Limit laws for the number of vertices in planar maps with constrained degrees

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We prove a general multi-dimensional central limit theorem for the expected number of vertices of a given degree in the family of planar maps whose vertex degrees are restricted to an arbitrary (finite or infinite) set of positive integers D. Our results rely on a classical bijection with mobiles (objects exhibiting a tree structure), combined with refined analytic tools to deal with the systems of equations on infinite variables that arise. We also discuss some possible extension to maps of higher genus.

Keywords: Planar maps, Central limit theorem, Analytic combinatorics, Mobiles

1 Introduction and Results

In this paper we study statistical properties of planar maps, which are connected planar graphs, possibly with loops and multiple edges, together with an embedding in the plane. Such objects are frequently used to describe topological features of geometric arrangements in two or three spatial dimensions. Thus, the knowledge of the structure and of properties of "typical" objects may turn out to be very useful in the analysis of particular algorithms that operate on planar maps. We say that map is *rooted* if an edge e is distinguished and oriented. It is called the root edge. The first vertex v of this oriented edge is called the root-vertex. The face to the right of e is called the root-face and is usually taken as the outer (or infinite) face. Similarly, we call a planar map *pointed* if just a vertex v is distinguished. However, we have to be really careful with the model. In rooted maps the root edge *destroys* potential symmetries, which is not the case if we consider pointed maps.

The enumeration of rooted maps is a classical subject, initiated by Tutte in the 1960's, see [11]. Among many other results, Tutte computed the number M_n of rooted maps with n edges, proving the formula

$$M_n = \frac{2(2n)!}{(n+2)!n!}3^n$$

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which directly provides the asymptotic formula

$$M_n \sim \frac{2}{\sqrt{\pi}} n^{-5/2} 12^n.$$

We are mainly interested in planar maps with degree restrictions. Actually, it turns out that this kind of asymptotic expansion is quite universal. Furthermore, there is always a (very general) central limit theorem for the number of vertices of given degree.

Theorem 1. *Suppose that D is an arbitrary set of positive integers but not a subset of $\{1,2\}$, let $\mathcal{M}_D$ be the class of planar rooted maps with the property that all vertex degrees are in D and let $M_{D,n}$ denote the number of maps in $\mathcal{M}_D$ with n edges. Furthermore, if D contains only even numbers, then set $d = \gcd\{i : 2i \in D\}$; set $d = 1$ otherwise.*

Then there exist positive constants c_D and ρ_D with

$$M_{D,n} \sim c_D n^{-5/2} \rho_D^{-n}, \qquad n \equiv 0 \bmod d. \tag{1}$$

Furthermore, let $X_n^{(d)}$ denote the random variable counting vertices of degree d ($\in D$) in maps in $\mathcal{M}_D$. Then $\mathbb{E}(X_n^{(d)}) \sim \mu_d n$ for some constant $\mu_d > 0$ and for $n \equiv 0 \bmod d$, and the (possibly infinite) random vector $\mathbf{X}_n = (X_n^{(d)})_{d \in D}$ ($n \equiv 0 \bmod d$) satisfies a central limit theorem, that is,

$$\frac{1}{\sqrt{n}} \left(\mathbf{X}_n - \mathbb{E}(\mathbf{X}_n) \right), \qquad n \equiv 0 \bmod d, \tag{2}$$

converges weakly to a centered Gaussian random variable $\mathbf{Z}$ (in ℓ^2).

Note that maps where all vertex degrees are 1 or 2 are very easy to characterize and are not really of interest, and that actually, their asymptotic properties are different from the general case. It is therefore natural to assume that D is not a subset of $\{1,2\}$.

Since we can equivalently consider dual maps, this kind of problem is the same as considering planar maps with restrictions on the face valencies. This means that the same results hold if we replace *vertex degree* by *face valency*. For example, if we assume that all face valencies equal 4, then we just consider planar quadrangulations (which have also been studied by Tutte [11]). In fact, our proofs will refer just to face valencies.

Theorem 1 goes far beyond known results. Only in the Eulerian case where all vertex degrees are even there are some general results. First, the asymptotic expansion (1) is known for Eulerian maps by Bender and Canfield [2]. Furthermore, a central limit theorem of the form (2) is known for all Eulerian maps (withouth degree restrictions) [9]. However, in the non-Eulerian case there are almost no results of this kind; there is only a one-dimensional central limit theorem for $X_n^{(d)}$ for all planar maps [10].

