *Assume $+, 0, 1 \in \mathcal{L}$, $m \in \mathbb{M}$, $m > 0$, $\mathcal{F} \subseteq \mathcal{H}$ is an $\mathcal{L}$-closed family of functions with $c_m \in \mathcal{F}$ and*

$$\Gamma_m = \{\exists \bar{z} \varphi(x, \bar{z}) \mid \varphi(x, \bar{z}) \in \Delta_0^{<\tilde{n}}(\mathcal{L}(\mathcal{F})) \text{ and } \varphi(x, \bar{z}) \text{ is } m\text{-good for } \mathcal{F}\}.$$

Assume further that $\mathcal{V}$ is a filter and $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae are Łos for $\mathcal{F}/\mathcal{V}$. Then $\mathcal{F}/\mathcal{V} \models \exists \Delta_0^{<\tilde{n}}(\mathcal{L}) - \mathrm{IND}^{\leq c_m^{\mathcal{V}}}$ whenever $\mathcal{F}/\mathcal{V} \models \Gamma_m - \mathrm{IND}^{\leq c_m^{\mathcal{V}}}$.

Proof. Assume $\psi(x, \bar{y}, \bar{z}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})$, $\bar{f} \in \mathcal{F}$ and $\mathcal{F}/\mathcal{V} \models \exists \bar{z} \psi(c_0, \bar{f}^{\mathcal{V}}, \bar{z}) \wedge \forall \bar{z} \neg \psi(c_m^{\mathcal{V}}, \bar{f}^{\mathcal{V}}, \bar{z})$. We have to show that there is $\alpha \in \mathcal{F}$ with $\mathcal{F}/\mathcal{V} \models \alpha < c_m^{\mathcal{V}} \wedge \exists \bar{z} \psi(\alpha, \bar{f}^{\mathcal{V}}, \bar{z}) \wedge \forall \bar{z} \neg \psi(\alpha + 1, \bar{f}^{\mathcal{V}}, \bar{z})$.

To do so, let $\varphi(x, \bar{z})$ be the $\mathcal{L}(\mathcal{F})$-formula $x = c_0 \vee (x \neq c_m \wedge \psi(x, \bar{f}, \bar{z}))$ Then for some (in fact for any) $\bar{g} \in \mathcal{F}$, $\mathbb{M} \models \langle\langle \varphi(c_0, \bar{g}) \rangle\rangle_\Omega = \Omega$ and for any $\bar{h} \in \mathcal{H}$, $\mathbb{M} \models \langle\langle \varphi(c_m, \bar{h}) \rangle\rangle_\Omega = \emptyset$ and so $\varphi(x, \bar{z})$ is m-good for $\mathcal{F}$ and in particular $\varphi(x, \bar{z}) \in \Gamma_m$. Since $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae are Łos for $\mathcal{F}/\mathcal{V}$ this gives $\mathcal{F}/\mathcal{V} \models \exists \bar{z} \varphi(c_0^{\mathcal{V}}, \bar{z}) \wedge \forall \bar{z} \neg \varphi(c_m^{\mathcal{V}}, \bar{z})$. Since $\mathcal{F}/\mathcal{V} \models \exists \bar{z} \varphi(x, \bar{z}) - \mathrm{IND}^{\leq c_m^{\mathcal{V}}}$ by the assumption on Γ_m, there is $\alpha \in \mathcal{F}/\mathcal{V}$ with $\mathcal{F}/\mathcal{V} \models \alpha < c_m^{\mathcal{V}} \wedge \exists \bar{z} \varphi(\alpha, \bar{z}) \wedge \forall \bar{z} \neg \varphi(\alpha + 1, \bar{z})$. But then $\mathcal{F}/\mathcal{V} \models \exists \bar{z} \psi(\alpha, \bar{f}^{\mathcal{V}}, \bar{z}) \wedge \forall y \neg \psi(\alpha + 1, \bar{f}^{\mathcal{V}}, \bar{z})$ by the definition of $\varphi(x, \bar{z})$ which finishes the argument. □

In the following paragraphs we describe the main idea of the construction which is deeply inspired by [Gar15, Theorem 3.4].

Suppose $+, 0, 1 \in \mathcal{L}$, $m, \ell \in \mathbb{M}$ are non-standard $\{c_w \mid w \leq \max(n, m)\} = \mathcal{X} \subseteq \mathcal{F} \subseteq \mathcal{H}$ with $\mathcal{X}, \mathcal{F} \in \mathbb{M}$ and $\mathcal{T}^{\mathcal{F}}$ is an $\mathcal{L}$-tree of height ℓ over $\mathcal{X}$ in $\mathcal{F}$. Assume $\mathcal{D}$ is a family of $\daleth$-dense sets on $\mathcal{B}$ for some $\daleth : [0, \#(\Omega)]_{\mathbb{Q}} \to [0, \#(\Omega)]_{\mathbb{Q}}$. Finally assume that a certain relation (E) between $\#(\Omega), \#(\mathcal{F}), \ell, n, m$ and $\daleth$ holds.

We will describe the main idea behind the construction of an $\mathcal{L}$-closed set $\mathcal{K}$ with $\mathcal{X} \subseteq \mathcal{K} \subseteq \mathcal{F}$ and a filter $\mathcal{V}$ interesting all sets from $\mathcal{D}$ such that $\mathcal{K}/\mathcal{V} \models \exists \Delta_0^{<\tilde{n}}(\mathcal{L}) - \mathrm{IND}^{\leq c_m^{\mathcal{V}}}$.

Assuming Γ_m is as in the Observation 2.2.5 we get that to ensure $\mathcal{K}/\mathcal{V} \models \exists\Delta_0^{<\tilde{n}}(\mathcal{L}) - \mathrm{IND}^{\leq c_m^{\mathcal{V}}}$ it suffices to ensure that $\mathcal{K}/\mathcal{V} \models \Gamma_m - \mathrm{IND}^{\leq c_m^{\mathcal{V}}}$ and that $\Delta_0^{<\tilde{n}}(\mathcal{L})$ formulae are Łos for $\mathcal{K}/\mathcal{V}$.

To do so, we will inductively construct a decreasing sequence $(A_k)_{k\in\mathbb{N}}$ of sets from $\mathcal{B}$ and a $\sqsubseteq$-increasing sequence $(s_k)_{k\in\mathbb{N}}$ of elements from $\mathcal{T}^{\mathcal{F}}$ such that we can let $\mathcal{K} = \mathcal{X} \cup \bigcup_{k\in\mathbb{N}} Fct(s_k)$ and $\mathcal{V}$ be the filter generated by $\{A_k\}_{k\in\mathbb{N}}$.[2]

Suppose $\{D_k\}_{k\in\mathbb{N}}$ is some enumeration of $\mathcal{D}$. By (E) we will wlog asume that $\mathcal{D}_{\Delta_0^{<\tilde{n}}(\mathcal{L})-\text{Łos}} \subseteq \mathcal{D}$. Once we will be done, this will ensure that $\Delta_0^{<\tilde{n}}(\mathcal{L})$ formulae will be Łos for $\mathcal{K}/\mathcal{V}$. Let $\{\varphi_k(x,\bar{z})\}_{k\in\mathbb{N}}$ be some enumeration of $\Delta_0^{<\tilde{n}}(\mathcal{L}(\mathcal{F}))$-formulae which are m-good for $\mathcal{F}$ such that every formula appears infinitely many times. Moreover let H be the set of tuples $(\bar{h}, F)$ where $\bar{h} \in \mathcal{F}$, $F \in \mathcal{L}$ is a function symbol and the length of $\bar{h}$ match the arity of F. Finally let $\{(\bar{h}, F)_k\}_{k\in\mathbb{N}}$ be an enumeration of H such that every tuple appears infinitely many times.

We start with $A_0 = \Omega$ and $s_0 = \emptyset$. At each step $k+1$

(a) we try to ensure that $\mathcal{K}/\mathcal{V} \models \exists\bar{z}\varphi_k(x,\bar{z}) - \mathrm{IND}^{c_m^{\mathcal{V}}}$ (assuming $\varphi_k(x,\bar{z})$ is an $\mathcal{L}(\mathcal{K})$ formula)

(b) for $(\bar{h}, F)_k$ we ensure that $F^{\mathbb{M}} \circ \bar{h} \in \mathcal{K}$ whenever $\bar{h} \in Fct(s_k) \cup \mathcal{X}$ and finally

(c) we ensure that D_k gets intersected by $\mathcal{V}$.

For (a) we check whether $\varphi(x,\bar{z})$ is m-good for $\mathcal{X} \cup Fct(s_k)$. If not then we skip to (b). Otherwise we will use induction in $\mathbb{M}$ for a suitable formula with parameters $s_k, A_k, \Omega, \mathcal{T}^F, \mathcal{X}$. It will be crucial for this step that $\mathcal{T}^F[s_k], \mathcal{X} \in \mathbb{M}$ (and of course $\Omega \in \mathbb{M}$) and that (E) holds. Using the suitable formula we will show there is $B_k \subseteq A_k$ such that if $B_k \in \mathcal{V}$ and $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae are Łos for $\mathcal{K}/\mathcal{V}$ then $\mathcal{K}/\mathcal{V} \models \exists z\varphi_k(x,\bar{z}) - \mathrm{IND}^{c_m^{\mathcal{V}}}$ (assuming $\varphi(x,\bar{z})$ is an $\mathcal{L}(\mathcal{K})$ formula). This will also give us some extension $s'_{k+1} \in \mathcal{T}^{\mathcal{F}}$ of s_k. The purpose of the extension is that if it happens that $\mathcal{K}/\mathcal{V} \models \exists\bar{z}\varphi(0,\bar{z}) \wedge \forall\bar{z}\neg\varphi(c_m^{\mathcal{V}},\bar{z})$ then there will be $w < m$ and a witness $\bar{h} \in Fct(s_{k'+1}) \cup \mathcal{X}$ for $\mathcal{K}/\mathcal{V} \models \exists\bar{z}\varphi(c_w^{\mathcal{U}},\bar{z}) \wedge \forall\bar{z}\neg\varphi(c_w^{\mathcal{U}}+1,\bar{z})$.

To ensure (b) we simply go to an extension $s_{k+1} = s'_{k+1}{}^\frown F^{\mathbb{M}} \circ \bar{h} \in \mathcal{T}^{\mathcal{F}}$ which exists by the definition of $\mathcal{T}^{\mathcal{F}}$ whenever $\bar{h} \in Fct(s_k) \cup \mathcal{X}$ (where $s'_{k+1} = s_k$ in the case that (a) was skipped). Finally to ensure that D_k will be intersected by $\mathcal{V}$ we let $A_{k+1} \in D_k$ be such that $A_{k+1} \subseteq B_k$ and $\#(A_{k+1}) \geq \daleth(\#(B_k))$ where $B_k = A_k$ in case that the step (a) was skipped.

By the yet unspecified relation (E) we will get that $\ell - len(s_{k+1})$ is non-standard and that the B_k can be chosen to be big enough so that $\#(A_{k+1})$ is non-standard as well. This will ensure that the induction can be proceeded for infinitely many steps.

Finally $\mathcal{K}/\mathcal{V} \models \exists\Delta_0^{<\tilde{n}}(\mathcal{L}) - \mathrm{IND}^{\leq c_m^{\mathcal{V}}}$ will follow by $\{A_k\}_{k\in\mathbb{N}} \subseteq \mathcal{V}$ or more precisely by $\{B_k\}_{k\in\mathbb{N}} \subseteq \mathcal{V}$ and the Observation 2.2.5 since $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae will be Łos for $\mathcal{K}/\mathcal{V}$ by the assumption on $\mathcal{D}$ made at the beginning.

The following lemma will be used to ensure the step (a) in the construction described above. The idea of this proof is thanks to [Gar15].

Lemma 2.2.6. *Suppose $\mathcal{X} \subseteq \mathcal{F} \subseteq \mathcal{H}$, $\mathcal{X}, \mathcal{F} \in \mathbb{M}$ and $\mathcal{T}^{\mathcal{F}}$ is an $\mathcal{L}$-tree of height ℓ over $\mathcal{X}$ in $\mathcal{F}$. Let $m \in \mathbb{M}$ and let further $t : \mathbb{Q}_{\mathbb{M}} \times [0,m]_{\mathbb{Q}} \to \mathbb{Q}_{\mathbb{M}}$ and $s : [0,m] \to [0,\ell]$*

[2] We will need to construct one more auxiliary sequence of elements from $\mathbb{M}$. However, the role of this sequence is only technical and so we will not comment on this here.

be functions definable in $\mathbb{M}$ *such that for any* $A \in \mathcal{B}$ *and* $w < m$*:* $\#(A) \geq t(\#(A), 0)$*,* $t(\#(A), m) > 0$ *and* $s(w+1) \geq s(w)$*.*

Now suppose $u \in \mathcal{T}^{\mathcal{F}}$ *with* $len(u) = s(0)$ *and* $\varphi(x, \bar{z})$ *is an* $\mathcal{L}(\mathcal{H})$*-formula which is* m*-good for* $Fct(u) \cup \mathcal{X}$*. Then for any* $A \in \mathcal{B}$ *there is* $B \in \mathcal{B}$ *and* $w < m$ *such that:*

(i) $B \subseteq A$*,*

(ii) there is $s \in \mathcal{T}^{\mathcal{F}}_{s(w)}[u]$ *and* $\bar{g} \in Fct(s) \cup \mathcal{X}$ *such that* $B \subseteq \langle\langle \varphi(c_w, \bar{g}) \rangle\rangle_\Omega$*,*

(iii) for any $s' \in \mathcal{T}^{\mathcal{F}}_{s(w+1)}[u]$ *and* $\bar{h} \in Fct(s') \cup \mathcal{X}$*:* $B \cap \langle\langle \varphi(c_{w+1}, \bar{h}) \rangle\rangle_\Omega = \emptyset$ *and*

(iv) $\#(B) \geq \min_{w<m}\{t(\#(A), w) - \#(\mathcal{F})^r t(\#(A), w+1)\}$ *where* r *is the length of* $\bar{z}$*.*

Proof. Let $A \in \mathcal{B}$ be given. Since $\varphi(x, \bar{z})$ is m-good for $Fct(u) \cup \mathcal{X}$ and $len(u) = s(0)$ i.e. $u \in T^{\mathcal{F}}_{s(0)}[u]$ we have that

$$\mathbb{M} \models \exists s' \in \mathcal{T}^{\mathcal{F}}_{s(0)}[u] \exists \bar{h} \in Fct(s') \cup \mathcal{X} : \#(A \cap \langle\langle \varphi(0, \bar{h}) \rangle\rangle_\Omega\ ) = \#(A)$$

and

$$\mathbb{M} \models \forall s' \in \mathcal{T}^{\mathcal{F}}_{s(m)}[u] \forall \bar{h} \in Fct(s') \cup \mathcal{X} : \#(A \cap \langle\langle \varphi(c_m, \bar{h}) \rangle\rangle_\Omega\ ) = 0.$$

Since $\#(A) \geq t(\#(A), 0)$ and $t(\#(A), m) > 0$ we have that

$$\mathbb{M} \models \exists s' \in \mathcal{T}^{\mathcal{F}}_{s(0)}[u] \exists \bar{h} \in Fct(s') \cup \mathcal{X} : \#(A \cap \langle\langle \varphi(0, \bar{h}) \rangle\rangle_\Omega\ ) \geq t(\#(A), 0)$$

and

$$\mathbb{M} \models \forall s' \in \mathcal{T}^{\mathcal{F}}_{s(m)}[u] \forall \bar{h} \in Fct(s') \cup \mathcal{X} : \#(A \cap \langle\langle \varphi(c_m, \bar{h}) \rangle\rangle_\Omega\ ) < t(\#(A), m).$$

Thus we can use induction in $\mathbb{M}$ for the formula

$$\psi(x) = \exists s' \in \mathcal{T}^{\mathcal{F}}_{s(x)}[u] \exists \bar{h} \in Fct(s') \cup \mathcal{X} : \#(A \cap \langle\langle \varphi(c_x, \bar{h}) \rangle\rangle_\Omega\ ) \geq t(\#(A), x)^3$$

to get $w < m$ and $s, \bar{g} \in \mathbb{M}$ with

$$\mathbb{M} \models s \in \mathcal{T}^{\mathcal{F}}_{s(w)}[u] \wedge \bar{g} \in Fct(s) \cup \mathcal{X} \wedge \#(A \cap \langle\langle \varphi(c_w, \bar{g}) \rangle\rangle_\Omega\ ) \geq t(\#(A), w)$$

and

$$\mathbb{M} \models \forall s' \in \mathcal{T}^{\mathcal{F}}_{s(w+1)}[u] \forall \bar{h} \in Fct(s') \cup \mathcal{X} : \#(A \cap \langle\langle \varphi(c_{w+1}, \bar{h}) \rangle\rangle_\Omega\ ) < t(\#(A), w+1).$$

Now let $r \in \mathbb{N}$ be the length of the tuple $\bar{z}$ in $\varphi(x, \bar{z})$ and set $C = \{\bar{h} \in \mathcal{F}^r \mid \bar{h} \in Fct(s') \cup \mathcal{X}$ for some $s' \in \mathcal{T}^{\mathcal{F}}_{s(w+1)}[u]\}$ and

$$B = A \cap \langle\langle \varphi(c_w, \bar{g}) \rangle\rangle_\Omega - \bigcup\{\langle\langle \varphi(c_{w+1}, h) \rangle\rangle_\Omega \mid h \in C\}$$

then (i),(ii),(iii) clearly holds. Finally to show (iv) consider the following estimations for $\#(B)$:

$$\#(B) \geq \#(A \cap \langle\langle \varphi(c_w, \bar{g}) \rangle\rangle_\Omega) - \sum_{h \in C} \#(A \cap \langle\langle \varphi(c_{w+1}, h) \rangle\rangle_\Omega)$$

$$\geq t(\#(A), w) - \#(C) t(\#(A), w+1) \geq \min_{w<m}\{t(\#(A), w) - \#(\mathcal{F})^r t(\#(A), w+1)\}$$

by $C \subseteq \mathcal{F}^r$ and so $\#(C) \leq \#(\mathcal{F})^r$. ∎

[3] we are using the convention discussed at the beginning and writting t(#(A),x) in place of t(#(A),x/1)

Now we can finally state the theorem of this section:

Theorem 2.2.7. *Assume* $+, 0, 1 \in \mathcal{L}$, $m, \ell \in \mathbb{M}$ *and* $\{c_v \mid v \leq \max(m,n)\} \subseteq \mathcal{X} \subseteq \mathcal{F} \subseteq \mathcal{H}$ *with* $\mathcal{X}, \mathcal{F} \in \mathbb{M}$. *Suppose there exists an* $\mathcal{L}$*-tree of height* ℓ *over* $\mathcal{X}$ *in* $\mathcal{F}$. *Moreover suppose:*

- $\gimel : [0, \#(\Omega)]_{\mathbb{Q}} \to [0, \#(\Omega)]_{\mathbb{Q}}$ *is a non-decreasing function with* $\gimel(x) \leq x/(n+1)$ *on* $[0, \#(\Omega)]_{\mathbb{Q}}$,

- $\mathcal{D}$ *is a countable family of* $\gimel$*-dense sets on* $\mathcal{B}$,

- $\daleth : [0, \#(\Omega)]_{\mathbb{Q}} \to [0, \#(\Omega)]_{\mathbb{Q}}$ *is such that* $\daleth(x) = \gimel(x\frac{\#(\mathcal{F})^{a-1}-1}{\#(\mathcal{F})^{am-1}})$ *for some* $\mathbb{M}$*-rational* $a > 1$.

Finally assume that (E):

$$\textit{for any } k \in \mathbb{N} : \daleth^{(k)}(\#(\Omega)) > \mathbb{N} \textit{ and } \frac{\ell}{m^k} > \mathbb{N}.$$

Then there is an $\mathcal{L}$*-closed* $\mathcal{K} \subseteq \mathcal{H}$ *with* $\mathcal{X} \subseteq \mathcal{K} \subseteq \mathcal{F}$ *and a filter* $\mathcal{V}$ *intersecting all sets from* $\mathcal{D}$ *such that* $\Delta_0^{<\tilde{n}}(\mathcal{L})$*-formulae are Łos for* $\mathcal{K}/\mathcal{V}$ *and*

$$\mathcal{K}/\mathcal{V} \models \exists\Delta_0^{<\tilde{n}} - \mathrm{IND}^{\leq c_m^{\mathcal{V}}}.$$

Moreover if $\alpha \in \mathcal{K}/\mathcal{V}$ *is such that* $\mathcal{K}/\mathcal{V} \models \alpha \leq c_n^{\mathcal{V}}$ *then there is* $v \leq n$ *such that* $\mathcal{K}/\mathcal{V} \models \alpha = c_v^{\mathcal{V}}$.

Proof. First we wlog assume $\mathcal{D}_{\Delta_0^{<\tilde{n}}-\text{Ł}os} \subseteq \mathcal{D}$. This is possible as $\gimel(x) \leq x/(n+1)$ on $[0, \#(\Omega)]_{\mathbb{Q}}$ and $\mathcal{D}_{\Delta_0^{<\tilde{n}}-\text{Ł}os}$ is a countable family of $x/(n+1)$-dense sets by the Observation 2.1.6 and its definition.

Let $\mathcal{D} = \{D_k\}_{k\in\mathbb{N}}$ be some enumeration of $\mathcal{D}$. Denote by G the a set of formulae $\varphi(x, \bar{z}) \in \Delta_0^{<\tilde{n}}(\mathcal{L}(\mathcal{F}))$ which are m-good for $\mathcal{F}$. Let further $\{(\varphi_k(x, \bar{z})\}_{k\in\mathbb{N}}$ be an enumeration of G such that every formula appears infinitely many times. Moreover denote by H the set of tuples $(\bar{h}, F)$ such that $F \in \mathcal{L}$ is an r-ary function symbol for some r and $\bar{f} \in \mathcal{F}$ an r-tuple. Finally let $\{(\bar{h}, F)_k\}_{k\in\mathbb{N}}$ be an enumeration of H such that every tuple appears infinitely many times. [4] Note that by the assumption on $\mathcal{L}$ made in the beginning of this section $\mathcal{L}$ is countable. Moreover as $\mathcal{H}$ is countable as well (see the comment on the beginning of this section) we get that G and H are countable and so such enumerations exist.

Let $\mathcal{T}^{\mathcal{F}}$ be an $\mathcal{L}$-tree of height ℓ over $\mathcal{X}$ in $\mathcal{F}$. We construct a decreasing sequence $(A_k)_{k\in\mathbb{N}}$ of elements from $\mathcal{B}$, an $\sqsubseteq$-increasing sequences $(s_k)_{k\in\mathbb{N}}$ of elements from $\mathcal{T}^{\mathcal{F}}$ and a sequence $(r_k)_{k\in\mathbb{N}}$ of elements $\leq \ell$ from $\mathbb{M}$ such that for every $k \in \mathbb{N}$:

(a) $r_k - len(s_k) \geq \frac{\ell}{m^{2k}}$ and if $k > 0$ then $s_{k-1} \sqsubseteq s_k$, $r_k \leq r_{k-1}$

(b) if $k > 0$ then $A_k \in D_{k-1}$

(c) if $k > 0$ and $\varphi_{k-1}(x, \bar{z})$ is m-good for $Fct(s_{k-1}) \cup \mathcal{X}$ then there is $\bar{g} \in Fct(s_k) \cup \mathcal{X}$ and $w < m$ such that $A_k \subseteq \langle\langle\varphi(c_w, \bar{g})\rangle\rangle_\Omega$ and for any $s' \in \mathcal{T}^{\mathcal{F}}_{r_k}[s_k]$ and $\bar{h} \in Fct(s') \cup \mathcal{X}$, $A_k \cap \langle\langle\varphi(c_{w+1}, \bar{h})\rangle\rangle_\Omega = \emptyset$.

(d) if $k > 0$ and $(\bar{h}, F)_k$ is such that $\bar{h} \in Fct(s_{k-1}) \cup \mathcal{X}$ then $F^{\mathbb{M}} \circ \bar{h} \in s_k$ and

(e) there is $q \in \mathbb{N}$ such that $\#(A_k) \geq \daleth^{(q)}(\#(\Omega))$

Then we let $\mathcal{K} = \mathcal{X} \cup \bigcup_{k\in\mathbb{N}} Fct(s_k)$ and $\mathcal{V}$ be the filter generated by $\{A_k\}_{k\in\mathbb{N}}$.

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[4] Taking $\varphi(x, \bar{z}) \in \Delta_0^{<\tilde{n}}(\mathcal{L}(\mathcal{H}))$ in place of $\varphi(x, \bar{z}) \in \Delta_0^{<\tilde{n}}(\mathcal{L}(\mathcal{F}))$ with "$\varphi(x, \bar{z})$ m-good for $\mathcal{H}$" in place of "$\varphi(x, \bar{z})$ m-good for $\mathcal{F}$" and $\bar{f} \in \mathcal{H}$ in place of $\bar{f} \in \mathcal{F}$ would make no difference.

Base of induction: Let $A_0 = \Omega$, $s_0 = \emptyset$ and $r_0 = m^s$ for some non-standard $s \in \mathbb{M}$ with $m^s \leq \ell$ which exists by the assumption on ℓ. Then clearly $(a) - (e)$ holds for A_0.

Induction step: Having A_k, s_k, r_k satisfying properties $(a) - (e)$ constructed we construct $A_{k+1}, s_{k+1}, r_{k+1}$ satisfying $(a) - (e)$. To do so, we will use the previous lemma. Let $s(x)$ be the function with domain $[0, m]$ defined by $s(w) = len(s_k) + w\lfloor \frac{r_k - len(s_k)}{m} \rfloor$ for every $w \leq m$. Then $s(0) = len(s_k)$ and for every $w < m$, $s(w+1) - s(w) = \lfloor \frac{r_k - len(s_k)}{m} \rfloor \geq \frac{r_k - len(s_k)}{m} - 1 \geq \ell/m^{2k+1} - 1 > \mathbb{N}$ by the induction assumption on s_k, r_k. Let the function $t : [0, \#(\Omega)]_{\mathbb{Q}} \times [0, m]_{\mathbb{Q}} \to \mathbb{M}$ be defined by $t(\#(B), w) = \#(B)/\#(F)^{paw}$ for any $B \in \mathcal{B}$, $w \in [0, m]_{\mathbb{Q}}$ where a is from the assumption of the theorem and $p \in \mathbb{N}$ is the length of $\bar{z}$ in $\varphi_k(x, \bar{z})$. It is not hard to see that $s(x), t(x, y) \in \mathbb{M}$ and that $s(x), t(x, y)$ satisfies assumptions of the previous lemma. Moreover for any $B \in \mathcal{B}$:

$$\min_{w<m}\{t(\#(B), w) - \#(\mathcal{F})^p t(\#(B), w+1)\} = \#(B)(\frac{1}{\#(F)^{pa(m-1)}} - \frac{\#(F)^p}{\#(F)^{pam}})$$
$$= \#(B)\frac{\#(\mathcal{F})^{pa} - \mathcal{F}^p}{\#(\mathcal{F})^{pam}} = \#(B)\frac{\#(\mathcal{F})^{p(a-1)} - 1}{\#(\mathcal{F})^{pam-p}} \geq \#(B)(\frac{\#(\mathcal{F})^{a-1} - 1}{\#(\mathcal{F})^{am-1}})^p.$$

Now if $\varphi_k(x, \bar{z})$ is not m-good for $Fct(s_k) \cup \mathcal{X}$ then let $B = A_k$, $s'_{k+1} = s_k$, $r_{k+1} = s(1)$ and skip to the next paragraph. Otherwise apply the previous lemma with the functions $t(x, y), s(x)$ for $\varphi_k(x, \bar{z}), s_k$ and A_k to find a set $B \in \mathcal{B}$ and $w < m$ such that:

(i) $B \subseteq A_k$,

(ii) there is $s \in \mathcal{T}^{\mathcal{F}}_{s(w)}[s_k]$ and $\bar{g} \in Fct(s) \cup \mathcal{X}$ such that $B \subseteq \langle\langle \varphi(c_w, \bar{g}) \rangle\rangle_\Omega$,

(iii) for any $s' \in \mathcal{T}^{\mathcal{F}}_{s(w+1)}[s_k]$ and $\bar{h} \in Fct(s') \cup \mathcal{X}$: $B \cap \langle\langle \varphi(c_{w+1}, \bar{h}) \rangle\rangle_\Omega = \emptyset$ and

(iv) $\#(B) \geq \min_{w<m}\{t(\#(A_k), w) - \#(F)^p t(\#(A_k), w+1)\} \geq \#(A_k)(\frac{\#(\mathcal{F})^{a-1}-1}{\#(\mathcal{F})^{am-1}})^p$.

Finally let $s'_{k+1} = s$ for the s from (ii) and $r_{k+1} = s(w+1)$.

In the both cases we get $r_{k+1} - len(s'_{k+1}) \geq \ell/m^{2k+1} - 1$.

To ensure (d) consider $(\bar{h}, F)_k$. If $\bar{h} \in Fct(s_k) \cup \mathcal{X}$ then let $s_{k+1} = s'_{k+1}{}^\frown F^{\mathbb{M}} \circ \bar{h}$ otherwise let $s_{k+1} = s'_{k+1}$. By the definition of $\mathcal{T}^{\mathcal{F}}$ we get $s_{k+1} \in \mathcal{T}^{\mathcal{F}}$. Moreover $r_k - len(s_{k+1}) \geq r_k - len({s'}_{k+1}) - 1 \geq \ell/m^{2k+1} - 2 \geq \ell/m^{2k+2}$ and also $r_k \geq r_{k+1}$ and $s_k \sqsubseteq s_{k+1}$ thus (a) and (d) holds for $k+1$.

