Enumeration and analysis of models of planar maps via the bijective method

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under the supervision of Éric Fusy
and Gilles Schaeffer

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Outline

Introduction to planar maps

I. Maps with boundaries of prescribed length

I.1. Constellations

I.2. Quasi-constellations

II. Simple maps

II.1. Bijection between outertriangular simple maps and Eulerian triangulations

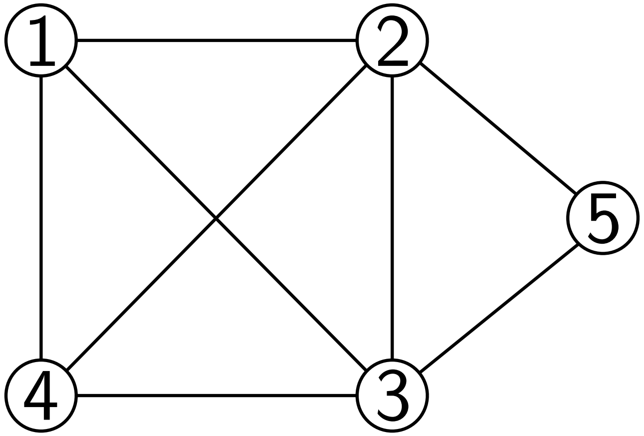
II.2. Enumeration and random sampling

II.3. Convergence of the distance-profile

Introduction to planar maps

Graphs and maps

Map = **Graph (vertices + edges)** + Drawing without crossing



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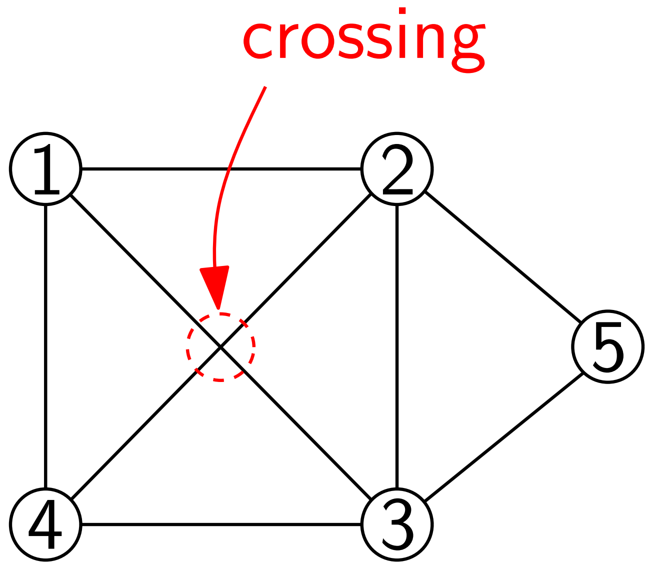

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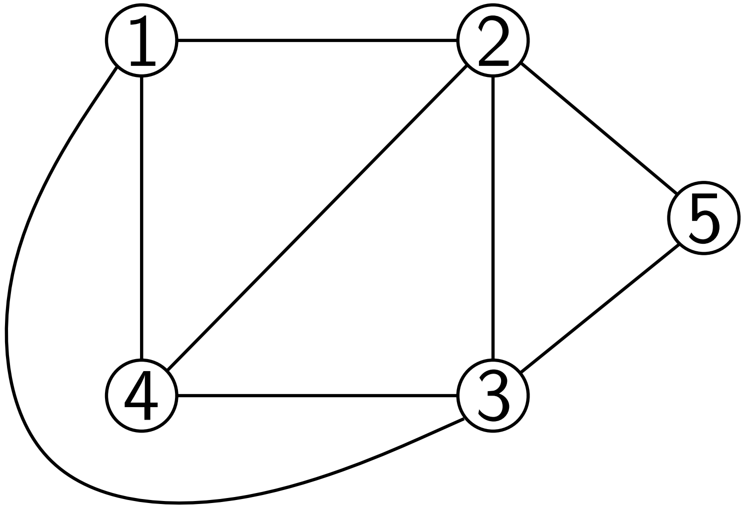
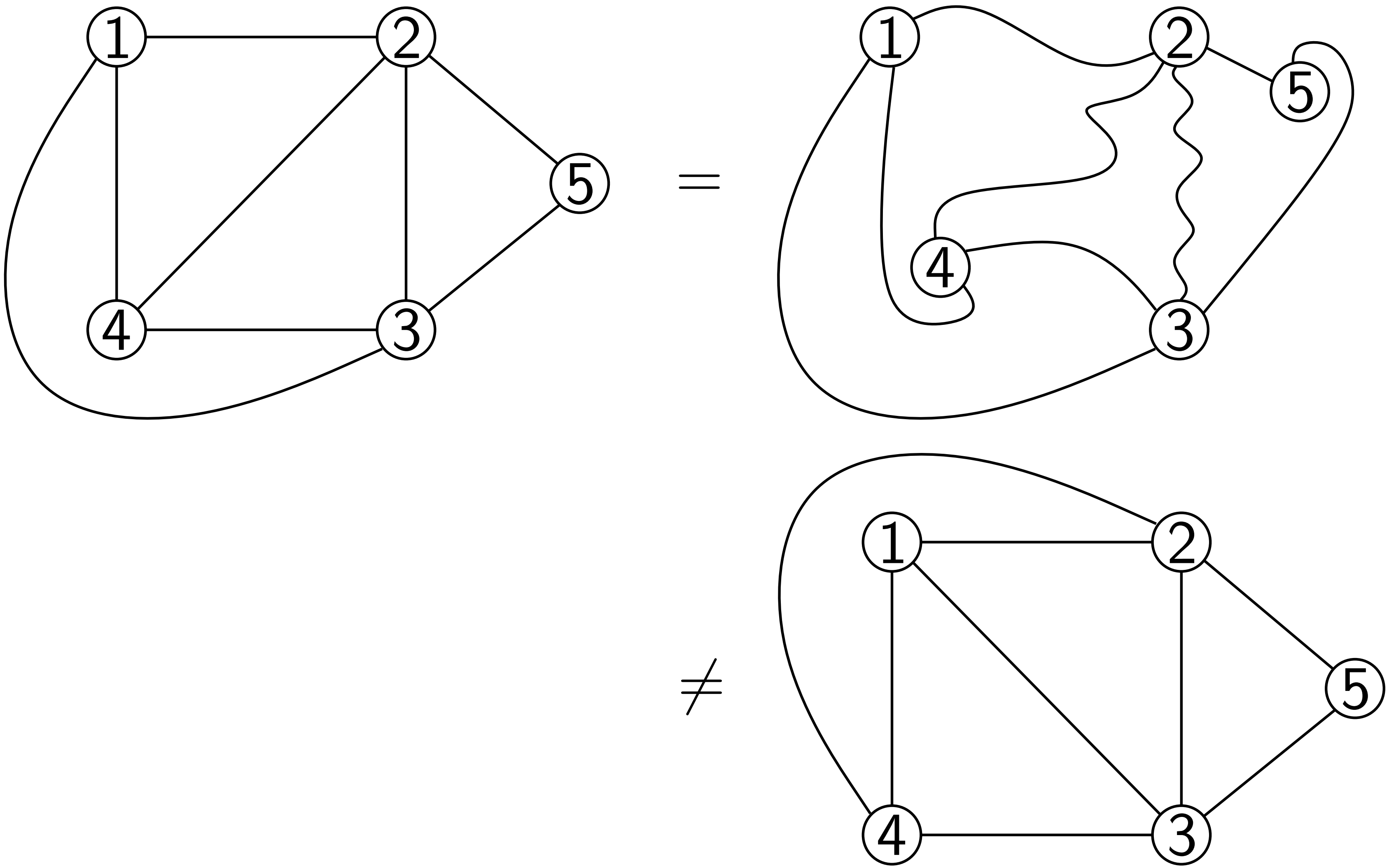


Graphs and maps

Map = Graph (vertices + edges) + Drawing without crossing
up to continuous deformation



Graphs and maps

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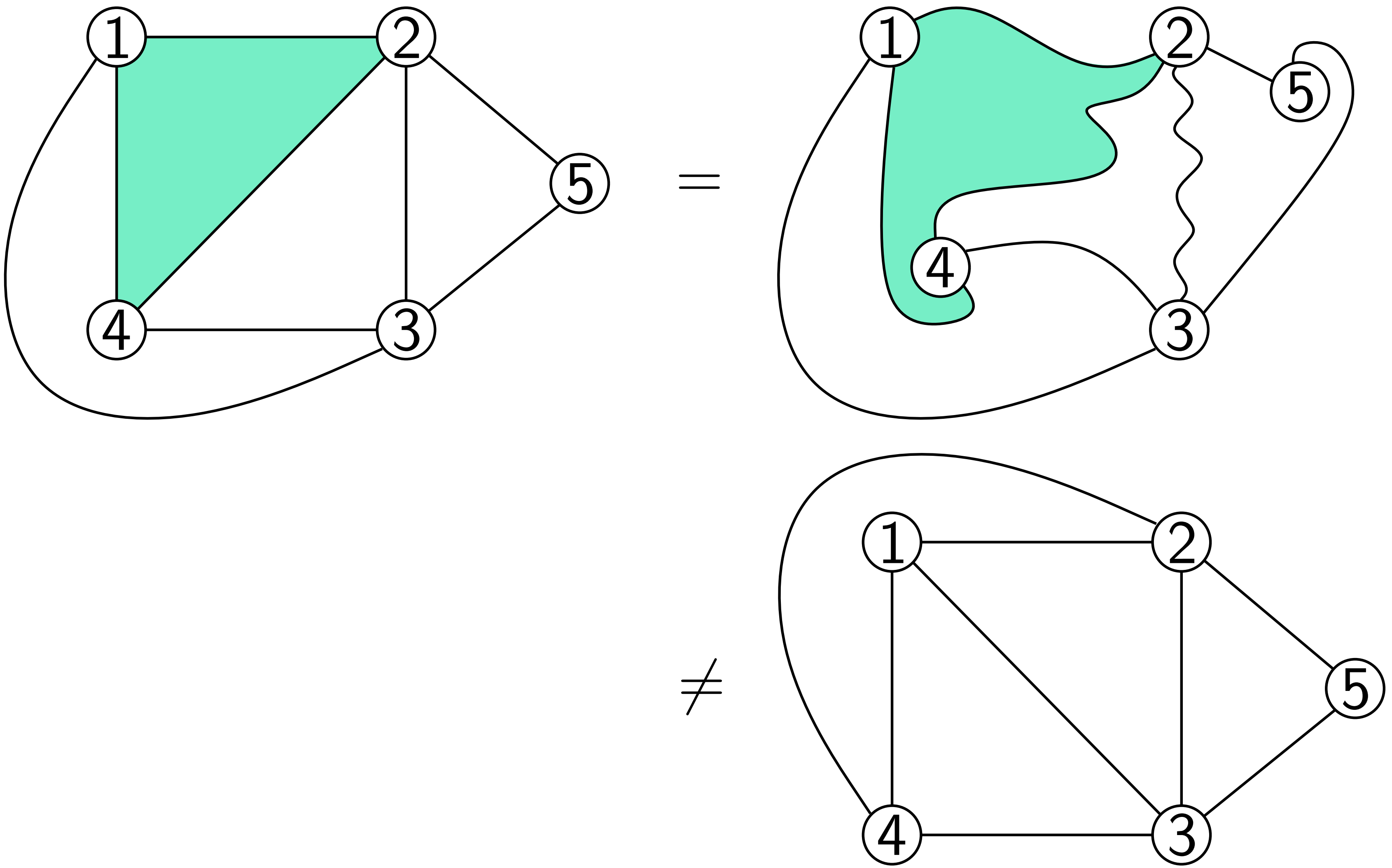
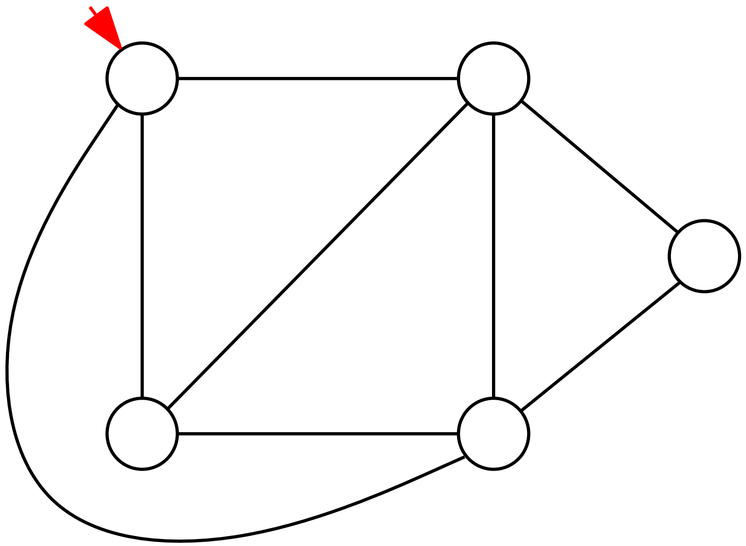


Graphs and maps

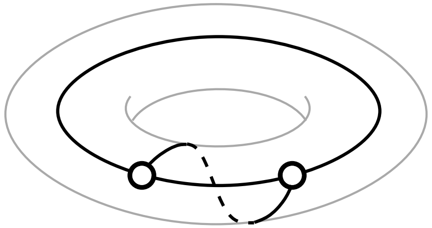
Map = Graph (vertices + edges) + Drawing without crossing
+ faces up to continuous deformation

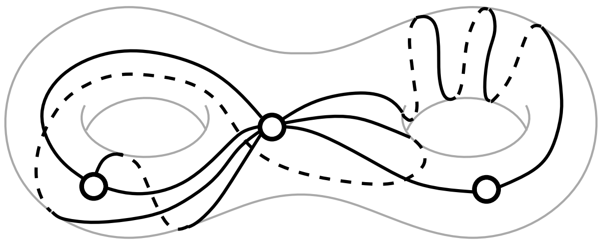
Rooted map = unlabelled map with a marked corner



Planar map = map drawn on the plane ($\simeq$ sphere, genus 0)

Map of genus $g = 1, 2, \ldots$





Enumerative methods

Loop equations, core extraction (Tutte, Eynard...)

Matrix integrals ('t Hooft...)