To ensure (b) let A_{k+1} be such that $A_{k+1} \in D_k$, $\#(A_{k+1}) \geq \beth(\#(B))$ and $A_{k+1} \subseteq B$. To check that (c) holds for A_{k+1} is easy by (i)-(iii) since $A_{k+1} \subseteq B$. To check (e) we first observe that $\daleth^{(p)}(x) \leq \beth(x \cdot d^p)$ where $d = (\frac{\#(\mathcal{F})^{a-1}-1}{\#(\mathcal{F})^{am-1}})$.[5]

[5]By induction on $e \geq 1$. The case for $e = 1$ follows by the definition of $\daleth$. For $e+1$ we have: $\beth(x \cdot d^{e+1}) \geq \beth(\beth(xd^e)d) = \daleth(\beth(x \cdot d^e)) \geq \daleth(\daleth^{(e)}(x)) = \daleth^{(e+1)}(x)$ where the first is by $xd^e \geq \beth(xd^e)$ i.e. $xd^{e+1} \geq \beth(xd^e)d$ using that $\beth$ is non-decreasing and $y \geq \beth(y)$ for $y \in [0, \#(\Omega)]_{\mathbb{Q}}$, then we used definition of $\daleth$ and for the last inequality we used the induction assumption for e and that $\beth$ and so $\daleth$ is non-decreasing.

Then we have:

$$\#(A_{k+1}) \quad \geq \quad \gimel(\#(B)) \quad \geq \quad \gimel(\#(A_k)(\frac{\#(\mathcal{F})^{a-1}-1}{\#(\mathcal{F})^{am-1}})^p) \quad \geq \quad \daleth^{(p)}(\#(A))$$

where we used (iv) for B and the observation from the previous paragraph. But by the induction assumption on A_k there is some $q \in \mathbb{N}$ with $\#(A_k) \geq \daleth^{(q)}(\#(\Omega))$ and so $\#(A_{k+1}) \geq \daleth^{(p)}(\daleth^{(q)}(\#(\Omega))) \geq \daleth^{(p+q)}(\#(\Omega))$ because $\daleth$ is non-decreasing since $\gimel$ is non-decreasing. Thus (e) holds for A_{k+1} and so $A_{k+1}, s_{k+1}, r_{k+1}$ satisfying conditions $(a)-(e)$ has been constructed.

Now let $\mathcal{K} = \mathcal{X} \cup \bigcup_{k\in\mathbb{N}} Fct(s_k)$ and $\mathcal{V}$ be the filter generated by $\{A_k\}_{k\in\mathbb{N}}$.

By (d) the set $\mathcal{K}$ is $\mathcal{L}$-closed. Indeed, if $\bar{h} \in \mathcal{K}$ and a function symbol $F \in \mathcal{L}$ are given, then $\bar{h} \in Fct(s_k) \cup \mathcal{X}$ for some $k \in \mathbb{N}$ and thus $(\bar{h}, F)$ was treated at some stage $k' > k$ with the result $F^{\mathbb{M}} \circ \bar{f} \in s_{k'}$.

Since $\mathcal{V}$ clearly intersects all dense sets from $\mathcal{D}$ by the condition (b) of the construction and we assumed $\mathcal{D}_{\Delta_0^{<\tilde{n}}-\text{Łos}} \subseteq \mathcal{D}$ we have by the Lemma 2.1.7 that $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae are Łos for $\mathcal{K}/\mathcal{V}$. Moreover the same lemma gives that if $\alpha \in \mathcal{K}/\mathcal{U}$ is such that $\mathcal{K}/\mathcal{V} \models \alpha \leq c_n^{\mathcal{V}}$ then there is $v \leq n$ such that $\mathcal{K}/\mathcal{V} \models \alpha = c_v^{\mathcal{V}}$.

Now we can finally show that $\mathcal{K}/\mathcal{V} \models \exists\Delta_0^{<\tilde{n}} - \text{IND}^{\leq c_m^{\mathcal{V}}}$. Let $\varphi(x, \bar{y}, \bar{z}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})$ and $\bar{f} \in \mathcal{K}$ be given. By the Observation 2.2.5 we can wlog assume that $\varphi(x, \bar{f}, \bar{z})$ is m-good for $\mathcal{K}$. Assume $\mathcal{K}/\mathcal{V} \models \exists\bar{z}\varphi(c_0^{\mathcal{V}}, \bar{f}^{\mathcal{V}}, \bar{z}) \wedge \forall\bar{z}\neg\varphi(c_m^{\mathcal{V}}, \bar{f}^{\mathcal{V}}, \bar{z})$. We will find $w < m$ and $\bar{g} \in \mathcal{K}$ such that $\mathcal{K}/\mathcal{V} \models \varphi(c_w^{\mathcal{V}}, \bar{f}^{\mathcal{U}}, \bar{g}) \wedge \forall\bar{z}\neg\varphi(c_w^{\mathcal{V}}+1, \bar{f}^{\mathcal{V}}, \bar{z})$. To do so, let $\bar{g_0} \in \mathcal{K}$ and $k_0 \in \mathbb{N}$ be such that $\langle\langle\varphi(0, \bar{f}, \bar{g_0})\rangle\rangle_\Omega = \Omega$ and $\bar{g_0} \in Fct(s_{k_0}) \cup \mathcal{X}$. Then $\varphi(x, \bar{f}, \bar{z})$ is m-good for $Fct(s_{k_0}) \cup \mathcal{X}$. Now let $k \in \mathbb{N}$ be such that $k \geq k_0$ and $\varphi(x, \bar{f}, \bar{z})$ has index k. Since $\varphi(x, \bar{f}, \bar{z})$ is m-good for $Fct(s_k) \cup \mathcal{X}$ we constructed A_{k+1} satisfying the conclusions of (c). But $\mathcal{K} \subseteq \mathcal{X} \cup \{h \in \mathcal{F} \mid h \in s \in \mathcal{T}^{\mathcal{F}}_{r_{k+1}}[s_{k+1}]\}$ and $A_{k+1} \in \mathcal{V}$ thus by the Łos property of $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae for $\mathcal{K}/\mathcal{V}$ we get that $\mathcal{K}/\mathcal{V} \models \varphi(c_w^{\mathcal{V}}, \bar{f}^{\mathcal{V}}, \bar{g}^{\mathcal{V}}) \wedge \forall\bar{z}\neg\varphi(c_w^{\mathcal{V}}+1, \bar{f}^{\mathcal{V}}, \bar{z})$ for some $\bar{g} \in Fct(s_{k+1}) \cup \mathcal{X}$ and $w < m$. This finishes the proof. □

The following corollary shows how can this construction be used to give independence of $\forall\exists$ $\mathcal{L}$-sentences[6] for strict$\Sigma_1^b(\mathcal{L}) - \text{LIND}$ or strict$\Sigma_1^b(\mathcal{L}) - \text{LLIND}$. Moreover we will use the following corollary to derive the Construction B of Michal Garlík in the next section.

Corollary 2.2.8. *Suppose $+, 0, 1 \in \mathcal{L}$, $m \in \mathbb{M}$ with $m \leq n$, $s \in \mathbb{M}$ with $s > \mathbb{N}$ and $\{id\} \cup \{c_v \mid v \leq \max(m, n)\} \subseteq \mathcal{X} \subseteq \mathcal{F} \subseteq \mathcal{H}$ with $\mathcal{X}, \mathcal{F} \in \mathbb{M}$. Suppose there exists an $\mathcal{L}$-tree of height m^s over $\mathcal{X}$ in $\mathcal{F}$ and call the tree $\mathcal{T}^{\mathcal{F}}$. Assume further $\varphi(x, \bar{y}, \bar{z}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})$, $\bar{h} \in \mathcal{X}$, there is an $\mathbb{M}$-rational q with:*

$$(Q) \qquad \textit{for any } \bar{f} \in \mathcal{F} : \#(\langle\langle\varphi(id, \bar{f}, \bar{h})\rangle\rangle_\Omega)/\#(\Omega) \leq q$$

and there is a non-standard $r \in \mathbb{M}$ with:

$$(q): \ \frac{n}{\#(\mathcal{F})^{2mr}} > q \textit{ and } \ (\Omega): \ \#(\Omega) \geq r\#(\mathcal{F})^{2mr-1}(n+1).$$

[6] i.e. the $\mathcal{L}$-sentences of the form $\forall\bar{x}\exists\bar{y}\varphi(\bar{x}, \bar{y})$ where $\varphi(\bar{x}, \bar{y})$ is an open $\mathcal{L}$-formula

Then there is an $\mathcal{L}$-closed $\mathcal{K} \subseteq \mathcal{H}$ with $\mathcal{X} \subseteq \mathcal{K} \subseteq \mathcal{F}$ and a filter $\mathcal{V}$ such that $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae are Łos for $\mathcal{K}/\mathcal{V}$,

$$\mathcal{K}/\mathcal{V} \models \exists\Delta_0^{<\tilde{n}} - \mathrm{IND}^{\leq c_m^{\mathcal{V}}} \text{ and } \mathcal{K}/\mathcal{V} \models \forall\bar{y}\neg\varphi(id^{\mathcal{V}}, \bar{y}, \bar{h}^{\mathcal{V}}).$$

Moreover if $\alpha \in \mathcal{K}/\mathcal{V}$ is such that $\mathcal{K}/\mathcal{V} \models \alpha \leq c_n^{\mathcal{V}}$ then there is $v \leq n$ such that $\mathcal{K}/\mathcal{V} \models \alpha = c_v^{\mathcal{V}}$.

In particular, if for any r-ary function symbol $F \in \mathcal{L}$ there are $c, k \in \mathbb{N}$ with

$$\mathbb{M} \models \forall y_0, y_1, \ldots, y_{r-1}(|F(y_0, y_1, \ldots, y_{r-1})| \leq c(c + |y_1| + \ldots + |y_r|)^k)$$

and there is $k \in \mathbb{N}$ such that for every h_i from the tuple $\bar{h}$, $\mathbb{M} \models \forall x \in \Omega : |h_i(x)| < n^k$ and $\mathbb{M} \models \forall x \in \Omega : |x| < n^k$, then

(i) $m = n$ implies $\mathrm{strict}\Sigma_1^b(\mathcal{L}) - \mathrm{LIND} + \exists x, \bar{z}\forall\bar{y}\neg\varphi(x, \bar{y}, \bar{z})$ *is consistent and*

(ii) $m = |n|$ implies $\mathrm{strict}\Sigma_1^b(\mathcal{L}) - \mathrm{LLIND} + \exists x, \bar{z}\forall\bar{y}\neg\varphi(x, \bar{y}, \bar{z})$ *is consistent.*

Proof. Let $\mathcal{D} = \mathcal{D}_{Err_\Omega^{\mathcal{F}}(\varphi(x,\bar{y},\bar{h}))}$ and $\gimel : [0, \#(\Omega)]_{\mathbb{Q}} \to [0, \#(\Omega)]_{\mathbb{Q}}$ be defined by $\gimel(x) = (x - q\#(\Omega))/(n+1)$. Then clearly $\gimel(x) \leq x/(n+1)$ (on $[0, \#(\Omega)]_{\mathbb{Q}}$). Moreover by the Lemma 2.1.8 and the assumption (Q) the set $\mathcal{D}$ is a countable family of $(x - q\#(\Omega))$-dense sets and thus a countable family of $\gimel$-dense sets.

We want to use the previous theorem for $\mathcal{F}, \mathcal{T}^{\mathcal{F}}, n, m, \mathcal{D}, \ell = m^s$ which suffices to finish the argument. Indeed, the only property of $\mathcal{K}/\mathcal{V}$ obtained in this way which is not directly given by the previous theorem is that $\mathcal{K}/\mathcal{V} \models \forall\bar{y}\neg\varphi(id^{\mathcal{V}}, \bar{y}, \bar{h}^{\mathcal{V}})$. But this will follow by the Lemma 2.1.8 as $\mathcal{V}$ intersects all sets from $\mathcal{D}_{Err_\Omega^{\mathcal{F}}(\varphi(x,\bar{y},\bar{h}))}$. The in particular part is a direct application of the Lemma 2.2.1 since the assumption on $\bar{h}$ and id gives $\bar{h}^{\mathcal{V}}, id^{\mathcal{V}} \in \{\alpha \in \mathcal{K}/\mathcal{V} \mid \mathcal{K}/\mathcal{V} \models \alpha < \tilde{n}^k \text{ for some } k \in \mathbb{N}\} = \mathcal{K}/\mathcal{V}'$ and $\varphi(x, \bar{y}, \bar{z})$ is absolute between $\mathcal{K}/\mathcal{V}'$ and $\mathcal{K}/\mathcal{V}$ as $\{c_w^{\mathcal{V}} \mid w \leq n\}$ is an initial segment of $\mathcal{K}/\mathcal{V}$ and $\mathcal{K}/\mathcal{V}'$.

To use the previous theorem in this setting we need to ensure the inequalities (E). But since $s > \mathbb{N}$ we have that $m^s/m^k > \mathbb{N}$ for every $k \in \mathbb{N}$ and thus the first inequality holds. To ensure the second inequality we will pick $a = 2$ and make some estimations on $\daleth$ (see the previous theorem) in other words:

Let $f(x) = \#(\mathcal{F})^x$, $d = \frac{f(1)-1}{f(2m-1)}$ and $\daleth(x) = \gimel(xd) = (xd - q\#(\Omega))/(n+1)$. Once we show that for every $k \in \mathbb{N}$, $\daleth^{(k)}(\#(\Omega)) > \mathbb{N}$ the argument will be finished.

Claim 2.2.9. *Let $c = d/(n+1)$ then for any $k \in \mathbb{N}$,*

$$\daleth^{(k)}(\#(\Omega)) \geq \#(\Omega)(c^k - \frac{q}{n+1}\frac{1-c^k}{1-c}).$$

Proof. Induction on k. The case for $k = 0$ is clear. For $k+1$ we have:

$$\begin{aligned}
&\daleth^{(k+1)}(\#(\Omega)) = \daleth(\daleth^{(k)}(\#(\Omega))) = [d\daleth^{(k)}(\#(\Omega)) - q\#(\Omega)]/(n+1) \\
&= \#(\Omega)[d(c^k - \frac{q}{n+1}\frac{1-c^k}{1-c}) - q]/(n+1) = \#(\Omega)[\frac{d}{n+1}(c^k - \frac{q}{n+1}\frac{1-c^k}{1-c}) - \frac{q}{n+1}] \\
&= \#(\Omega)[c^{k+1} - c\frac{q}{n+1}\frac{1-c^k}{1-c} - \frac{q}{n+1}] = \#(\Omega)[c^{k+1} - \frac{q}{n+1}(c\frac{1-c^k}{1-c} + 1)] \\
&= \#(\Omega)(c^{k+1} - \frac{q}{n+1}\frac{1-c^{k+1}}{1-c}).
\end{aligned}$$

□

We will need those claims:

Claim 2.2.10. *For any non-standard $r \in \mathbb{M}$ and $k \in \mathbb{N}$:*

$$\#(\Omega)(c^k - \frac{q}{n+1}\frac{1-c^k}{1-c}) \geq \#(\Omega)(c^r - \frac{q}{n+1}\frac{1-c^r}{1-c})$$

Proof. The function $d(x) = \#(\Omega)(c^x - \frac{q}{n+1}\frac{1-c^x}{1-c})$ is definable in $\mathbb{M}$. It is easy to see that $\mathbb{M} \models \forall x(d(x) \geq d(x+1))$ since $\mathbb{M} \models \forall x > 0(c^x < 1)$ and so $\mathbb{M} \models \forall x, y(x < y \rightarrow d(x) \geq d(y))$. □

Claim 2.2.11. *For any $r \in \mathbb{M}$ with $r > 1$,*

$$\#(\Omega)(c^r - \frac{q}{n+1}\frac{1-c^r}{1-c}) \geq \#(\Omega)f(1)(f(-2mr) - \frac{q}{(n+1)}).$$

Proof. Using $f(1) \geq n+1$ we have that

$$\frac{1-c^r}{1-c} \leq \frac{1}{1-c} = \frac{f(2m-1)(n+1)}{f(2m-1)(n+1)-f(1)+1} \leq \frac{f(2m)}{f(2m-1)(n+1)-f(1)} \leq f(1)$$

and using $(n+1)^r \leq f(r)$

$$c^r = \frac{(f(1)-1)^r}{f(r(2m-1))(n+1)^r} \geq \frac{f(1)}{f(r(2m-1))(n+1)^r} \geq \frac{f(1)}{f(2mr)} = f(1-2mr).$$

And so

$$\#(\Omega)(c^r - \frac{q}{n+1}\frac{1-c^r}{1-c}) \geq \#(\Omega)(f(1-2mr) - \frac{qf(1)}{n+1}) = \#(\Omega)f(1)(f(-2mr) - \frac{q}{(n+1)})$$

□

Thus to ensure that for any $k \in \mathbb{N}$, $\daleth^{(k)}(\#(\Omega)) > \mathbb{N}$ it suffices to ensure that there is $r, s > \mathbb{N}$ with

$$\#(\Omega)f(1)(f(-2mr) - \frac{q}{(n+1)}) = \#(\Omega)f(1)(\frac{(n+1)-qf(2mr)}{f(2mr)(n+1)}) > s.$$

To do so, assume r is from the assumption of this theorem. Since (q) holds we have that $(n+1) - qf(2mr) > 0$ and so the last inequality is equivalent to

$$\#(\Omega)f(1) > \frac{sf(2mr)(n+1)}{(n+1)-qf(2mr)}.$$

Since (q) also gives $(n+1) - qf(2mr) > 1$ we have that to ensure the last inequality it suffices to ensure $\#(\Omega)f(1) \geq sf(2mr)(n+1)$ which is equivalent to

$$\#(\Omega) \geq sf(2mr-1)(n+1).$$

But since (Ω) holds we can take $s = r$ and we are done.

□

In the Section 3 we will give an example of a construction from [Gar15] with a set of functions defined by Straight-line programs for which the estimations (q) and (Ω) from the corollary above can be established. Now we will give one more statement connected to the theorem above which we will also need in the Section 3.

Assume $\mathcal{K} \subseteq \mathcal{H}$ is an $\mathcal{L}$-closed family of functions and $a \in \Omega$. Then we denote by $\langle a\rangle_{\mathcal{K}}$ the substructure of $\mathbb{M}$ (in language $\mathcal{L}$) with domain $\{f(a) \mid f \in \mathcal{K}\}$. It is easy to see that if we let $\mathcal{V}$ to be an ultrafilter generated by the set $\{a\}$ then $\mathcal{K}/\mathcal{V} \cong \langle a\rangle_{\mathcal{K}}$ via $i : f^{\mathcal{V}} \mapsto f(a)$. Thus the following lemma can be considered as a special case of the previous theorem. Note that we will not need $\mathcal{F} \in \mathbb{M}$.

Lemma 2.2.12. *Assume* $+, 0, 1 \in \mathcal{L}$, $\ell, m \in \mathbb{M}$, $\{id\} \cup \{c_v \mid v \leq max(n,m)\} \subseteq \mathcal{X} \subseteq \mathcal{F} \subseteq \mathcal{H}$ *with* $\mathcal{X} \in \mathbb{M}$. *Suppose* $\mathcal{T}^{\mathcal{F}}$ *is an* $\mathcal{L}$*-tree of height* ℓ *over* $\mathcal{X}$ *in* $\mathcal{F}$ *and suppose further that for every* $k \in \mathbb{N}$: $\frac{\ell}{m^k} > \mathbb{N}$.

Then for any $a \in \Omega$ *there is* $\mathcal{K} \subseteq \mathcal{H}$ *with* $\mathcal{X} \subseteq \mathcal{K} \subseteq \mathcal{F}$ *such that* $\langle a\rangle_{\mathcal{K}} \models \exists\Delta_0^{<\tilde{n}} - \mathrm{IND}^{\leq m}$ *and* $[0, n]$ *is an initial segment of* $\langle a\rangle_{\mathcal{K}}$.

Proof. Denote by G the a set of formulae $\varphi(x, \bar{f}, \bar{z})$ where $\varphi(x, \bar{y}, \bar{z}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})$, $\bar{f} \in \mathcal{F}$ and $\varphi(x, \bar{f}, \bar{z})$ is m-good for $\mathcal{F}$. Let further $\{(\varphi_k(x, \bar{f}, \bar{z})\}_{k\in\mathbb{N}}$ be an enumeration of G such that every formula appears infinitely many times. Moreover denote by H the set of tuples $(\bar{h}, F)$ such that $F \in \mathcal{L}$ is an r-ary function symbol for some r and $\bar{f} \in \mathcal{F}$ an r-tuple. Finally let $\{(\bar{h}, F)_k\}_{k\in\mathbb{N}}$ be an enumeration of H such that every tuple appears infinitely many times.

To find a suitable set of functions $\mathcal{K}$ we proceed similarly as in the proof of the previous theorem. Namely we construct an $\sqsubseteq$-increasing sequences $(s_k)_{k\in\mathbb{N}}$ of elements from $\mathcal{T}^{\mathcal{F}}$ and a sequence $(r_k)_{k\in\mathbb{N}}$ of elements $\leq \ell$ from $\mathbb{M}$ such that for every $k \in \mathbb{N}$:

(a) $r_k - len(s_k) \geq \frac{\ell}{m^{2k}}$ and if $k > 0$ then $s_{k-1} \sqsubseteq s_k$, $r_k \leq r_{k-1}$

(b) if $k > 0$ and $\varphi_{k-1}(x, \bar{f}, \bar{z})$ is m-good for $Fct(s_k) \cup \mathcal{X}$ then there is $\bar{g} \in Fct(s_k) \cup \mathcal{X}$ and $w < m$ such that $\mathbb{M} \models \varphi_{k-1}(w, \bar{f}(a), \bar{g}(a))$ and for any $s' \in \mathcal{T}^{\mathcal{F}}_{r_k}[s_k]$ and $\bar{h} \in Fct(s') \cup \mathcal{X}$, $\mathbb{M} \models \neg\varphi_{k-1}(w+1, \bar{f}(a), \bar{h}(a)$.

(c) if $k > 0$ and $(\bar{h}, F)_{k-1}$ is such that $\bar{h} \in Fct(s_{k-1}) \cup \mathcal{X}$ then $F^{\mathbb{M}} \circ \bar{h} \in s_k$

Then we let $\mathcal{K} = \mathcal{X} \cup \bigcup_{k\in\mathbb{N}} Fct(s_k)$.

Let $s_0 = \emptyset, r_0 = \ell$, then s_0, r_0 satisfies $(a) - (c)$. Assume s_k, r_k are constructed, to construct s_{k+1}, r_{k+1} let $s(x) \in \mathbb{M}$ be the function with domain $[0, m]$ defined as $s(w) = len(s_k) + w\lfloor\frac{r_k - len(s_k)}{m}\rfloor$ for every $w \leq m$. If $\varphi_k(x, \bar{f}, \bar{z})$ is not m-good for $Fct(s_k) \cup \mathcal{X}$ then let $s'_{k+1} = s_k$, $r_{k+1} = s(1)$ and skip the rest of this paragraph. Otherwise, since $\varphi_k(x, \bar{f}, \bar{z})$ is m-good for $Fct(s_k) \cup \mathcal{X}$ we have that

$$\mathbb{M} \models \exists s' \in \mathcal{T}^{\mathcal{F}}_{s(0)}[s_k]\exists\bar{g} \in Fct(s') \cup \mathcal{X} : \langle\langle\varphi(c_0, \bar{f}, \bar{g})\rangle\rangle_{\Omega} = \Omega$$

and

$$\mathbb{M} \models \forall s' \in \mathcal{T}^{\mathcal{F}}_{s(m)}[s_k]\forall \bar{g} \in Fct(s') \cup \mathcal{X} : \langle\langle \varphi(c_m, \bar{f}, \bar{g})\rangle\rangle_\Omega = \emptyset.$$

This gives

$$\mathbb{M} \models \exists s' \in \mathcal{T}^{\mathcal{F}}_{s(0)}[s_k]\exists \bar{g} \in Fct(s') \cup \mathcal{X} : \varphi(0, \bar{f}(a), \bar{g}(a))$$

and

$$\mathbb{M} \models \forall s' \in \mathcal{T}^{\mathcal{F}}_{s(m)}[s_k]\forall \bar{g} \in Fct(s') \cup \mathcal{X} : \neg\varphi(m, \bar{f}(a), \bar{g}(a)).$$

Thus we can use induction in $\mathbb{M}$ for the formula

$$\psi(x) = \exists s' \in \mathcal{T}^{\mathcal{F}}_{s(x)}[s_k]\exists \bar{g} \in Fct(s') \cup \mathcal{X} : \varphi(x, \bar{f}(a), \bar{g}(a))$$

to get $w < m$ and $s, \bar{g} \in \mathbb{M}$ such that

$$\mathbb{M} \models s \in \mathcal{T}^{\mathcal{F}}_{s(w)}[s_k] \wedge \bar{g} \in Fct(s) \cup \mathcal{X} \wedge \varphi(w, \bar{f}(a), \bar{g}(a))$$

and

$$\mathbb{M} \models \forall s' \in \mathcal{T}^{\mathcal{F}}_{s(w+1)}[s_k]\forall \bar{h} \in Fct(s') \cup \mathcal{X} : \neg\varphi(w+1, \bar{f}, \bar{h}(a)).$$

Finally we can let $s'_{k+1} = s$ and $r_{k+1} = s(w+1)$.

In both cases we get $r_{k+1} - len(s'_{k+1}) \geq \lfloor (r_k - len(s_k))/m \rfloor \geq \lfloor \ell/m^{2k+1} \rfloor \geq \ell/m^{2k+1} - 1$ using the induction assumption on s_k, r_k. To ensure (c) consider $(\bar{h}, F)_k$ and assume $\bar{h} \in Fct(s_k) \cup \mathcal{X}$. Then we let $s_{k+1} = {s'}^{\frown}_{k+1}(F^{\mathbb{M}} \circ \bar{h})$ which is in $\mathcal{T}^{\mathcal{F}}$ by the definition of $\mathcal{T}^{\mathcal{F}}$. Then $r_k - len(s_{k+1}) \geq r_k - len({s'}_{k+1}) - 1 \geq \ell/m^{2k+1} - 2 \geq \ell/m^{2k+2}$ and clearly $s_k \sqsubseteq s_{k+1}, r_k \geq r_{k+1}$ and so (a) holds for $k+1$. It is easy to see that (b) and (c) holds for $k+1$. Thus the sequences described above has been constructed.

Now we can let $\mathcal{K} = \mathcal{X} \cup \bigcup_{k \in \mathbb{N}} Fct(s_k)$. To see it is $\mathcal{L}$-closed is easy by (c) and so $\langle a \rangle_{\mathcal{K}}$ is a well-defined substructure of $\mathbb{M}$. Since $\{c_v \mid v \leq \max(m, n)\} \subseteq \mathcal{X}$ we get $[0, m] \subseteq \langle a \rangle_{\mathcal{K}}$. Moreover $[0, n]$ is an initial segment of $\langle a \rangle_{\mathcal{K}}$ and so we get that $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae are absolute between $\langle a \rangle_{\mathcal{K}}$ and $\mathbb{M}$. To show that $\langle a \rangle_{\mathcal{K}} \models \exists\Delta_0^{<\tilde{n}}(\mathcal{L}) - \mathrm{IND}^{\leq m}$ let a $\varphi(x, \bar{y}, \bar{z}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})$-formula and $\bar{b} \in \langle a \rangle_{\mathcal{K}}$ be given. Suppose $\langle a \rangle_{\mathcal{K}} \models \varphi(0, \bar{b}, \bar{c}) \wedge \forall \bar{x} \neg\varphi(m, \bar{b}, \bar{x})$ for some $\bar{c} \in \langle a \rangle_{\mathcal{K}}$. Let $\bar{f}, \bar{g} \in \mathcal{K}$ be such that $\bar{b} = \bar{f}(a)$ and $\bar{c} = \bar{g}(a)$ and let further $k \in \mathbb{N}$ be such that $\bar{g} \in Fct(s_k) \cup \mathcal{X}$. Since $\langle a \rangle_{\mathcal{K}} \cong \mathcal{K}/\mathcal{V}$ for an ultrafilter $\mathcal{V}$ with $\{a\} \in \mathcal{V}$ we can apply the Lemma 2.2.5 on $\langle a \rangle_{\mathcal{K}}$ and wlog assume $\varphi(x, \bar{f}, \bar{z})$ is m-good and so $\varphi(x, \bar{f}, \bar{z})$ is m-good for $Fct(s_k) \cup \mathcal{X}$. Let $k' > k$ be such that $\varphi(x, \bar{f}, \bar{z})$ has index k'. Then at step $k'+1$ we found $w < m$ such that there is $\bar{g}' \in Fct(s_{k'+1}) \cup \mathcal{X}$ with $\mathbb{M} \models \varphi(w, \bar{f}(a), \bar{g}'(a))$ and for any $s' \in \mathcal{T}^{\mathcal{F}}_{r_{k'+1}}[s_{k'+1}]$ and any $\bar{h} \in Fct(s') \cup \mathcal{X}$, $\mathbb{M} \models \neg\varphi(w+1, \bar{f}(a), \bar{h}(a))$. But since $\mathcal{K} \subseteq \mathcal{X} \cup \{h \in s \mid s \in \mathcal{T}^{\mathcal{F}}_{r_{k'+1}}[s_{k'+1}]\}$ and $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae are absolute between $\langle a \rangle_{\mathcal{K}}$ and $\mathbb{M}$ this gives $\langle a \rangle_{\mathcal{K}} \models \varphi(w, \bar{b}, \bar{g}'(a)) \wedge \forall \bar{z} \neg\varphi(w+1, \bar{b}, \bar{z})$ and we are done. $\square$

In the following lemma we denote by $\mathcal{L}_{BUSS}(g)$ the language $\mathcal{L}_{BUSS} \cup \{g\}$ where g is some function symbol its interpretation in $\mathbb{M}$ is an $\mathbb{M}$-definable function such that there is $c, k \in \mathbb{N}$ with $\mathbb{M} \models \forall x \in \Omega : |g(x)| \leq c + c|x|^k$).