Bijections: Cori-Vauquelin-Schaeffer,
Bouttier-Di Francesco-Guitter

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↳ Planar maps $\simeq$ Decorated (blossoming/labelled) trees

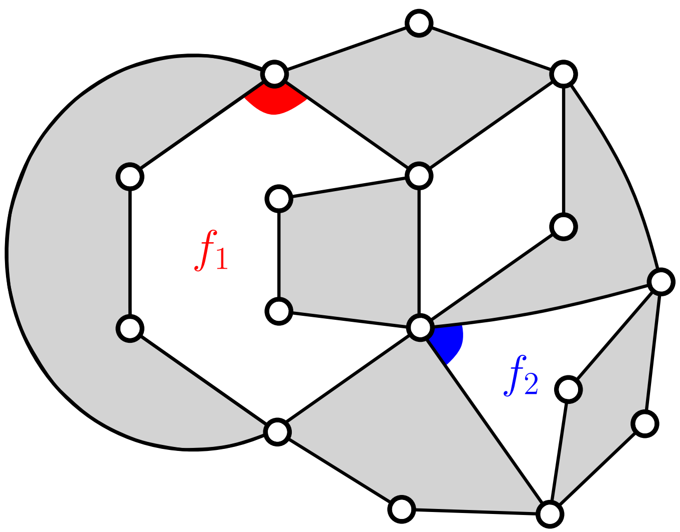
↳ easier to count, to sample

↳ preserve some metric properties

↳ known behaviour when size $\to \infty$

I. Maps with boundaries of prescribed length

Constellations and boundaries

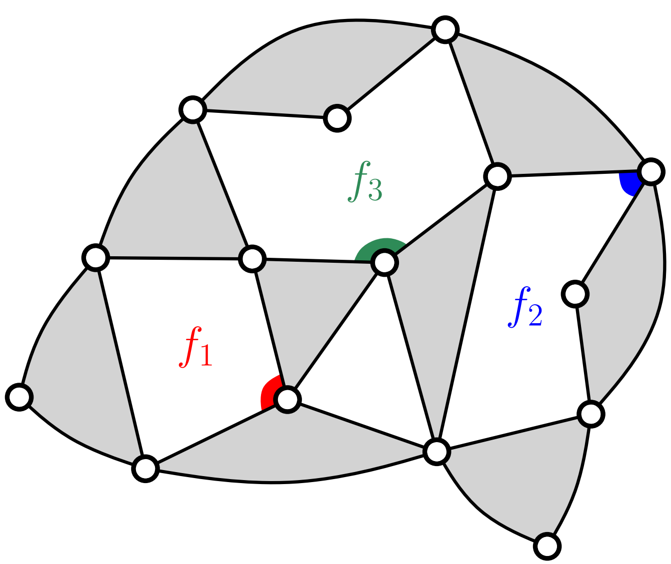


p-constellation $(p \geq 2) =$

planar map with:

- dark faces of length p
- light faces of length $pm, m \geq 0$

boundary $=$ light face with a marked corner



quasi-p-constellation $(p \geq 2) =$

2 boundaries of length $\not\equiv 0[p]$

RQ : $pm + d$ et $pm' - d, 0 < d < p$

Constellations and boundaries

$\mathcal{G}_{pa_1,\dots,pa_r}$ = p-constellations with r boundaries of respective lengths $pa_1, \dots, pa_r$

$\mathcal{G}_{pa_1+d,pa_2-d,pa_3\dots,pa_r}$ = quasi-p-constellations with r boundaries of resp. lengths $pa_1 + d, pa_2 - d, pa_3 \dots, pa_r$

Question :

Count $\mathcal{G}_{pa_1,\dots,pa_r}$ and $\mathcal{G}_{pa_1+d,pa_2-d,pa_3\dots,pa_r}$, for any $r \geq 1$

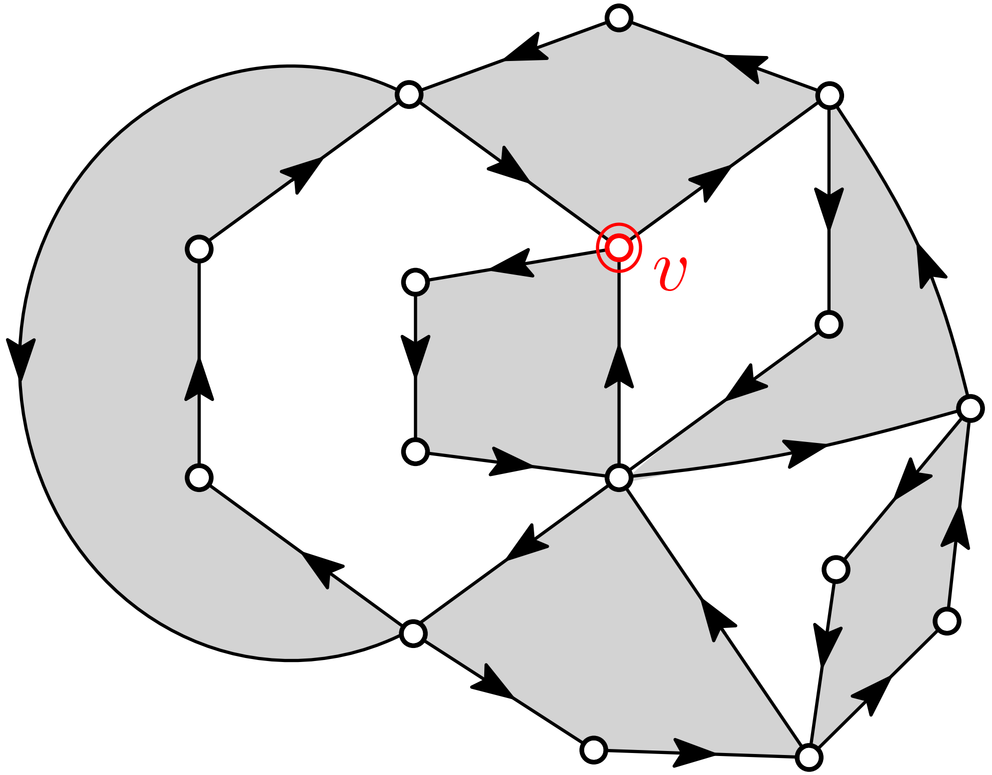
Related works:

$\rightarrow$ Slicing formula (Tutte)

$\rightarrow$ Eynard for $(p = 2, r = 2, 3)$ + general method

$\rightarrow$ Bouttier and Guitter: irreducible maps with boundaries

From p-constellations to p-mobiles [BDG' 04]



4-constellation with a pointed vertex v

$+$ natural orientation

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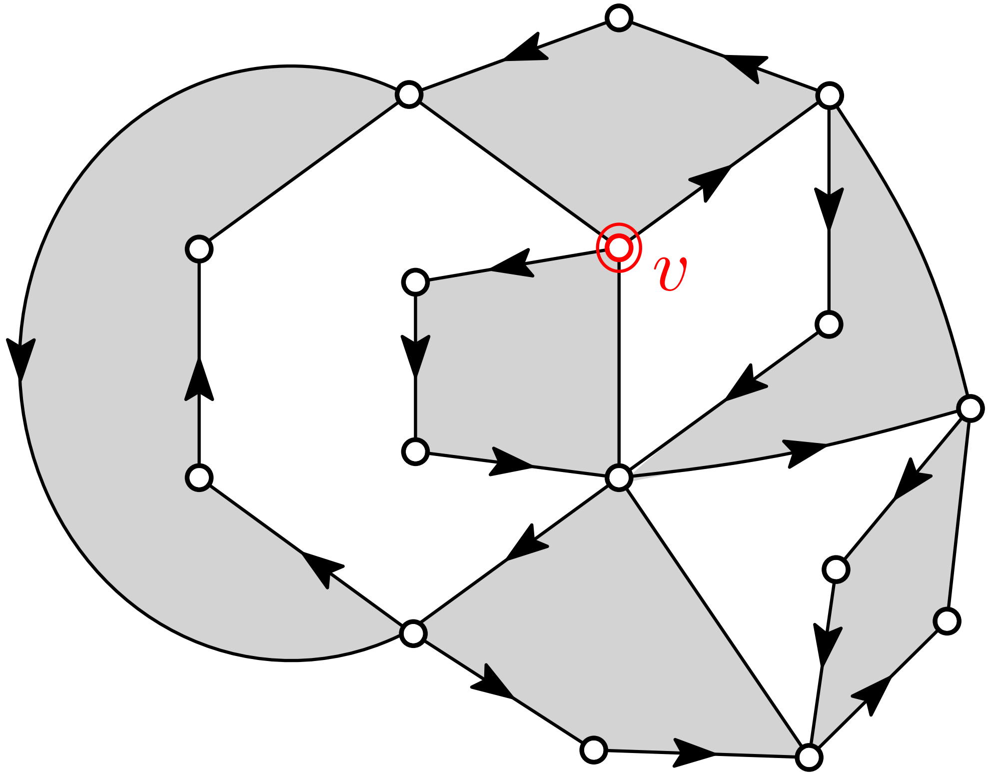

4-constellation with a pointed vertex v

$+$ geodesic orientation from v

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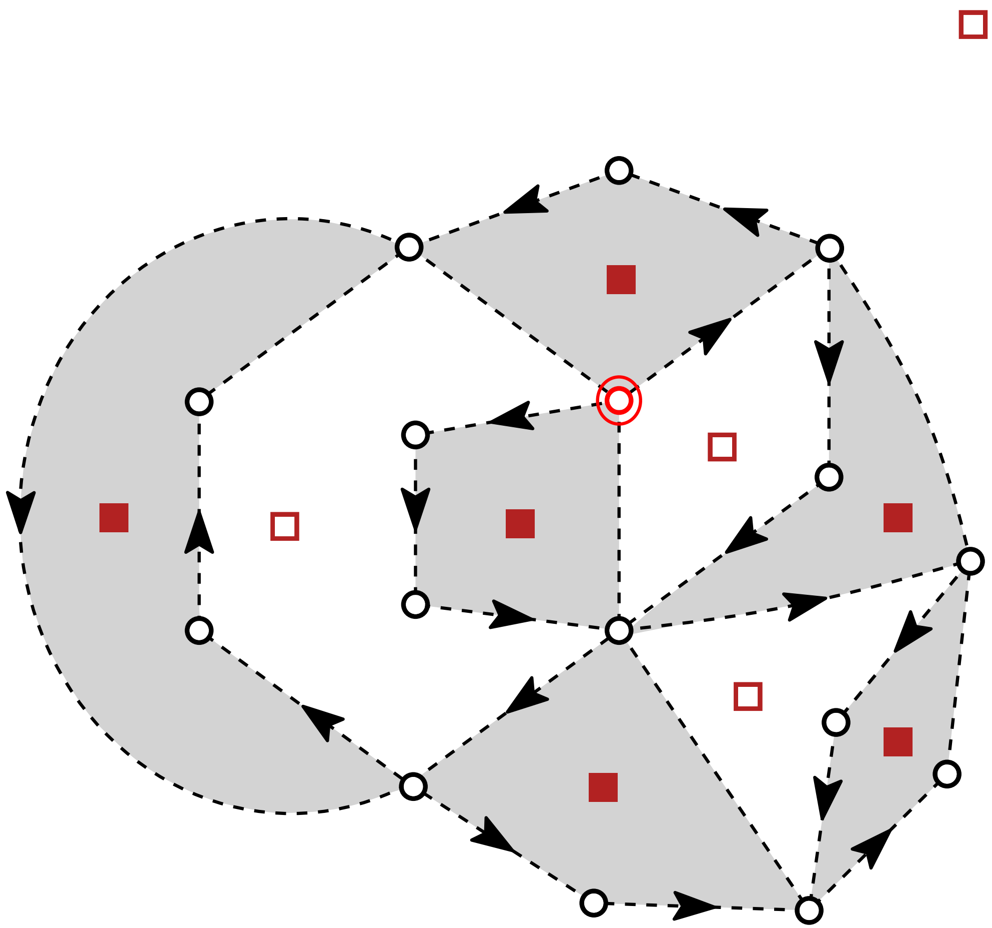
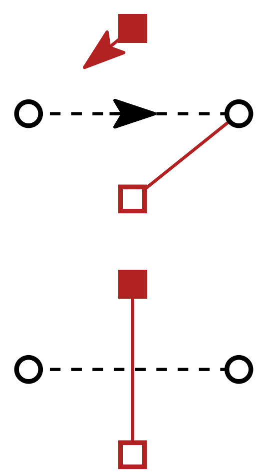


Local rule:



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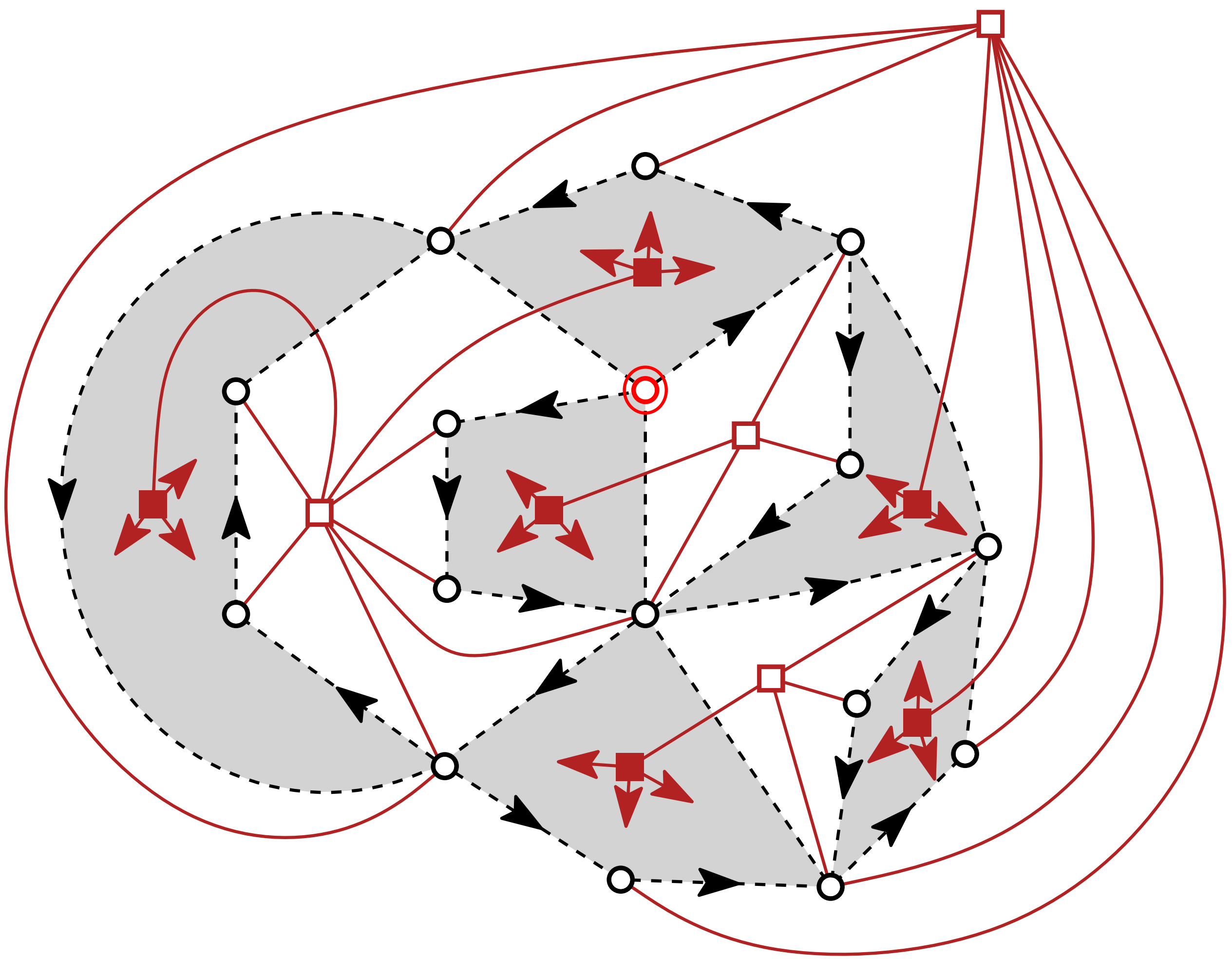

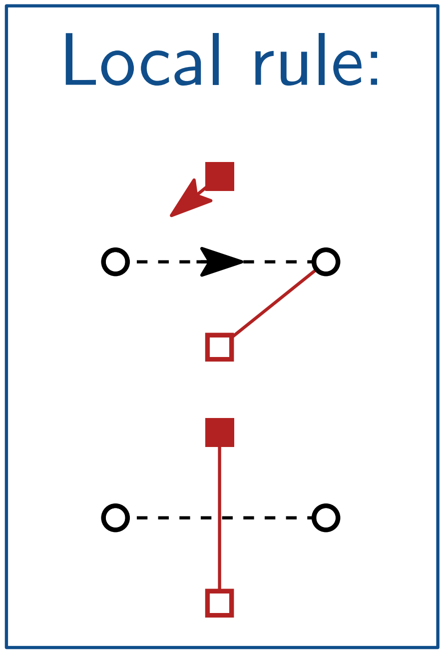

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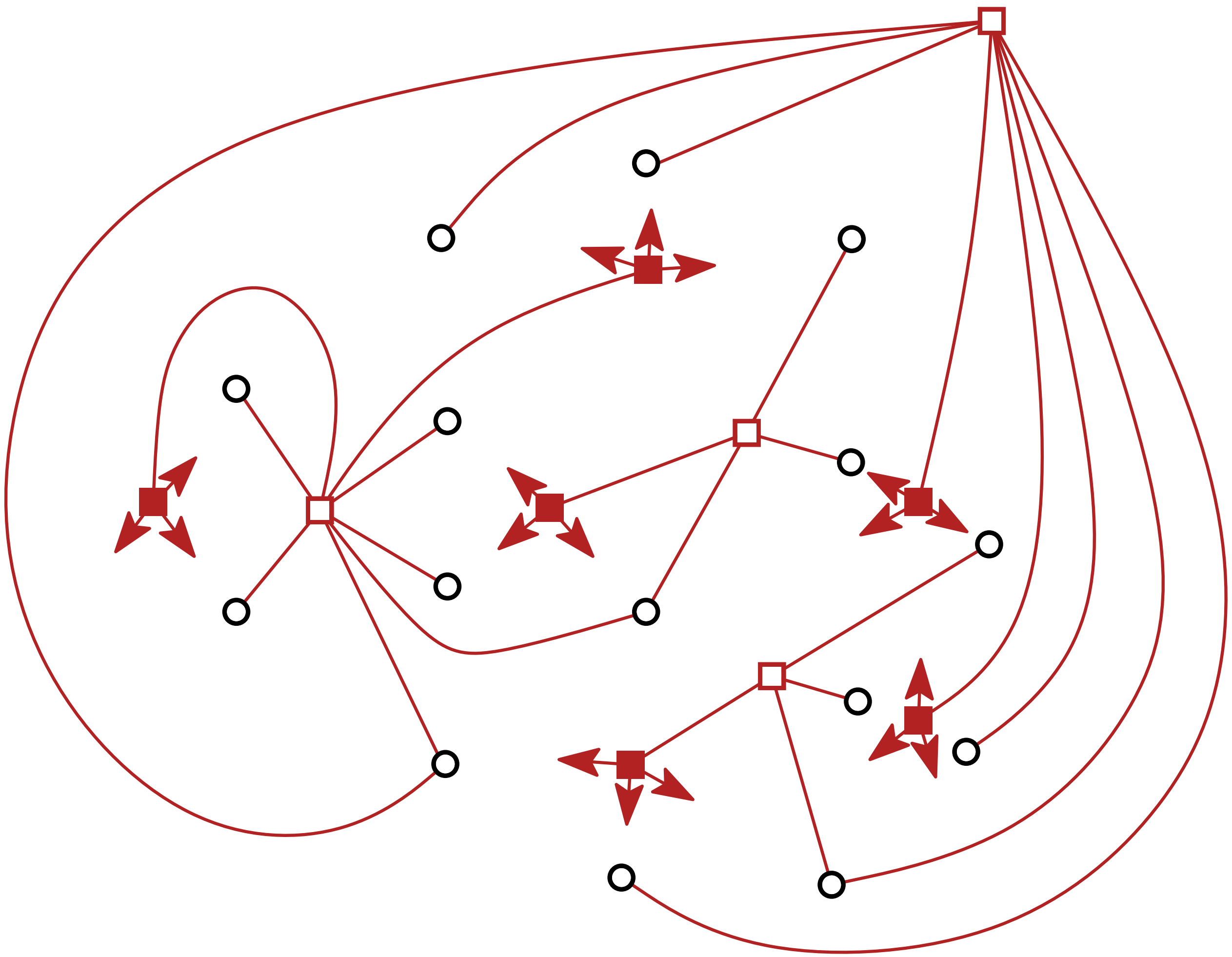
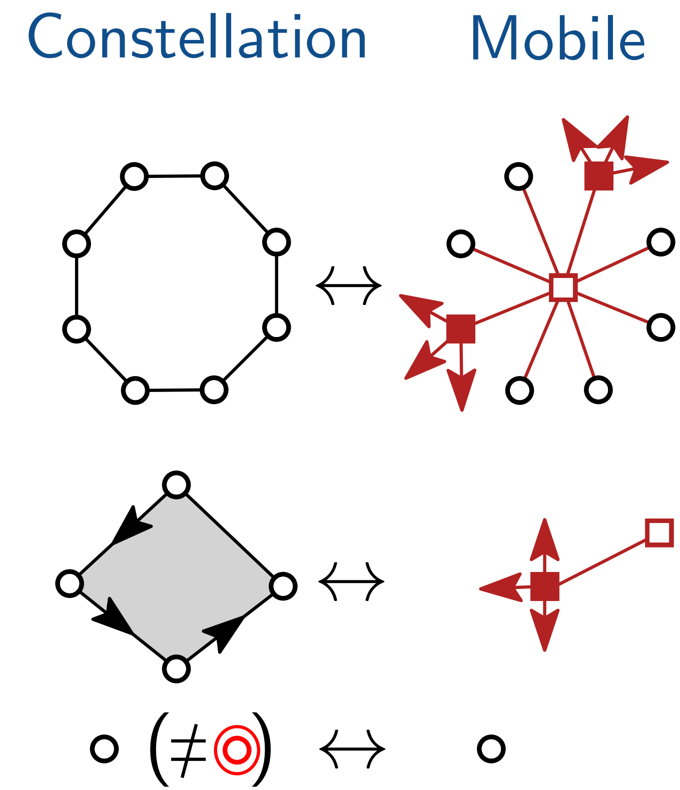


4-mobile



From p-constellations to p-mobiles [BDG' 04]

Let $R_p \equiv R_p(t; x_1, x_2, \ldots)$, the generating series for p-mobiles rooted on the corner of a round vertex, given by:

$$R_p = t + \sum_{i \geq 1} x_i \binom{pi-1}{i} R_p^{(p-1)i}$$

Let $G_{pa} \equiv G_{pa}(t; x_1, \ldots)$, the generating series for $\mathcal{G}_{pa}$.

Proposition : $\frac{d}{dt} G_{pa} = \binom{pa}{a} R_p^{(p-1)a}$

Proof:

1. Point a vertex in the constellation
2. Apply bijection
3. Decompose p-mobile along the light square mapped to boundary

Formula for p-constellations with boundaries

Let $G_{pa_1,\ldots,pa_r} \equiv G_{pa_1,\ldots,pa_r}(t; x_1, x_2, \ldots)$, the generating series for $\mathcal{G}_{pa_1,\ldots,pa_r}$.

Proposition : $s = (p-1)\sum_{i=1}^{r} a_i$,

$$\frac{d}{dt} G_{pa_1,\ldots,pa_r} = \frac{1}{s}\left(\prod_{i=1}^{r} pa_i \binom{pa_i - 1}{a_i}\right) \frac{d^{r-1}}{dt^{r-1}} R_p^s$$

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$\Rightarrow$ Generalizes Eynard's formula for the bipartite case ($p = 2$) with 2 and 3 boundaries

$\Rightarrow$ Recover Tutte's slicing formula: soit $A_{pa_1,\dots,pa_r}$ the number of p-constellations with r numbered light faces of length $pa_1, \dots, pa_r$, with a marked corner in each:

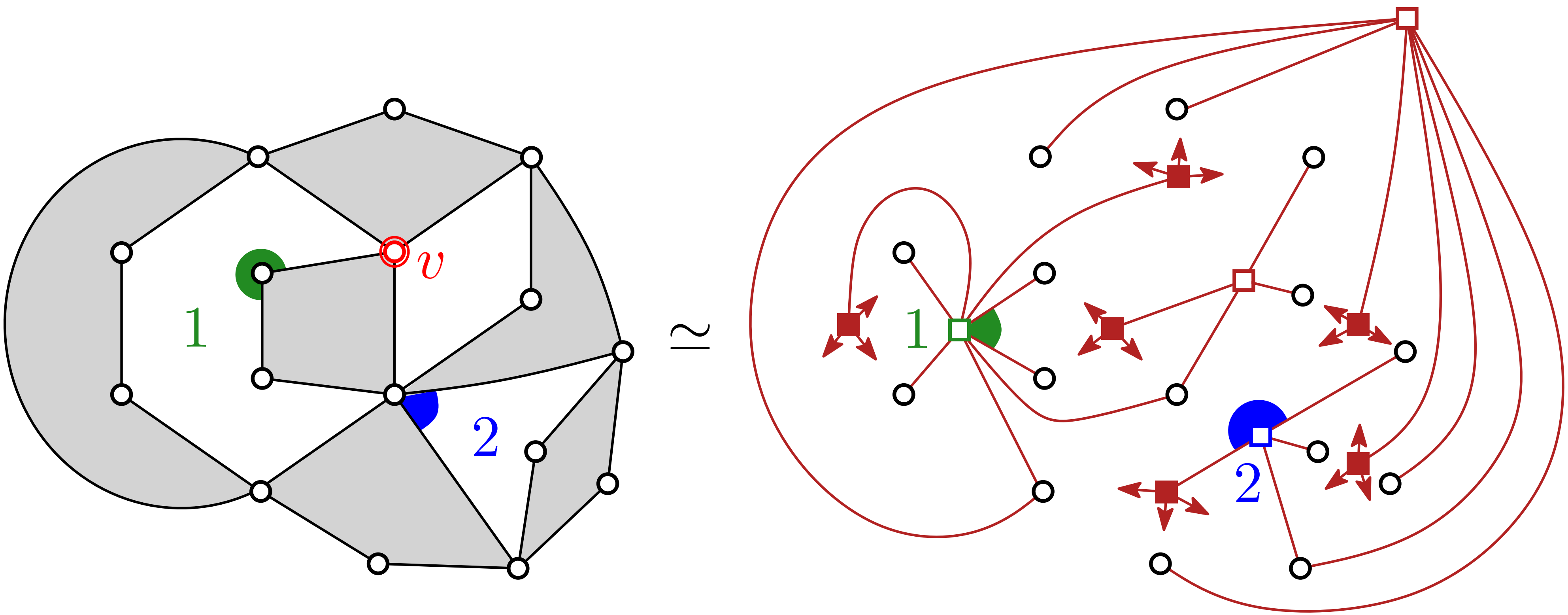
$$A_{pa_1,\dots,pa_r} = \frac{(e-d-1)!}{v!} \prod_{i=1}^{r} pa_i \binom{pa_i - 1}{a_i}$$

$e = \#$ edges
$v = \#$ vertices
$d = \#$ dark faces

$\rightarrow$ Set $x_i = 0$, $\forall i$.

Formula for p-constellations with boundaries

$$\frac{d}{dt} G_{pa_1,\ldots,pa_r} = \left(\prod_{i=1}^{r} pa_i \binom{pa_i - 1}{a_i} \right) \frac{1}{s} \frac{d^{r-1}}{dt^{r-1}} R_p^s, \qquad s = (p-1) \sum_{i=1}^{r} a_i$$



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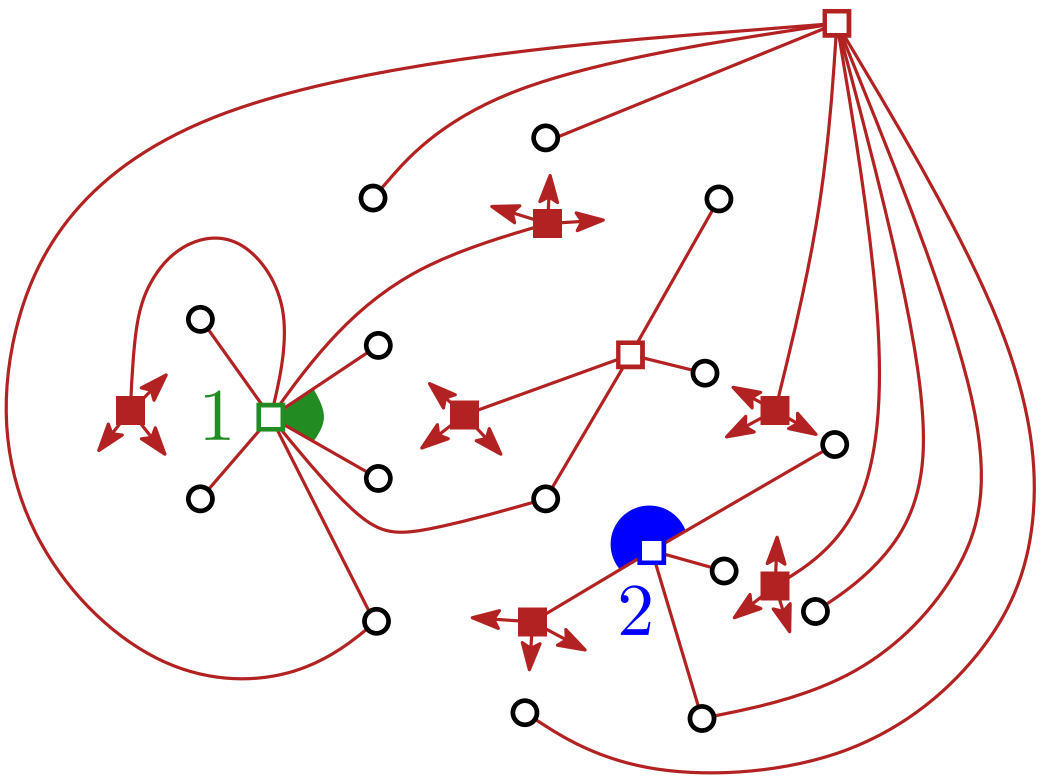
marked corner and
adjacent dark squares
around each boudary