Corollary 2.2.13. *Let $g \in \mathcal{L}_{all}$ be a function symbol, $\ell \in \mathbb{M}$, $\{id\} \cup \{c_v \mid v \leq \max(n, m)\} \subseteq \mathcal{X} \subseteq \mathcal{F} \subseteq \mathcal{H}$ with $\mathcal{X} \in \mathbb{M}$ and assume there exists an $\mathcal{L}$-tree of height*

ℓ over $\mathcal{X}$ in $\mathcal{F}$. Finally assume $\varphi(x, y)$ is an $\mathcal{L}_{BUSS}(g)$-formula such that there is $a \in \mathbb{M}$ and $k \in \mathbb{N}$ with:

$$\mathbb{M} \models |a| < n^k \textit{and for any } f \in \mathcal{F} : \mathbb{M} \models \neg\varphi(a, f(a)).$$

Then

(i) if for any $k \in \mathbb{N} : \ell/|n|^k > \mathbb{N}$ then $\mathrm{strict}\Sigma_1^b(\mathcal{L}_{BUSS}(g)) - \mathrm{LIND} + \exists x \forall y \neg\varphi(x, y)$ *is consistent and*

(ii) if for any $k \in \mathbb{N} : \ell/n^k > \mathbb{N}$ then $\mathrm{strict}\Sigma_1^b(\mathcal{L}_{BUSS}(g)) - \mathrm{LLIND} + \exists x \forall y \neg\varphi(x, y)$ *is consistent.*

Proof. Using the lemma above with $\mathcal{L} = \mathcal{L}_{BUSS}(g) \cup \{\tilde{n}\}$ and $m = n$ in case (i) and $m = |n|$ in case (ii) there is an $\mathcal{L}_{BUSS}(g)$-closed $\mathcal{K} \subseteq \mathcal{H}$ with $\langle a \rangle_{\mathcal{K}} \models \exists\Delta_0^{<\tilde{n}} - \mathrm{IND}^{\leq n}$ in case (i) and $\langle a \rangle_{\mathcal{K}} \models \exists\Delta_0^{<\tilde{n}} - \mathrm{IND}^{\leq |n|}$ in case (ii). By the assumption on a and $\mathcal{F}$ in both cases $\langle a \rangle_{\mathcal{K}} \models \forall y \neg\varphi(a, y)$. But since in both cases $\langle a \rangle_{\mathcal{K}} \models |a| < \tilde{n}^k$ as $\mathbb{M} \models |a| < n^k$, (i) and (ii) follows from the Lemma 2.2.1. □

2.3 Proof of the Lemma 2.2.1

Recall the statement of the Lemma 2.2.1:

Let $\mathcal{L}_{all} \supseteq \mathcal{L}' \supseteq \mathcal{L}_{\mathrm{BUSS}}$ and $\mathbb{K}$ be an infinite $\mathcal{L}'$-structure with $m \in \mathbb{K}$ such that

$$\mathbb{K} \models \mathrm{Th}_{\forall\Delta_0^{<\tilde{n}}(\mathcal{L}')}(\mathbb{M}) \text{ and } \mathbb{K} \models \mathrm{BASIC} \text{ and } \mathbb{K} \models \exists\Delta_0^{<\tilde{n}}(\mathcal{L}') - \mathrm{IND}^{\leq m}.$$

Suppose further that *(L)*:

For any r-ary function symbol $F \in \mathcal{L}'$ there are $c, k \in \mathbb{N}$ with

$$\mathbb{M} \models \forall y_0, y_1, \ldots, y_{r-1}(|F(y_0, y_1, \ldots, y_{r-1})| \leq c(c + |y_1| + \ldots + |y_r|)^k).$$

Then $K' = \{b \in \mathbb{K} \mid \mathbb{K} \models |b| < \tilde{n}^k \text{ for some } k \in \mathbb{N}\}$ is a domain of a structure $\mathbb{K}' \leq \mathbb{K}$ and

(i) if $\mathbb{K} \models \tilde{n} = m$ then $\mathbb{K}' \models \mathrm{strict}\Sigma_1^b(\mathcal{L}') - \mathrm{LIND}$ and

(ii) if $\mathbb{K} \models |\tilde{n}| = m$ then $\mathbb{K}' \models \mathrm{strict}\Sigma_1^b(\mathcal{L}') - \mathrm{LLIND}$.

Proof. We separate the proof of this lemma into four claims. The first claim shows the main proof idea:

Claim 2.3.1. *Assume*

(a) $\mathbb{K} \models \exists\Delta_0^{<\tilde{n}}(\mathcal{L}') - \mathrm{IND}^{\leq m^k}$ for all $k \in \mathbb{N}$ and

(b) for any $w, \bar{p} \in \mathbb{K}'$ and any $\varphi(\bar{x}, \bar{y}) \in$ $\mathrm{strict}\Sigma_1^b(\mathcal{L}')$ *there is a $\psi(\bar{x}, \bar{y}) \in \exists\Delta_0^{<\tilde{n}}(\mathcal{L}')$ such that $\mathbb{K} \models \forall \bar{x} < w(\varphi(\bar{x}, \bar{p}) \leftrightarrow \psi(\bar{x}, \bar{p}))$ and*

(c) bounded $\mathcal{L}'$-formulae are absolute between $\mathbb{K}'$ and $\mathbb{K}$.

Then (i) and (ii) holds.

Proof. To show (i) assume $\mathbb{K}' \models \tilde{n} = m$ and let $\varphi(x, \bar{y}) \in \mathrm{strict}\Sigma_1^b(\mathcal{L}')$ and $\bar{p} \in \mathbb{K}'$ be given. To show $\mathbb{K}' \models \varphi(x, \bar{p}) - \mathrm{LIND}$ it suffices to show that for any $w \in \mathbb{K}'$:

$$\mathbb{K}' \models \varphi(x, \bar{p}) - \mathrm{IND}^{\leq |w|} \text{ i.e. } \mathbb{K}' \models \varphi(0, \bar{p}) \land \forall x < |w|(\varphi(x, \bar{p}) \rightarrow \varphi(x+1, \bar{p})) \rightarrow \varphi(|w|, \bar{p}).$$

Since $\varphi(x,\bar{p}) - \mathrm{IND}^{\leq |w|}$ is a bounded $\mathcal{L}'(\mathbb{K}')$-sentence, this holds by (c) if and only if for any $w \in \mathbb{K}'$, $\mathbb{K} \models \varphi(x,\bar{p}) - \mathrm{IND}^{\leq |w|}$.

To show this let $w \in \mathbb{K}'$ be given. As $\bar{p}, w \in \mathbb{K}'$ there is by (b) a formula $\psi(x,\bar{y}) \in \exists\Delta_0^{<\tilde{n}}(\mathcal{L}')$ such that $\mathbb{K} \models \forall x < w(\varphi(x,\bar{p}) \leftrightarrow \psi(x,\bar{p}))$. By (a) we get $\mathbb{K} \models \psi(x,\bar{y}) - \mathrm{IND}^{\leq \tilde{n}^k}$ for any $k \in \mathbb{N}$ (using the assumption $\mathbb{K} \models \tilde{n} = m$). Finally since $\mathbb{K} \models |w| < \tilde{n}^k$ for some $k \in \mathbb{N}$ by the definition of K' we have that $\mathbb{K} \models \psi(x,\bar{y}) - \mathrm{IND}^{\leq |w|}$ i.e $\mathbb{K} \models \psi(x,\bar{p}) - \mathrm{IND}^{\leq |w|}$ and so $\mathbb{K} \models \varphi(x,\bar{p}) - \mathrm{IND}^{\leq |w|}$ which finishes the argument for (i).

The argument for (ii) is similar but with $\varphi(x,\bar{p}) - \mathrm{IND}^{\leq ||w||}$ in place of $\varphi(x,\bar{p}) - \mathrm{IND}^{\leq |w|}$ where $\mathbb{K}' \models |\tilde{n}| = m$ gives $||w|| < k|\tilde{n}|$ for some $k \in \mathbb{N}$ and (a) gives $\mathbb{K} \models \exists\Delta_0^{<\tilde{n}}(\mathcal{L}') - \mathrm{IND}^{\leq |\tilde{n}|^k}$ for all $k \in \mathbb{N}$ i.e. $\mathbb{K} \models \varphi(\bar{x},\bar{p}) - \mathrm{IND}^{\leq ||w||}$. □

Thus once we show that (a),(b) and (c) holds we are done.

Claim 2.3.2 ((a) holds)**.** *For all $k \in \mathbb{N}$, $\mathbb{K} \models \exists\Delta_0^{<\tilde{n}}(\mathcal{L}') - \mathrm{IND}^{\leq m^k}$.*

Proof. Since we will give a folklore argument we only sketch the proof idea and leave it to the reader to fill in the details.

Observe that it suffices to show that for any $a, b \in \mathbb{K}$, if $\exists\Delta_0^{<\tilde{n}}(\mathcal{L}')$-induction holds in $\mathbb{K}$ up to a and up to b then it holds in $\mathbb{K}$ up to $a \cdot b$. Assume $a, b \in \mathbb{K}$ and $\Delta_0^{<\tilde{n}}(\mathcal{L}')$-induction holds in $\mathbb{K}$ up to a and up to b. To show $\mathbb{K} \models \exists\Delta_0^{<\tilde{n}}(\mathcal{L}') - \mathrm{IND}^{\leq ab}$ let $\varphi(x) \in \exists\Delta_0^{<\tilde{n}}(\mathcal{L}'(\mathbb{K}))$ be given and assume $\mathbb{K} \models \varphi(0) \wedge \neg\varphi(a \cdot b)$. Let $\psi(x) = \varphi(a \cdot x)$, then either $\mathbb{K} \models \neg\varphi(a)$ and thus we can use induction up to a in $\mathbb{K}$ for $\varphi(x)$ to get $w \leq$ with $\mathbb{K} \models \varphi(w) \wedge \neg\varphi(w+1)$ and we are done. Or $\mathbb{K} \models \varphi(a)$ and then $\mathbb{K} \models \psi(1) \wedge \neg\psi(b)$. It is not hard to see that induction up to b in $\mathbb{K}$ for a suitable $\psi' \in \exists\Delta_0^{<\tilde{n}}(\mathcal{L}')$ gives $w < b$ such that $\mathbb{K} \models \psi(w) \wedge \neg\psi(w+1)$ and so $\mathbb{K} \models \varphi(a \cdot w) \wedge \neg\varphi(a \cdot (w+1))$. But since the interval $[a \cdot w, a \cdot (w+1)]$ is of length $a+1$ we can find using induction up to a in $\mathbb{K}$ (again by going to some suitable $\psi'' \in \exists\Delta_0^{<\tilde{n}}(\mathcal{L}')$) a witness $w' \in [a \cdot w, a \cdot (w+1)]$ such that $\mathbb{K} \models \varphi(w') \wedge \neg\varphi(w'+1)$ and we are done.

□

For the next claim we will need the following observation where D denotes the set of $\mathcal{L}'$-formulae with each quantifier strictly bounded by $\tilde{n}^k$ for some $k \in \mathbb{N}$ (i.e. the k can differ among the quantifiers of a formula from D).

Observation 2.3.3. *For any $\varphi(\bar{x}) \in D$ there is $\psi(\bar{x}) \in \Delta_0^{<\tilde{n}}(\mathcal{L}')$ such that $\mathbb{K} \models \forall\bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.*

Proof. We will proceed by induction on a number of quantifiers q in the formula $\varphi(\bar{x}) \in D$ which we can wlog assume is in a prenex form. The case when $q = 0$ is clear as then $\varphi(\bar{x})$ is an open formula. Now assume $\varphi(\bar{x}) = \exists y < \tilde{n}^k \psi(y,\bar{x})$ for some $k \in \mathbb{N}$ and for some $\psi(y,\bar{x})$ for which the induction hypothesis holds. Let $s_k(y_0,\ldots,y_{k-1}) = y_0 + y_1 \cdot \tilde{n} + \ldots + y_{k-1} \cdot \tilde{n}^{k-1}$. It is not hard to see that

$$\mathbb{M} \models \forall y[y < \tilde{n}^k \leftrightarrow \exists y_0 < \tilde{n}, \ldots \exists y_{k-1} < \tilde{n}(y = s_k(y_0,\ldots,y_{k-1}))]$$

which is a $\forall\Delta_0^{<\tilde{n}}(\mathcal{L}')$-sentence and thus also valid in $\mathbb{K}$. Now let $\psi'(y,\bar{x}) \in \Delta_0^{<\tilde{n}}(\mathcal{L}')$ be such that $\mathbb{K} \models \forall x, \bar{y}(\psi(x,\bar{y}) \leftrightarrow \psi'(x,\bar{y}))$ which exists by the induction assumption on $\psi(y,\bar{x})$. Then

$$\mathbb{K} \models \forall \bar{x}(\exists y < \tilde{n}^k \psi(y,\bar{x}) \leftrightarrow \exists y_0 < \tilde{n}, \ldots \exists y_{k-1} < \tilde{n}\psi'(s_k(y_0,\ldots,y_{k-1}),\bar{x})).$$

But the formula on the right-hand side of the equivalence is $\Delta_0^{<\tilde{n}}(\mathcal{L}')$-formula and so we are done. The case when $\varphi(\bar{x}) = \forall y < \tilde{n}^k\psi(y,\bar{x})$ for some $\psi(y,\bar{x})$ follows by the above step by going to $\neg\varphi(\bar{x})$. This finishes the argument. $\square$

Claim 2.3.4 ((b) holds)**.** *For any $w,\bar{p} \in \mathbb{K}'$ and any $\varphi(\bar{x},\bar{y}) \in \mathrm{strict}\Sigma_1^b(\mathcal{L}')$ there is a $\psi(\bar{x},\bar{y}) \in \exists\Delta_0^{<\tilde{n}}(\mathcal{L}')$ such that $\mathbb{K} \models \forall\bar{x} < w(\varphi(\bar{x},\bar{p}) \leftrightarrow \psi(\bar{x},\bar{p}))$.*

Proof. We first show that for any $\varphi(\bar{x},\bar{y}) \in \Delta_0^b(\mathcal{L}')$ we have $(*)$: for any $w,\bar{p} \in \mathbb{K}'$ there is $\psi(\bar{x},\bar{y}) \in \Delta_0^{<\tilde{n}}(\mathcal{L}')$ such that

$$\mathbb{K} \models \forall\bar{x} < w(\varphi(\bar{x},\bar{p}) \leftrightarrow \psi(\bar{x},\bar{p})).$$

To do so, we proceed by induction on number of quantifiers q in the formula $\varphi(\bar{x},\bar{y}) \in \Delta_0^b(\mathcal{L}')$ which is wlog assumed to be in prenex form. The case when $q = 0$ is clear as then $\varphi(\bar{x},\bar{y})$ is an open formula and so $\varphi(\bar{x},\bar{y}) \in \Delta_0^{<\tilde{n}}(\mathcal{L}')$.

Now assume that for any $\varphi(\bar{x},\bar{y}) \in \Delta_0^b(\mathcal{L}')$ with q quantifiers $(*)$ holds. Suppose $\varphi(\bar{x},\bar{y}) \in \Delta_0^b(\mathcal{L}')$ has $q+1$ many quantifiers, is wlog in prenex form and $\varphi(\bar{x},\bar{y}) = \exists z \le |t(\bar{x},\bar{y})|\theta(z,\bar{x},\bar{y})$ for some $\theta(z,\bar{x},\bar{y}) \in \Delta_0^{<\tilde{n}}(\mathcal{L}')$.

Now observe that by (L) we have that $\mathbb{K} \models \forall\bar{x} < w(|t(\bar{x},\bar{p})| \le c(c+r\cdot|w|+|\bar{p}|)^{k_0})$ for some $c, k_0 \in \mathbb{N}$ where r is the number of elements in the tuple $\bar{x}$. But as $c(c+r\cdot|w|+|\bar{p}|)^{k_0} \in \mathbb{K}'$ and $\mathbb{K}' \models$ BASIC there is $w' \in \mathbb{K}'$ with $c(c+r\cdot|w|+|\bar{p}|)^{k_0} \le |w'|$ thus $|w'| < \tilde{n}^k$ for some $k \in \mathbb{N}$ and $\mathbb{K} \models \forall\bar{x} < w(|t(\bar{x},\bar{p})| < \tilde{n}^k)$. Now it is not hard to see that

$$\mathbb{K} \models \forall\bar{x} < w(\varphi(\bar{x},\bar{p}) \leftrightarrow \exists z < \tilde{n}^k(z \le |t(\bar{x},\bar{p})| \wedge \theta(z,\bar{x},\bar{p}))).$$

Finally let $c = \max(w,\tilde{n}^k)$ and let $\theta'(z,\bar{x},\bar{y})$ be such that

$$\mathbb{K} \models \forall\bar{x}, z < c(\theta(z,\bar{x},\bar{p}) \leftrightarrow \theta'(z,\bar{x},\bar{p}))).$$

Then clearly

$$\mathbb{K} \models \forall\bar{x} < w(\varphi(\bar{x},\bar{p}) \leftrightarrow \exists z < \tilde{n}^k(z \le |t(\bar{x},\bar{p})| \wedge \theta'(z,\bar{x},\bar{p}))$$

and we are done. Indeed, the formula $\exists z < \tilde{n}^k(z \le |t(\bar{x},\bar{y})| \wedge \theta'(z,\bar{x},\bar{y})$ is in D and thus by the observation above equivalent to some $\Delta_0^{<\tilde{n}}(\mathcal{L}'))$-formula.

The case when the first quantifier of $\varphi(\bar{x},\bar{y})$ is universal follows from the previous by going to negation of $\varphi(\bar{x},\bar{y})$.

To finish the proof let $\varphi(\bar{x},\bar{y}) \in \mathrm{strict}\Sigma_1^b(\mathcal{L}')$ be be of the form $\exists z_1 \le t_1(\bar{x},\bar{y})\exists z_2 \le t_2(z_1,\bar{x},\bar{y})\ldots\exists z_{k-1} \le t_k(z_1,\ldots z_k,\bar{x},\bar{y})\theta(\bar{z},\bar{x},\bar{y})$ for some $\theta(\bar{z},\bar{x},\bar{y}) \in \Delta_0^b(\mathcal{L}')$ where $\bar{z} = (z_1,\ldots,z_k)$. Then for any $w,\bar{p} \in \mathbb{K}'$ there is some $w' \in \mathbb{K}'$ such that by (L) and a similar argument as above

$$\mathbb{K}' \models \forall\bar{x} < w\forall z_1 \le t(\bar{x},\bar{p})\forall z_2 \le t(z_1,\bar{x},\bar{p})\ldots\forall z_k \le t_k(z_1,\ldots z_{k-1},\bar{x},\bar{p})(\bar{z} < w').$$

If we let $c = \max(w, w')$ then by what was proven above there is $\theta'(\bar{z}, \bar{x}, \bar{y}) \in \Delta_0^{<\tilde{n}}(\mathcal{L}')$ such that $\mathbb{K}' \models \forall \bar{x}, \bar{z} < c(\theta(\bar{z}, \bar{x}, \bar{p}) \leftrightarrow \theta'(\bar{z}, \bar{x}, \bar{p}))$ and so

$$\mathbb{K}' \models \forall x < w(\varphi(\bar{x}, \bar{p}) \leftrightarrow \exists \bar{z}(z_1 \leq t(\bar{x}, \bar{p}) \wedge \ldots \wedge z_k \leq t_k(z_1, \ldots z_{k-1}, \bar{x}, \bar{p}) \wedge \theta'(\bar{z}, \bar{x}, \bar{p})))$$

where the formula on the right-hand side of the equivalence is in $\exists \Delta_0^{<\tilde{n}}(\mathcal{L}')$ and we are done.

□

Claim 2.3.5 ((c) holds)**.** *Bounded $\mathcal{L}'$-formulae are absolute between $\mathbb{K}'$ and $\mathbb{K}$.*

Proof. To show this we proceed by induction on complexity of a bounded $\mathcal{L}'$-formula $\varphi(\bar{x})$. The case when $\varphi(\bar{x})$ is an open $\mathcal{L}'$-formula is clear as $\mathbb{K}' \leq \mathbb{K}$ thus the only case to consider is if $\varphi(\bar{x}) = \exists y < t(\bar{x})\psi(y, \bar{x})$ for some $\psi(y, \bar{x})$ which is absolute between $\mathbb{K}'$ and $\mathbb{K}$.

Assume $\varphi(\bar{x})$ is of the form described above, $\bar{p} = (p_0, p_1, \ldots, p_{r-1}) \in \mathbb{K}$ is given and $\mathbb{K} \models \exists y \leq t(\bar{p})\psi(y, \bar{p})$ i.e. $\mathbb{K} \models q \leq t(\bar{p}) \wedge \psi(q, \bar{p})$ for some $q \in \mathbb{K}$. Thus if we show that $q \in \mathbb{K}'$ the rest will follow by induction assumption. Since by the definition of K' there is $k_0 \in \mathbb{N}$ with $|p_i| < \tilde{n}^{k_0}$ for any $i < r$ we get by (L) a natural number k with $|t(\bar{p})| < \tilde{n}^k$. But then $|q| < \tilde{n}^k$ and so $q \in \mathbb{K}'$.

□

□

Chapter 3

Garlík's construction

It was shown in the Corollary 3.9, the Theorem 3.9 and the Theorem 4.1 of [Gar15] using the Construction B given in the Theorem 3.4 ibid that three pairs of theories relevant to the Complexity theory are not (logically) equivalent under a certain Complexity theoretic assumption. In this section we show that the Construction B of Michal Garlík can be derived from the Corollary 2.2.8. The author consider this as a plausible argument that the construction developed in the previous section can indeed be used for reasoning about arithmetical theories.

From now on let $\mathbb{M}$ be a countable model of TA in the language $\mathcal{L}_{all}$ and:

- $n \in \mathbb{M} - \mathbb{N}$,
- $\Omega \subseteq \{m \in \mathbb{M} \mid \mathbb{M} \models m < 2^n\}$ a definable (and so coded) infinite set
- $\mathcal{B}$ an algebra of $\mathbb{M}$-definable subsets of Ω,
- $\mathcal{L}$ first order countable language with binary relation symbol $\leq$ and a finite number of function symbols containing a unary function symbol S and a constants $0, 1, \tilde{n}$. Interpretations of all function symbols from $\mathcal{L}$ is some $\mathbb{M}$-definable function with S interpreted as the successor function, $\tilde{n}$ by n and 0, 1, $\leq$ as usual. Finally let
- $\mathcal{X} = \{h_1, \ldots, h_d\} \cup \{id\} \cup \{c_v \mid v \in \mathbb{M} \text{ and } \mathbb{M} \models v \leq n\}$ for $d \in \mathbb{N}$ be a set of definable functions from Ω to $\mathbb{M}$ where id is the identity function on Ω and for any $v \in \mathbb{M}$, $c_v \in {}^{\Omega}\mathbb{M}$ is the function which is constant v on Ω.

Recall the definition of a Straight-line program given in the Section 2:

Definition 3.0.1. *Suppose $\ell \in \mathbb{M}$. We say that a straight-line program (*SLP *for short) over $\mathcal{L}$ and $\mathcal{X}$ of size ℓ is a sequence of functions $y_0, y_1, \ldots, y_{\ell-1}$ of the following form: for $i < \ell$ the $i-th$ function y_i equals to $F^{\mathbb{M}} \circ (y_{i_0}, \ldots, y_{i_{r-1}}, f_0, \ldots, f_{k-1})$ where $F \in F_{\mathcal{L}}$ is some $(r+k)$-ary function, $i_j < i$ for all $j < r$ and $f_0, f_1, \ldots, f_{k-1} \in \mathcal{X}$. Moreover we let $\mathrm{SLP}_\ell(\mathcal{X})$ be the set of all* SLP *programs over $\mathcal{L}$ and $\mathcal{X}$ of size ℓ and $\mathrm{SLP}_{\leq \ell}(\mathcal{X})$ to be the set of all* SLP *programs over $\mathcal{L}$ and $\mathcal{X}$ of size $\leq \ell$. If $P \in \mathrm{SLP}_{\leq \ell}(\mathcal{X})$ then we define $Fct(P)$ to be the set of all functions $f : \Omega \to \mathbb{M}$ such that either $f \in \mathcal{X}$ or $f = y_i$ for some $i < \ell$ with y_i in P. Finally we let $FCT_\ell(\mathcal{X}) = \bigcup_{P \in \mathrm{SLP}_\ell} Fct(P)$.*

Where we changed the notion of Fct from the Section 2 according to [Gar15]. Note that for any $\ell \in \mathbb{M}$ this definition can be formalised in $\mathbb{M}$ since $\mathcal{X} \in \mathbb{M}$ and the number of function symbols from $\mathcal{L}$ is a standard number and so $FCT_\ell(\mathcal{X}), \mathrm{SLP}_{\leq \ell}(\mathcal{X}) \in \mathbb{M}$. This is a crucial point for the whole construction.

A reader familiar with the [Gar15] might recognised that we gave a "semantic" definition of the "syntactic" notion of SLP defined in [Gar15]. More precisely, the SLP of length ℓ according to [Gar15] is a sequence of instructions of the form $F \circ (y_{i_0}, \ldots, y_{i_{r-1}}, f_0, \ldots, f_{k-1})$ with the same meaning of $F, \bar{y}, \bar{f}$ as in the definition above (i.e. F in place of $F^{\mathbb{M}}$). The set of functions defined by such an SLP consists from functions which can be obtained by interpreting the instructions in $\mathbb{M}$ or which are elements of $\mathcal{X}$. However, this difference is immaterial as it will be obvious from the statement of the Theorem of Garlík. This is because $FCT_\ell(\mathcal{X})$ from above is equal to $FCT_\ell(\mathcal{X})$ defined in the way taken in [Gar15] and the Theorem is independent of the particular definition of SLP as far as it defines the same set of functions $FCT_\ell(\mathcal{X})$.

Now for a non-standard $m, s \in \mathbb{M}$ with $m < n$ and for $\psi(x, y, \bar{x}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})$ and an $\mathbb{M}$-rational number q with $q \in (0,1)$ we denote by (H) the following hypothesis:

$$\text{For every } f \in FCT_{m^s} : \#(\langle\langle \psi(id, f, h_0, h_1, \ldots, h_{d-1})\rangle\rangle_\Omega)/\#(\Omega) < q.^{1}$$

Further we denote by (R) the following relation between q, n, m:

$$\text{For every } k \in \mathbb{N} : q < \frac{1}{n^{m^k}}.$$

Then the Theorem of Garlík states:

Theorem 3.0.2. *[Gar15, Theorem 3.4]*

Let $\mathbb{M}$, Ω, n, $\mathcal{L}$, $\mathcal{X}$ be as above. Let $\psi(x, y, \bar{z}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})$ be a formula, $m, s \in \mathbb{M}$ non-standard with $m < n$ and an $\mathbb{M}$-rational $q \in (0,1)$ be such that the hypothesis (H) and (R) holds.

Then there is an $\mathcal{L}$-closed $\mathcal{K} \subseteq FCT_{m^s}(\mathcal{X})$ with $\mathcal{X} \subseteq \mathcal{K}$ and a filter $\mathcal{V}$ on $\mathcal{B}$ such that:

(1) if $\alpha \in \mathcal{K}/\mathcal{V}$ is such that $\mathcal{K}/\mathcal{V} \models \alpha \leq c_n^{\mathcal{V}}$ then there is $v \leq n$ such that $\mathcal{K}/\mathcal{V} \models \alpha = c_v^{\mathcal{V}}$,

(2) $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae are Łos for $\mathcal{K}/\mathcal{V}$,

(3) $\mathcal{K}/\mathcal{V} \models \forall y \neg\psi(id^{\mathcal{V}}, y, h_1^{\mathcal{V}}, \ldots, h_d^{\mathcal{V}})$ and

(4) $\mathcal{K}/\mathcal{V} \models \exists\Delta_0^{<\tilde{n}}(\mathcal{L}) - \mathrm{IND}^{\leq c_m^{\mathcal{V}}}$.

We will show that this theorem can be derived from the Corollary 2.2.8 where $\mathcal{H}$ denotes the set of all $\mathbb{M}$-definable functions from Ω to $\mathbb{M}$:

Corollary 2.2.8. Suppose $+, 0, 1 \in \mathcal{L}$, $m \in \mathbb{M}$ with $m \leq n$, $s \in \mathbb{M}$ with $s > \mathbb{N}$, $\mathcal{F} \subseteq \mathcal{H}$, and $\{id\} \cup \{c_v \mid v \leq \max(m, n)\} \subseteq \mathcal{X} \subseteq \mathcal{F}$ with $\mathcal{X}, \mathcal{F} \in \mathbb{M}$. Suppose there exists an $\mathcal{L}$-tree of height m^s over $\mathcal{X}$ in $\mathcal{F}$ and call the tree $\mathcal{T}^{\mathcal{F}}$. Assume further $\psi(x, y, \bar{z}) \in \Delta_0^{<\tilde{n}}(\mathcal{L})$, $\bar{h} \in \mathcal{X}$, there is an $\mathbb{M}$-rational q with:

$$(Q) \qquad \text{for any } f \in \mathcal{F} : \#(\langle\langle \psi(id, f, \bar{h})\rangle\rangle_\Omega)/\#(\Omega) \leq q$$

and there is a non-standard $r \in \mathbb{M}$ with:

$$(q) : \frac{n}{\#(\mathcal{F})^{2mr}} > q \text{ and } (\Omega) : \#(\Omega) \geq r\#(\mathcal{F})^{2mr-1}(n+1).$$

[1] For the discussion about $\mathbb{M}$-rationals and definition of the function $\#(\cdot)$ see beginning of the previous chapter.

Then there is an $\mathcal{L}$-closed $\mathcal{K} \subseteq \mathcal{H}$ with $\mathcal{X} \subseteq \mathcal{K} \subseteq \mathcal{F}$ and a filter $\mathcal{V}$ such that $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae are Łos for $\mathcal{K}/\mathcal{V}$,

$$\mathcal{K}/\mathcal{V} \models \exists\Delta_0^{<\tilde{n}} - \mathrm{IND}^{\leq c_m^{\mathcal{V}}} \text{ and } \mathcal{K}/\mathcal{V} \models \forall y \neg\psi(id^{\mathcal{V}}, y, \bar{h}^{\mathcal{V}}).$$

Moreover if $\alpha \in \mathcal{K}/\mathcal{V}$ is such that $\mathcal{K}/\mathcal{V} \models \alpha \leq c_n^{\mathcal{V}}$ then there is $v \leq n$ such that $\mathcal{K}/\mathcal{V} \models \alpha = c_v^{\mathcal{V}}$.

Note that the $\mathcal{K}/\mathcal{V}$ from the corollary above posses all properties (1)-(4). Note also the assumption (H) holds for any $s' < s$ and $\mathrm{SLP}_{\leq m^{s'}}(\mathcal{X})$ is an $\mathcal{L}$-tree of height $m^{s'}$ over $\mathcal{X}$ in $FCT_{m^{s'}}(\mathcal{X})$ for any $s' \in \mathbb{M}$. Thus once we find a suitable non-standard $s' \leq s$ such that that (q) and (Ω) can be satisfied for some $r > \mathbb{N}$ and $\mathcal{F} = FCT_{m^{s'}}(\mathcal{X})$, we can use the corollary above for $\mathcal{F} = FCT_{m^{s'}}(\mathcal{X})$ with $q, \Omega, \psi(x, y, \bar{z}), \bar{h}$ and $\mathcal{L}$ from the theorem of Garlík and we will be done.