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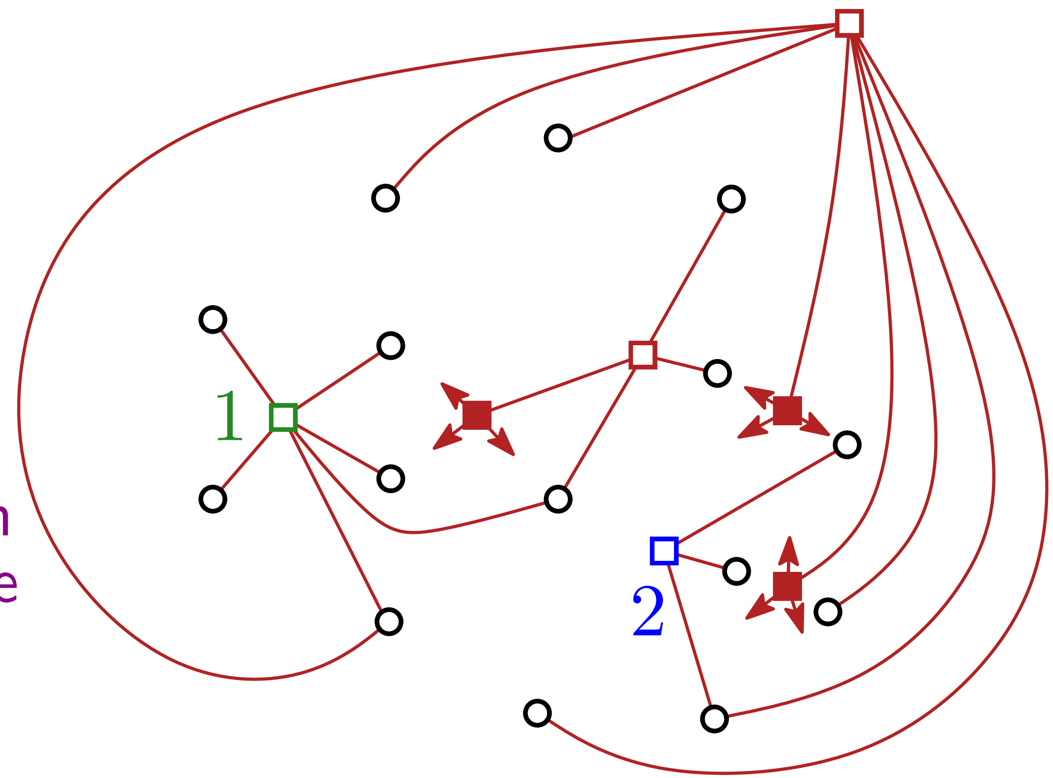
marked corner and adjacent dark squares around each boudary

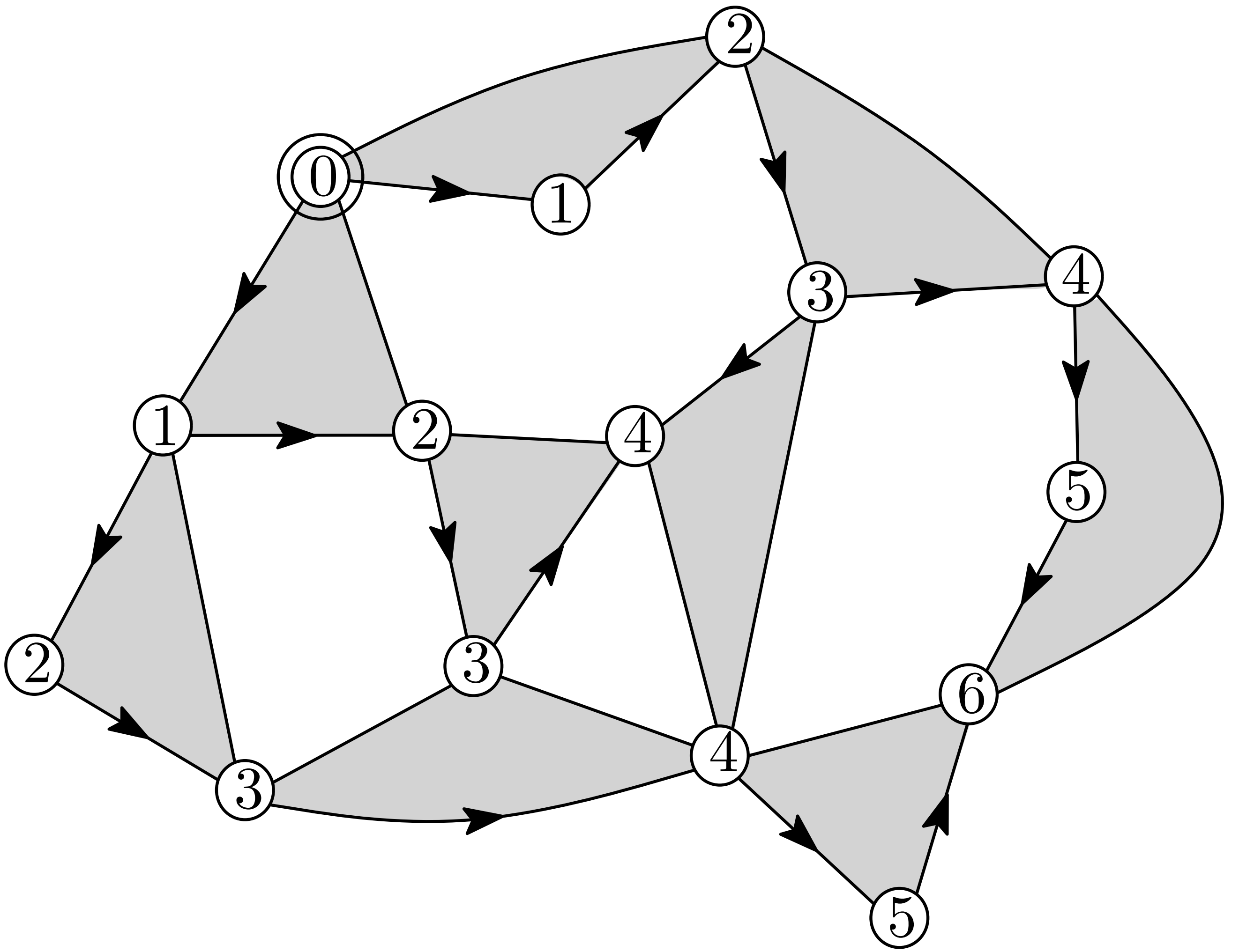
"pruned" p-Mobile



- r numbered light squares with no adjacent squares, of degree $(p-1)a_1, \ldots, (p-1)a_r$
- rooted on the corner of one marked square

$\rightarrow$ obtained by aggregating rooted p-mobiles on marked squares

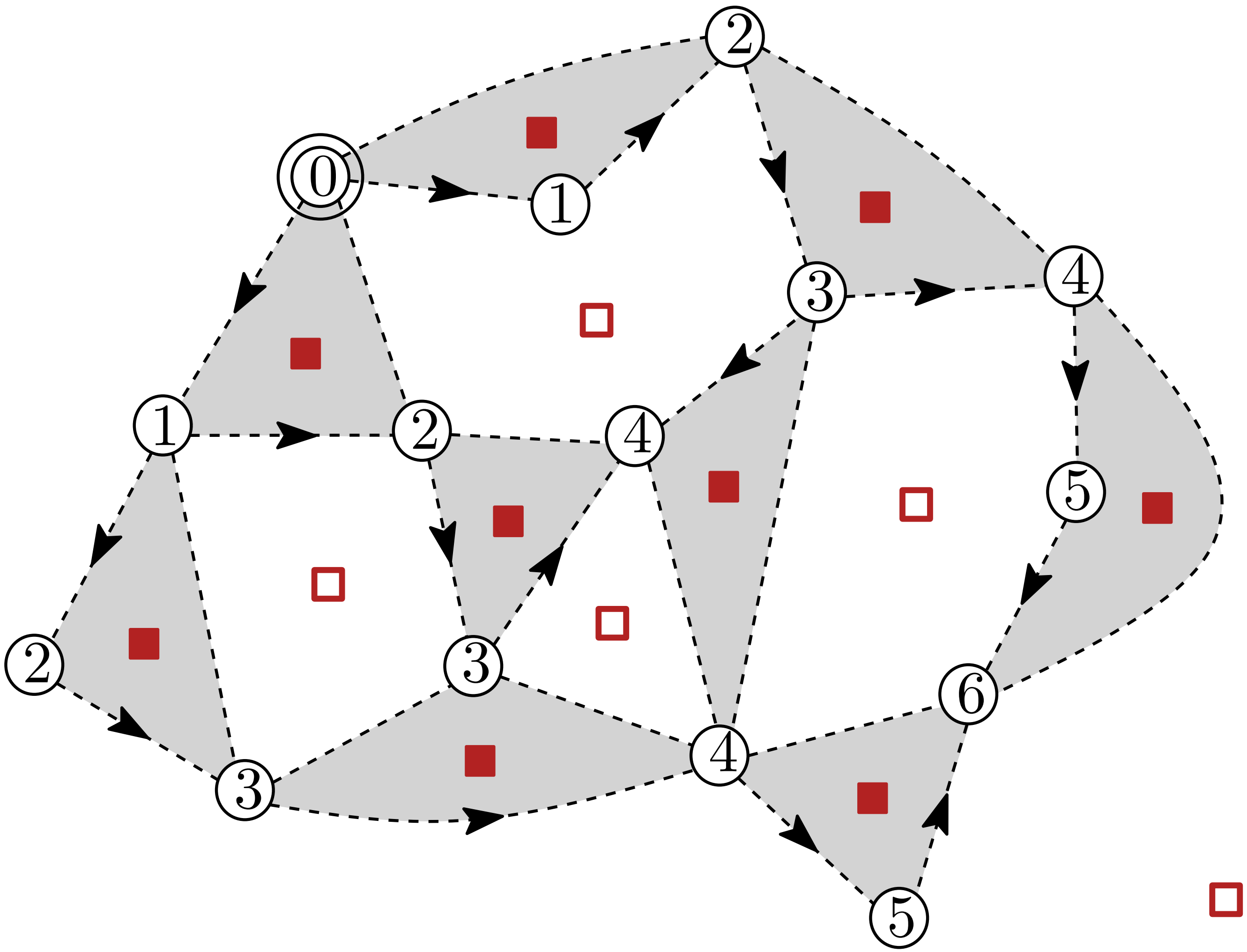
From quasi-p-constellations to quasi-p-mobiles

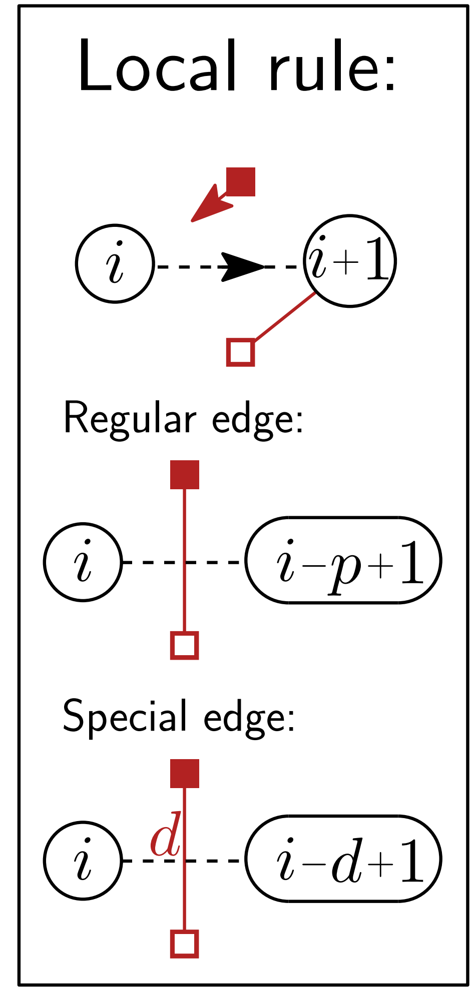


Quasi-3-constellation with a pointed vertex

$+$ geodesic orientation **and labelling**

From quasi-p-constellations to quasi-p-mobiles





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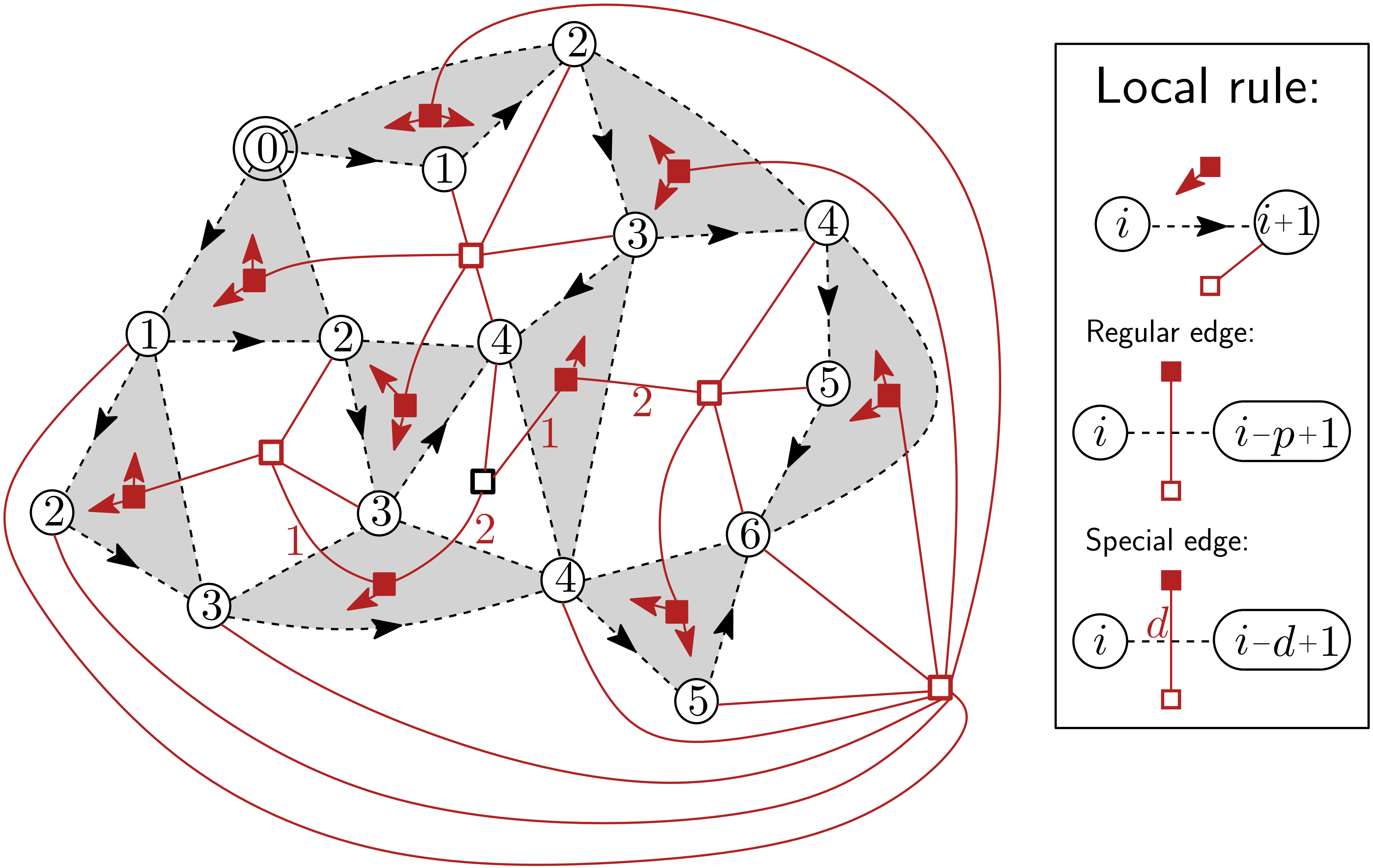

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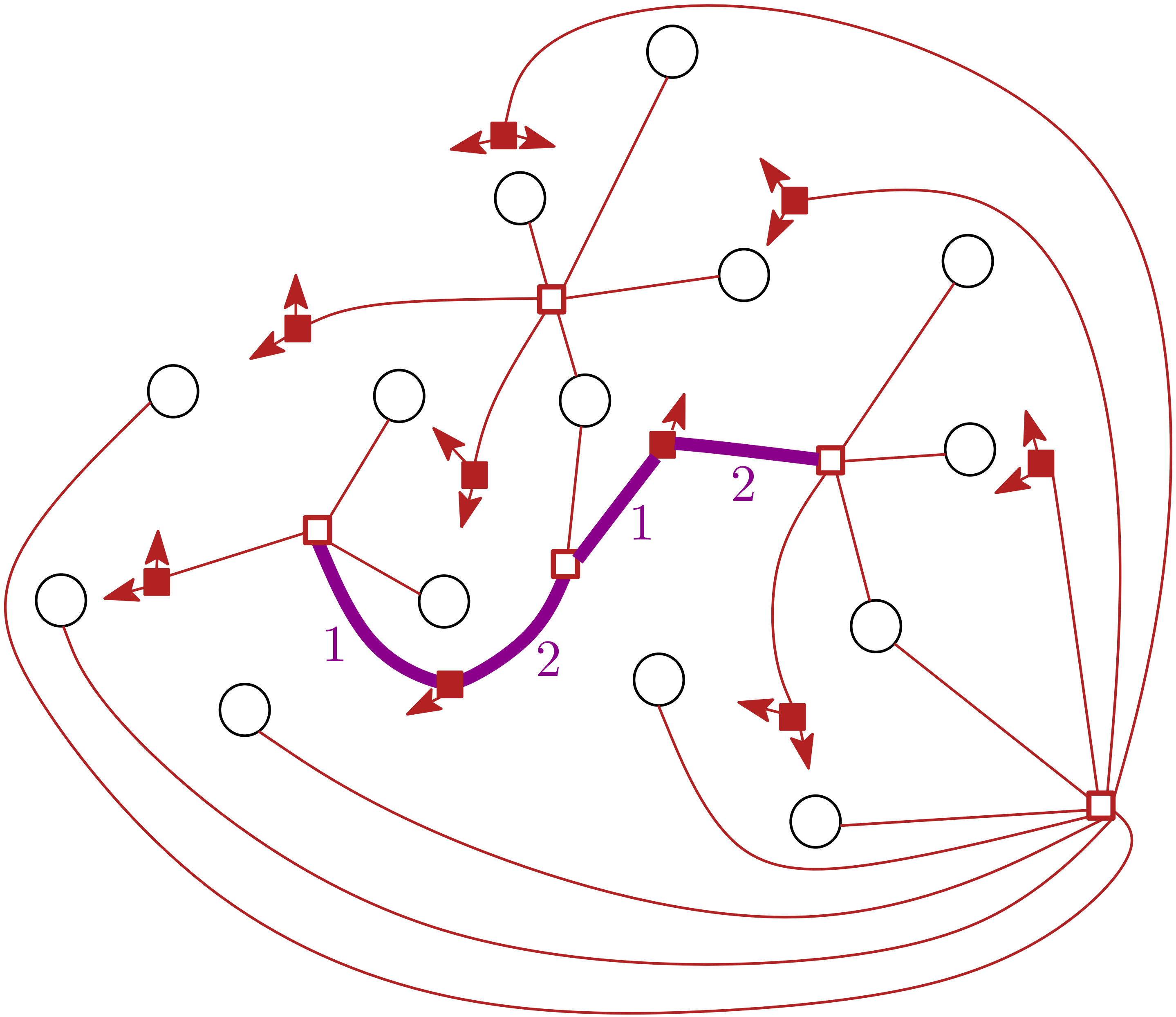

Alternating path (d,p-d) formed by the special edges

quasi-3-mobile

Formula for the quasi-p-constellations

Let $G^{(p)}_{\ell_1,\dots,\ell_r} \equiv G^{(p)}_{\ell_1,\dots,\ell_r}(t;x_1,\dots)$, the generating series for $\mathcal{G}^{(p)}_{\ell_1,\dots,\ell_r}$, where two $\ell_i \not\equiv 0[p]$.

Proposition : Soit $s = \frac{p-1}{p}(\ell_1 + \ldots + \ell_r)$,

et $\alpha(\ell) = \frac{\ell!}{\lfloor \ell/p \rfloor ! (\ell - \lfloor \ell/p \rfloor - 1)!}$,

$$\frac{d}{dt} G^{(p)}_{\ell_1,\dots,\ell_r} = \frac{p-1}{s} \left(\prod_{i=1}^{r} \alpha(\ell_i) \right) \frac{d^{r-1}}{dt^{r-1}} R_p^s$$

Proof:

Idea (Cori, Chapuy) :

$\mathcal{R}_{pa_1+d,pa_2-d,\ell_3,\dots,\ell_r} \simeq (p-1)\mathcal{R}_{pa_1,pa_2,\ell_3,\dots,\ell_r}$

Transfer degree along the path formed by the special edges

Can we go further?

Tutte slicing formula only holds for at most 2 non-regular faces

Coefficients do not seem to behave nicely (large prime factors)

Why?

$\rightarrow$ non-regular edges form a forest

$\rightarrow$ things get nasty when there are more than two

II. Simple maps

Definitions

A *simple planar map* (also called a plane graph) is a planar map with no loops nor multiple edges, i.e., an embedded (simple) planar graph

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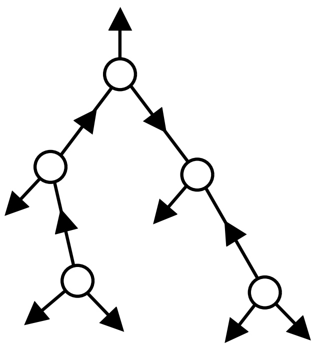
The series $M \equiv M(t)$ of rooted simple maps (by edges) is given by

$$M = \frac{(1+2u)^2}{(1+u)^3},$$



with $u = t(1+2u)^2$ the series of rooted oriented binary trees

$$M(t) = 1 + t + 2t^2 + 6t^3 + 23t^4 + 103t^5 + 512t^6 + 2740t^7 + 15485t^8 + \cdots$$

Rk: appears in Sloane, #1342-avoiding permutations of size n [Bona'97]

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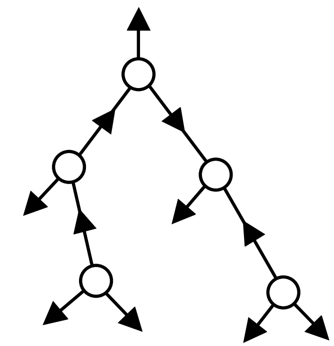

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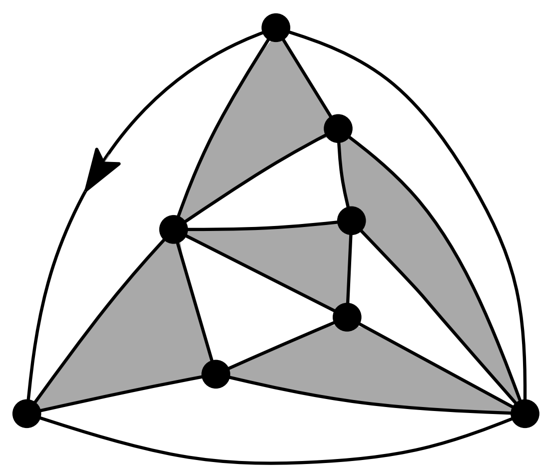
Rk: the GF $E(t)$ of rooted eulerian triangulations is expressed in terms of u

$E = 1 + u - u^2$, with $u = t(1+2u)^2$

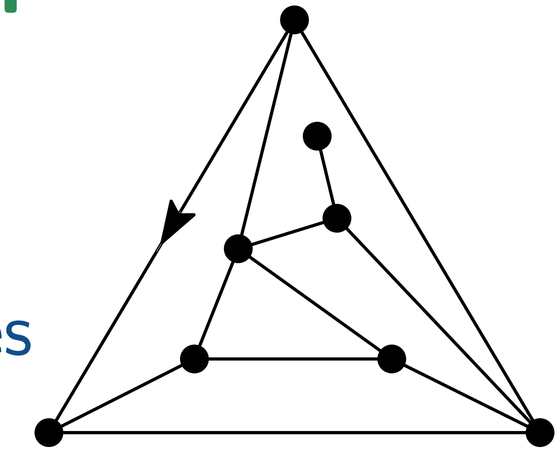
(also $E = 1 + \sum_{n\geq 1} 3 \cdot 2^{n-1} \frac{(2n)!}{n!(n+2)!} t^n$)

$$\Rightarrow M(t) = \frac{1}{1 - tE(t)}$$



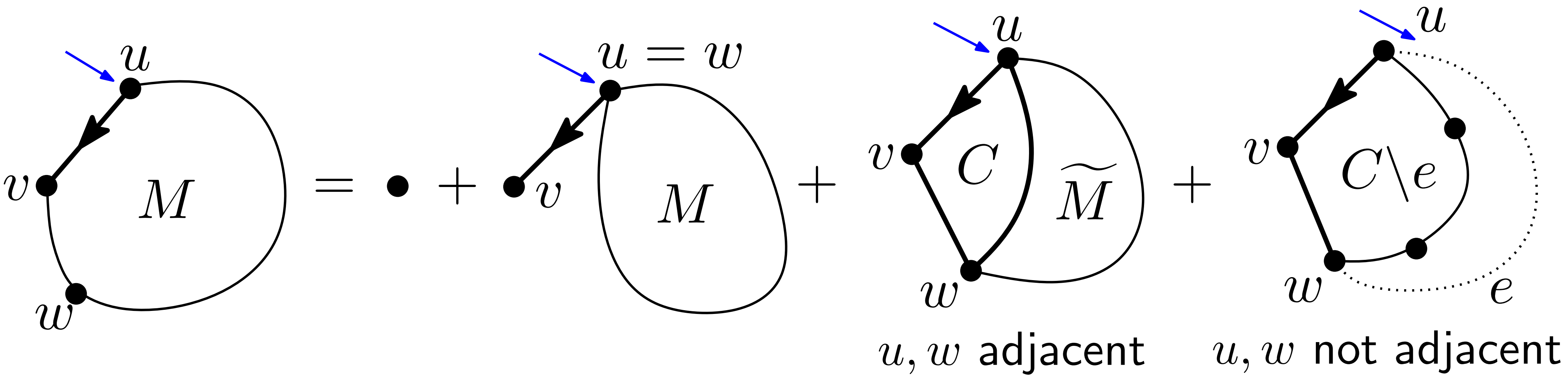
Focus on outertriangular simple maps

Consider subfamily $\mathcal{C}$ of **outertriangular** simple maps



$C(z)$ generating series for rooted ones according to edges

- Decomposition of $\mathcal{M} = 1 + \widetilde{\mathcal{M}}$ in terms of $\mathcal{C}$

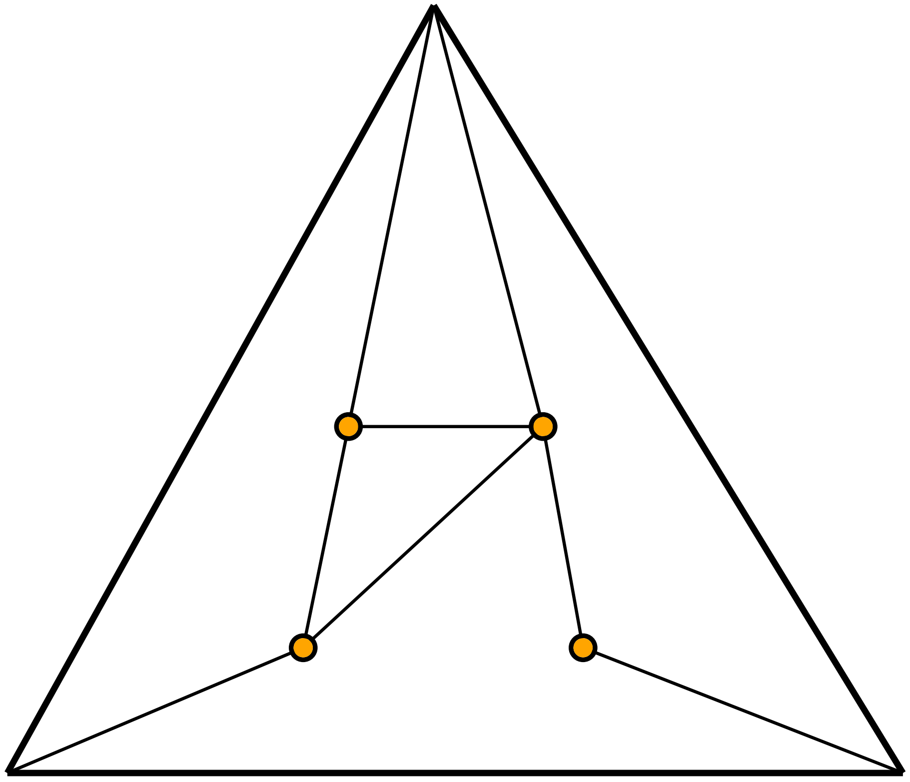


$$\Rightarrow M(t) = 1 + tM(t) + \tfrac{1}{t}C(t)(M(t)-1) + \tfrac{1}{t}C(t)$$

$$\Rightarrow \boxed{M(t) = \frac{1}{1 - t(1 + C(t)/t^2)}} \Rightarrow$$

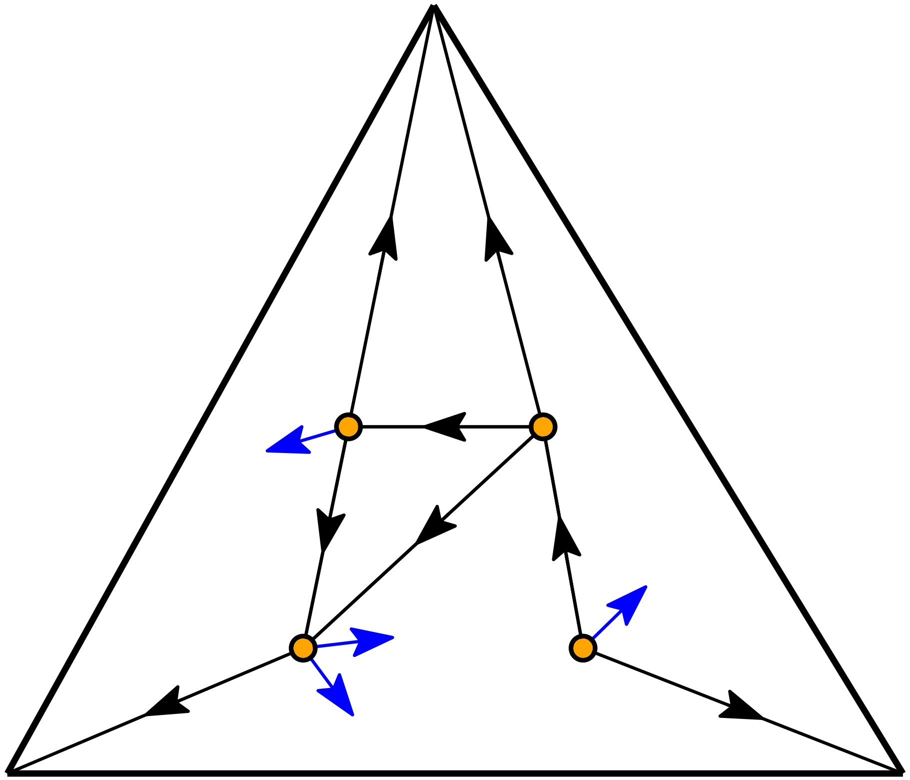
Have to prove bijectively that $E(t) = 1 + C(t)/t^2$