Unfortunately, we will have to handle one obstacle. To proceed as described above, we will have to derive some lower bound on $\#(\Omega)$. However, the assumption (R) on n, m, q in Garlíks theorem does not give explicit lower bound on $\#(\Omega)$. This can be solved by the following considerations:

If there is a function $f \in FCT_{m^s}(\mathcal{X})$ such that $\langle\langle\psi(id, f, \bar{h})\rangle\rangle_\Omega \neq \emptyset$ then q must be at least $\frac{1}{\#(\Omega)}$. But since (R) gives $q < \frac{1}{n^{m^k}}$ for all $k \in \mathbb{N}$ we will get that $\#(\Omega) \geq n^{m^k}$ for all $k \in \mathbb{N}$ which will be sufficient to ensure the estimation (Ω).

On the other hand, if for any $f \in FCT_{m^s}(\mathcal{X})$, $\langle\langle\psi(id, f, \bar{h})\rangle\rangle_\Omega = \emptyset$ then we will use Lemma 2.2.12 to derive the Theorem of Garlík directly.

We will need an upper bounds on the size of $FCT_\ell(\mathcal{X})$ for any non-standard $\ell \in \mathbb{M}$. Our upper bound is quite generous but still sufficient for the latter estimations.

Observation 3.0.3. *Let $\ell \in \mathbb{M}$ be non-standard. Then $\#(FCT_\ell(\mathcal{X})) \leq (\#(\mathcal{X})+\ell)^{\ell^2}$.*

Proof. Recall that for any $i < \ell$ the $i-th$ function y_i of an SLP P of size ℓ is a function equal to $F^{\mathbb{M}} \circ (y_{i_0}, \ldots, y_{i_{r-1}}, f_0, \ldots, f_{k-1})$ where $i_j < i$ for all $j < r$, $F \in \mathcal{L}$ for some $(r+k)$-ary function symbol and $f_0, f_1, \ldots, f_{k-1} \in \mathcal{X}$. Let w be the maximal arity between function symbols from $\mathcal{L}$ and $symb$ the number of function symbols in $\mathcal{L}$. Then the number of tuples $(y_{i_0}, \ldots, y_{i_{r-1}}, f_0, \ldots, f_{k-1})$ with $i_0, i_1, \ldots, i_{r-1} < \ell$, $f_0, f_1, \ldots, f_{k-1} \in \mathcal{X}$ and $r+k \leq w$ is at most $\sum_{i \leq w} \#(\mathcal{X}+\ell)^i < (\#(\mathcal{X}+\ell))^{w^2}$ where the inequality holds as $\#(X)+\ell$ is non-standard. Thus the number of sequences of length ℓ of the functions defined as described above is at most $(symb \cdot (\#(\mathcal{X})+\ell)^{w^2})^\ell$. Since any $P \in \mathrm{SLP}_{ell}$ is of such form we get $\#(\mathrm{SLP}_\ell(\mathcal{X})) \leq (symb \cdot (\#(\mathcal{X})+\ell)^{w^2})^\ell$ and together with $\#(Fct(P)) \leq \#(\mathcal{X})+\ell$ for every $P \in \mathrm{SLP}_\ell(\mathcal{X})$ we get that:

$$\begin{aligned}
\#(FCT_\ell(\mathcal{X})) \leq \sum_{P \in \mathrm{SLP}_\ell(\mathcal{X})} \#(Fct(P)) &\leq \sum_{P \in \mathrm{SLP}_\ell(\mathcal{X})} (\#(\mathcal{X})+\ell) \\
= \#(\mathrm{SLP}_\ell(\mathcal{X}))(\#(\mathcal{X})+\ell) &\leq (symb \cdot (\#(\mathcal{X})+\ell)^{w^2})^\ell(\#(\mathcal{X})+\ell) \\
&\leq (\#(\mathcal{X})+\ell)^{(1+w^2)\ell+1} \leq (\#(\mathcal{X})+\ell)^{\ell^2}
\end{aligned}$$

where we used $symb < \#(\mathcal{X})+\ell$ and $w^2+1 < \ell$ by $\ell > \mathbb{N}$ and $symb, w \in \mathbb{N}$. □

To ensure the estimations (q) and (Ω) we need this technical observation.

Observation 3.0.4. *Let $s, r, m, n > \mathbb{N}$ and $f(x) = \#(FCT_{m^s}(\mathcal{X}))^x$. Then*

$$\frac{n}{f(2mr)} > q \text{ and } \#(\Omega) \geq rf(2mr-1)(n+1)$$

whenever $s+1, 2r \leq m < n$ and $r(n+1) \leq \#(FCT_{m^s}(\mathcal{X}))$ and $q < \frac{1}{n^{m^{2s+3}}}$ and $\#(\Omega) \geq q^{-1}$.

Proof. By the previous observation, $\#(FCT_{m^s}(\mathcal{X})) \leq (\#(\mathcal{X})+m^s)^{m^{2s}} = (2+d+n+m^s)^{m^{2s}}$ where $d \in \mathbb{N}$. Thus if $n \leq m^s$ then $\#(FCT_{m^s}(\mathcal{X})) \leq (m^{s+1})^{m^{2s}} = m^{(s+1)m^{2s}} \leq m^{m^{2s+1}}$ where the last is by $s+1 \leq m$ and so $f(2mr) \leq m^{m^{2s+1}2mr} \leq m^{m^{2s+3}}$ since $2r \leq m$. If $m^s \leq n$ then $\#(FCT_{m^s}(\mathcal{X})) \leq (n^2)^{m^{2s}} = n^{2m^{2s}} \leq n^{m^{2s+1}}$ and so $f(2mr) \leq n^{m^{2s+1}2mr} \leq n^{m^{2s+3}}$ again since $2r \leq m$.

Now let $e = m$ if $m^s \geq n$ and $e = n$ otherwise.

Then the first inequality $\frac{n}{f(2mr)} \geq q$ holds if $\frac{n}{e^{m^{2s+3}}} \geq q$ which holds if $\frac{1}{e^{m^{2s+3}}} \geq q$. Since $m < n$ we have $\frac{1}{n^{m^{2s+3}}} < \frac{1}{m^{m^{2s+3}}}$ and so the last inequality holds under the assumption $q < \frac{1}{n^{m^{2s+3}}}$.

The second inequality $\#(\Omega) \geq rf(2mr-1)(n+1)$ holds if $\#(\Omega) \geq f(2mr)$ under the assumption $r(n+1) \leq \#(FCT_{m^s}(\mathcal{X}))$. By the first paragraph $\#(\Omega) \geq f(2mr)$ holds if $\#(\Omega) \geq e^{m^{2s+3}}$. Finally by the assumption $q \leq \frac{1}{n^{m^{2s+3}}}$ and $\#(\Omega) \geq q^{-1}$ and $m < n$ we get $\#(\Omega) \geq n^{m^{2s+3}} > m^{m^{2s+3}}$ and thus $\#(\Omega) \geq e^{m^{2s+3}}$ holds. □

Now we can derive the Theorem of Garlík:

Proof of the Theorem of Garlík. We will distinguish between two cases:

Case 1: Assume that for every non-standard s' there is a function $f \in Fct_{m^{s'}}(\mathcal{X})$ such that $\langle\langle \psi(id, f, \bar{h}) \rangle\rangle_\Omega \neq \emptyset$.

Let $t > \mathbb{N}$ be non-standard such that $q < \frac{1}{n^{m^t}}$ and let $s' > \mathbb{N}$ we arbitrary such that $t > 2s'+3$ and $m > s'$. Observe that (H) holds for $FCT_{m^{s'}}(\mathcal{X})$ as it holds for any subset of $FCT_{m^s}(\mathcal{X})$. Moreover $\mathrm{SLP}_{\leq m^{s'}}(\mathcal{X})$ is an $\mathcal{L}$-tree of height $m^{s'}$ over $\mathcal{X}$ in $FCT_{m^{s'}}(\mathcal{X})$. Thus the only condition we have to establish to use the Corollary 2.2.8 for $FCT_{m^{s'}}(\mathcal{X}), s', n, m, \mathcal{X}, \psi(x, z, \bar{h})$ and $\mathcal{L}$ as described above is to find $r > \mathbb{N}$ with

$$(q) \quad \frac{n}{\#(FCT_{m^{s'}}(\mathcal{X}))^{2mr}} > q \text{ and } (\Omega) \quad \#(\Omega) > r\#(FCT_{m^{s'}}(\mathcal{X}))^{2mr-1}(n+1).$$

But by the choice of s' and t we have: $q < \frac{1}{n^{m^t}} < \frac{1}{n^{m^{2s'+3}}}$, $s' < m < n$ and by the assumption of the Case 1 we also have $\#(\Omega) > q^{-1}$. Thus if we let $r > \mathbb{N}$ be such that $r < \min(\lfloor m/2 \rfloor, \lfloor \#(FCT_{m^{s'}}(\mathcal{X}))/(n+1) \rfloor)$ we get by the Observation 3.0.4 that (q) and (Ω) holds for the r.

Case 2: Assume there is $s' > \mathbb{N}$ such that for every $f \in FCT_{m^{s'}}(\mathcal{X})$ we have $\langle\langle \psi(id, f, \bar{h}) \rangle\rangle_\Omega = \emptyset$ and wlog $s' < s$.

By the Lemma 2.2.12[2] for $\mathcal{F} = FCT_{m^{s'}}(\mathcal{X}), \ell = m^{s'}, \mathcal{X}$ we get an $\mathcal{L}$-closed $\mathcal{K} \subseteq \mathcal{H}$ with $\mathcal{X} \subseteq \mathcal{K} \subseteq FCT_{m^{s'}}(\mathcal{X}) \subseteq FCT_{m^s}(\mathcal{X})$ such that $\langle a \rangle_{\mathcal{K}} \models \exists\Delta_0^{<\tilde{n}} - \mathrm{IND}^{\leq c_m}$, where $\langle a \rangle_{\mathcal{K}}$ is the substructure of $\mathbb{M}$ with the domain $\{f(a) \mid f \in \mathcal{K}\}$.

Now let $\mathcal{V}$ be an ultrafilter on Ω generated by $\{a\}$. Observe that $\mathcal{K}/\mathcal{V} \cong \langle a \rangle_{\mathcal{K}}$ via $i : f^{\mathcal{V}} \mapsto f(a)$ thus we get (4) for $\mathcal{K}/\mathcal{V}$. Since $\{c_v \mid v \leq n\} \subseteq \mathcal{X} \subseteq \mathcal{K}$ we have that $[0, n]$ is an initial segment of $\langle a \rangle_{\mathcal{K}}$. But this gives (1) and also that $\Delta_0^{<n}(\mathcal{L})$-formulae are absolute between $\langle a \rangle_{\mathcal{K}}$ and $\mathbb{M}$. Then to show (2) we have: for any $\varphi(\bar{x}) \in \Delta_0^{<n}(\mathcal{L})$ and $\bar{f} \in \mathcal{F} : \mathcal{K}/\mathcal{V} \models \varphi(\bar{f}^{\mathcal{V}})$ if and only if $\langle a \rangle_{\mathcal{K}} \models \varphi(\bar{f}(a))$ if and only if $\mathbb{M} \models \varphi(\bar{f}(a))$ if and only if $\langle\langle \varphi(\bar{f}) \rangle\rangle_\Omega \in \mathcal{V}$. So $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae are Łos for $\mathcal{K}/\mathcal{V}$.

Finally to show (3) assume for a contradiction that for some $f \in \mathcal{F}$: $\mathcal{K}/\mathcal{V} \models \psi(id^{\mathcal{V}}, f^{\mathcal{V}}, \bar{h}^{\mathcal{V}})$. Since by (2) $\Delta_0^{<\tilde{n}}(\mathcal{L})$-formulae are Łos for $\mathcal{K}/\mathcal{V}$ we get $\langle\langle \psi(id, f, \bar{h}) \rangle\rangle_\Omega \in \mathcal{V}$ and thus $\mathbb{M} \models \psi(a, f(a), \bar{h}(a))$ contradicting the assumption of the Case 2. This finishes the proof. $\square$

[2] Assume $+, 0, 1 \in \mathcal{L}$, $\ell, m \in \mathbb{M}$, $\{id\} \cup \{c_v \mid v \leq max(n, m)\} \subseteq \mathcal{X} \subseteq \mathcal{F} \subseteq \mathcal{H}$ with $\mathcal{X} \in \mathbb{M}$. Suppose $\mathcal{T}^{\mathcal{F}}$ is an $\mathcal{L}$-tree of height ℓ over $\mathcal{X}$ in $\mathcal{F}$ and suppose further that for every $k \in \mathbb{N}$: $\frac{\ell}{m^k} > \mathbb{N}$.

Then for any $a \in \Omega$ there is $\mathcal{K} \subseteq \mathcal{H}$ with $\mathcal{X} \subseteq \mathcal{K} \subseteq \mathcal{F}$ such that $\langle a \rangle_{\mathcal{K}} \models \exists\Delta_0^{<\tilde{n}} - \mathrm{IND}^{\leq m}$ and $[0, n]$ is an initial segment of $\langle a \rangle_{\mathcal{K}}$.

Chapter 4

A variation on the theorem of Hirschfeld

We will give an a theorem of Hirschfeld from [Hir75, section 3.7]. To do so, some preparations are in order.

Definition 4.0.1. *We say that a structure* $\mathbb{M}$ *in language* $\mathcal{L}$ is generated by one point (in language $\mathcal{L}$) *if there is* $c \in \mathbb{M}$ *such that*

$$\mathbb{M} = \{t^{\mathbb{M}}(c) \mid t(x) \textit{ a term of } \mathcal{L}\}.$$

Moreover if c *is as above then we say that* $\mathbb{M}$ *is* generated by c (in language $\mathcal{L}$).

Note that the isomorphism type of such models is only dependent on the set of open $\mathcal{L}$-formulae satisfied by c. More precisely by the Observation 1:

Observation 4.0.2. *Assume* $\mathbb{M}_1, \mathbb{M}_2$ *are* $\mathcal{L}$ *structures of the same size,* $\mathbb{M}_1$ *is generated by* c_1 *in* $\mathcal{L}$*,* $\mathbb{M}_2$ *is generated by* c_2 *in* $\mathcal{L}$*. Assume further that for any open* $\mathcal{L}$*-formula* $\varphi(x)$*,* $\mathbb{M}_1 \models \varphi(c_1)$ *if and only if* $\mathbb{M}_2 \models \varphi(c_2)$*. Then* $\mathbb{M}_1 \cong \mathbb{M}_2$*.*

We show latter that the the following condition on a theory T ensures that any (countable for the case (ii)) model of the theory T has an extension which is generated by one point.

Definition 4.0.3. *Let* T *be a theory in language* $\mathcal{L}$*.*

(i) We say that T admits one point extensions *if for any countable model* $\mathbb{M}$ *of* T *there is a countable set of* $\mathcal{L}$*-terms* $\{t_m\}_{m \in \mathbb{M}}$ *such that the set of formulae* $\{t_m(x) = m \mid m \in \mathbb{M}\}$ *is finitely satisfiable in* $\mathbb{M}$*.*

(ii) We say that T admits coding of standard length tuples *if for every* $k, i \in \mathbb{N}$ *there is a* k*-ary function symbol* $(x_0, x_1, \ldots, x_{k-1}) \in \mathcal{L}$ *and a unary function symbol* $\pi_i \in \mathcal{L}$ *such that if* $i < k$ *then* $T \vdash \pi_i((x_0, x_1, \ldots, x_{k-1})) = x_i$

Observation 4.0.4. *Assume* T *is a theory in language* $\mathcal{L}$*. Then* T *admits one point extensions whenever* T *admits coding of standard length tuples.*

Proof. Let $\mathbb{M} \models T$ be countable and $\{m_i\}_{i \in \omega}$ some enumeration of $\mathbb{M}$. Then $\{\pi_i(x) = m_i \mid i \in \omega\}$ is finitely satisfiable in $\mathbb{M}$ as for any $k \in \mathbb{N}$ there is $m \in \mathbb{M}$ with $m = (m_0, m_1, \ldots, m_k)^{\mathbb{M}}$ and $\mathbb{M} \models \wedge_{i \leq k} \pi_i(m) = m_i$. $\square$

Note that the property (ii) in the definition above holds for all so called *sequential theories* which were studied in [HP93, p. 150-151] and in particular for $\widetilde{\mathrm{PV}}$, $\mathrm{S}^i_2(\mathrm{PV})$ for any $i \geq 1$.

Next we define another type of ultrapowers called *limit ultrapowers* whose special type first appeared in [Hir75]. We only note in advance that the name "limit" signalise the fact that if the ground model satisfies enough arithmetical theory one can view limit ultrapowers as a union of an increasing chain of ultrapowers which we will show latter.

We first settle some notation regarding partitions:

Definition 4.0.5. *Let M be a set.*

1. *We say that a family P of subsets of M is a* partition of M *if $\forall p, q \in P$: $p \cap q = \emptyset$ and $\bigcup P = M$. Moreover for any $A \subseteq M$ we let P_A* - the restriction of P on A - *to be the set $\{p \cap A \mid p \in P\}$*

2. *If P, Q are partitions of M then we say that P* is coarser then *Q or equivalently Q* is finer then *P and write $Q \leq P$ if for any $q \in Q$ there is $p \in P$ with $q \subseteq p$. We say that a partition R of M is a* common refinement *of partitions Q, P of M if $P \geq R$ and $Q \geq R$ and we denote by $Q \wedge P$ the set $\{q \cap p \mid q \in Q$ and $p \in P\}$.*[1]

3. *We say that a set $\mathcal{G}$ of partitions of M is a* partition filter on *M if for any two partitions from $\mathcal{G}$ its common refinement is in $\mathcal{G}$ and if any partition coarser than a partition from $\mathcal{G}$ is in $\mathcal{G}$.*

4. *Let $f : M \to M$ be a function and let $=_f \subseteq M \times M$ be an equivalence relation defined by $x =_f y$ if and only if $f(x) = f(y)$. Then we denote by $\langle f \rangle$ the partition generated by an equivalence relation $=_f$ i.e. the set $M/{=_f}$.*

Observation 4.0.6. *Let M be a set, $f, g : M \to M$, $P^0, \ldots, P^{n-1}, R^0, \ldots, R^{n-1}$ partitions of M, $B_0, B_1, \ldots, B_{n-1}, A \subseteq M$ and $B = \cap_{i<n} B_i$. Then:*
(i) $\langle f \circ g \rangle \geq \langle g \rangle$ and if f is injective then $\langle f \circ g \rangle = \langle g \rangle$,
(ii) $(\bigwedge_{i<n} P^i)_A = \bigwedge_{i<n} P^i_A$,
(iii) if for every $i < n$,$P^i_{B_i} \geq R^i_{B_i}$ then $(\bigwedge_{i<n} P^i)_B \geq (\bigwedge_{i<n} R^i)_B$.

Proof. (i) is trivial. To show (ii) we have $(\bigwedge_{i<n} P^i)_A = \{A \cap \bigcap_{i<n} p_i \mid p_i \in P^i$ for all $i < n\} = \{\bigcap_{i<n}(p_i \cap A) \mid p_i \in P^i$ for all $i < n\} = \bigwedge_{i<n} P^i_A$. Finally to show (iii) let $r \in (\bigwedge_{i<n} R^i)_B$ be given. Let for $i < n$, $r_i \in R^i$ be such that $r = B \cap \bigcap_{i<n} r_i$. For every $i < n$ there is by the assumption $p_i \in P^i$ with $p_i \cap B_i \supseteq r_i \cap B_i$ and thus $\bigcap_{i<n} p_i \cap B \supseteq \bigcap_{i<n} r_i \cap B$ i.e. $B \cap \bigcap_{i<n} p_i \supseteq B \cap \bigcap_{i<n} r_i = r$ where the former set is in $(\bigwedge_{i<n} P^i)_B$ which finishes the argument. $\square$

Definition and Lemma 4.0.7. *Let $\mathbb{M}$ be a structure in language $\mathcal{L}$, $\mathcal{F} \subseteq {}^{\mathbb{M}}\mathbb{M}$, $\mathcal{L}$-closed and $\mathcal{U}$ an ultrafilter on $\mathbb{M}$. Suppose further $\mathcal{G}$ is a partition filter on $\mathbb{M}$. Let*

$$\mathcal{U}_{Open} = \{A \in \mathcal{U} \mid A \text{ is in } \mathbb{M} \text{ definable by an open } \mathcal{L}\text{-formula without parameters}\}$$

[1] Note that this coincides with the common use of $\wedge$ for lattices as one can easily see that $\{q \cap p \mid q \in Q$ and $p \in P\}$ is the maximal common refinement of Q and P.

and assume

$$D = \{f^{\mathcal{U}} \in \mathcal{F}/\mathcal{U} \mid \textit{there is } A \in \mathcal{U}_{Open} \textit{ and } P \in \mathcal{G} \textit{ such that } \langle f\rangle_A \geq P_A\}$$

is non-empty. Then the substructure of $\mathcal{F}/\mathcal{U}$ *with the domain* D *is called the* limit ultrapower of $\mathcal{F}/\mathcal{U}$ generated by $\mathcal{G}$ *and will be denoted by* $\mathcal{F}/\mathcal{U}/\mathcal{G}$. *Moreover if* D *is non-empty then we will say that the limit ultrapower of* $\mathcal{F}/\mathcal{U}$ *is defined for* $\mathcal{G}$.

Justification: We have to check that for any function symbol $F(\bar{x}) \in \mathcal{L}_{PV}$ and $\bar{\alpha} \in D$, $F^{\mathcal{F}/\mathcal{U}}(\bar{\alpha}) \in D$. To do so, let an r-ary function symbol $F \in \mathcal{L}$ and $f_0^{\mathcal{U}}, f_1^{\mathcal{U}}, \ldots, f_{r-1}^{\mathcal{U}} \in \mathcal{F}/\mathcal{U}/\mathcal{G}$ be given. Then for any $i \leq r$ there is $B_i \in \mathcal{U}_{Open}$ and $P^i \in \mathcal{G}$ such that $\langle f_i\rangle_{B_i} \geq P^i_{B_i}$. Thus for $B = \bigcap_{i\leq r} B_i \in \mathcal{U}_{Open}$ and $P = \bigwedge_{i\leq r} P^i \in \mathcal{G}$ we have $(\bigwedge_{i\leq r}\langle f_i\rangle)_B \geq P_B$. Let $g^{\mathcal{U}} \in \mathcal{F}/\mathcal{U}$ be such that $\mathcal{F}/\mathcal{U} \models F(f_0^{\mathcal{U}}, f_1^{\mathcal{U}}, \ldots, f_{n-1}^{\mathcal{U}}) = g^{\mathcal{U}}$ i.e. $A = \langle\langle F(f_0, f_1, \ldots, f_{r-1}) = g\rangle\rangle_{\mathbb{M}} \in \mathcal{U}_{Open}$. Since $\langle F \circ (f_0, f_1, \ldots, f_{r-1})\rangle \geq \bigwedge_{i\leq r}\langle f_i\rangle$ and $\langle g\rangle_A = \langle F \circ (f_0, f_1, \ldots, f_{r-1})\rangle_A$ we get that $\langle g\rangle_{A\cap B} \geq (\bigwedge_{i\leq r}\langle f_i\rangle)_{A\cap B} \geq P_{A\cap B}$. Thus $A \cap B \in \mathcal{U}_{Open}$ and $P \in \mathcal{G}$ give $g^{\mathcal{U}} \in \mathcal{F}/\mathcal{U}/\mathcal{G}$ and we are done. □

In the context of the previous definition and lemma note that the limit ultrapower of $\mathcal{F}/\mathcal{U}$ is defined for any $\mathcal{G}$ whenever there is a constant function in $\mathcal{F}$ or in particular a constant symbol in $\mathcal{L}$. Indeed, if $f : \mathbb{M} \to \mathbb{M}$ is a constant function then $\langle f\rangle_A \geq P_A$ for any $A \in \mathcal{U}_{open}$ and $P \in \mathcal{G}$ and thus $f^{\mathcal{U}} \in D$. Moreover if $\mathcal{L}$ contains a constant symbol then $\mathcal{F}$ contains a constant function as it it $\mathcal{L}$-closed. Finally the following last definition:

Definition and Lemma 4.0.8. *Assume* $\mathbb{M}$ *is a structure in a non-empty countable language* $\mathcal{L}$. *Let*

$$\mathcal{F} = \{t^{\mathbb{M}} \in {}^{\mathbb{M}}\mathbb{M} \mid t(x) \textit{ is a term of } \mathcal{L}\}$$

and let $\mathcal{U}$ *be an ultrafilter on* $\mathbb{M}$. *We say that the ultrapower* $\mathcal{F}/\mathcal{U}$ *in language* $\mathcal{L}$ *is an* ultrapower over the language $\mathcal{L}$ wrt to $\mathbb{M}$ *and denote it as* $\mathbb{M}(\mathcal{L},\mathcal{U})$. *We will omit the "wrt to* $\mathbb{M}$*" if the model* $\mathbb{M}$ *is known from the context.*

Justification: $\mathcal{F}$ is non-empty since $x^{\mathbb{M}} \in \mathcal{F}$ and it is also $\mathcal{L}$-closed by the definition thus $\mathbb{M}(\mathcal{L},\mathcal{U})$ is a well-defined ultrapower for any ultrafilter $\mathcal{U}$ over $\mathbb{M}$.

□

Recall that an $\mathcal{L}$-structure $\mathbb{M}$ is existentially closed in an $\mathcal{L}$-structure $\mathbb{K}$ (in symbols $\mathbb{M} \leq_{\exists} \mathbb{K}$) if $\mathbb{M} \subseteq \mathbb{K}$ and for any open $\mathcal{L}$-formula φ and $\bar{m} \in \mathbb{M}$: $\mathbb{K} \models \exists\bar{x}\varphi(\bar{x}, \bar{m})$ implies $\mathbb{M} \models \exists\bar{x}\varphi(\bar{x}, \bar{m})$. Moreover recall that we do not notationally distinguish between a model $\mathbb{M}$ in language $\mathcal{L}$ and its natural expansion into the language $\mathcal{L}(\mathbb{M})$.

Observation 4.0.9. *Assume* $\mathbb{M}$ *is a countable* $\mathcal{L}$*-structure and* $\mathcal{U}$ *an ultrafilter over* $\mathbb{M}$. *Let id denote the function which is identity on* $\mathbb{M}$. *Then* $id^{\mathcal{U}} \in \mathbb{M}(\mathcal{L},\mathcal{U})$ *and* $\mathbb{M}(\mathcal{L},\mathcal{U})$ *is generated by* $id^{\mathcal{U}}$ *in the language* $\mathcal{L}$. *Moreover up to an isomorphism* $\mathbb{M} \leq_{\exists} \mathbb{M}(\mathcal{L}(\mathbb{M}),\mathcal{U})$ *for any ultrafilter* $\mathcal{U}$ *over* $\mathbb{M}$.

Proof. We have that $\{t^{\mathbb{M}} \in {}^{\mathbb{M}}\mathbb{M} \mid t(x)$ is a term of $\mathcal{L}\}$ contains an identity function on $\mathbb{M}$ represented by the interpretation of a free variable i.e by $x^{\mathbb{M}}$. Moreover since open formulae are Łos for $\mathbb{M}(\mathcal{L},\mathcal{U})$ we have that for any $\mathcal{L}$-term $t(x)$, $\mathbb{M}(\mathcal{L},\mathcal{U}) \models$

$t(id^{\mathcal{U}}) = [t^{\mathbb{M}}]^{\mathcal{U}}$ if and only if $\langle\langle t(id) = t^{\mathbb{M}}\rangle\rangle_{\mathbb{M}} \in \mathcal{U}$ but the latter is always true as $\langle\langle t(id) = t^{\mathbb{M}}\rangle\rangle_{\mathbb{M}} = \mathbb{M}$.

For the moreover part let an ultrafilter $\mathcal{U}$ over $\mathbb{M}$ be given. Note that for any $m \in \mathbb{M}$ and the function c_m which is constant m on $\mathbb{M}$ we have $c_m^{\mathcal{U}} \in \mathbb{M}(\mathcal{L}(\mathbb{M},\mathcal{U}))$ and thus if we identify $\{c_m^{\mathcal{U}} \mid m \in \mathbb{M}\}$ with $\mathbb{M}$ we get $\mathbb{M} \leq \mathbb{M}(\mathcal{L}(\mathbb{M}),\mathcal{U})$. But if $\mathbb{M}(\mathcal{L}(\mathbb{M}),\mathcal{U}) \models \exists \bar{x}\varphi(\bar{x})$ for an open $\mathcal{L}(\mathbb{M})$ formula $\varphi(\bar{x})$ then $\langle\langle \varphi(\bar{g})\rangle\rangle_{\mathbb{M}} \in \mathcal{U}$ for some $\bar{g}^{\mathcal{U}} \in \mathbb{M}(\mathcal{L}(\mathbb{M}),\mathcal{U})$ and thus $\mathbb{M} \models \varphi(\bar{g}(\omega))$ for some $\omega \in \langle\langle \varphi(\bar{g})\rangle\rangle_{\mathbb{M}}$. □

4.1 Variation on the theorem of Hirschfeld

Now we can state our theorem:

Theorem 4.1.1. *Let $\mathbb{N}$ be a countably infinite $\mathcal{L}$-structure and let T be the universal theory of $\mathbb{N}$ in language $\mathcal{L}$. Assume further that T admits one point extensions.*

Then for any countable model $\mathbb{M}$ of T there is a partition filter $\mathcal{G}$ on $\mathbb{M}$ and an ultrafilter $\mathcal{U}$ on $\mathbb{M}$ such that $\mathbb{M} \cong \mathbb{N}(\mathcal{L},\mathcal{U})/\mathcal{G}$. On the other hand for every partition filter $\mathcal{G}$ on $\mathbb{M}$ and an ultrafilter $\mathcal{U}$ on $\mathbb{M}$, $\mathbb{N}(\mathcal{L},\mathcal{U})/\mathcal{G} \models T$ whenever the limit ultrapower of $\mathbb{N}(\mathcal{L},\mathcal{U})$ is defined for $\mathcal{G}$.