Orientation for outertriangular simple maps



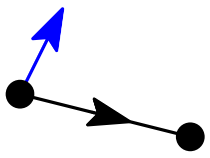
Orientation for outertriangular simple maps

[Bernardi, Fusy]



Canonical orientation:

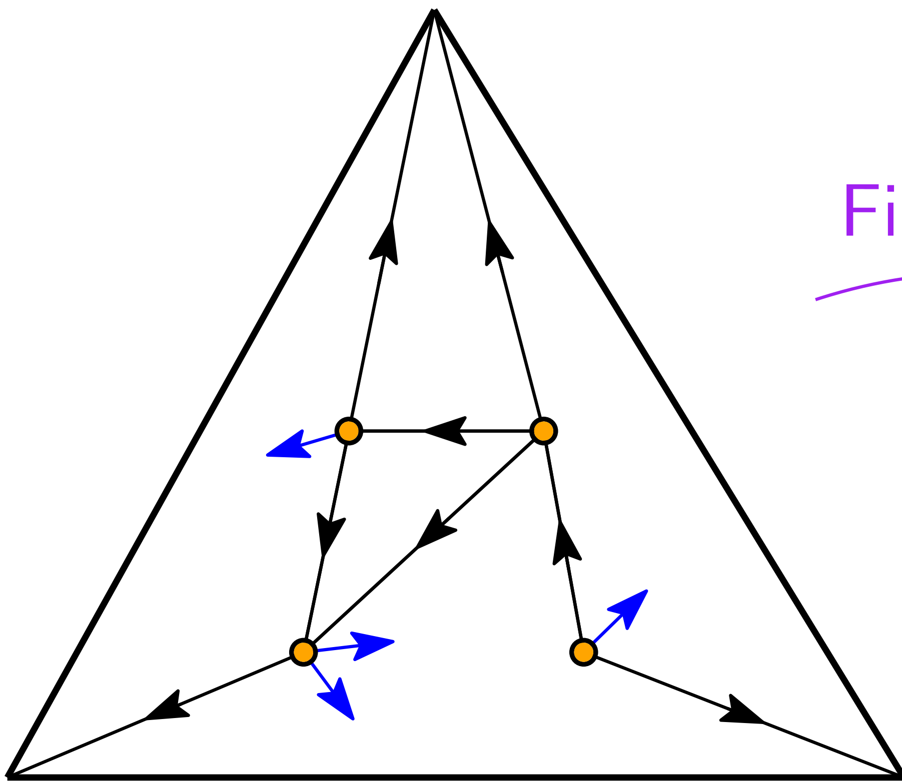
3 outgoing edges

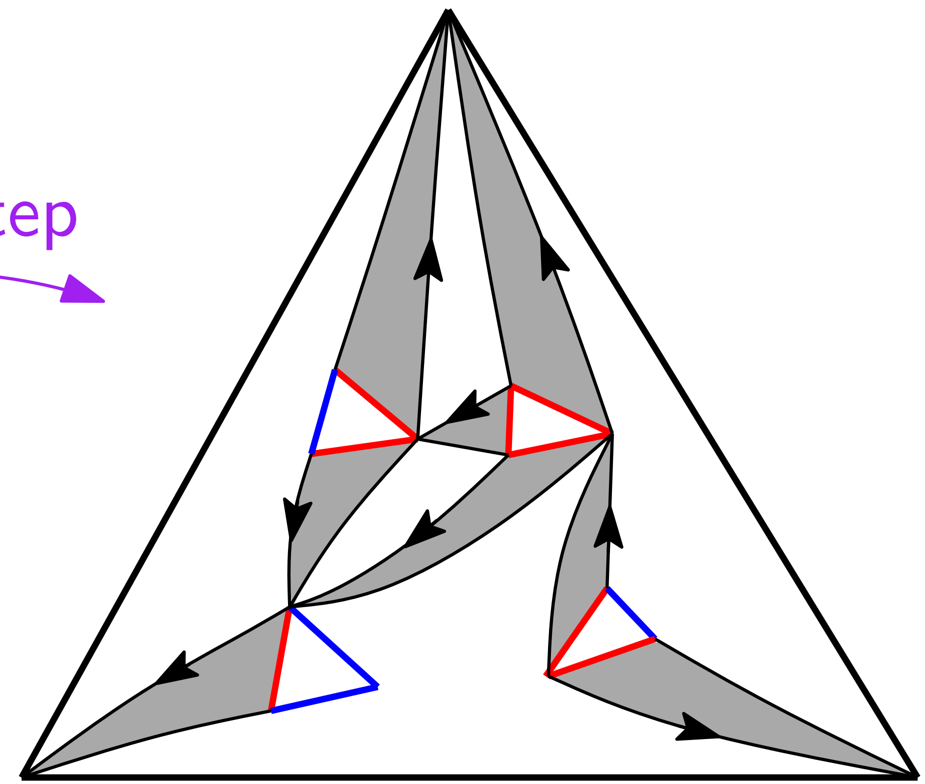
outer-accessibility + 

no clockwise circuit

face of degree $d + 3$: d arrows

Bijection for outertriangular simple maps





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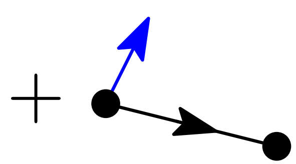

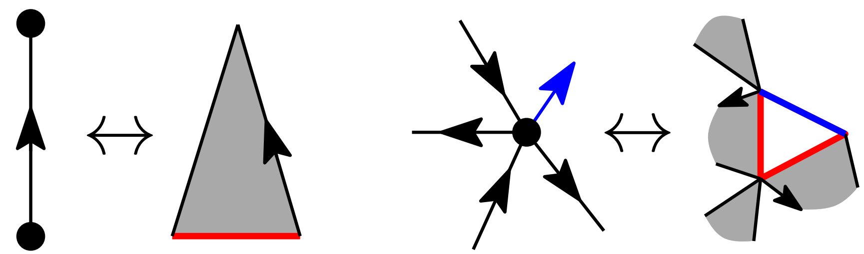

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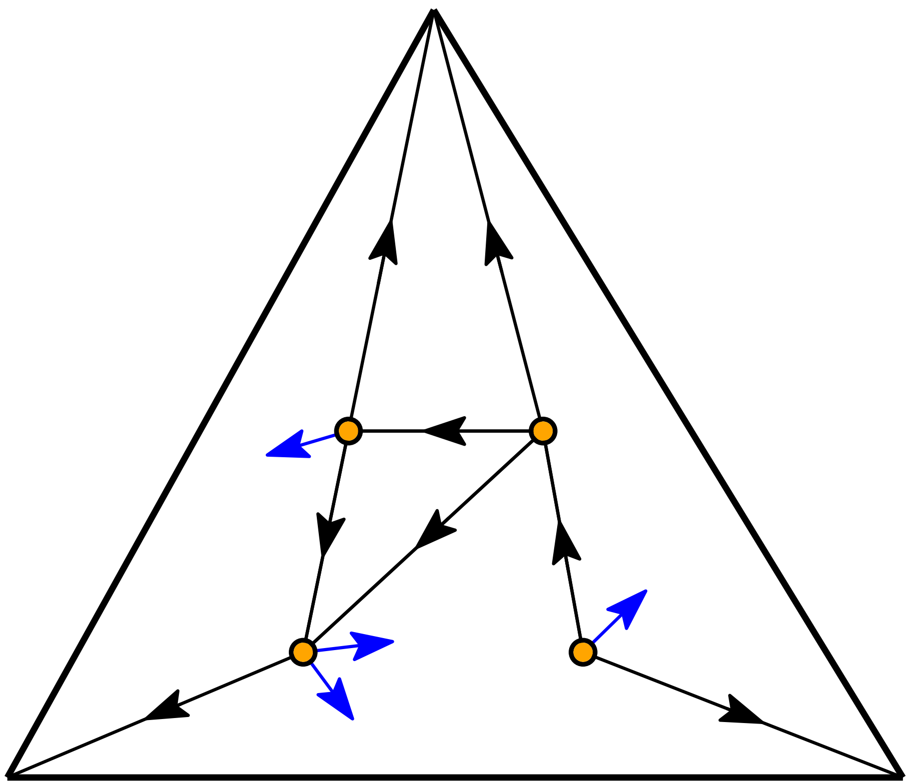

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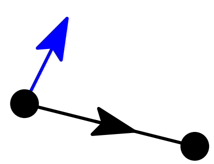
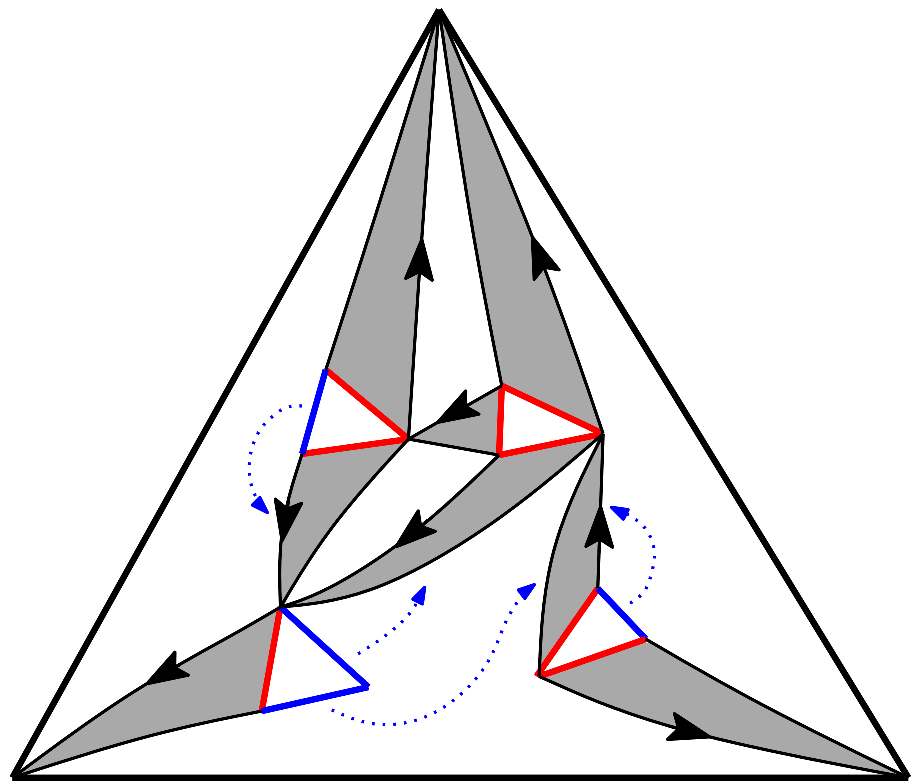
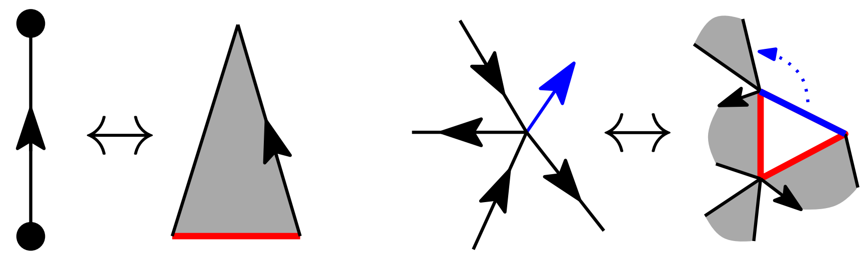
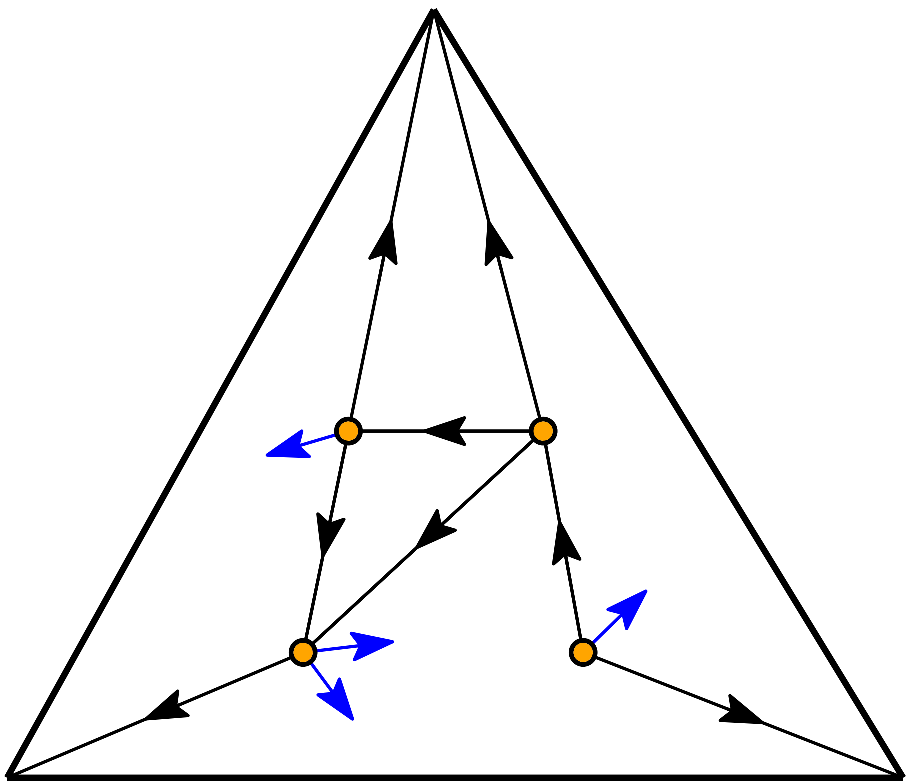
$+$ 

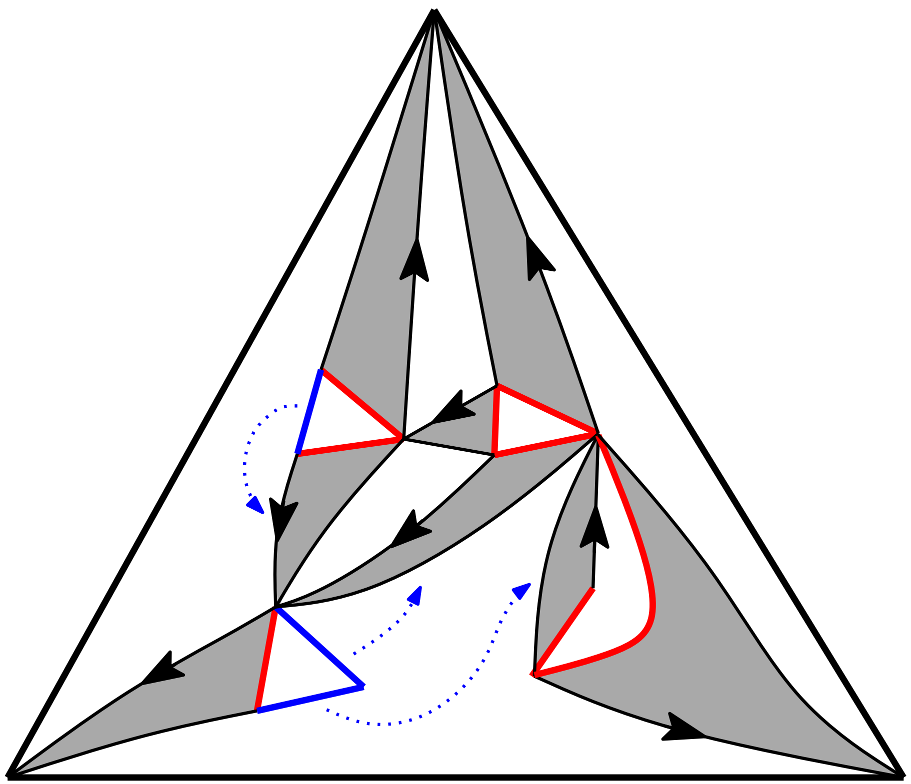




Second step $=$ merging

Bijection for outertriangular simple maps





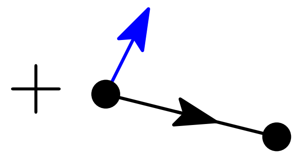
Canonical orientation:

3 outgoing edges

outer-accessibility

no clockwise circuit

face of degree $d + 3$: d arrows



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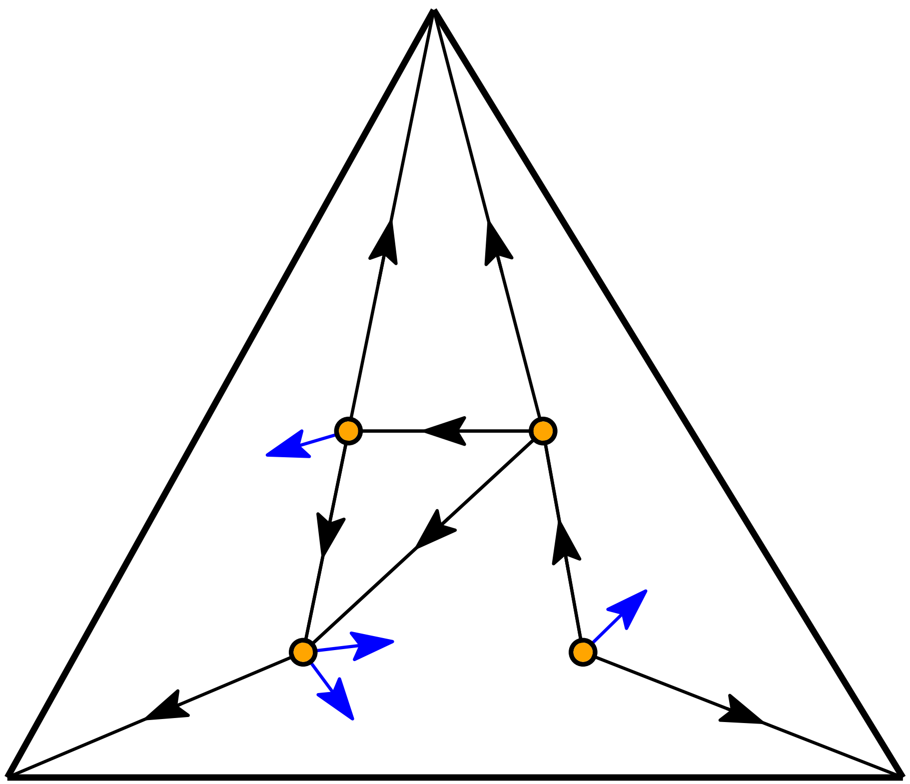

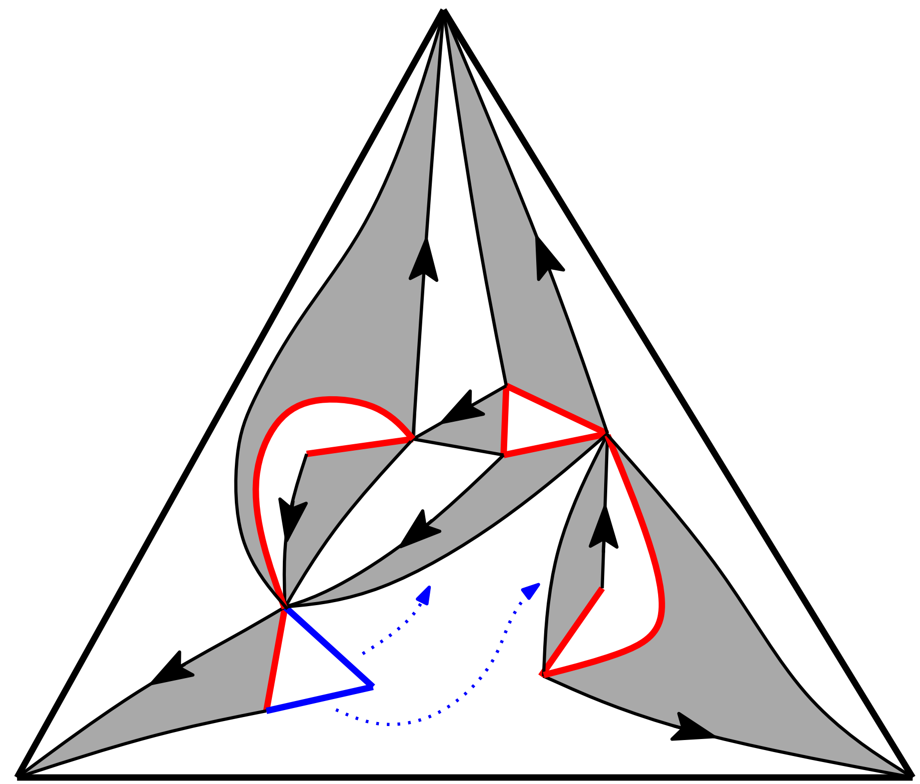

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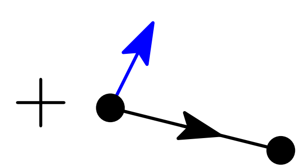

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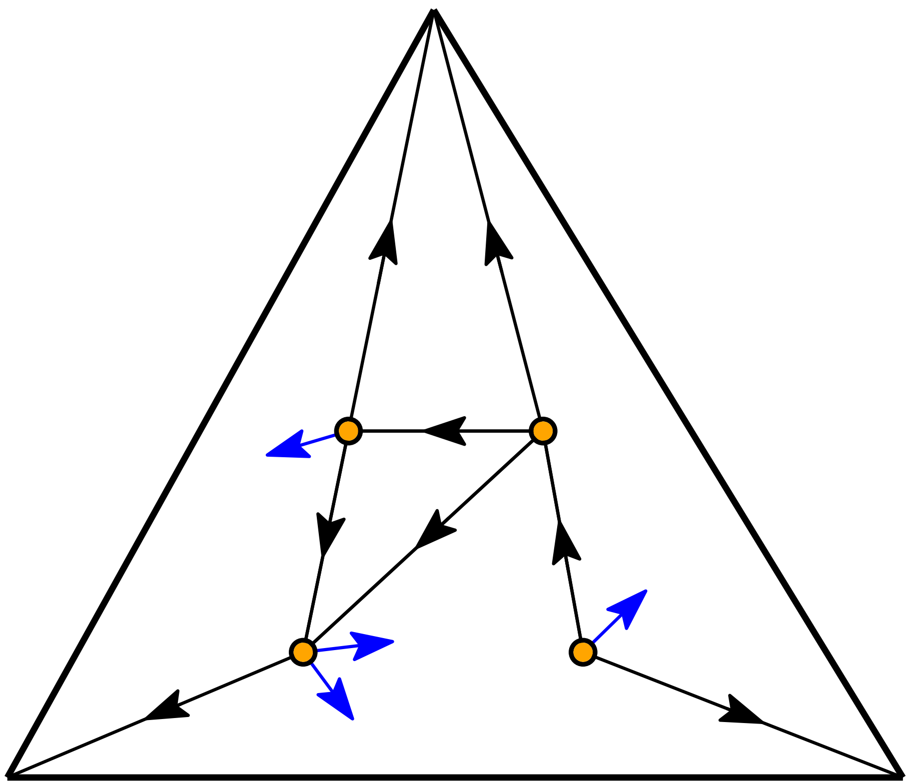

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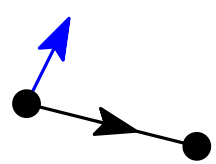
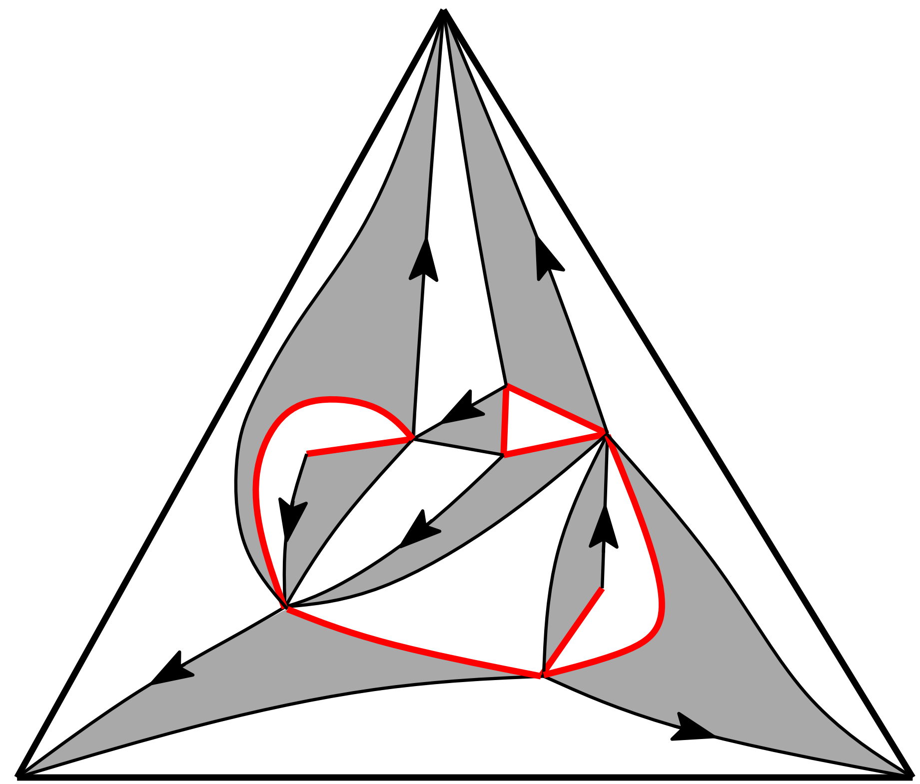
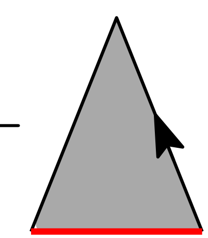




Canonical orientation:

1 outgoing edge at vertices
oriented forest
with 3 components
on outer vertices

$+$



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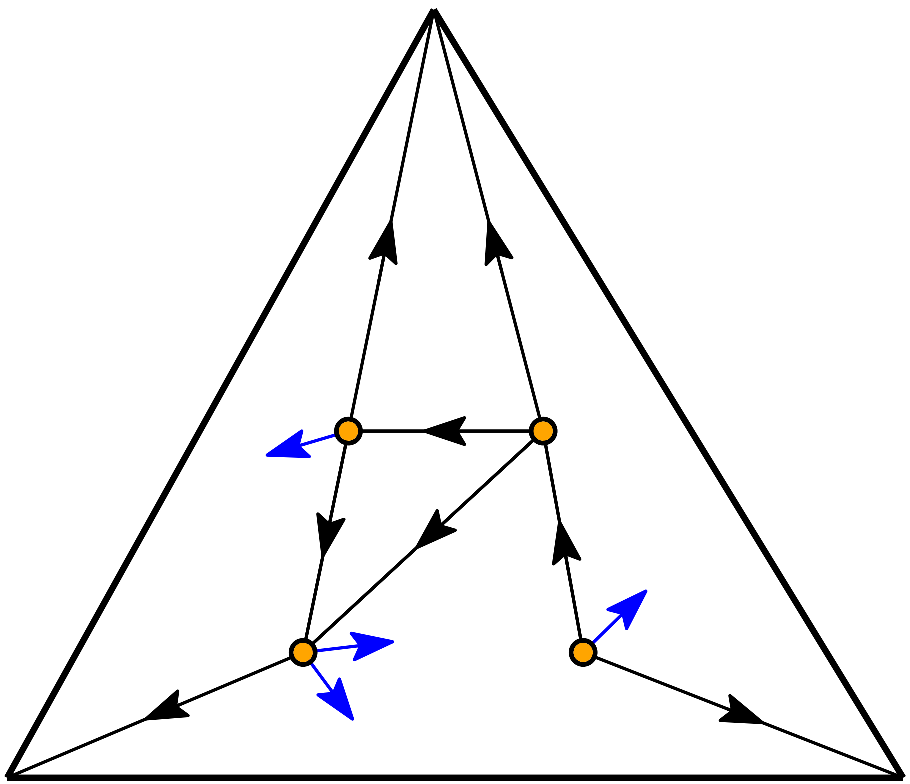