As a direct corollary of this theorem we get a variation on the theorem of Hirschfeld from [Hir75, Section 3.7]:

Corollary 4.1.2. *Let $\mathbb{N}$ be the standard model of True arithmetic in the language $\mathcal{L}_{\mathcal{R}}$ which is $\mathcal{L}_{\mathrm{PA}}$ augment by a function symbol for every recursive function with the natural interpretation. Suppose $\mathcal{R}$ is the set of unary recursive functions on $\mathbb{N}$. Then a countable model $\mathbb{M}$ in the language $\mathcal{L}_{\mathcal{R}}$ is a model of $\mathrm{Th}_{\forall}(\mathbb{N}, \mathcal{L}_{\mathcal{R}})$ if and only if there is a partition filter $\mathcal{G}$ on $\mathbb{N}$ and an ultrafilter $\mathcal{U}$ over $\mathbb{N}$ such that $\mathbb{M} \cong \mathcal{R}/\mathcal{U}/\mathcal{G}$.*

Assume from now on that $\mathbb{N}, \mathcal{L}$ and T are as in the statement of the Theorem 4.1.1. We split the proof of this theorem into three lemmas.

Lemma 4.1.3. *Let $\mathbb{M}$ be a countable model of T. Then up to an isomorphism, $\mathbb{M}$ has an extension which is generated by one point and in which $\mathbb{M}$ is existentially closed.*

Proof. We will show that there is an ultrafilter $\mathcal{U}$ over $\mathbb{M}$ such that up to an isomorphism, $\mathbb{M} \leq_{\exists} \mathbb{M}(\mathcal{L}(\mathbb{M}),\mathcal{U})$ and the latter model is generated by $id^{\mathcal{U}}$ in the language $\mathcal{L}$.

For every $m \in \mathbb{M}$ let $c_m \in \mathcal{L}(\mathbb{M}) - \mathcal{L}$ be a constant symbol such that $c_m^{\mathbb{M}} = m$. Since T admits one point extensions there is a set of $\mathcal{L}$-terms $\{t_m\}_{m\in\mathbb{M}}$ such that the set $A = \{\langle\langle t_m^{\mathbb{M}} = c_m\rangle\rangle_{\mathbb{M}} \mid m \in \mathbb{M}\}$ has a finite intersection property. Thus there is an ultrafilter $\mathcal{U}$ over $\mathbb{M}$ extending A. By the Observation 4.0.9 we have that $\mathbb{M} \leq_{\exists} \mathbb{M}(\mathcal{L}(\mathbb{M}),\mathcal{U})$ (up to an isomorphism of $\mathbb{M}$). Further $\mathbb{M}(\mathcal{L}(\mathbb{M}),\mathcal{U})$ is by the Observation 4.0.9 generated by $id^{\mathcal{U}}$ in the language $\mathcal{L}(\mathbb{M})$. But since for every constant symbol c_m we have that $\langle\langle t_m^{\mathbb{M}} = c_m\rangle\rangle_{\mathbb{M}} = \langle\langle t_m(id) = c_m\rangle\rangle_{\mathbb{M}} \in \mathcal{U}$ we get $\mathbb{M}(\mathcal{L}(\mathbb{M}),\mathcal{U}) \models t_m(id^{\mathcal{U}}) = c_m$ and so $\mathbb{M}(\mathcal{L}(\mathbb{M}),\mathcal{U})$ is generated by $id^{\mathcal{U}}$ in the language $\mathcal{L}$.

□

Lemma 4.1.4. *Let $\mathbb{S}$ be a countable model of T generated by one point. Then there is an ultrapower over $\mathcal{L}$ wrt to $\mathbb{N}$ which is isomorphic to $\mathbb{S}$.*

Proof. For an open $\mathcal{L}$-formula $\varphi(x)$ let $\varphi(\mathbb{S}) = \{s \in \mathbb{S} \mid \mathbb{S} \models \varphi(s)\}$ and similarly for $\varphi(\mathbb{N})$. Let $c \in \mathbb{S}$ be the generator of $\mathbb{S}$. We claim that (a):

$$B = \{\varphi(\mathbb{N}) \mid \varphi(x) \text{ open } \mathcal{L}\text{-formula and } \mathbb{S} \models \varphi(c)\}$$

has a finite intersection property and that (b): if we let $\mathcal{U} \supseteq B$ be an ultrafilter then $\mathbb{N}(\mathcal{L},\mathcal{U}) \cong \mathbb{S}$.

For (a) assume $\varphi_0(\mathbb{N}), \dots, \varphi_{n-1}(\mathbb{N}) \in B$ are given. Then $\mathbb{S} \models \bigwedge_{i<n} \varphi_i(c)$ i.e. $\mathbb{S} \models \exists x \bigwedge_{i<n} \varphi_i(x)$ i.e. $T \not\vdash \forall x \bigvee_{i \leq n} \neg\varphi_i(x)$ i.e. $\mathbb{N} \models \exists x \bigwedge_{i \leq n} \varphi_i(x)$ i.e. B has finite intersection property.

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ extending B. To show (b) note that by the Observation 4.0.9, $\mathbb{N}(\mathcal{L},\mathcal{U})$ is generated by $id^{\mathcal{U}}$ and so by the the Observation 4.0.2 it suffices to show that for every open $\mathcal{L}$- formula $\varphi(x)$, $\mathbb{S} \models \varphi(s)$ if and only if $\mathbb{N}(\mathcal{L},\mathcal{U}) \models \varphi(id^{\mathcal{U}})$. But since open $\mathcal{L}$-formulae are Łos for $\mathbb{N}(\mathcal{L},\mathcal{U})$ we have: $\mathbb{S} \models \varphi(s)$ if and only if $\varphi(\mathbb{N}) \in B$ if and only if $\langle\langle \varphi(id) \rangle\rangle_\Omega \in \mathcal{U}$ if and only if $\mathbb{N}(\mathcal{L},\mathcal{U}) \models \varphi(id^{\mathcal{U}})$ and we are done. □

Now we can make a comment on the intuition behind the word "limit" in the definition of limit ultrapower. Assume a countable model $\mathbb{S}$ is not generated by one point in language $\mathcal{L}$. Then we can decompose it into countably many models $\{\mathbb{S}_i\}_{i\in\omega}$ each generated by one point. Namely, fix some well ordering $<$ of $\mathbb{S}$ and let $c_0 = \min_< \mathbb{S}$ and let $\mathbb{S}_0 = \{t^{\mathbb{S}}(c_0) \mid t(x) \text{ is a term of } \mathcal{L}\}$. For $i+1$ let $c_{i+1} = \min_<(\mathbb{S} - \bigcup_{j\leq i} \mathbb{S}_j)$ and $\mathbb{S}_{i+1} = \{t^{\mathbb{S}}(c_{i+1}) \mid t(x) \text{ is a term of } \mathcal{L}\}$. Then for any $i \in \omega$, $\mathbb{S}_i \leq \mathbb{S}$ and so $\mathbb{S}_i \models \mathrm{Th}_\forall(\mathbb{S})$. Assume for a simplicity that T admits coding of standard length tuples. Then we can let $\mathbb{R}_0 = \mathbb{S}_0$ and $\mathbb{R}_{i+1} = \{t^{\mathbb{S}}((c_0, c_1, \dots, c_{i+1})) \mid t(x) \text{ is a term of } \mathcal{L}\}$. Since $\mathbb{S} \models \pi_j((c_0, \dots, c_i)) = c_j$ for every $j \leq i$ we have that $\mathbb{S}_j \subseteq \mathbb{R}_i$ for every $j \leq i$. Then also $\mathbb{R}_i \subseteq \mathbb{R}_{i+1}$ for every $i \in \omega$ and thus $\mathbb{S}$ is the limit $\bigcup_{i\in\omega} \mathbb{R}_i$ of the increasing chain of models $\{\mathbb{R}_i\}_{i\in\omega}$. Since for every $i \in \omega$, $\mathbb{R}_i$ is generated by one point we have that for every $i \in \omega$ there is an ultrafilter $\mathcal{U}_i$ over $\mathbb{N}$ such that $\mathbb{R}_i \cong \mathbb{N}(\mathcal{L},\mathcal{U}_i)$. Whence one can see $\mathbb{S}$ as a limit of some increasing chain of ultrapowers over $\mathbb{N}$. The next lemma shows that the limit of this ultrapower construction can be achieved in one step by finding suitable partition filter on $\mathbb{N}$ and an ultrafilter on $\mathbb{N}$.

Lemma 4.1.5. *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let $\mathbb{M}$ be a substructure of $\mathbb{N}(\mathcal{L},\mathcal{U})$ which is existentially closed in $\mathbb{N}(\mathcal{L},\mathcal{U})$. Then there is a partition filter $\mathcal{G}$ on $\mathbb{N}$ such that the limit ultrapower of $\mathbb{N}(\mathcal{L},\mathcal{U})$ is defined for $\mathcal{G}$ and $\mathbb{M} = \mathbb{N}(\mathcal{L},\mathcal{U})/\mathcal{G}$.*

Proof. Let $\mathcal{G}$ be a partition filter generated[2] by the set $G = \{\langle f \rangle \mid f^{\mathcal{U}} \in \mathbb{M}\}$. Then clearly $\mathbb{M} \subseteq \mathbb{N}(\mathcal{L},\mathcal{U})/\mathcal{G}$.

For the other inclusion assume $g^{\mathcal{U}} \in \mathbb{N}(\mathcal{L},\mathcal{U})/\mathcal{G}$ i.e. there is $A \in \mathcal{U}_{Open}$ and $\langle f_0 \rangle, \dots, \langle f_{n-1} \rangle \in G$ such that $\langle g \rangle_A \geq (\bigwedge_{i<n} \langle f_i \rangle)_A = \bigwedge_{i<n} \langle f_i \rangle_A$. By the definition of G we have $f_0^{\mathcal{U}}, f_1^{\mathcal{U}}, \dots, f_{n-1}^{\mathcal{U}} \in \mathbb{M}$. Let φ be the formula defining A in $\mathbb{N}$. Since $A = \langle\langle \varphi(id) \rangle\rangle_{\mathbb{N}} \in \mathcal{U}$ and thus $\mathbb{N}(\mathcal{L},\mathcal{U}) \models \varphi(id)$ we have

$$\mathbb{N}(\mathcal{L},\mathcal{U}) \models \varphi(id^{\mathcal{U}}) \wedge g(id^{\mathcal{U}}) = g^{\mathcal{U}} \wedge \bigwedge_{j<n} f_j(id^{\mathcal{U}}) = f_j^{\mathcal{U}}.$$

[2] i.e. $A \in \mathcal{G}$ if and only if $A \geq B_1 \wedge B_2 \wedge \ldots \wedge B_n$ for some $B_1, B_2, \dots, B_n \in G$

Since $\mathbb{M}$ is existentially closed in $\mathbb{N}(\mathcal{L},\mathcal{U})$ there are some $i^{\mathcal{U}}, b^{\mathcal{U}} \in \mathbb{M}$ with

$$\mathbb{M} \models \varphi(i^{\mathcal{U}}) \wedge g(i^{\mathcal{U}}) = b^{\mathcal{U}} \wedge \bigwedge_{j<n} f_j(i^{\mathcal{U}}) = f_j^{\mathcal{U}}.$$

But then also

$$\mathbb{N}(\mathcal{L},\mathcal{U}) \models \varphi(i^{\mathcal{U}}) \wedge g(i^{\mathcal{U}}) = b^{\mathcal{U}} \wedge \bigwedge_{j<n} f_j(i^{\mathcal{U}}) = f_j^{\mathcal{U}}$$

thus $B = \langle\langle \varphi(i) \wedge g(i) = b \wedge \bigwedge_{j<n} f_j(i) = f_j \rangle\rangle_{\mathbb{N}} \in \mathcal{U}$. To finish the proof we show that b and g equals on $B \cap A$ and thus $g^{\mathcal{U}} = b^{\mathcal{U}} \in \mathbb{M}$. Assume $\omega \in B \cap A$ is given. Then by $\omega \in B$ we have

$$\mathbb{N} \models g(i(\omega)) = b(\omega) \wedge \bigwedge_{j<n} f_j(i(\omega)) = f_j(\omega)$$

and so $\omega =_{f_j} i(\omega)$ for every $j < n$. Since $\omega \in A \cap B$ this implies by $\langle g \rangle_A \geq \bigwedge_{i<n} \langle f_i \rangle_A$ and $\mathbb{N} \models \varphi(i(\omega))$ i.e. $i(\omega) \in A$ that $\omega =_g i(\omega)$ i.e. $g(i(\omega)) = g(\omega)$. Finally by $\omega \in B$ we get $b(\omega) = g(i(\omega))$. Whence g equals to b on $B \cap A$ and we are done.

□

Proof of the Theorem: Let $\mathbb{M}$ be a countable model of T. Then by the Lemma 4.1.3 there is an extension $\mathbb{S}$ of $\mathbb{M}$ generated by one point in which $\mathbb{M}$ is existentially closed (up to an isomorphism of $\mathbb{M}$). Moreover by the Lemma 4.1.4 there is an ultrafilter $\mathcal{U}$ over $\mathbb{N}$ such that $\mathbb{S} \cong \mathbb{N}(\mathcal{L},\mathcal{U})$. Thus $\mathbb{M}$ is isomorphic to a substructure $\mathbb{M}'$ of $\mathbb{N}(\mathcal{L},\mathcal{U})$ which is existentially closed in $\mathbb{N}(\mathcal{L},\mathcal{U})$. By the last lemma there is a partition filter $\mathcal{G}$ on $\mathbb{N}$ such that $\mathbb{M}' = \mathbb{N}(\mathcal{L},\mathcal{U})/\mathcal{G}$ and so $\mathbb{M} \cong \mathbb{N}(\mathcal{L},\mathcal{U})/\mathcal{G}$.

On the other hand if an ultrafilter $\mathcal{U}$ over $\mathbb{N}$ is given then any substructure of $\mathbb{N}(\mathcal{L},\mathcal{U})$ is a model of T. Indeed, T is a universal theory, open $\mathcal{L}$-formulae are Łos for $\mathbb{N}(\mathcal{L},\mathcal{U})$ since $\mathcal{U}$ is an ultrafilter and so $\mathbb{N}(\mathcal{L},\mathcal{U}) \models T$ which finishes the argument.

□

Corollary 4.1.6. *Let* $\mathbb{N}$ *be an* $\mathcal{L}$*-structure,* $T = \mathrm{Th}_\forall(\mathbb{N})$ *and* T *admits one-point extensions. Then for any countable* $\mathcal{L}$*-structure* $\mathbb{M}$, $\mathbb{M} \models T$ *if and only if* $\mathbb{M} \cong \mathcal{H}/\mathcal{U}$ *for some* $\mathcal{L}$*-closed* $\mathcal{H} \subseteq \mathcal{F} = \{t^{\mathbb{N}} \mid t(x)$ *is a unary term of* $\mathcal{L}\}$ *and an ultrafilter* $\mathcal{U}$ *on* $\mathbb{N}$*.*

Proof. The right-left implication is clear since T is a universal theory and $\mathcal{H}/\mathcal{U} \leq \mathcal{F}/\mathcal{U}$ for any $\mathcal{L}$-closed set of functions $\mathcal{H} \subseteq \mathcal{F}$. For the left-right implication let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and $\mathcal{G}$ a partition filter on $\mathbb{N}$ such that $\mathbb{M} \cong \mathbb{N}(\mathcal{L},\mathcal{U})/\mathcal{G}$. Then (up to an isomorphism of $\mathbb{M}$) $\mathbb{M} \leq \mathbb{N}(\mathcal{L},\mathcal{U}) = \mathcal{F}/\mathcal{U}$ and thus $\mathbb{M} \cong \mathcal{H}/\mathcal{U}$ for some $\mathcal{L}$-closed family of functions $\mathcal{H} \subseteq \mathcal{F}$ by the Lemma 1.2.4 . □

4.2 Some corollaries for $\widetilde{\mathrm{PV}}$

In this section we investigate some corollaries of the Theorem 4.1.1 with focus on the theory $\widetilde{\mathrm{PV}}$. Although most of the statements from this chapter can be easily generalised for any theory T in language $\mathcal{L}$ satisfying the assumptions of the Theorem 4.1.1 we will state them for $\widetilde{\mathrm{PV}}$ and $\mathcal{L}_{PV}$ to make them more explicit.

Since $\widetilde{\mathrm{PV}}$ is a theory defined to reason about polynomial-time one may be interested in some analogue to polynomial-time in non-standard models of $\widetilde{\mathrm{PV}}$. We will give two possible definitions of polynomial-time in non-standard models of $\widetilde{\mathrm{PV}}$.

Let $\mathbb{M}$ be a countable model of $\widetilde{\mathrm{PV}}$ and $H \subseteq$, $H(\mathbb{M})$ be sets of unary terms of $\mathcal{L}_{PV}$ and $\mathcal{L}_{PV}(\mathbb{M})$ respectively. Then we define the sets:

$$\mathcal{P}^{\mathbb{M}} = \{f^{\mathbb{M}} \mid f \in \mathcal{L}_{PV} \text{ is a unary function symbol}\}$$

$$\mathcal{P}(\mathbb{M}) = \{t^{\mathbb{M}} \mid t(x) \text{ is an } \mathcal{L}_{PV}(\mathbb{M})\text{-term}\}.$$

Moreover we omit the model $\mathbb{M}$ in the definition if $\mathbb{M}$ is the standard model $\mathbb{N}$. Thus we will write $\mathcal{P}$ for $\mathcal{P}(\mathbb{N})$ or $\mathcal{P}^{\mathbb{N}}$. This is correct as clearly $\mathcal{P}(\mathbb{N}) = \mathcal{P}^{\mathbb{N}}$. Note that since $\widetilde{\mathrm{PV}}$ admits coding of standard length tuples, the set $\mathcal{P}(\mathbb{M})$ remains the same even if we allow only terms containing at most one symbol form $\mathcal{L}(\mathbb{M})$-$\mathcal{L}$. We also have that for any ultrafilter $\mathcal{U}$ on $\mathbb{M}$, $\mathcal{P}^{\mathbb{M}}/\mathcal{U} = \mathbb{M}(\mathcal{L}_{PV},\mathcal{U})$ and $\mathcal{P}(\mathbb{M})/\mathcal{U} = \mathbb{M}(\mathcal{L}_{PV}(\mathbb{M}),\mathcal{U})$. This also gives that $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \widetilde{\mathrm{PV}}$ and $\mathcal{P}^{\mathbb{M}}/\mathcal{U} \models \widetilde{\mathrm{PV}}$ for any ultrafilter $\mathcal{U}$ over $\mathbb{M}$ and we will call those ultrapowers *polynomial ultrapowers* over $\mathbb{M}$.

Assume from now on that $\mathbb{N}$ is the standard model of True arithmetic in the language $\mathcal{L}_{PV}$. We can start with one simple observation:

Observation 4.2.1. *Assume there is a countable model of* $\mathrm{S}^1_2(\mathrm{PV}) + \widetilde{\mathrm{PV}}$ *which is generated by one point (in the language* $\mathcal{L}_{PV}$*) and not isomorphic to* $\mathbb{N}$*.*

Then for any $\mathcal{L}_{PV}$*-sentence* σ*,* $\mathrm{S}^1_2(\mathrm{PV}) + \widetilde{\mathrm{PV}} + \sigma$ *is consistent whenever* $\mathcal{P}/\mathcal{U} \models \sigma$ *for all non-principal ultrafilters* $\mathcal{U}$ *over the standard model* $\mathbb{N}$*.*

Proof. Let $\mathbb{M} \models \mathrm{S}^1_2(\mathrm{PV}) + \widetilde{\mathrm{PV}}$ be a countable model generated by one point with $\mathbb{M} \ncong \mathbb{N}$. Then $\mathbb{M} \cong \mathcal{P}/\mathcal{U}$ for some ultrafilter $\mathcal{U}$ and $\mathcal{U}$ is non-principal as $\mathbb{M} \ncong \mathbb{N}$. Thus $\mathbb{M} \models \sigma$ as $\mathcal{P}/\mathcal{U} \models \sigma$ and we are done. $\square$

Assume $\mathbb{M} \models \widetilde{\mathrm{PV}}$. We defined $\mathcal{P}^{\mathbb{M}}$ and $\mathcal{P}(\mathbb{M})$ with an intention to use them as a set of functions for power constructions over $\mathbb{M}$. We can immediately make two observations about this type of constructions.

First, since for any ultrafilter $\mathcal{U}$ over $\mathbb{M}$ we have $\mathcal{P}^{\mathbb{M}}/\mathcal{U} = \mathbb{M}(\mathcal{L}_{PV},\mathcal{U})$ we get by the Observation 4.0.9 that $\mathcal{P}^{\mathbb{M}}/\mathcal{U}$ is generated by $id^{\mathcal{U}}$ in the language $\mathcal{L}_{PV}$. Thus by the Lemma 4.1.4 it is isomorphic to $\mathbb{N}(\mathcal{L}_{PV},\mathcal{U}_{\mathbb{N}}) = \mathcal{P}/\mathcal{U}_{\mathbb{N}}$ for some ultrafilter $\mathcal{U}_{\mathbb{N}}$ over $\mathbb{N}$. Checking the proof of this Lemma we get that $\mathcal{U}_{\mathbb{N}}$ extends the set $\{\varphi(\mathbb{N}) \mid \varphi(x) \text{ is an open } \mathcal{L}_{PV}\text{-formula and } \varphi(\mathbb{M}) \in \mathcal{U}\}$. Thus there is no practical difference between working with $\mathcal{P}^{\mathbb{M}}$ over $\mathbb{M}$ and $\mathcal{P}$ over $\mathbb{N}$.

On the other hand, for an ultrafilter $\mathcal{U}$ over $\mathbb{M}$ the ultrapower $\mathcal{P}(\mathbb{M})/\mathcal{U} = \mathbb{M}(\mathcal{L}_{PV}(\mathbb{M}),\mathcal{U})$ is generated by $id^{\mathcal{U}}$ in the language $\mathcal{L}_{PV}(\mathbb{M})$ but not always in the language $\mathcal{L}_{PV}$. By the Corollary 4.1.6, $\mathcal{P}(\mathbb{M})/\mathcal{U} \cong \mathcal{H}/\mathcal{V}$ for some ultrafilter $\mathcal{V}$ over $\mathbb{N}$ and any $\mathcal{L}_{PV}$-closed $\mathcal{H} \subseteq \mathcal{P}$. By the previous considerations we very often get $\mathcal{H} \subsetneq \mathcal{P}$ and thus we can reach models which are not generated by one point. One can see this as an advantage since to show some properties of $\mathcal{H}/\mathcal{V}$ may be harder then to show properties of $\mathcal{P}(\mathbb{M})/\mathcal{U}$ as in the latter case one have access to all polynomial functions wrt to $\mathbb{M}$ (i.e. interpretations of function symbols from $\mathcal{L}_{PV}$with parameters from $\mathbb{M}$). In the next chapter we show that when using the set of functions $\mathcal{P}(\mathbb{M})$ it is useful to consider special type of models $\mathbb{M}$ of $\widetilde{\mathrm{PV}}$.

Herbrand saturated models in the role of groundmodel

In accord with the article [Avi02] we say that a model $\mathbb{M}$ in language $\mathcal{L}$ is *Herbrand Saturated* if for every open $\mathcal{L}(\mathbb{M})$ formula φ, $\mathbb{M} \models \exists\bar{x}\forall\bar{y}\varphi(\bar{x},\bar{y})$ whenever $\exists\bar{x}\forall\bar{y}\varphi(\bar{x},\bar{y})$ is consistent with $\mathrm{Th}_\forall(\mathbb{M},\mathcal{L}(\mathbb{M}))$.

By [Avi02, Theorem 3.2] every (countable) consistent universal theory has a (countable) Herbrand saturated model. The reason why we are going to use a Herbrand saturated model is the following property which is a direct corollary of [Avi02, Theorem 3.3]:

Theorem 4.2.2. *Let* $\mathbb{M} \models \widetilde{\mathrm{PV}}$ *be a Herbrand saturated model in language* $\mathcal{L}_{PV}$*,* $\varphi(\bar{x},y,\bar{z})$ *an open* $\mathcal{L}_{PV}$*-formula and* $\bar{m}$ *a tuple of parameters from* $\mathbb{M}$*. Assume that*

$$\mathbb{M} \models \forall\bar{x}\exists y\varphi(\bar{x},y,\bar{m})$$

then there is a function symbol $f \in \mathcal{L}_{PV}$ *and a parameter* $c \in \mathbb{M}$ *i.e.* $f^{\mathbb{M}}(\cdot,c) \in \mathcal{P}(\mathbb{M})$ *such that*

$$\mathbb{M} \models \forall\bar{x}\varphi(\bar{x},f(\bar{x},c),\bar{m}).$$

Let $\mathbb{M} \models \widetilde{\mathrm{PV}}$ be Herbrand saturated. Since $\Delta^b_0(\mathcal{L}_{PV})$-formulae are $\widetilde{\mathrm{PV}}$-provably equivalent to open $\mathcal{L}_{PV}$-formulae the theorem above applies to $\varphi \in \Delta^b_0(\mathcal{L}_{PV})$ as well. Thus it also follows that unary functions which are $\Sigma^b_1(\mathcal{L}_{PV})$-definable in $\mathbb{M}$ are in $\mathcal{P}(\mathbb{M})$[3]. Since it is not known whether the unary functions which are $\Sigma^b_1(\mathcal{L}_{PV})$-definable in $\mathbb{N}$ are in $\mathcal{P}$, having a Herbrand saturated model as a ground-model instead of having the standard model $\mathbb{N}$ as a groundmodel may give us certain advantage. An example of a construction where this is the case can be found in the Section 5.

Observation 4.2.3. *Let* $\mathbb{M},\mathbb{K}$ *be countable models of* $\widetilde{\mathrm{PV}}$*,* $\mathbb{M}$ *Herbrand saturated and* $\mathbb{M} \leq_\exists \mathbb{K}$*. Then* $\mathrm{Th}_{\exists\forall\exists}(\mathbb{M}) \subseteq \mathrm{Th}_{\exists\forall\exists}(\mathbb{K})$.[4]

Proof. Let $\varphi(x,y,z)$ be an open $\mathcal{L}_{PV}$-formula with $\mathbb{M} \models \exists x\forall y\exists z\varphi(x,y,z)$. Let $m \in \mathbb{M}$ be such that $\mathbb{M} \models \forall y\exists z\varphi(m,y,z)$. Since $\mathbb{M}$ is Herbrand saturated there is a function symbol $F \in \mathcal{L}_{PV}$ and $d \in \mathbb{M}$ with $\mathbb{M} \models \forall y\varphi(m,y,F(y,d))$. Since $\mathbb{M} \leq_\exists \mathbb{K}$ it is not the case that $\mathbb{K} \models \exists y\neg\varphi(m,y,F(y,d))$ and thus $\mathbb{K} \models \forall y\varphi(m,y,F(y,d))$ i.e. $\mathbb{K} \models \exists x\forall y\exists z\varphi(x,y,z)$. □

Lemma 4.2.4. *Let* $\mathbb{M}$ *be a countable Herbrand saturated model of* $\widetilde{\mathrm{PV}}$*. Then:*
(i) for any ultrafilter $\mathcal{U}$ *over* $\mathbb{M}$*,* $\mathrm{Th}_{\exists\forall\exists}(\mathbb{M}) \subseteq \mathrm{Th}_{\exists\forall\exists}(\mathcal{P}(\mathbb{M})/\mathcal{U})$ *and*
(ii) there is an ultrafilter $\mathcal{U}$ *over* $\mathbb{N}$ *such that* $\mathrm{Th}_{\exists\forall\exists}(\mathbb{M}) \subseteq \mathrm{Th}_{\exists\forall\exists}(\mathcal{P}/\mathcal{U})$*.*

Proof. To see (i) observe that $\mathbb{M} \leq_\exists \mathcal{P}(\mathbb{M})/\mathcal{U}$ (up to an isomorphism of $\mathbb{M}$) for any ultrafilter $\mathcal{U}$ over $\mathbb{M}$ thus (i) follows from the previous observation.

To see (ii) let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ such that up to an isomorphism $\mathbb{M} \leq_\exists \mathbb{N}(\mathcal{L}_{PV},\mathcal{U}) = \mathcal{P}/\mathcal{U}$ and use the observation above.

□

[3] if $\exists w\psi(x,y,w)$ for $\psi(x,y,z) \in \Delta^b_1(\mathcal{L}_{PV}(\mathbb{M}))$ defines a (unary) function F in $\mathbb{M}$ then there is a binary function symbol $f \in \mathcal{L}_{PV}$ and $c \in \mathbb{M}$ such that $\mathbb{M} \models \forall x\psi(\bar{x},\pi_0(f(x,c)),\pi_1(f(x,c)))$ and so the function $F = \pi_0^{\mathbb{M}} \circ f^{\mathbb{M}}(\cdot,c) \in \mathcal{P}(\mathbb{M})$

[4] the set of $\exists\forall\exists$ $\mathcal{L}_{PV}$-sentences is the set of sentences of the form $\exists x\forall y\exists z\varphi(x,y,z)$ with an open $\mathcal{L}_{PV}$-formula $\varphi(x,y,z)$.

Corollary 4.2.5. *Assume σ is $\forall\exists\forall$ sentence in language $\mathcal{L}_{\widetilde{PV}}$ such that for any non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$, $\mathcal{P}/\mathcal{U} \models \sigma$. Then $\mathrm{S}^1_2(\mathrm{PV}) + \widetilde{\mathrm{PV}} + \sigma$ is consistent.*

Proof. By [Avi02] or [Kra95, Theorem 7.6.3] any Herbrand saturated model of $\widetilde{\mathrm{PV}}$ is a model of $\mathrm{S}^1_2(\mathrm{PV})$. Moreover by the same reference there is a countable Herbrand saturated model $\mathbb{M} \models \widetilde{\mathrm{PV}}$. By the previous lemma there is a non-principal ultrafilter $\mathcal{U}$ such that $\mathrm{Th}_{\exists\forall\exists}(\mathbb{M}) \subseteq \mathrm{Th}_{\exists\forall\exists}(\mathcal{P}/\mathcal{U})$. But then by $\mathcal{P}/\mathcal{U} \models \sigma$ we get that $\mathbb{M} \models \sigma$ and so $\mathbb{M} \models \mathrm{S}^1_2(\mathrm{PV}) + \widetilde{\mathrm{PV}} + \sigma$. $\square$

It is not known to the author whether for every ultrapower $\mathcal{P}/\mathcal{U}$ with $\mathcal{U}$ non-principal there is a Herbrand saturated model $\mathbb{M}$ of $\widetilde{\mathrm{PV}}$ such that $\mathbb{M} \leq_{\exists} \mathcal{P}/\mathcal{U}$. In particular, it is not known to the author whether one can state the corollary above with "for some non-principal ultrafilter $\mathcal{U}$" instead of "for any non-principal ultrafilter $\mathcal{U}$".

Chapter 5

Unprovability of circuit upper bounds in $\widetilde{\mathrm{PV}}$

Recall that $\widetilde{\mathrm{PV}}$ denotes the set of all universal $\mathcal{L}_{PV}$- sentences valid in the standard model $\mathbb{N}$. For a binary function $g \in \mathcal{L}_{PV}$, a unary function $h \in \mathcal{L}_{PV}$ and a natural numbers c, k we define the following three formulae:

$$\begin{aligned}\mathrm{UP}_k(g(\cdot, y)) : \exists c \forall \ell > 0 \exists C [&\mathit{Circuit}(C, |\ell|) \wedge \mathit{size}(C) \leq |c| \cdot |\ell|^k \\ &\wedge \forall i(|\ell| = |i| \rightarrow (g(i, y) = 0 \leftrightarrow \mathit{eval}(C, i) = 0))]\end{aligned}$$

$$\begin{aligned}\mathrm{UP}_k(h) : \exists c \forall \ell > 0 \exists C [&\mathit{Circuit}(C, |\ell|) \wedge \mathit{size}(C) \leq |c| \cdot |\ell|^k \\ &\wedge \forall i(|\ell| = |i| \rightarrow (h(i) = 0 \leftrightarrow \mathit{eval}(C, i) = 0))]\end{aligned}$$

$$\begin{aligned}\mathrm{UP}_{c,k}(h) : \forall \ell > 0 \exists C [&\mathit{Circuit}(C, |\ell|) \wedge \mathit{size}(C) \leq c|\ell|^k \\ &\wedge \forall i(|\ell| = |i| \rightarrow (h(i) = 0 \leftrightarrow \mathit{eval}(C, i) = 0))]\end{aligned}$$

where $\mathrm{UP}_k(g(\cdot, y))$ contains a free variable y. The functions and predicates $\mathit{Circuit}(C, n)$, $\mathit{size}(C)$, $\mathit{eval}(C, i)$ have its natural meaning i.e to check whether C is a circuit with n inputs, to output the number of gates of a circuit C and to evaluate a circuit C on input i where a precise definition of those functions will be given latter.

It was shown in [KO17] that for every natural number $k \geq 1$ there is a unary function symbol $h \in \mathcal{L}_{PV}$ such that the theory $\widetilde{\mathrm{PV}} \cup \{\neg \mathrm{UP}_{c,k}(h) \mid c \in \mathbb{N}\}$ is consistent. We answer the question from [KO17, Remark 2.2] whether for every natural number $k \geq 1$ there is a unary function symbol $h \in \mathcal{L}_{PV}$ such that $\widetilde{\mathrm{PV}} + \neg \mathrm{UP}_k(h)$ is consistent. Using polynomial ultrapower over a Herbrand saturated model of $\widetilde{\mathrm{PV}}$ we show that the answer is positive.

After the following section with notational conventions and definitions which are necessary to present our proof idea we remind the proof idea of [KO17] and present the main idea of our construction. We encourage a reader with basic knowledge on ultrapower constructions who is familiar with traditional concepts regarding models of $\widetilde{\mathrm{PV}}$ to skip to the Section 5.2.

5.1 Preparation

Recall that the universal diagram of a structure $\mathbb{M}$ in language $\mathcal{L}$ is the set of all universal sentences in language $\mathcal{L}(\mathbb{M})$ which are valid in $\mathbb{M}$. In accord with the article [Avi02] we say that a model $\mathbb{M}$ in language $\mathcal{L}$ is *Herbrand Saturated* if for every open $\mathcal{L}(\mathbb{M})$ formula φ, $\mathbb{M} \models \exists\bar{x}\forall\bar{y}\varphi(\bar{x},\bar{y})$ whenever $\exists\bar{x}\forall\bar{y}\varphi(\bar{x},\bar{y})$ is consistent with the universal diagram of $\mathbb{M}$.

By [Avi02, Theorem 3.2] every (countable) consistent universal theory has a (countable) Herbrand saturated model. The reason why we are going to use a Herbrand saturated model in our construction is the following property which is a direct corollary of [Avi02, Theorem 3.3]:

Theorem 5.1.1. *Let $\mathbb{M} \models \widetilde{\mathrm{PV}}$ be a Herbrand saturated model in language $\mathcal{L}_{PV}$, $\varphi(\bar{x}, y, \bar{z})$ an open $\mathcal{L}_{PV}$-formula and $\bar{m}$ a tuple of parameters from $\mathbb{M}$. Assume that*

$$\mathbb{M} \models \forall\bar{x}\exists y\varphi(\bar{x}, y, \bar{m})$$

then there is a function symbol $f \in \mathcal{L}_{PV}$ and a parameter $c \in \mathbb{M}$ such that

$$\mathbb{M} \models \forall\bar{x}\varphi(\bar{x}, f(\bar{x}, c), \bar{m}).$$

Let $\mathbb{M}$ be a (countable) model of $\widetilde{\mathrm{PV}}$. For a function symbol $f \in \mathcal{L}_{PV}$ we will denote its interpretation in the standard model $\mathbb{N}$ as $f^{\mathbb{N}}$ and similarly for $f^{\mathbb{M}}$ and $\mathbb{M}$. We will frequently use the phrase *"Let $f \in \mathcal{L}_{PV}$ be a function symbol corresponding to the following algorithm (or Turing machine)..."* meaning that $f^{\mathbb{N}}$ is in $\mathbb{N}$ computed by the described algorithm (or Turing machine) and $\widetilde{\mathrm{PV}}$ proof about f all the properties claimed about the algorithm (or Turing machine).

Recall that we say that $A \subseteq \mathbb{M}$ is $\mathbb{M}$-definable if there is an $\mathcal{L}_{PV}(\mathbb{M})$ formula $\varphi(x)$ such that $A = \{m \in \mathbb{M} \mid \mathbb{M} \models \varphi(m)\}$ and in this case we will often write $x \in A$ in place of $\varphi(x)$. We follow the usual notation and denote by $\mathrm{Log}\mathbb{M}$ the set $\{|m|^{\mathbb{M}} \mid m \in \mathbb{M}\}$ of lengths of $\mathbb{M}$. Moreover for an $\mathcal{L}_{PV}(\mathbb{M})$-formula $\varphi(x)$ we will write $\exists n \in \mathrm{Log} : \varphi(n)$ for $\exists N, n(n = |N| \wedge \varphi(n))$ and $\forall n \in \mathrm{Log} : \varphi(n)$ for $\forall n, N(n = |N| \rightarrow \varphi(n))$. For a number $n \in \mathrm{Log}\mathbb{M}$ we will denote by $1^{(n)}$ the number $(1\#N) - 1$ where $N \in \mathbb{M}$ is such that $|N|^{\mathbb{M}} = n$. Thus for any $n \in \mathrm{Log}M$ and $m \in \mathbb{M}$ we have $m = 1^{(n)}$ if and only if $m = (1\#m) - 1$. Finally for a $\mathcal{L}_{PV}$-formula $\varphi(x, y, \bar{z})$ we will often write $\forall n \in \mathrm{Log} : \varphi(n, 1^{(n)}, \bar{z})$ as a shorthand for $\forall n, N(n = |N| \rightarrow \varphi(n, 1\#N - 1, \bar{z}))$. Now we can make one technical observation considering our notational convention:

Observation 5.1.2. *Assume $\mathbb{M}$ is Herbrand staurated model of $\widetilde{\mathrm{PV}}$, $\varphi(x, y, z, \bar{w})$ is an open $\mathcal{L}_{PV}$-formula, $\bar{a} \in \mathbb{M}$ and $\mathbb{M} \models \forall n \in \mathrm{Log}\exists z : \varphi(n, 1^{(n)}, z, \bar{a})$.*

Then there is a binary function $f \in \mathcal{L}_{PV}$ and $b \in \mathbb{M}$ such that $\mathbb{M} \models \forall n \in \mathrm{Log} : \varphi(n, 1^{(n)}, f(1^{(n)}, b), \bar{a})$.

Proof. By our notational convention $\mathbb{M} \models \forall n \in \mathrm{Log}\exists z : \varphi(n, 1^{(n)}, z, \bar{a})$ stands for

$$\mathbb{M} \models \forall n, N(n = |N| \rightarrow \exists z\varphi(n, 1\#N - 1, z, \bar{a})).$$

This gives

$$\mathbb{M} \models \forall n, N(n = |N| \wedge 1\#N - 1 = N \to \exists z \varphi(n, 1\#N - 1, z, \bar{a})).$$

By the Herbrand saturation there is a ternary function symbol $f_0 \in \mathcal{L}_{PV}$ and $b \in \mathbb{M}$ such that

$$\mathbb{M} \models \forall n, N(n = |N| \wedge 1\#N = N \to \varphi(n, 1\#N - 1, f_0(n, N, b), \bar{a}))$$

i.e.

$$\mathbb{M} \models \forall n, N(n = |N| \wedge 1\#N = N \to \varphi(n, 1\#N - 1, f_0(n, 1\#N - 1, b), \bar{a})).$$

Thus we get

$$\mathbb{M} \models \forall n, N(n = |N| \to \varphi(n, 1\#N - 1, f_0(n, 1\#N - 1, b), \bar{a}))$$

because $\mathbb{M} \models \forall N_1, N_2(1\#N_1 - 1 = 1\#N_2 - 1)$. But using the notational convention the last reads as $\mathbb{M} \models \forall n \in \mathrm{Log} : \varphi(n, 1^{(n)}, f_0(n, 1^{(n)}, b), \bar{a})$. Now if we let $f \in \mathcal{L}_{PV}$ be a binary function symbol such that $\widetilde{\mathrm{PV}} \vdash \forall x, p : f(x, p) = f_0(|x|, x, p)$ then we get $\mathbb{M} \models \forall n \in \mathrm{Log} : \varphi(n, 1^{(n)}, f(1^{(n)}, b), \bar{a})$ and we are done. $\square$

We define the following set of functions

$$\mathcal{P}(\mathbb{M}) = \{f^{\mathbb{M}}(\cdot, p) : \mathbb{M} \to \mathbb{M} \mid f \in \mathcal{L}_{\mathrm{PV}} \text{ a binary function symbol and } p \in \mathbb{M}\}.$$

$$\mathcal{P}_{all}(\mathbb{M}) = \{f^{\mathbb{M}}(\cdot, \ldots, \cdot, p) : \mathbb{M}^r \to \mathbb{M} \mid r \in \mathbb{N}, \\ f \in \mathcal{L}_{\mathrm{PV}} \text{ an (r+1)-ary function symbol and } p \in \mathbb{M}\}.$$

If $\mathbb{M}$ is a model of $\mathcal{L}_{PV}$ we always implicitly assume we are working with its expansion into the language $\mathcal{L} \supseteq \mathcal{L}_{PV}$ augment by a symbol for each function from $\mathcal{P}_{all}(\mathbb{M})$ with its natural interpretation. This is correct as any model of $\widetilde{\mathrm{PV}}$ clearly has such expansion. By the expression $f^{\mathbb{M}} \in \mathcal{P}_{all}(\mathbb{M})$ we will always mean that $f^{\mathbb{M}}$ is an interpretation of a function symbol f from the language $\mathcal{L}$. In case we will demand $f^{\mathbb{M}} \in \mathcal{P}_{all}(\mathbb{M})$ to be such that f is a function symbol from the language $\mathcal{L}_{PV}$ we will always state it explicitly. Moreover we will say that $f^{\mathbb{M}} \in \mathcal{P}_{all}(\mathbb{M})$ *does not contain a parameter* if there is a function symbol $g \in \mathcal{L}_{PV}$ of an arity matching the arity of $f^{\mathbb{M}}$ such that $g^{\mathbb{M}} = f^{\mathbb{M}}$. Recall that we say that an $\mathcal{L}_{PV}$-function symbol $f(\bar{x})$ is a Boolean function in $\widetilde{\mathrm{PV}}$ if $\widetilde{\mathrm{PV}} \vdash f(\bar{x}) = 0 \vee f(\bar{x}) = 1$ and that $g^{\mathbb{M}} \in \mathcal{P}_{all}(\mathbb{M})$ is a Boolean function in $\mathbb{M}$ if $\mathbb{M} \models \forall \bar{x}(g(\bar{x}) = 1 \vee g(\bar{x}) = 0)$. Finally let $id \in \mathcal{P}(\mathbb{M})$ denote the identity function on $\mathbb{M}$ and for every $a \in \mathbb{M}$ let $c_a \in \mathcal{P}(\mathbb{M})$ be the function which is constant a.

Observation 5.1.3. $\mathcal{P}(\mathbb{M})$ *is closed under definitions by distinction by cases by open* $\mathcal{L}_{PV}$*-formulae i.e:*

Suppose $\varphi_0(x,\bar a_0),\dots,\varphi_k(x,\bar a_k)$ are open $\mathcal{L}_{PV}$-formulae with tuples of parameters $\bar a_0,\dots \bar a_k\in\mathbb{M}$, $f_0^{\mathbb{M}}(\cdot,b_0),\dots,f_k^{\mathbb{M}}(\cdot,b_k)\in\mathcal{P}(\mathbb{M})$ with $f_0,f_1,\dots,f_{k-1}\in\mathcal{L}_{PV}$ and that a function F on $\mathbb{M}$ is defined by:

$$F(m)=\begin{cases} f_0^{\mathbb{M}}(m,b_0) & \textit{if } \mathbb{M}\models\varphi_0(m,\bar a_0)\\ f_1^{\mathbb{M}}(m,b_1) & \textit{if } \mathbb{M}\models\varphi_1(m,\bar a_1)\wedge\neg\varphi_0(m,\bar a_0)\\ \quad\vdots & \qquad\qquad\vdots\\ f_{k-1}^{\mathbb{M}}(m,b_{k-1}) & \textit{if } \mathbb{M}\models\varphi_{k-1}(m,\bar a_{k-1})\wedge\bigwedge_{i<k-1}\neg\varphi_i(m,\bar a_i)\\ f_k^{\mathbb{M}}(m,b_k) & \textit{otherwise.}\end{cases}$$

Then there is $f^{\mathbb{M}}\in\mathcal{P}(\mathbb{M})$ such that $F=f^{\mathbb{M}}$.

Proof. Let for every $i\le k$, r_i be length of the tuple $\bar a_i$. Let $f\in\mathcal{L}_{PV}$ be a binary function symbol corresponding to the following algorithm: given input x,p check whether p is a code of a tuple of the form $(b_0,b_1,\dots,b_k,\tilde a_0,\dots,\tilde a_k)$ where $\tilde a_i$ is a code of some tuple $\bar a_i$ of length r_i. If p is not of this form then output 0. Otherwise try to find minimal $i<k$ such that $\varphi_i(x,\bar a_i)$ holds. If such i exists output $f_i^{\mathbb{N}}(x,b_i)$, otherwise output $f_k^{\mathbb{N}}(x,b_k)$.

Now let $b\in\mathbb{M}$ be such that b is a code of the tuple $(b_0,\dots,b_k\tilde a_0,\dots,\tilde a_k)$ where $\tilde a_i$ is a code of the tuple $\bar a_i$ for every $i\le k$. Then clearly $F=f^{\mathbb{M}}(\cdot,b)$. □

We will leave it to the reader to check that for the functions from $\mathcal{P}(\mathbb{M})$ we also have the following:

Observation 5.1.4. *Let $f^{\mathbb{M}}\in\mathcal{P}(\mathbb{M})$. Then there is $s\in\mathbb{N}$ and $d\in\mathbb{M}$ such that $\mathbb{M}\models\forall x>d:|f(x)|\le|x|^s$.*

Note that d is dependent on the parameter from $f^{\mathbb{M}}$ thus in general it is well possible that $d\in\mathbb{M}-\mathrm{Log}\mathbb{M}$. This is one of the reasons why we will choose the domain of our ultrapower Ω to be unbounded in $\mathbb{M}$:

Let $\Omega=\{1^{(n)}\mid n\in\mathrm{Log}\mathbb{M}\}$. Then by the definition of $1^{(n)}$, Ω is $\mathbb{M}$-definable by an open $\mathcal{L}_{PV}$-formula $x=1\#x-1$. Thus for every open $\mathcal{L}_{PV}$-formula φ and $\bar f^{\mathbb{M}}\in\mathcal{P}(\mathbb{M})$ the set $\langle\langle\varphi(\bar f)\rangle\rangle_\Omega=\{1^{(n)}\in\Omega\mid\mathbb{M}\models\varphi(\bar f(1^{(n)}))\}$ is definable by an open $\mathcal{L}_{PV}$-formula. Recall that:

Definition 5.1.5. *Let $A\subseteq\mathbb{M}$ be non-empty, we say that an ultrafilter $\mathcal{U}$ on A is unbounded if for any $b\in\mathbb{M}:\{a\in A\mid a\le^{\mathbb{M}} b\}\notin\mathcal{U}$.*

Lemma 5.1.6. *Let Ω be as defined above. Then there exists an unbounded ultrafilter on Ω.*

Proof. For every $b\in\mathbb{M}$ let $[b,\infty)$ denote the set $\{a\in\mathbb{M}\mid b\le^{\mathbb{M}} a\}$. Since Ω is cofinal in $\mathbb{M}$ the set $B=\{[b,\infty)\cap\Omega\mid b\in\mathbb{M}\}$ has finite intersection property thus there is an ultrafilter $\mathcal{U}$ on Ω extending B. This ultrafilter is unbounded on Ω by the definition of B. □

For unbounded ultrafilters on Ω we can make one simple observation:

Observation 5.1.7. *Suppose $\mathcal{U}$ is an unbounded ultrafilter on Ω, $f_0^{\mathbb{M}}, f_1^{\mathbb{M}}, \dots, f_{k-1}^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ and $\varphi(\bar{x})$ an open $\mathcal{L}_{PV}$-formula. Let $\alpha_0, \alpha_1, \dots, \alpha_{k-1} \in \mathcal{P}(\mathbb{M})/\mathcal{U}$ be equivalence classes of $f_0^{\mathbb{M}}, f_1^{\mathbb{M}}, \dots, f_{k-1}^{\mathbb{M}}$ respectively.*

Then $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \varphi(\alpha_0, \alpha_1, \dots, \alpha_{k-1})$ whenever there is $d \in \mathrm{Log}\mathbb{M}$ such that $\mathbb{M} \models \forall n \in \mathrm{Log}(n > d \to \varphi(f_0(1^{(n)}), \dots, f_{k-1}(1^{(n)})))$.

Proof. Since $\mathbb{M} \models \forall n \in \mathrm{Log}(n > d \to \varphi(f_0(1^{(n)}), \dots, f_{k-1}(1^{(n)})))$ for some $d \in \mathrm{Log}\mathbb{M}$ we have that $\{1^{(n)} \in \mathbb{M} \mid n >^{\mathbb{M}} d\} \subseteq \langle\langle \varphi(f_0, \dots, f_{k-1})\rangle\rangle_\Omega$. As Ω is unbounded in $\mathbb{M}$ we also have that $\{1^{(n)} \in \mathbb{M} \mid n >^{\mathbb{M}} d\} \neq \emptyset$. Thus as $\mathcal{U}$ does not contain bounded sets we get $\{1^{(n)} \in \mathbb{M} \mid n >^{\mathbb{M}} d\} \in \mathcal{U}$ and so $\langle\langle \varphi(f_0, \dots, f_{k-1})\rangle\rangle_\Omega \in \mathcal{U}$. Finally since $\mathcal{U}$ is an unbounded ultrafilter we have that open $\mathcal{L}_{PV}$-formulae are Łos for $\mathcal{P}(\mathbb{M})/\mathcal{U}$ and so $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \varphi(\alpha_0, \alpha_1, \dots, \alpha_{k-1})$. $\square$

The following will be convenient for working with the formulae $\mathrm{UP}_k(g(\cdot, y))$ and $\mathrm{UP}_k(h)$. Let $k \in \mathbb{N}$ and $g \in \mathcal{L}_{PV}$ be a binary function symbol. We will denote by $\mathrm{UP}'_k(g(\cdot, y))$ the formula:

$$\exists c \forall \ell > 0 \exists C[\mathit{Circuit}(C, |\ell|) \wedge \mathit{size}(C) \leq |c| \cdot |\ell|^k \wedge \forall i(|\ell| = |i| \to g(i, y) = \mathit{eval}(C, i))]$$

which has a free variable y. The formula $\mathrm{UP}'_k(h)$ is defined analogously. Then we have:

$$\widetilde{\mathrm{PV}} \vdash \forall y[\forall x(g(x, y) = 1 \vee g(x, y) = 0) \to (\mathrm{UP}_k(g(\cdot, y)) \leftrightarrow \mathrm{UP}'_k(g(\cdot, y)))].$$

Henceforth if $\mathbb{M}$ is a model of $\widetilde{\mathrm{PV}}$ with $a \in \mathbb{M}$ such that $g^{\mathbb{M}}(\cdot, a)$ is a Boolean function we will freely interchange $UP_k(g(\cdot, a))$ and $UP'_k(g(\cdot, a))$. Moreover if $g^{\mathbb{M}}(\cdot, a)$ is a Boolean function and $\mathcal{U}$ a ultrafilter on Ω then $g^{\mathcal{P}(\mathbb{M})/\mathcal{U}}(\cdot, c_a^{\mathcal{U}})$ is a Boolean function on $\mathcal{P}(\mathbb{M})/\mathcal{U}$. Indeed, we have that $\mathbb{M} \models \forall x(g(x, a) = 1 \vee g(x, a) = 0)$ and $c_a \in \mathcal{P}(\mathbb{M})$ thus it follows by Corollary 1.1.8.

Finally throughout this chapter we let ϵ to range exclusively over the standard rational numbers i.e. the numbers of the form s/t for some standard s, t. For any such ϵ and a standard number k we will fix a unary function symbol $\cdot^{k+\epsilon} \in \mathcal{L}_{PV}$ such that for every $n \in \mathbb{N}$, $n^{k+\epsilon^{\mathbb{N}}}$ is the lower part of $n^{k+\epsilon}$.

5.2 The proof idea of [KO17]

We remind the arful proof idea of [KO17]. For a given natural k this proof can be divided into two steps. Adapting the result of [SR14, Proposition 2.1] one first (provably in $\widetilde{\mathrm{PV}}$) find a suitable function symbol $g \in \mathcal{L}_{PV}$. The g has the property that if $\widetilde{\mathrm{PV}} \vdash \mathrm{UP}_{c,k}(h)$ for any $c \in \mathbb{N}$ and a unary function symbol $h \in \mathcal{L}_{PV}$ then for any uniform sequence of circuits $(C_n)_{n\in\mathbb{N}}$ of size $\leq n^k$ there is $d \in \mathbb{N}$ such that g can not be computed by $(C_n)_{n\in\mathbb{N}}$ on any length bigger then d. The next step is based on an argument that provability of $\mathrm{UP}_{c,k}(g)$ (for any c) from $\widetilde{\mathrm{PV}}$ gives via KPT Theorem a finite number of uniform sequences of circuits of size $\leq n^k$ which can together compute g on any length. But this contradicts the property of g and so the first or the second assumption fails which in both cases gives the consistency result claimed.

However, since $\mathrm{UP}_k(h)$ is an $\exists\forall\exists\forall$-formula one can not use KPT Theorem to get suitable polynomial algorithms and proceed similarly as in [KO17]. It is exactly this part of the proof which we show can be bypassed by an ultrapower construction over a Herbrand saturated model of $\widetilde{\mathrm{PV}}$.

5.3 The ultrapower construction

Let throughout this section $\mathbb{M}$ be a fixed countable Herbrand saturated model of $\widetilde{\mathrm{PV}}$ and $\Omega = \{1^{(n)} \mid n \in \mathrm{Log}M\}$. We will employ an ultrapower construction with the set of functions $\mathcal{P}(\mathbb{M})$ on the domain Ω modulo an unbounded ultrafilter $\mathcal{U}$ on Ω to show that for any given $k \in \mathbb{N}$ there is a binary function symbol $g \in \mathcal{L}_{PV}$ and $a \in \mathbb{M}$ such that $g^{\mathbb{M}}(\cdot, a)$ is a Boolean function on $\mathbb{M}$ and $\mathbb{M} \models \exists x \neg \mathrm{UP}_k(g(\cdot, x))$ or there is a binary function symbol $g \in \mathcal{L}_{PV}$ and $a \in \mathbb{M}$ such that $g^{\mathcal{P}(\mathbb{M})/\mathcal{U}}(\cdot, c_a^{\mathcal{U}})$ is a Boolean function on $\mathcal{P}(\mathbb{M})/\mathcal{U}$ and $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \exists x \neg \mathrm{UP}_k(g(\cdot, x))$. Since by the Corollary 1.1.7 we get $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \widetilde{\mathrm{PV}}$ this will give the consistency of $\widetilde{\mathrm{PV}} + \exists x \neg \mathrm{UP}_k(g(\cdot, x))$ for some binary function symbol $g \in \mathcal{L}_{PV}$ (and given k). After that we will show:

Lemma 5.5.4. *Let $\mathbb{M}$ be a model of $\widetilde{\mathrm{PV}}$, $k \in \mathbb{N}$ and $g_2 \in \mathcal{L}_{PV}$ a binary function symbol. Assume $a \in \mathbb{M}$ is such that $\mathbb{M} \models \neg \mathrm{UP}_k(g_2(\cdot, a))$ and $g_2^{\mathbb{M}}(\cdot, a)$ is a Boolean function. Then there is a unary function symbol $g_1 \in \mathcal{L}_{PV}$ such that $\mathbb{M} \models \neg \mathrm{UP}_k(g_1)$.*

and so the consisteny of $\widetilde{\mathrm{PV}} + \neg \mathrm{UP}_k(h)$ for some unary function symbol $h \in \mathcal{L}_{PV}$ (and given k) will follow.

We consider this the following observation as a first step in bypassing the KPT theorem from the proof given in [KO17] which will become clear latter.

Observation 5.3.1. *Let $k \in \mathbb{N}, g \in \mathcal{L}_{PV}$ be a binary function symbol, $a \in \mathbb{M}$ and $\mathcal{U}$ an unbounded ultrafilter on Ω. Assume that for every $s \in \mathbb{N}$ there is $t \in \mathbb{N}$ such that*

$$(*)\ \forall C[\mathit{Circuit}(C, |id^{\mathcal{U}}|^t) \wedge \mathit{size}(C) \leq |id^{\mathcal{U}}|^{kt+s} \to \exists i(|i| = |id^{\mathcal{U}}|^t \wedge g(i, c_a^{\mathcal{U}}) \neq \mathit{eval}(C, i))]$$

holds in $\mathcal{P}(\mathbb{M})/\mathcal{U}$. Then $\neg UP'_k(g(\cdot, c_a^{\mathcal{U}}))$ i.e.

$$\forall c \exists \ell > 0 \forall C[\mathit{Circuit}(C, |\ell|) \wedge \mathit{size}(C) \leq |c||\ell|^k \to \exists i(|i| = |\ell| \wedge g(i, c_a^{\mathcal{U}}) \neq \mathit{eval}(C, i))]$$

holds in $\mathcal{P}(\mathbb{M})/\mathcal{U}$.

Proof. Let $\alpha \in \mathcal{P}(\mathbb{M})/\mathcal{U}$ be given and let $f_\alpha^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ be some representative of α. By the Observation 5.1.4 there is $s \in \mathbb{N}$ and $d \in \mathbb{M}$ such that $\mathbb{M} \models \forall x > d |f_\alpha(x)| \leq |x|^s$ and so $\mathbb{M} \models \forall n \in \mathrm{Log}(n > |d| \to |f_\alpha(1^{(n)})| \leq n^s)$. Since $\mathcal{U}$ is unbounded we get by the Observation 5.1.7 that $\mathcal{P}(\mathbb{M})/\mathcal{U} \models |\alpha| \leq |id^{\mathcal{U}}|^s$. To find a witness for ℓ let $t \in \mathbb{N}$ be such that the assumption $(*)$ holds for s and t. Now choose for the variable ℓ the element $\beta = id^{\mathcal{U}} \#^{\mathbb{M}} \ldots \#^{\mathbb{M}} id^{\mathcal{U}}$ (t-times) of $\mathcal{P}(\mathbb{M})/\mathcal{U}$ i.e. $\mathcal{P}(\mathbb{M})/\mathcal{U} \models |id^{\mathcal{U}}|^t = |\beta|$. To sum up, we have $\mathcal{P}(\mathbb{M})/\mathcal{U} \models |\beta| = |id^{\mathcal{U}}|^t \wedge |\alpha||\beta|^k \leq |id^{\mathcal{U}}|^{s+kt}$.

Now we can show

$$\mathcal{P}(\mathbb{M})/\mathcal{U} \models \forall C[\mathit{Circuit}(C, |\beta|) \wedge \mathit{size}(C) \leq |\alpha||\beta|^k \to \exists i(|i| = |\beta| \wedge g(i, c_a^{\mathcal{U}}) \neq \mathit{eval}(C, i))].$$

Assume $\gamma \in \mathcal{P}(\mathbb{M})/\mathcal{U}$ is such that $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \mathit{Circuit}(\gamma, |\beta|) \wedge \mathit{size}(\gamma) \leq |\alpha||\beta|^k$ which gives $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \mathit{Circuit}(\gamma, |id^{\mathcal{U}}|^t) \wedge \mathit{size}(\gamma) \leq |id^{\mathcal{U}}|^{kt+s}$. Since $(*)$ holds for s and t this gives $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \exists i(|i| = |id^{\mathcal{U}}|^t \wedge g(i, c_a^{\mathcal{U}}) \neq \mathit{eval}(\gamma, i))$ i.e. $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \exists i(|i| = |\beta| \wedge g(i, c_a^{\mathcal{U}}) \neq \mathit{eval}(\gamma, i))$ and we are done. □

We present the next observation and lemma in greater generality to point out that this step is only dependent on a syntactic form of the formula and not on its particular meaning. The following definition will latter be used to define the notion of uniform sequences of circuits in $\mathbb{M}$:

Definition 5.3.2. *Let $\varphi(x, y)$ be an open $\mathcal{L}_{PV}$-formula with all free variables shown and $A \subseteq \Omega$ definable by an open $\mathcal{L}_{PV}$-formula.*

We say that a function $f^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ is $\varphi(x, y)$-uniform on A if $\mathbb{M} \models \forall z \in A : \varphi(f(z), z)$. Moreover we say that $f^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ is $\varphi(x, y)$-uniform if it is $\varphi(x, y)$-uniform on Ω.