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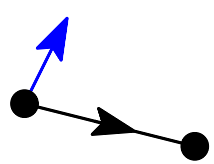
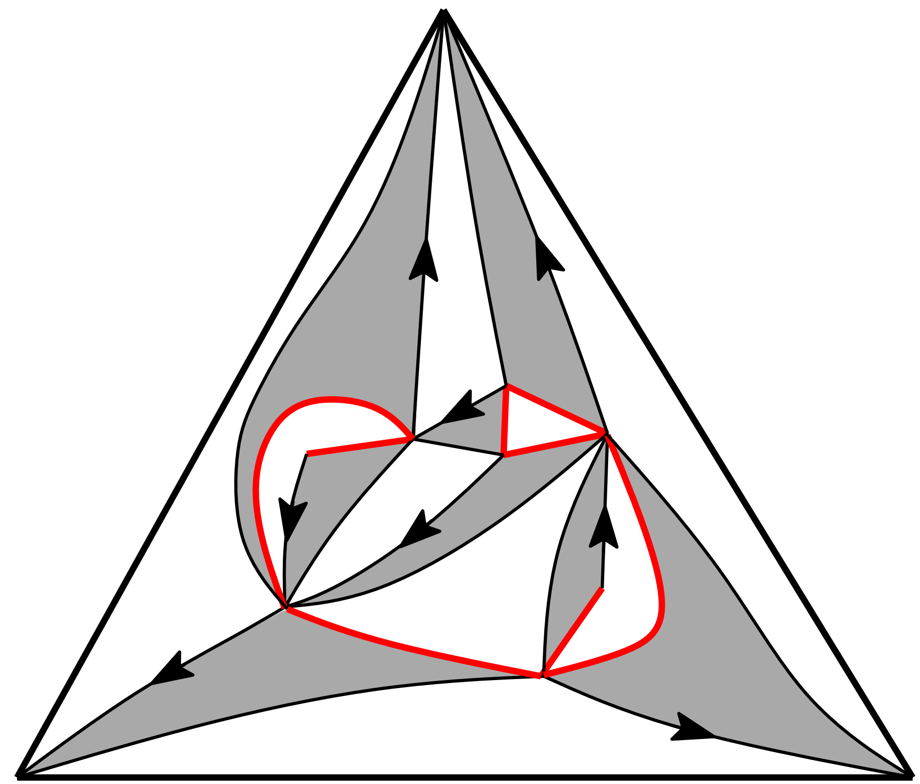
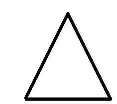
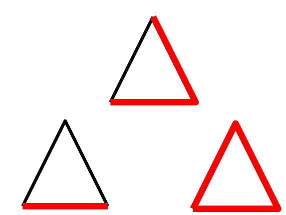
$+$ 

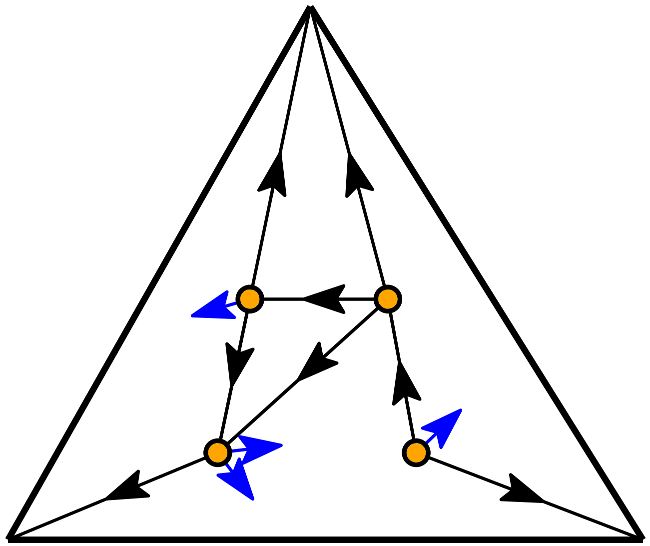


Face of the simple map: 

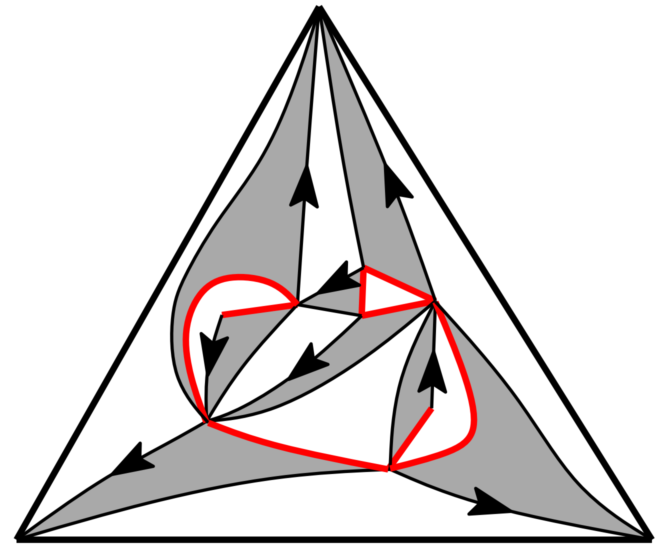
Inner vertex of the simple map: 

Summary

There is a bijection between outer-triangular simple maps with n inner edges and eulerian triangulations with n inner dark faces



$\leftrightarrow$



inner face	$\leftrightarrow$	white face with no red edge
inner vertex with $i \in \{0, 1, 2\}$ buds	$\leftrightarrow$	white face with $3 - i$ red edges

---

gives bijective proof of the formula

$$M(t) = \frac{1}{1 - tE(t)}$$

that links (the GFs of) rooted simple maps and eulerian triangulations

Counting results

- Exact bivariate enumeration,

  the series $M(t,x)$ for rooted simple maps by edges & vertices satisfies

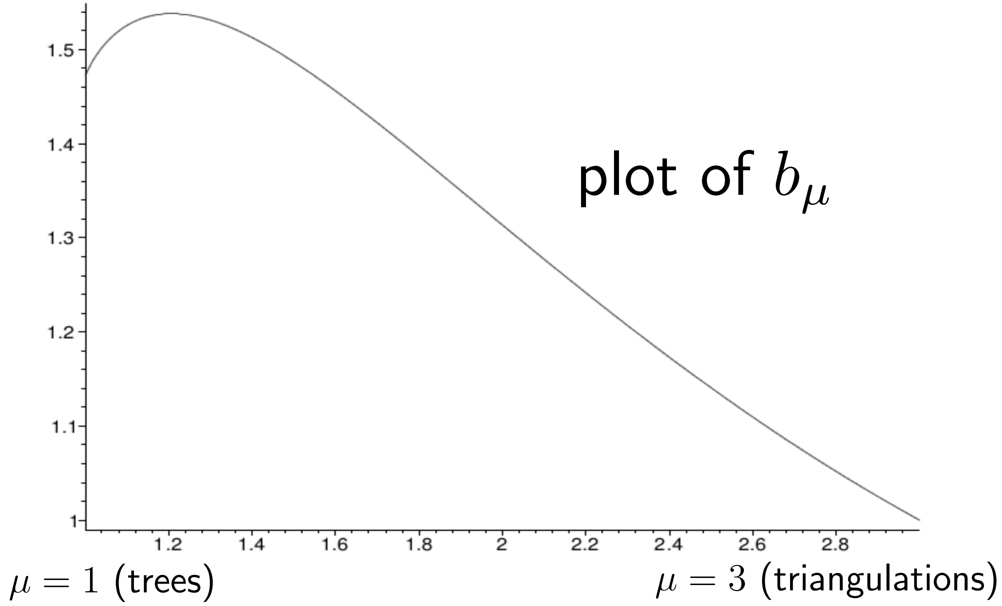
$$M = \frac{x^2 t + x^3 U \cdot (1 - V/t)}{1 - xt - xU \cdot (1 - V/t)}$$

where $\begin{cases} U = (t+V)^2 + 2xU(t+V)^2 + xU^2 \\ V = x(t+U+V)^2 \end{cases}$

- Asymptotic expected number of planar embeddings

Let $e_{n,m}$ = number of embeddings of a random (connected unembedded) planar graphs with n vertices and m edges

Then, for fixed $\mu \in (1,3)$
as $n \to \infty$ and $m/n \to \mu$
$E(e_{n,m}) = \frac{G_{n,m}}{M_{n,m}} \sim c_\mu b_\mu^n$
with c_μ, b_μ explicit



Maps as metric spaces

Study of the **shape** of planar maps:

$\rightarrow$ Chassaing–Schaeffer (2004):
typical distance in rooted plane quadrangulations
with n faces $\sim n^{1/4}$ when $n \rightarrow \infty$
$+$ distance profile converges towards **ISE**

$\rightarrow$ Le Gall (2007, 2012), Miermont (2012):
quadrangulations converge towards the **Brownian map**
when rescaled by $n^{-1/4}$
extended to $2p$-angulations and triangulations

$\rightarrow$ Addario-Berry–Albenque (2013):
simple triangulations and simple quadrangulations
converge towards the **Brownian map**

universal limit?

Distance profile of rooted simple maps

Distances in a simple map:

Let M be a simple map rooted at e_0 uniform with n edges
$\forall e \in E_M : d(e) =$ length of the shortest path from e to e_0

Profile : $(f_k)_{k\geq 1}$, where $f_k := \frac{1}{n}\sum_{e\in E_M}\delta_{d(e)}$
Radius : $r(M) := max(d(e), e \in E_M)$

Theorem [Bernardi, C., Fusy 2014]:

$f_{k/(2n)^{1/4}} \xrightarrow{(d)}$ ISE positively shifted

$r/(2n)^{1/4} \xrightarrow{(d)}$ width of ISE (also holds for moments)

Remark:

results similar to Chassaing–Schaeffer, with a scaling factor of $(2n)^{-1/4}$ instead of $(8n/9)^{-1/4}$.

Sketch of the proof of convergence

Idea 1: Rightmost paths are quasi-geodesic in 3-orientations
$\rightarrow$ [Addario-Berry&Albenque'13]

Idea 2: Rightmost paths in the simple map become oriented paths in the eulerian triangulation via the bijection

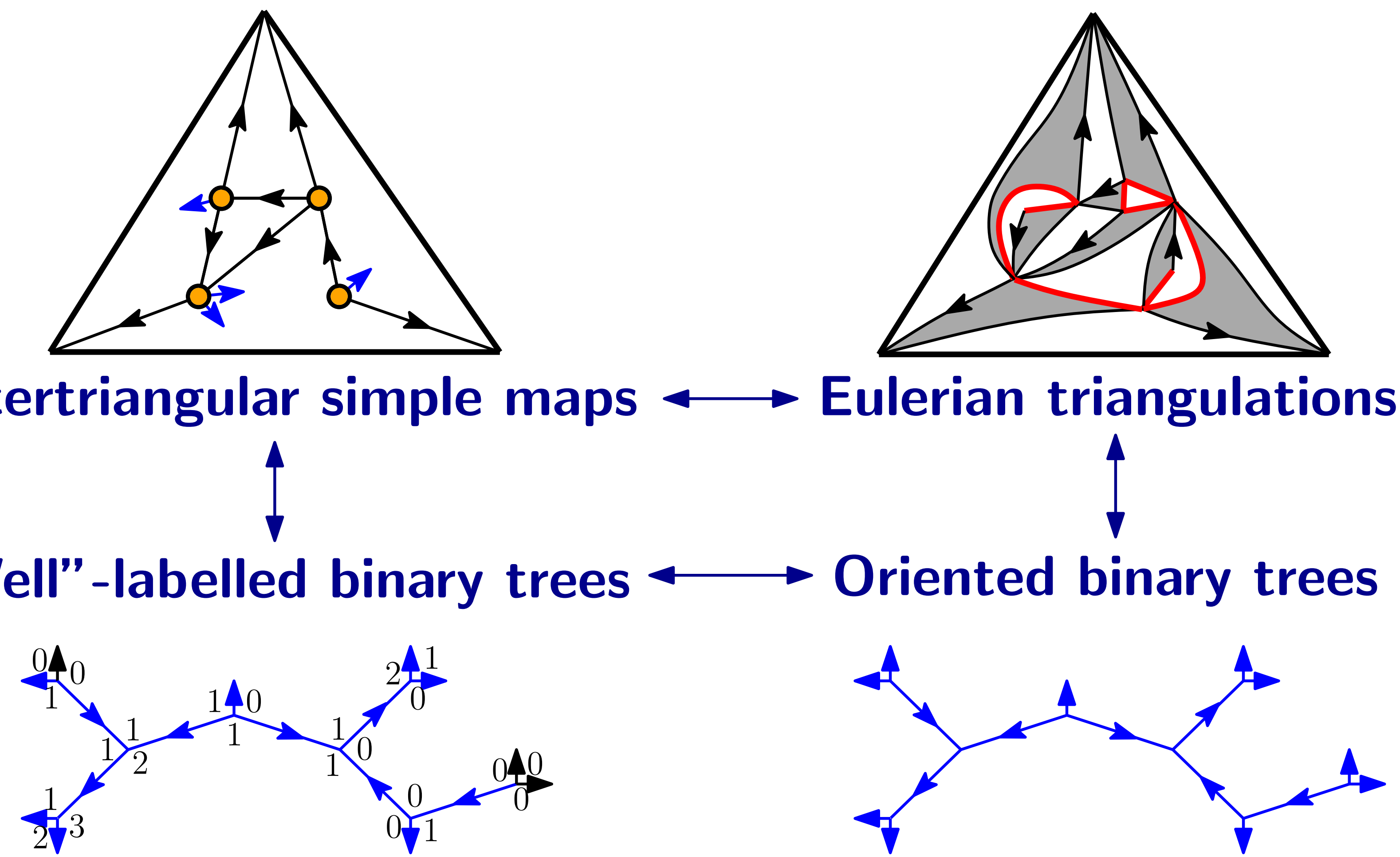
Idea 3: The profile for oriented paths of a rooted eulerian triangulation, rescaled by $(2n)^{-1/4}$, converges to positively shifted ISE.
$\rightarrow$ [Chassaing–Schaeffer'04][Le Gall'06]

Further results

Similar results hold for random loopless maps and general maps (recover [Bettinelli-Jacob-Miermont'13])

$\rightarrow$ core-extraction: loopless maps have a.s. a giant simple core (size $2n/3$)

$\rightarrow$ core-extraction: general maps have a.s. a giant loopless core (size $2n/3$)



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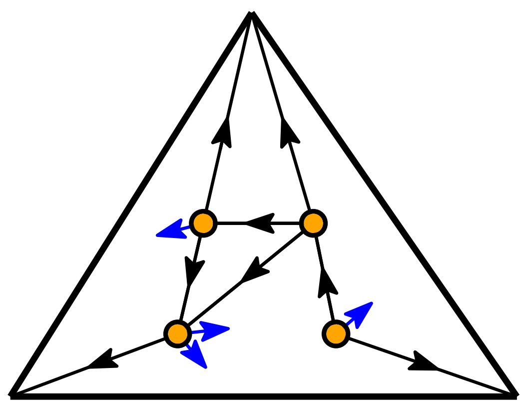

Outertriangular simple maps

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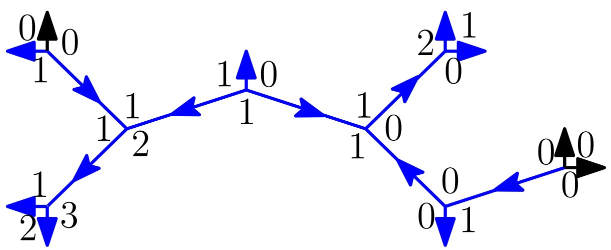
"Well"-labelled binary trees



Suitable candidate for Le Gall's criteria → convergence towards the Brownian map (current work with Marie Albenque)

Thank you!