The following observation and lemma is the second ingredient for bypassing the step of the proof from [KO17] which uses the KPT theorem.

Observation 5.3.3. *Assume $\varphi(x, y)$ is an open $\mathcal{L}_{PV}$-formula, $\mathbb{M} \models \exists x \forall y \in \Omega : \varphi(x, y)$, $\mathcal{U}$ is an ultrafilter on Ω and $\alpha \in \mathcal{P}(\mathbb{M})/\mathcal{U}$ such that $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \varphi(\alpha, id^{\mathcal{U}})$.*

Then there is $f_\alpha^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ which is a $\varphi(x, y)$-uniform representative of α.

Proof. Assume $\alpha \in \mathcal{P}(\mathbb{M})/\mathcal{U}$ is such that $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \varphi(\alpha, id^{\mathcal{U}})$. Let $g_\alpha \in \mathcal{P}(\mathbb{M})$ be a representative of α. Since $\mathcal{L}_{PV}$formulae are Łos we have $A = \langle\langle \varphi(g_\alpha, id) \rangle\rangle_\Omega \in \mathcal{U}$ and A is clearly definable by an open $\mathcal{L}_{PV}$ formula as Ω is. So we get that g_α is $\varphi(x, y)$-uniform on A. We have that $\mathcal{P}(\mathbb{M})$ is closed under the definition by distinction by cases by open $\mathcal{L}_{PV}(\mathbb{M})$-formulae. Thus there is a function $f_\alpha^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ such that $f_\alpha^{\mathbb{M}}(x) = g_\alpha^{\mathbb{M}}(x)$ if $x \in A$ and $f_\alpha^{\mathbb{M}}(x) = m$ otherwise for some fixed $m \in \mathbb{M}$ with $\mathbb{M} \models \forall y \varphi(m, y)$. Then f_α is $\varphi(x, y)$-uniform by the assumption on m. Since $\langle\langle f_\alpha = g_\alpha \rangle\rangle_\Omega = A \in \mathcal{U}$,f_α is a representative of α and we are done.

□

Lemma 5.3.4. *Assume $\varphi(x, y), \psi(x, y, z, w)$ are open $\mathcal{L}_{PV}$- formulae, $\mathbb{M} \models \exists x \forall y \in \Omega : \varphi(x, y)$ and let $h^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$. Assume further that:*

For every $f^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ which is $\varphi(x, y)$-uniform there is $d \in \mathrm{Log}\mathbb{M}$ such that

$$\mathbb{M} \models \forall n \in \mathrm{Log}(n > d \rightarrow \exists i \psi(f(1^{(n)}), 1^{(n)}, i, h(1^{(n)}))).$$

Then for any unbounded ultrafilter $\mathcal{U}$ on Ω

$$\mathcal{P}(\mathbb{M})/\mathcal{U} \models \forall x(\varphi(x, id^{\mathcal{U}}) \rightarrow \exists i \psi(x, id^{\mathcal{U}}, i, \beta))$$

where β denotes the equivalence class of $h^{\mathbb{M}}$.

Proof. Assume $\alpha \in \mathcal{P}(\mathbb{M})/\mathcal{U}$ is such that $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \varphi(\alpha, id^{\mathcal{U}})$. By the previous observation there is a representative $f_\alpha^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ of α which is $\varphi(x, y)$-uniform. Thus by the assumption $\mathbb{M} \models \forall n \in \mathrm{Log}(n > d \rightarrow \exists i \psi(f_\alpha(1^{(n)}), 1^{(n)}, i, h(1^{(n)}))$ for some $d \in \mathrm{Log}\mathbb{M}$. By the Observation 5.1.2 there is a binary function symbol $i \in \mathcal{L}_{PV}$ and $b \in \mathbb{M}$ i.e. $i^{\mathbb{M}}(\cdot, b) \in \mathcal{P}(\mathbb{M})$ with

$$\mathbb{M} \models \forall n \in \mathrm{Log}(n > d \to \psi(f_\alpha(1^{(n)}), 1^{(n)}, i(1^{(n)}, b), h(1^{(n)}))).$$

Thus by the Observation 5.1.7 we get that $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \psi(\alpha, id^{\mathcal{U}}, \iota, \beta)$ where $\iota \in \mathcal{P}(\mathbb{M})/\mathcal{U}$ denotes the equivalence class of $i^{\mathbb{M}}(\cdot, b)$ and we are done. □

Now we can define the uniform property relevant to our proof. For every $s, t, k \in \mathbb{N}$ let $\varphi^k_{s,t}(x, y)$ be the formula $Circuit(x, |y|) \wedge size(x) \leq |y|^{kt+s}$.

Definition 5.3.5. *We say that $f^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ is a uniform sequence of circuits (in $\mathbb{M}$) of size $\leq n^{kt+s}$ with n^t many inputs if $f^{\mathbb{M}}$ is $\varphi^k_{s,t}(x, y)$-uniform.*

Let $k, s, t \in \mathbb{N}$, $a \in \mathbb{M}$ and $g \in \mathcal{L}_{PV}$ be a binary function symbol. We will denote by $\mathrm{RULB}^{g(\cdot,a)}_k(s, t)$ the following meta-statement:

$g^{\mathbb{M}}(\cdot, a)$ is a Boolean function and for all $f^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ such that $f^{\mathbb{M}}$ is a uniform sequence of circuits of size $\leq n^{kt+s}$ with n^t many inputs there is $d \in \mathrm{Log}\mathbb{M}$ for which

$$\mathbb{M} \models \forall n \in \mathrm{Log}(n > d \to \exists i(|i| = n^t \wedge g(i, a) \neq eval(f(1^{(n)}), i))).$$

Where RULB stands for *Relative uniform lower bound*.

Using the previous observation and the lemma above we can show:

Theorem 5.3.6. *Let $k \in \mathbb{N}$ be given and assume that there is a binary function symbol $g \in \mathcal{L}_{PV}$ and $a \in \mathbb{M}$ such that for every $s \in \mathbb{N}$ there is $t \in \mathbb{N}$ such that $\mathrm{RULB}^{g(\cdot,a)}_k(s, t)$ holds.*

Then $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \neg\mathrm{UP}_k(g(\cdot, c^{\mathcal{U}}_a))$ for any unbounded ultrafilter $\mathcal{U}$ on Ω.

Proof. Fix an unbounded ultrafilter $\mathcal{U}$ on Ω. For every $t \in \mathbb{N}$ let $\psi_t(C, \ell, i, a)$ be the formula $|i| = |\ell|^t \wedge g(i, a) \neq eval(C, i)$. Since $g^{\mathbb{M}}(\cdot, a) \in \mathcal{P}(\mathbb{M})$ is a Boolean function the function $g^{\mathcal{P}(\mathbb{M})/\mathcal{U}}(\cdot, c^{\mathcal{U}}_a)$ is a Boolean function as well. Thus to show $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \neg UP_k(g(\cdot, c^{\mathcal{U}}_a))$ it suffices to show that $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \neg UP'_k(g(\cdot, c^{\mathcal{U}}_a))$. To do so, by the Observation 5.3.1 it suffices to show that for every $s \in \mathbb{N}$ there is $t \in \mathbb{N}$ such that $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \forall C(\varphi^k_{s,t}(C, id^{\mathcal{U}}) \to \exists i \psi_t(C, id^{\mathcal{U}}, i, c^{\mathcal{U}}_a))$.

To show this assume $s \in \mathbb{N}$ is given and let $t \in \mathbb{N}$ be such that $\mathrm{RULB}^{g(\cdot,a)}_k(s, t)$ holds. Following our notation the statement $\mathrm{RULB}^{g(\cdot,a)}_k(s, t)$ reads:
Whenever $f^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ is a $\varphi^k_{s,t}(x, y)$-uniform sequence then there is $d \in \mathrm{Log}\mathbb{M}$ such that

$$\mathbb{M} \models \forall n \in \mathrm{Log}(n > d \to \exists i \psi_t(f(1^{(n)}), 1^{(n)}, i, c_a(1^{(n)}))).$$

Since we can consider 0 as a code of a circuit of size 0 we get that for every $s, t, k \in \mathbb{N}$, $\mathbb{M} \models \forall x \in \Omega : \varphi^k_{s,t}(0, x)$. Thus by the Lemma 5.3.4 we get that $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \forall C(\varphi^k_{s,t}(C, id^{\mathcal{U}}) \to \exists i \psi_t(C, id^{\mathcal{U}}, i, c^{\mathcal{U}}_a))$ and we are done. □

Before we define one more meta-statement we will slightly extend our concept of uniform sequences of circuits in $\mathbb{M}$. Let for every $k \in \mathbb{N}$ and a standard rational ϵ let $\varphi^k_\epsilon(x, y)$ be the formula $Circuit(x, |y|) \wedge size(x) \leq |y|^{k+\epsilon}$. Then:

Definition 5.3.7. *We say that $f^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ is a uniform sequence of circuits (in $\mathbb{M}$) of size $\leq n^{k+\epsilon}$ if $f^{\mathbb{M}}$ is $\varphi^k_\epsilon(x, y)$-uniform.*

Now we can give the meta-statement:

Let $k \in \mathbb{N}$, $a \in \mathbb{M}$, $g \in \mathcal{L}_{PV}$ be a binary function symbol and ϵ a standard rational. We will denote by $\mathrm{ULB}_k^{g(\cdot,a)}(\epsilon)$ the meta-statement:

$g^{\mathbb{M}}(\cdot, a)$ is a Boolean function and for all $f^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ such that $f^{\mathbb{M}}$ is a uniform sequence of circuits of size $\leq n^{k+\epsilon}$ there is $d \in \mathbb{M}$ such that

$$\mathbb{M} \models \forall n \in \mathrm{Log}(n > d \to \exists i(|i| = n \wedge g(i,a) \neq \mathit{eval}(f(1^{(n)}), i))).$$

Where ULB stands for *Uniform lower bounds.* It is only a technicality which we will prove latter that:

Lemma 5.5.2. *Let $\mathbb{M}$ be a model of $\widetilde{\mathrm{PV}}$, $a \in \mathbb{M}$, $k \in \mathbb{N}$ and $g \in \mathcal{L}_{PV}$ be a binary function symbol. Assume there is a standard rational $\epsilon > 0$ such that $\mathrm{ULB}_k^{g(\cdot,a)}(\epsilon)$ holds. Then for every $s \in \mathbb{N}$ there is $t \in \mathbb{N}$ such that $\mathrm{RULB}_k^{g(\cdot,a)}(s,t)$ holds.*

Based on the previous paragraphs we can describe how the rest of our proof works. We will fix a Herbrand saturated model $\mathbb{M}$ of $\mathcal{L}_{PV}$. For a given $k \geq 1$ we will wlog assume that $k > 3$. This is possible as for any $k \in \mathbb{N}$ we have that $\widetilde{\mathrm{PV}} \vdash \exists x \neg \mathrm{UP}_{k+1}(g(\cdot,x) \to \exists x \neg \mathrm{UP}_k(g(\cdot,x)))$. To find the suitable function symbol $g \in \mathcal{L}_{PV}$ we formalize the proof of [KO17, Lemma 3.1] in $\mathbb{M}$. The symbol g will be such that if there is no binary function symbol $f \in \mathcal{L}_{PV}$ and no $b \in \mathbb{M}$ such that $f^{\mathbb{M}}(\cdot, b)$ is a Boolean function on $\mathbb{M}$ with $\mathbb{M} \models \exists x \neg \mathrm{UP}_k(f(\cdot,x))$. Then there is a standard rational $\epsilon > 0$ and $a \in \mathbb{M}$ such that $\mathrm{ULB}_k^{g(\cdot,a)}(\epsilon)$ holds in $\mathbb{M}$. To show this, we will give a proof almost identical to the proof of Lemma 3.2 ibid modulo the "$+\epsilon$" factor with an assumption $\epsilon < (k-3)/2$ and modulo formalisation in $\mathbb{M}$. Hence using the Lemma 5.5.4 we get that either the consistency result is witnessed by $\mathbb{M}$. Or using the theorem above with Lemma 5.5.2 and the Lemma 5.5.4 the consistency result will be witnessed by $\mathcal{P}(\mathbb{M})/\mathcal{U}$ for any unbounded $\mathcal{U}$.

Finally, we remark and proof latter that our construction will have the following corollary:

Corollary 5.5.7. *Assume there is a countable Herbrand saturated model $\mathbb{M}$ of $\widetilde{\mathrm{PV}}$ and an unbounded ultrafilter $\mathcal{U}$ on $\Omega = \{1^{(n)} \mid n \in \mathrm{Log}\mathbb{M}\}$ with $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \mathrm{S}_2^1(\mathrm{PV})$.*

Then for every natural nnumber $k \geq 0$ there is a binary function symbol $g \in \mathcal{L}_{PV}$ and a unary function symbol $h \in \mathcal{L}_{PV}$ such that

$$\mathrm{S}_2^1(\mathrm{PV}) + \widetilde{\mathrm{PV}} + \exists x \neg \mathrm{UP}_k(g(\cdot,x)) \text{ and } \mathrm{S}_2^1(\mathrm{PV}) + \widetilde{\mathrm{PV}} + \neg \mathrm{UP}_k(h)$$

are consistent.

5.4 Complexity in non-standard models of $\widetilde{\mathrm{PV}}$

Recall that an r-ary function $f : \mathbb{N}^r \to \mathbb{N}$ is in $\mathrm{DTIME}(n^d)$ for some $d \in \mathbb{N}$ if there is a constant $c \in \mathbb{N}$ and a Turing machine $\mathbb{A}$ computing f on each input $w_0, w_1, \ldots, w_{r-1}$ in less than $c(1 + |w_1| + \ldots + |w_r|)^d$ many steps. We define the deterministic time in $\mathbb{M}$ as follows: for any $d \in \mathbb{N}$

$$\begin{aligned}\mathrm{DTIME}^{\mathbb{M}}(n^d) = \{&f^{\mathbb{M}}(\cdot, \ldots, \cdot, p) : \mathbb{M}^r \to \mathbb{M} \mid r \in \mathbb{N},\ p \in \mathbb{M},\\ &f \in \mathcal{L}_{PV} \text{ an } (r+1)\text{-ary function symbol and } f^{\mathbb{N}} \in \mathrm{DTIME}(n^d)\}.\end{aligned}$$

In the following paragraphs we will clarify concepts of coding of strings and simulations of Turing machines in $\mathbb{M}$.

First we fix encoding functions:

$bit(\cdot,\cdot)$: Let $bit(x,i) \in \mathcal{L}_{PV}$ be a binary function symbol such that

$$bit(x,i)^{\mathbb{N}} = \begin{cases} \text{“the } i\text{-th bit in the binary expansion of } x\text{”} & \text{if } i < |x| \\ 0 & \text{otherwise} \end{cases}$$

where the 0-th bit is by convention the least significant bit. Thus a number $m \in \mathbb{M}$ codes the string '$bit(m,0)bit(m,1)\dots bit(m,|m|-1)$'. Moreover if $\ell, p \in \mathbb{M}$ and $\varphi(x,y) \in \Delta_0^b(\mathcal{L}_{PV})$ then there is a number $m \in \mathbb{M}$ such that $\mathbb{M} \models \forall i \leq |l| : bit(m,i) = 1 \leftrightarrow \varphi(i,p)$.

$\cdot * \cdot$: Let $\cdot * \cdot \in \mathcal{L}_{PV}$ be a binary function symbol definded by an algorithm which on input x, y decode the string w_x coded by x and w_y coded by y and output the number coding the string $w_x w_y$.

In other words, the following formulae are provable in $\widetilde{\mathrm{PV}}$ and thus its universal closure hold in $\mathbb{M}$

$$(i < |a| \rightarrow bit(a*b,i) = bit(a,i)) \wedge (|a| \leq i < |b| + |a| \rightarrow bit(a*b,i) = bit(b, i - |a|))^1$$

$$|a*b| = |a| + |b|$$

$(\cdot,\dots,\cdot)$: Let for any standard number k, $(\cdot,\dots,\cdot) \in \mathcal{L}_{PV}$ be an k-ary function symbol corresponding to an algorithm which given input $x_0, x_1, \dots, x_{k-1}$ interpret x_i as a code of a string w_i and output code of the k-tuple $(w_0, w_1, \dots, w_{k-1})$. This functions can be chosen such that for any $k \in \mathbb{N}$ there is $c \in \mathbb{N}$ such that $\widetilde{\mathrm{PV}}$ proves the formula

$$|(x_1,\dots,x_k)| \leq c(k + |x_1| + \ldots + |x_k|).$$

For the following definitions we fix some notion of coding of a sequence (in $\mathbb{N}$) such that any number codes some sequence.

$len(\cdot)$: Let $len(x) \in \mathcal{L}_{PV}$ be a unary function symbol definded by an algorithm which on input x outputs the number of items in the sequence coded by x.

For decoding sequences we have the following function:

$(\cdot)_i$: Let $(\cdot)_i \in \mathcal{L}_{PV}$ be a binary[2] function symbol definded by an algorithm which on input x, i outputs the i-th element in the sequence coded by x and 0 if the index i is bigger then the number of items in the sequence coded by x.

We can assume that the functions $(\cdot)_i^{\mathbb{N}}$, $len^{\mathbb{N}}(\cdot)$ and thus the notion of coding of a sequence (in $\mathbb{N}$) were chosen such that for some $c \in \mathbb{N}$ the following formula is provable in $\widetilde{\mathrm{PV}}$:

$$|m| \leq c(len(m) + \sum_{i<len(m)} |(m)_i|).$$

[1] where x-y is defined as usual if $x > y$ and equals 0 otherwise

[2] the second variable is i

Where the above sum is definable in $\widetilde{\mathrm{PV}}$ as $\widetilde{\mathrm{PV}} \vdash len(m) \leq |m|$. With aid of the function *bit* we can now easily code (in $\mathbb{M}$) sequences of n many k-tuples for any standard k and $n \in \mathrm{Log}\mathbb{M}$. Clearly the interpretations of the function symbols above in $\mathbb{N}$ can be chosen such that they all are in $\mathrm{DTIME}(n^2)$.

Next recall that by [AB09, Theorem 1.9] there is a universal Turing machine which given a code $c_{\mathbb{A}}$ of a Turing machine $\mathbb{A}$ and an input x simulates in $d_{\mathbb{A}} \log(T)T$ steps the first T steps of the computation of $\mathbb{A}$ on input x. The constant $d_{\mathbb{A}}$ is independent on $|x|$ and depends only on $\mathbb{A}$'s alphabet size, number of input tapes and number of states. Using the universal Turing machine we have:

Lemma 5.4.1. *For any $d \in \mathbb{N}, d \geq 2$ there is a 4-ary function symbol $un_d \in \mathcal{L}_{PV}$ with $un_d^{\mathbb{N}} \in \mathrm{DTIME}(\log^4(n)n^d)$ such that for any ternary function symbol $f \in \mathcal{L}_{PV}$ with $f^{\mathbb{N}} \in \mathrm{DTIME}(n^d)$ there is $c_f, l_f \in \mathbb{N}$ such that*

$$\text{(1)} \qquad \widetilde{\mathrm{PV}} \vdash \forall x > l_f \forall p, a(|a| = |x|^{2/3} \rightarrow un_d(x, a, p, c_f) = f(x, a, p)).$$

Proof. Let $un_d \in \mathcal{L}_{PV}$ be 4-ary function symbol corresponding to the following algorithm:

Given input x, a, p, c_f check whether c_f is a code of a Turing machine. If not output 0, otherwise let $\mathbb{A}$ be the Turing machine coded by c_f. Simulate computation of $\mathbb{A}$ on input x, a, p and stop the simulation after $\log(|x| + |a| + |p|)^4(|x| + |a| + |p|)^d$ many steps. If the simulated computation halts, output the computed value of $\mathbb{A}$ on x, a, p. Otherwise output 0.

We claim that the algorithm runs in $O(\log^4(n)n^d)$. Given input x, a, p, c_f with $|x| + |a| + |p| + |c_f| = n$ to check whether c_f is a code of a Turing machine can be done in $O(|c_f|^2)$ and so in $O(n^2)$ many steps. To count the number of steps of the simulation and proceed the simulation for $\log(|x| + |a| + |p|)^4(|x| + |a| + |p|)^d$ many steps is in $O(\log(n)^4 n^d)$. Thus for $d \geq 2$ we have $un_d^{\mathbb{N}} \in \mathrm{DTIME}(\log(n)^4 n^d)$.

Now we will show that for any ternary function symbol $f \in \mathcal{L}_{PV}$ with $f^{\mathbb{N}} \in \mathrm{DTIME}(n^d)$ there is $c_f, l_f \in \mathbb{N}$ such that (1) holds. Assume f is such a function symbol and let $c_f \in \mathbb{N}$ be a code of a Turing machine $\mathbb{A}$ computing $f^{\mathbb{N}}$ in time $O(n^d)$. Then for all x, a, p, $|x| + |a| + |p| = n$ with x big enough the computation of $\mathbb{A}$ on x, a, p halts in $\log(|x|)n^d$ many steps. By the previous paragraph there is $d_{\mathbb{A}}$ such that for any x, a, p the first $\log(|x|)n^d$ many steps of the computation of $\mathbb{A}$ on input x, a, p can be simulated in $d_{\mathbb{A}} \log(\log(|x|)n^d) \log(|x|)n^d$ many steps. For big enough x this is $\leq \log(|x|)^3 \log(n)n^d \leq \log(n)^4 n^d$. Thus if x, p, a for x big enough are given then the simulation of $\mathbb{A}$ on input x, p, a halts in less then $\log(|x| + |a| + |p|)^4(|x| + |a| + |p|)^d$ many steps and so the algorithm above outputs $f^{\mathbb{N}}(x, a, p)$ i.e. $un_d^{\mathbb{N}}(x, a, p, c_f) = f^{\mathbb{N}}(x, a, p)$. Moreover since we made the estimations for "big enough x" independent on size of a and p we indeed do not have to give any lower bound on a, p. □

The previous lemma gives us a possibility to use $un_d^{\mathbb{M}}$ for a certain kind of diagonalization in $\mathbb{M}$ against $\mathrm{DTIME}^{\mathbb{M}}(n^d)$:

Lemma 5.4.2. *For every standard $d \geq 2$ there a unary function symbol $g_{d+1} \in \mathcal{L}_{PV}$ such that $g_{d+1}^{\mathbb{M}} \in \mathrm{DTIME}^{\mathbb{M}}(n^{d+1})$ is a Boolean function and for every binary function*

$h^{\mathbb{M}} \in \mathrm{DTIME}^{\mathbb{M}}(n^d)$ there is $d_h \in \mathrm{Log}\mathbb{M}$ with:

$$(2) \qquad \mathbb{M} \models \forall n \in \mathrm{Log}\forall a(n > d_h \wedge |a| = n^{2/3} \to \exists x(|x| = n \wedge g_{d+1}(x) \neq h(x,a))).$$

Moreover if $h^{\mathbb{M}}$ does not contain parameter from $\mathbb{M}$ then d_h can be chosen to be in $\mathbb{N}$.

Proof. Let $g_{d+1} \in \mathcal{L}_{PV}$ be a function symbol corresponding to the following algorithm:

Given x interpret it as a code of a binary string w. Interpret first $||x||$ bits of w as a binary representation of a number c_h, interpret the next $|x|^{2/3}$ bits as a binary representation of an advice a and finally interpret the last bits as a binary representation of a parameter p. Output 1 if $un_d(x,p,a,c_h) = 0$ and 0 otherwise.

Since to compute $un_d^{\mathbb{N}}(x,p,a,c_h)$ is in time $O(\log(n)^4 n^d)$ and to code/decode strings is in time $O(n^2)$, the described algorithm runs in time $O(n^{d+1})$ and so $g_{d+1}^{\mathbb{M}} \in \mathrm{DTIME}^{\mathbb{M}}(n^{d+1})$.

Now assume a Boolean function $h^{\mathbb{M}}(\cdot,\cdot,p) \in \mathrm{DTIME}^{\mathbb{M}}(n^d)$ for $h \in \mathcal{L}_{PV}$ and $p \in \mathbb{M}$ is given. We are obligated to show that there is $d_h \in \mathbb{M}$ such that (2) holds. To do so, let $c_h, l_h \in \mathbb{N}$ be such that (1) holds for h. Let further $n > d_h$ and $a \in \mathbb{M}$ with $|a|^{\mathbb{M}} = (n^{2/3})^{\mathbb{M}}$ be given where d_h is big enough specified by the following paragraphs. It suffices to let $m \in \mathbb{M}$ be of length n coding a string w with the following property:

(i) the first $(|n|^{\mathbb{M}})$-many bits of w are 0's followed by binary representation of c_h,
(ii) the next $(n^{2/3})^{\mathbb{M}}$-many bits of w are binary representation of a and
(iii) the last part of w consists of 0's followed by a binary representation of p.

Such m clearly exists as it can be computed from a, p, c_h if a is big enough wrt p. Then by the definition of g_{d+1}, $g_{d+1}^{\mathbb{M}}(m) \neq un_d^{\mathbb{M}}(m,a,p,c_h)$ and so if in addition $n > l_h$ then by (1) we get $g_{d+1}^{\mathbb{M}}(m) \neq un_{d+1}^{\mathbb{M}}(m,a,p,c_h) = h^{\mathbb{M}}(m,a,p)$ which finishes the argument.

For the second part of the statement observe that if $h^{\mathbb{M}}$ does not contain any parameter then the element m from above does not have to code any parameter p. But then the length of m is only dependent on a and c_{h_0} and thus such m exists for any $n \in \log \mathbb{M}$ with $|n| > c_f \in \mathbb{N}$. □

Before stating the next lemma, we will discuss what uniform sequences of circuits, circuits and coding of circuits means in $\mathbb{M}$. We describe two ways of codding circuits and uniform families of circuits following [SR14] and [KO17].

Assume C is a (standard) circuit with n inputs then it can be coded as a sequence of 4-tuples

$$(1^{(n)}, u, v, w)$$

where u, v are gates of C such that there is a wire from the gate u to the gate v and w describes the type of a gate u. If S denotes the number of gates in C then $u, v \leq S$, $w \leq e$ for a fixed e independent on C and so $|(1^{(n)}, u, v, w)|$ is in $O(n + \log(S))$.

We let *Circuit*$(\cdot,\cdot) \in \mathcal{L}_{PV}$ be a binary function symbol corresponding to an algorithm which for input C, n outputs 1 if the given number C is a code of a sequence coding a circuit with n many inputs according to the coding above and outputs 0 otherwise. We will abuse the notation and use this function symbol as a predicate thus we will write *Circuit*(C,n) instead of *Circuit*$(C,n)=1$. Moreover we fix a unary function symbol $size \in \mathcal{L}_{PV}$ such that $size^{\mathbb{N}}$ assign to every circuit in $\mathbb{N}$ its size (i.e. number of gates) and 0 to any number which is not a code of a circuit. The symbols *Circuit* and *size*, can be chosen so that $Circuit^{\mathbb{N}}, size^{\mathbb{N}} \in \mathrm{DTIME}(n^2)$.

Finally we let $eval \in \mathcal{L}_{PV}$ be a binary function symbol corresponding to the following algorithm: given input C, i check whether C is a code of a circuit and output 0 if not. Otherwise let n be the number of inputs of the circuit coded by C and compute the value of C with the assignment of input variables of C where the j-th input variable of C get as an input the j-th bit of i if $j < |i|$ and 0 otherwise. To compute the value of a circuit C on input i can be done in $O(size(C)^2 + |i|)$ many steps. Thus the algorithm runs in $O(C^2 + |i|)$ for any C, i.

We say that $C \in \mathbb{M}$ is a circuit (in $\mathbb{M}$) if $\mathbb{M} \models \exists n\, Circuit(C,n)$. Recall that for a given standard number k and a standard rational ϵ we say that a function $f^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ is a uniform sequence of circuits (in $\mathbb{M}$) of size $\leq n^{k+\epsilon}$, if

$$\mathbb{M} \models \forall n \in \mathrm{Log} : Circuit(f(1^{(n)}), n) \wedge size(f(1^{(n)})) \leq n^{k+\epsilon}.$$

where the function symbols *Circuit* and *size* are now fixed by the previous paragraphs.

Finally we define a more efficient way of coding uniform sequences of circuits called the "succint" version which is thanks to [SR14] with a slight change following [KO17][3]. Let $f^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ be a uniform sequence of circuits of size $\leq n^{k+\epsilon}$ for some $k \in \mathbb{N}$ and a standard rational ϵ. Then for every n from $\mathrm{Log}M$ we can code the circuit $f^{\mathbb{M}}(1^{(n)})$ by a sequence of 5-tuples of the form

$$(n * 0 * 1^{(n^{1/(3k)})}, u, v, w, t)$$

with the same meaning of u, v, w as above and t is a code of a string of 1's to padd the length of the tuple to $n^{1/(2k)}$ if possible. The padding is possible for any big enough n since for some fixed $e \in \mathbb{N}$ $u, v \leq n^{k+\epsilon}$, $w \leq en$ and so $|(n * 0 * 1^{(n^{1/(3k)})}, u, v, w)|$ is $O(|n| + n^{1/(3k)} + |u| + |v| + |w|) = O(n^{1/(3k)} + \log(n)) = O(n^{1/(3k)})$.

We denote by $\chi^f_{succ} \in \mathcal{P}(\mathbb{M})/\mathcal{U}M$ the characteristic function of the language which consists of all 5-tuples that appear in the sequence coding the circuit $f^{\mathbb{M}}(1^{(n)})$ for some $n \in \mathrm{Log}\mathbb{M}$ with n big enough so that the tuples can be padded to the length $n^{1/(2k)}$. Since χ^f_{succ} can be defined using $f^{\mathbb{M}}$ we have that $\chi^f_{succ} \in \mathcal{P}(\mathbb{M})$.

Considering size of codes we will need the following claim:

Claim 5.4.3. *Let $k > 3$ be a standard number and assume $\{D_n\}_{n \in \mathrm{Log}\mathbb{M}}$ is such that for any $n \in \mathrm{Log}\mathbb{M}$, $\mathbb{M} \models Circuit(D_n, n) \wedge size(D_n) \leq (n^{1/(2k)})^k$. Then there is $d \in \mathbb{N}$ such that $\mathbb{M} \models |D_n| \leq n^{2/3}$ for any $n \in \log\mathbb{M}$ with $n > d$.*

Proof. Let $n \in \mathrm{Log}\mathbb{M}$ be given and let D be the circuit coded by the number D_n. Recall that if D has $r \leq n^{1/(2k)}$ many inputs it is coded by a sequence of 4-tuples $(1^{(r)}, u, v, w)$ where for each wire of D there is exactly one 4-tuple in the sequence. For each gate of D there are at most two incoming wires and there is at most $(n^{1/(2k)})^k$ many gates. Using that each wire is an incoming wire of some edge the number of

[3] in [SR14] the 5-tuples defined bellow are not padded to exact length

wires in D and so the length of the sequence coding D is in $O(n^{1/2})$. Since each 4-tuple is coded by a number of length $O(r+\log(n^{1/(2k)}))$ and so $O(n^{1/(2k)})$ by $r \leq n^{1/(2k)}$ we get that D can be coded by a number of length $O(n^{1/2}n^{1/(2k)}) = O(n^{1/2+1/(2k)})$. But then D can be coded by a number of length $\leq n^{2/3}$ for all big enough n as $1/2k < 1/6$ by the assumption on k and thus $O(n^{1/2+1/(2k)}) < O(n^{2/3})$. Thus $|D_n| \leq n^{2/3}$ for all big enough $n \in \mathrm{Log}\mathbb{M}$. Since the estimations holds for big enough standard numbers, we can have $d \in \mathbb{N}$. □

The proof of the following Lemma is almost identical with the proof of [KO17, Lemma 3.2] modulo the factor "$+\epsilon$" and formalisation in a model of $\widetilde{\mathrm{PV}}$.

Lemma 5.4.4. *Let $k > 3$ be a standard natural number. Then there is unary function symbol $g \in \mathcal{L}_{PV}$ such that $g \in \mathrm{DTIME}^{\mathbb{M}}(n^{3k})$ is a Boolean function and whenever $f^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ is a sequence of uniform circuits of size $\leq n^{k+\epsilon}$ for $\epsilon < (k-3)/2$ and $\mathbb{M} \models \mathrm{UP}'_k(\chi^f_{succ})$[4] then there is $d_f \in \mathbb{N}$ such that:*

$$\mathbb{M} \models \forall n \in \mathrm{Log}[n > d_f \to \exists x(|x| = n \wedge g(x) \neq eval(f(1^{(n)}), x))].$$

Proof. Let g be the g_{d+1} from the Lemma 5.4.2 for $d = 3k-1$. We show that there is a binary function symbol $h \in \mathcal{L}_{PV}$ with $h^{\mathbb{M}}(\cdot,\cdot) \in \mathrm{DTIME}^{\mathbb{M}}(n^{3k-1})$, a set of advices $\{D_n\}_{n \in \mathrm{Log}\mathbb{M}}$ in $\mathbb{M}$ and $d \in \mathbb{N}$ such that:

For any $n > d$ from $\mathrm{Log}\mathbb{M}$,

$$\mathbb{M} \models |D_n| = n^{2/3} \wedge \forall x(|x| = n \to h(x, D_n) = eval(f(1^{(n)}), x)).$$

Using this and assuming conclusion of this lemma fails, we will derive a contradiction with the Lemma 5.4.2.

Let $h \in \mathcal{L}_{PV}$ be a binary function symbol corresponding to the following algorithm: Given x, a where $|x| = n$, if a is not a (padded) code of a circuit of size $\leq (n^{1/(2k)})^k$ with $(n^{1/(2k)})$ many inputs then reject. Otherwise let D be the circuit coded by a. Try all possible 5-tuples of the form $(n * 0 * 1^{(n^{1/(3k)})}, u, v, w, t)$ of length $n^{1/(2k)}$ with $u, v \leq n^{k+\epsilon}$, $w \leq e$ and t coding string of one's and using D check whether they are in a sequence defining a circuit. After trying all possible 5-tuples check whether the constructed sequence of accepted 5-tuples correctly defines a circuit of size $\leq n^{k+\epsilon}$, if not - reject. Otherwise call C the circuit defined by the sequence and output the value of C on input x.

We first show that:

Claim 5.4.5. *The algorithm above runs in $O(n^{3k-1})$ and thus $h^{\mathbb{M}} \in \mathrm{DTIME}^{\mathbb{M}}(n^{3k-1})$.*

Proof. Let an input x, a where $|x| = n$ be given. Since $u, v \leq n^{k+\epsilon}$ and $w \leq en$ the number of relevant 5-tuples is $\leq en^{2k+1+2\epsilon}$. To evaluate a circuit D of size $\leq (n^{1/(2k)})^k$ i.e. $O(n^{1/2})$ is in time $O(n)$. Thus to create the sequence of accepted 5-tuples is in $O(n \cdot n^{2k+1+2\epsilon}) = O(n^{2k+2+2\epsilon})$ many steps. To evaluate the computed circuit C of size $\leq n^{k+\epsilon}$ on input x with $|x| = n$ is in time $O(n^{2k+2\epsilon})$. Altogether gives that the above algorithm runs in $O(n^{2k+2+2\epsilon} + n^{2k+2\epsilon})$ which is $O(n^{3k-1})$ for $\epsilon < (k-3)/2$ on any input tuple x, a. Thus $h^{\mathbb{N}} \in \mathrm{DTIME}(n^{3k-1})$ and so $h^{\mathbb{M}} \in \mathrm{DTIME}^{\mathbb{M}}(n^{3k-1})$. □

[4] i.e. $\mathbb{M} \models \mathrm{UP}'_k(\chi(\cdot, b))$ for some binary function symbol $\chi \in \mathcal{L}_{PV}$ and $b \in \mathbb{M}$ with $\chi^{\mathbb{M}}(\cdot, b) = \chi^f_{succ}$

Now for any $n \in \mathrm{Log}\mathbb{M}$ let D_n be a circuit of size $\leq (n^{1/(2k)})^k$ with $n^{1/(2k)}$ many inputs computing χ^f_{succ} on inputs of length $n^{1/(2k)}$. Such circuits exist for every $n \in \mathrm{Log}\mathbb{M}$ by the assumption $\mathbb{M} \models \mathrm{UP}'_k(\chi^f_{succ})$. We can wlog assume there is $d_0 \in \mathbb{N}$ such that for any $n > d_0$ with $n \in \mathrm{Log}\mathbb{M}$: $\mathbb{M} \models |D_n| = n^{2/3}$. This is correct because by the Claim 5.4.3 as $k > 3$ there is $d_0 \in \mathbb{N}$ such that for any $n \in \mathrm{Log}M$, $\mathbb{M} \models |D_n| \leq n^{2/3}$ and we can padd D_n if necessary to be of size $n^{2/3}$ for any $n \in \mathrm{Log}M$ with $n > d_0$. Moreover let $d_1 \in \mathbb{N}$ be such that for any $n \in \mathrm{Log}\mathbb{M}$ with $n > d_1$ the 5-tuples $(n * 0 * 1^{(n^{1/(3k)})}, u, v, w, t)$ can be padded to length $n^{1/(2k)}$. Finally set $d = \max(d_0, d_1)$.

Now we can show:

Claim 5.4.6. *For any $n \in \mathrm{Log}\mathbb{M}$ with $n > d$:*

$$\mathbb{M} \models |D_n| = n^{2/3} \wedge \forall x(|x| = n \rightarrow h(x, D_n) = \mathit{eval}(f(1^{(n)}), x)).$$

Proof. Assume $n \in \mathrm{Log}\mathbb{M}$ with $n > d$ is given. Since $n > d_0$ we have that $|D_n| = n^{2/3}$. Since $n > d_1$ we also have that the 5-tuples corresponding to the circuit $f^{\mathbb{M}}(1^{(n)})$ are padded to length $n^{1/(2k)}$. Thus by definition of χ^f_{succ} and definition of D_n, the circuit coded by D_n can decide whether given 5-tuple of length $n^{1/(2k)}$ corresponds to the circuit $f^{\mathbb{M}}(1^{(n)})$. Now assume an input $i \in \mathbb{M}$ with $|i| = n$ is given. Since the circuit coded by D_n is of size $\leq (n^{1/(2k)})^k$ the algorithm above will on input i, D_n correctly compute the sequence coding the circuit $f^{\mathbb{M}}(1^{(n)})$. Since $\mathit{size}^{\mathbb{M}}(f^{\mathbb{M}}(1^{(n)})) \leq n^{k+\epsilon}$ it will continue and output $\mathit{eval}^{\mathbb{M}}(f^{\mathbb{M}}(1^{(n)}), i)$.

□

Now we can derive the promised contradiction. First, by the definition of g together with $h^{\mathbb{M}} \in \mathrm{DTIME}^{\mathbb{M}}(n^{3k-1})$ there is by the Lemma 5.4.2 some $c \in \mathbb{N}$ (h does not use parameter from $\mathbb{M}$) with

$$\mathbb{M} \models \forall n \in \mathrm{Log} \forall a(n > c \wedge |a| = n^{2/3} \rightarrow \exists x(|x| = n \wedge g(x) \neq h(x, a))).$$

Assuming the conclusion of this lemma fails there is $n > \max(c, d)$ from $\mathrm{Log}\mathbb{M}$ such that $\mathbb{M} \models \forall x(|x| = n \rightarrow g(x) = \mathit{eval}(f(1^{(n)}), x))$. But by the previous paragraph as $n > d$ also $\mathbb{M} \models |D_n| = n^{2/3} \wedge \forall x(|x| = n \rightarrow h(x, D_n) = \mathit{eval}(f(1^{(n)}), x))$ and thus $\mathbb{M} \models \forall x(|x| = n \rightarrow g(x) = h(x, D_n))$ contradicting $n > c$. Hence we can let $d_f = \max(c, d)$ i.e. $d_f \in \mathbb{N}$ and we are done. □

In the context of the previous lemma we remark that it is not known to the author whether one can under the assumption that $\mathbb{M} \models \mathrm{UP}'_k(\chi^f_{succ})$ construct explicitly a function $i^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ which can compute witness for x in the statement

$$\mathbb{M} \models \forall n \in \mathrm{Log}[n > d_f \rightarrow \exists x(|x| = n \wedge g(x) \neq \mathit{eval}(f(1^{(n)}), x))].$$

In particular, having such function explicitly constructed for every given f of the form as above assuming $\mathbb{M} \models \mathrm{UP}'_k(\chi^f_{succ})$ one could avoid the assumption that $\mathbb{M}$ is Herbrand saturated in the construction.

5.5 The result

Recall the definition of the meta-statements $\mathrm{ULB}_k^{g(\cdot,a)}(\epsilon)$ and $\mathrm{RULB}_k^{g(\cdot,a)}(s,t)$ from the Section 5.3. As a direct corollary of the last lemma we get:

Corollary 5.5.1. *Let $\mathbb{M}$ be a model of $\widetilde{\mathrm{PV}}$ and let $k > 3$ be a natural number. Assume there is no binary function symbol $f \in \mathcal{L}_{PV}$ and $a \in \mathbb{M}$ such that $f^{\mathbb{M}}(\cdot,a)$ is a Boolean function with $\mathbb{M} \models \neg\mathrm{UP}'_k(f(\cdot,a))$. Then there is a binary function symbol $g \in \mathcal{L}_{PV}$ and $a \in \mathbb{M}$ such that $\mathrm{ULB}_k^{g(\cdot,a)}(\epsilon)$ holds for any standard rational ϵ with $\epsilon < (k-3)/2$.*

The following lemma is the last missing bit of our construction:

Lemma 5.5.2. *Let $\mathbb{M}$ be a model of $\widetilde{\mathrm{PV}}$ and let $k \in \mathbb{N}$ be given. Assume a binary function symbol $g \in \mathcal{L}_{PV}$ and $a \in \mathbb{M}$ is such that $\mathrm{ULB}_k^{g(\cdot,a)}(\epsilon)$ holds for some standard rational $\epsilon > 0$. Then for every $s \in \mathbb{N}$ there is $t \in \mathbb{N}$ such that $\mathrm{RULB}_k^{g(\cdot,a)}(s,t)$ holds.*

Proof. The proof idea of this lemma is straightforward: Assume no uniform sequence of circuits of size $\leq n^{k+\epsilon}$ can compute $g^{\mathbb{M}}(\cdot,a)$ on all big enough lengths and $f^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ is a uniform sequence of circuits of size $\leq n^{kt+s}$ with with n^t many inputs and $s/t \leq \epsilon$. Then $f^{\mathbb{M}}$ cannot on input $1^{(n)}$ with n big enough produce a circuit computing $g^{\mathbb{M}}(\cdot,a)$ on inputs of length n^t. Indeed, the size of the circuit computed by $f^{\mathbb{M}}$ from $1^{(n)}$ wrt to $m = n^t$ is $\leq n^{kt+s} \leq m^{k+s/t} \leq m^{k+\epsilon}$. But then we could use $f^{\mathbb{M}}$ to define a uniform sequence of circuits of size $\leq n^{k+\epsilon}$ computing $g^{\mathbb{M}}(\cdot,a)$ on all big enough lengths. Formally the proof goes as follows:

Let ϵ be such that $\mathrm{ULB}_k^{g(\cdot,a)}(\epsilon)$ holds. Let $s \in \mathbb{N}$ be given and choose a $t \in \mathbb{N}$ such that $s/t < \epsilon$. Assume $f^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ is a uniform sequence of circuits of size $\leq n^{kt+s}$ with n^t many inputs.

Let $h^{\mathbb{M}} \in \mathcal{P}(\mathbb{M})$ be any such that $\mathbb{M} \models \forall n \in \mathrm{Log} : h(1^{(n)}) = f(1^{(n^{1/t})})$. Then clearly $\mathbb{M} \models \forall n \in \mathrm{Log} : \mathit{Circuit}(h(1^n),(n^{1/t})^t)$. Moreover we can wlog assume that $\mathbb{M} \models \forall n \in \mathrm{Log} : \mathit{Circuit}(h(1^{(n)}),n)$.[5] This is possible as $\mathbb{M} \models \forall n \in \mathrm{Log} : (n^{1/t})^t \leq n$ and so we can add to each circuit computed by $h^{\mathbb{M}}$ on $1^{(n)}$, $\mathit{add}(n) = n-(n^{1/t})^t$ many inputs not connected to any other gate. Then $\mathbb{M} \models \forall n \in \mathrm{Log} : \mathit{size}(h(1^{(n)})) \leq n^{k+\epsilon}$. Indeed, reasoning in $\mathbb{M}$: if $n \in \mathrm{Log}$ then $\mathit{size}(h(1^{(n)})) \leq \mathit{size}(f(1^{(n^{1/t})})) + \mathit{add}(n) \leq (n^{1/t})^{kt+s} + \mathit{add}(n) \leq ((n^{1/t})^t + \mathit{add}(n))^k n^{s/t} \leq n^{k+s/t} \leq n^{k+\epsilon}$. Hence $h^{\mathbb{M}}$ is a uniform sequence of circuits of size $\leq n^{k+\epsilon}$.

Using the assumption $\mathrm{ULB}_k^{g(\cdot,a)}(\epsilon)$ for $h^{\mathbb{M}}$, there is $d \in \mathrm{Log}M$ such that :

$$\mathbb{M} \models \forall n \in \mathrm{Log}(n > d \rightarrow \exists i(|i| = n \wedge g(i,a) \neq \mathit{eval}(h(1^{(n)}),i))).$$

This implies

$$\mathbb{M} \models \forall n \in \mathrm{Log}(n > d \rightarrow \exists i(|i| = n^t \wedge g(i,a) \neq \mathit{eval}(h(1^{(n^t)}),i)))$$

but then

$$\mathbb{M} \models \forall n \in \mathrm{Log}(n > d \rightarrow \exists i(|i| = n^t \wedge g(i,a) \neq \mathit{eval}(f(1^{(n)}),i)))$$

since $\mathbb{M} \models h(1^{(n^t)}) = f(1^{(n)})$ for any $n \in \mathrm{Log}\mathbb{M}$ and we are done. □

[5] as defined in the beginning of this chapter we use the convention that $n^{1/t}$ is the lower part of $n^{1/t}$ and so in general it is not true that $(n^{1/t})^t = n$

Now we can put everything together and state:

Theorem 5.5.3. *For any natural number $k \geq 1$ there is a binary function symbol $g \in \mathcal{L}_{\mathrm{PV}}$ such that*

$$\widetilde{\mathrm{PV}} + \exists x \neg \mathrm{UP}_k(g(\cdot, x))$$

is consistent.

Moreover in the context of the previous paragraphs, either there is a binary function symbol $g \in \mathcal{L}_{PV}$ such that $\mathbb{M} \models \neg \mathrm{UP}_k(g(\cdot, a))$ for some $a \in \mathbb{M}$ and $g^{\mathbb{M}}(\cdot, a)$ is a Boolean function. Or there is a binary function symbol $g \in \mathcal{L}_{PV}$ such that $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \neg \mathrm{UP}_k(g(\cdot, c_a^{\mathcal{U}}))$ for some $a \in \mathbb{M}$ and $g^{\mathcal{P}(\mathbb{M})/\mathcal{U}}(\cdot, c_a^{\mathcal{U}})$ is a Boolean function.

Proof. Let $\mathbb{M}$ be a countable Herbrand saturated model of $\widetilde{\mathrm{PV}}$ and let a natural number $k \geq 1$ be given. Since for any binary function symbol $g \in \mathcal{L}_{PV}$ we clearly have $\widetilde{\mathrm{PV}} \vdash \exists y \neg \mathrm{UP}'_{k+1}(g(\cdot, y)) \rightarrow \exists y \neg \mathrm{UP}'_k(g(\cdot, y))$ we can wlog assume $k > 3$. We can also assume there is no binary function symbol $f \in \mathcal{L}_{PV}$ and no $a \in \mathbb{M}$ such that $f^{\mathbb{M}}(\cdot, a)$ is a Boolean function and $\mathbb{M} \models \neg \mathrm{UP}'_k(f(\cdot, a))$ as otherwise we are done. Indeed, if $f^{\mathbb{M}}(\cdot, a)$ is a Boolean function then $\mathbb{M} \models \mathrm{UP}'_k(f(\cdot, a)) \leftrightarrow \mathrm{UP}_k(f(\cdot, a))$.

Then by the Corollary 5.5.1 there is a function symbol $g \in \mathcal{L}_{PV}$ and a parameter $a \in \mathbb{M}$ such that $\mathrm{ULB}_k^{g(\cdot,a)}(\epsilon)$ holds for some standard rational $\epsilon > 0$ (the function g we constructed is in fact unary but we can wlog assume it has a parameter a). But then by the previous lemma for every $s \in \mathbb{N}$ there is $t \in \mathbb{N}$ such that $\mathrm{RULB}_k^{g(\cdot,a)}(s, t)$ holds. Finally using the Theorem 5.3.6 we get that for any unbounded ultrafilter $\mathcal{U}$ on Ω, $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \neg \mathrm{UP}'_k(g(\cdot, c_a^{\mathcal{U}}))$.

The moreover part follows by the assumption made on $\mathbb{M}$ and since the $g^{\mathbb{M}}(\cdot, a)$ and thus $g^{\mathcal{P}(\mathbb{M})/\mathcal{U}}(\cdot, c_a^{\mathcal{U}})$ is a Boolean function. $\square$

Lemma 5.5.4. *Let $\mathbb{M}$ be a model of $\widetilde{\mathrm{PV}}$, $k \in \mathbb{N}$ and $g_2 \in \mathcal{L}_{PV}$ a binary function symbol. Assume $a \in \mathbb{M}$ is such that $\mathbb{M} \models \neg \mathrm{UP}_k(g_2(\cdot, a))$ and $g_2^{\mathbb{M}}(\cdot, a)$ is a Boolean function. Then there is a unary function symbol $g_1 \in \mathcal{L}_{PV}$ such that $\mathbb{M} \models \neg \mathrm{UP}_k(g_1)$.*

Proof. Let k be given. Since $g_2^{\mathbb{M}}(\cdot, a)$ is a Boolean function on $\mathbb{M}$, we have $\mathbb{M} \models \mathrm{UP}'_k(g_2(\cdot, a))$. We will show there is $g_1 \in \mathcal{L}_{PV}$ such that $g_1^{\mathbb{M}}$ is a Boolean function with $\mathbb{M} \models \mathrm{UP}'_k(g_1)$ and so $\mathbb{M} \models \mathrm{UP}_k(g_1)$.

Let $\langle \cdot, \cdot \rangle \in \mathcal{L}_{PV}$ be a binary function symbol corresponding to the following algorithm:

Given input x, a let $i_0 i_1 \dots i_{|x|-1}$ and $j_0 j_1 \dots j_{|a|-1}$ be the strings coded by x and a respectively. Output the code of the string $i_0 i_0 i_1 i_1 \dots i_{|x|-1} i_{|x|-1} 01 j_0 j_0 j_1 j_1 \dots j_{|a|-1} j_{|a|-1}$.

We claim that we can let $g_1 \in \mathcal{L}_{PV}$ be a unary function symbol such that

$$\widetilde{\mathrm{PV}} \vdash \forall x, a (g_1(\langle x, a \rangle) = g_2(x, a)) \wedge \forall y (g_1(y) = 0 \vee g_1(y) = 1).$$

We will need the following claim which is the only trick of this proof.

Claim 5.5.5. *Let $a, N \in \mathbb{M}$ be given and let C be a circuit in $\mathbb{M}$ with $2|N| + 2 + 2|a|$ many inputs such $\mathbb{M} \models \forall x (|x| = |N| \rightarrow eval(C, \langle x, a \rangle) = g_1(\langle x, a \rangle))$. Then there is a circuit C_a in $\mathbb{M}$ of size $\leq size^{\mathbb{M}}(C)$ with $|N|$ many inputs such that $\mathbb{M} \models \forall x (|x| = |N| \rightarrow eval(C_a, x) = g_2(x, a))$.*

Proof. We will argue in $\mathbb{M}$: Assume a, N and C satisfying the assumptions are given. Let $a_0a_1 \dots a_{|a|-1}$ be the string coded by a. By the definition of $\langle \cdot, \cdot, \rangle$ we have that for any x with $|x| = |N|$, $\langle x, a \rangle$ is of the form

$$i_0i_0i_1i_1 \dots i_{|N|-1}i_{|N|-1}01a_0a_0a_1a_1 \dots a_{|a|-1}a_{|a|-1}$$

where $i_m \in \{0, 1\}$ for any $m < |N|$. Now we construct the circuit C_a from C as follows:

- If q, p are the input gates of C that get evaluated by i_m, i_m for some $m < |N|$, then substitute q, p by one input gate q' which is connected to all wires of q and p.
- If q is the input gate of C that get evaluated by the constant 0 then substitute q by the constant 0 (and keep the wires of q).
- If q is the input gate of C that get evaluated by the constant 1 then substitute q by the constant 1 (and keep the wires of q).
- If q, p are the input gates of C that get evaluated by a_m, a_m for some $m < |a|$. Then substitute q, p by the constant 0 (or 1) which is connected to all wires of q and p if $a_m = 0$ (or if $a_m = 1$).

It is easy to see that $size(C_a) \leq size(C)$, the number of inputs of c_a is $|N|$ and for any x with $|x| = |N|$, $eval(C, \langle x, a \rangle) = eval(C_a, x)$. □

For the rest of this proof we fix a $d \in \mathbb{N}$ such that

$$\widetilde{\mathrm{PV}} \vdash N > 0 \to |e|(2|N| + 2 + 2|a|)^k \leq d(|e| + 1)(|a| + 1)^k|N|^k.$$

Now we will argue in $\mathbb{M}$ as follows: Assume that $\neg\mathrm{UP}'_k(g_2(\cdot, a))$ for some a and assume for a contradiction that $\mathrm{UP}'_k(g_1)$. Let e be such that

$$\forall \ell > 0 \exists C[\mathit{Circuit}(C, |\ell|) \wedge \mathit{size}(C) \leq |e| \cdot |\ell|^k \wedge \forall i(|\ell| = |i| \to (g_1(i) = \mathit{eval}(C, i)))]$$

then by $\neg\mathrm{UP}'_k(g_2(\cdot, a))$ (choosing for c an element of length $d(|e|+1)(|a|+1)^k$) there is some $N > 0$ such that

$$\forall C[\mathit{Circuit}(C, |N|) \wedge \mathit{size}(C) \leq d(|e| + 1)(|a| + 1)^k|N|^k \\ \to \exists i(|N| = |i| \wedge (g_2(i, a) \neq \mathit{eval}(C, i)))].$$

But then by

$$\forall \ell > 0 \exists C[\mathit{Circuit}(C, |\ell|) \wedge \mathit{size}(C) \leq |e| \cdot |\ell|^k \wedge \forall i(|\ell| = |i| \to (g_1(i) = \mathit{eval}(C, i)))]$$

(choosing for ℓ an element of length $2|N| + 2 + 2|a|$) there is a circuit D of size $\leq |e|(2|N| + 2 + 2|a|)^k$ with $2|N| + 2 + 2|a|$ many inputs such that

$$\forall i(2|N| + 2 + 2|a| = |i| \to (g_1(i) = \mathit{eval}(D, i))).$$

Now since for any x of length $|N|$, $|\langle x, a \rangle| = 2|N| + 2 + 2|a|$ we get that for any x of length $|N|$, $g_1(\langle x, a \rangle) = \mathit{eval}(D, \langle x, a \rangle)$. Then by the the claim above there is a circuit D_a with $|N|$ many inputs such that for any x of length $|N|$, $g_2(x, a) = \mathit{eval}(D_a, x)$. But since $\mathit{size}(D) \leq |e|(2|N|+2+2|a|)^k$ and $N > 0$ we get $\mathit{size}(D_a) \leq d(|e|+1)(|a|+1)^k|N|^k$ contradicting the choice of N.

So we have shown that $\mathbb{M} \models \neg\mathrm{UP}'_k(g_1)$ and since $g_1^{\mathbb{M}}$ is a Boolean function we get $\mathbb{M} \models \neg\mathrm{UP}_k(g_1)$.

□

Corollary 5.5.6. *For any natural number $k \geq 1$ there is a unary function symbol $h \in \mathcal{L}_{PV}$ such that*

$$\widetilde{\mathrm{PV}} + \neg \mathrm{UP}_k(h)$$

is consistent.

Proof. Apply the lemma above to the previous theorem. $\square$

Corollary 5.5.7. *Assume there is a countable Herbrand saturated model $\mathbb{M}$ of $\widetilde{\mathrm{PV}}$ and an unbounded ultrafilter $\mathcal{U}$ on $\Omega = \{1^{(n)} \mid n \in \mathrm{Log}\mathbb{M}\}$ with $\mathcal{P}(\mathbb{M})/\mathcal{U} \models \mathrm{S}^1_2(\mathrm{PV})$.*

Then for every natural number $k \geq 1$ there is a binary function symbol $g \in \mathcal{L}_{PV}$ and a unary function symbol $h \in \mathcal{L}_{PV}$ such that

$$\mathrm{S}^1_2(\mathrm{PV}) + \widetilde{\mathrm{PV}} + \exists x \neg \mathrm{UP}_k(g(\cdot, x)) \text{ and } \mathrm{S}^1_2(\mathrm{PV}) + \widetilde{\mathrm{PV}} + \neg \mathrm{UP}_k(h)$$

are consistent.

Proof. It is only necessary to note that any Herbrand saturated model of $\widetilde{\mathrm{PV}}$ is a model of $S^1_2(PV)$ (see [Avi02] or [Kra95, Theorem 7.6.3]). Thus by the assumption we can choose $\mathcal{U}$ such that $\mathbb{M}$ and $\mathcal{P}(\mathbb{M})/\mathcal{U}$ are models of $\mathrm{S}^1_2(\mathrm{PV}) + \widetilde{\mathrm{PV}}$. The rest follows by the same argument as in the proof of the last theorem using the lemma above. $\square$

We only note that it is not known to the author whether there is any model of $\widetilde{\mathrm{PV}}$ such that the assumption of the previous corollary without "Herbrand saturated" holds. We also note that it is not known to the author whether the assumption is strictly weaker than $\widetilde{\mathrm{PV}} \vdash \mathrm{S}^1_2(\mathrm{PV})$.

Chapter 6

Conclusion

In the Chapter 1 we gave a general definition of the (ultra)power constructions. In the Theorem 1.3.3 we showed that by (ultra)power constructions one can reach all models of the universal theory of the groundmodel for any countable groundmodel. We left open whether one can generalise this theorem for uncountable cardinalities and use it for reasoning about model-theoretic problems.

In the Chapter 2 we generalised the Construction B of [Gar15] which lead to a general technique of a construction of models of weak forms of induction. We augmented the construction of Michal Garlík by some sort of density arguments that gives more flexibility to this technique. However, a general construction leading to models of stronger form of induction or a construction of this type in more elegant framework is still missing.

In the Section 4.1 we gave a variation on the Theorem of Hirschfeld from [Hir75] for a rich class of universal theories which can prove basic facts about coding of (standard) finite tuples. The generalisation has direct implications for the theory $\widetilde{\mathrm{PV}}$. Although we did not use this theorem to derive some new results we believe this result gives a new inside in the behaviour of countable models of theories like $\widetilde{\mathrm{PV}}$.

Finally in the Section 5 we did an ultrapower construction to answer an open question from the article [KO17]. The Corollary 5.5.7 of this construction gives a conditional answer to another question from ibid. However, the strength of the condition given in this corollary is unknown and the author is sceptical in this direction. A question whether a similar construction can be made for other theories then $\mathrm{S}^1_2(\mathrm{PV})$ asked in the paper is open.

We believe that this thesis gave good examples of a use of the technique of power constructions in reasoning about complexity theory, bounded arithmetic and arithmetic and that more research in this direction could help to foster the current state of knowledge in bounded arithmetics and related areas.

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