Section 2 introduces planar mobiles which, being in bijection with pointed planar maps, will reduce our analysis to simpler objects with a tree structure. Their asymptotic behaviour is derived in Section 3, first for the simpler case of bipartite maps (i.e., when D contains only even integers), then for families of maps without constraints on D. Section 4 is devoted to the proof of the central limit theorem using analytic tools from [8, 9]. Finally, in Section 5 we discuss the combinatorics of maps on orientable surface of higher genus. The expressions we obtain are much more involved than in the planar case, but it is expected to lead to similar analytic results.

2 Mobiles

Instead of investigating planar maps themselves, we will follow the principle presented in [5], whereby pointed planar maps are bijectively related to a certain class of trees called mobiles. (Their version of mobiles differ from the definition originally given in [3]; the equivalence of the two definitions is not shown explicitly in [5], but [7] gives a straightforward proof.)

Definition 1. A *mobile* is a planar tree – that is, a map with a single face – such that there are two kinds of vertices (black and white), edges only occur as black–black edges or black–white edges, and black vertices additionally have so-called "legs" attached to them (which are not considered edges), whose number equals the number of white neighbor vertices.
A *bipartite mobile* is a mobile without black–black edges.
The *degree* of a black vertex is the number of half-edges plus the number of legs that are attached to it.
A mobile is called *rooted* if an edge is distinguished and oriented.

The essential observation is that mobiles are in bijection to pointed planar maps.

Theorem 2. *There is a bijection between mobiles that contain at least one black vertex and pointed planar maps, where white vertices in the mobile correspond to non-pointed vertices in the equivalent planar map, black vertices correspond to faces of the map, and the degrees of the black vertices correspond to the face valencies. This bijection induces a bijection on the edge sets so that the number of edges is the same. (Only the pointed vertex of the map has no counterpart.)*

Similarly, rooted mobiles that contain at least one black vertex are in bijection to rooted and vertex-pointed planar maps.

Finally, bipartite mobiles with at least two vertices correspond to bipartite maps with at least two vertices, in the unrooted as well as in the rooted case.

Proof: For the proof of the bijection between mobiles and pointed maps we refer to [7], where the bipartite case is also discussed. It just remains to note that the induced bijection on the edges can be directly used to transfer the root edge together with its direction.

2.1 Bipartite Mobile Counting

We start with bipartite mobiles since they are more easy to *count*, in particular if we consider rooted bipartite mobiles, see [7].

Proposition 1. *Let $R = R(t, z, x_1, x_2, \ldots)$ be the solution of the equation*

$$R = tz + z \sum_{i\geq 1} x_{2i} \binom{2i-1}{i} R^i. \tag{3}$$

Then the generating function $M = M(t, z, x_1, x_2, \ldots)$ of bipartite rooted maps satisfies

$$\frac{\partial M}{\partial t} = 2\left(R/z - t\right), \tag{4}$$

where the variable t corresponds to the number of vertices, z to the number of edges, and x_{2i}, $i \geq 1$, to the number of faces of valency $2i$.

Proof: See Appendix.

Remark 1. It can be easily checked that Formula (4) can be specialized to count M_D, for any subset D of even positive integers: it suffices to set to 0 every x_{2i} such that $2i \in D$.

2.2 General Mobile Counting

We now proceed to develop a mechanism for general mobile counting that is adapted from [5]. For this, we will require Motzkin paths.

Definition 2. A *Motzkin path* is a path starting at 0 and going rightwards for a number of steps; the steps are either diagonally upwards $(+1)$, straight (0) or diagonally downwards (-1). A *Motzkin bridge* is a Motzkin path from 0 to 0. A *Motzkin excursion* is a Motzkin bridge which stays non-negative.

We define generating functions in the variable t and u, which count the number of steps of type 0 and -1, respectively. (Explicitly counting steps of type 1 is then unnecessary, of course.) The ordinary generating functions of Motzkin bridges, Motzkin excursions, and Motzkin paths from 0 to $+1$ shall be denoted by $B(t,u)$, $E(t,u)$ and $B^{(+1)}(t,u)$, respectively.

Continuing to follow the presentation of [5] and decomposing these three types of paths by their last passage through 0, we arrive at the equations:

$$\begin{aligned} E &= 1 + tE + uE^2, \\ B &= 1 + (t + 2uE)B, \\ B^{(+1)} &= EB. \end{aligned}$$

In what follows we will also make use of bridges where the first step is either of type 0 or -1. Clearly, their generating function $\overline{B}$ is given by

$$\overline{B} = tB + uB^{(+1)} = B(t + uE).$$

When Motzkin bridges are not constrained to stay non-negative, they can be seen as a random arrangement of a given number of steps $+1, 0, -1$. It is then possible to obtain explicit expressions for

$$B_{\ell,m} = [t^\ell u^m]B(t,u) = \binom{l+2m}{l,m,m}, \tag{5}$$

$$B^{(+1)}_{\ell,m} = [t^\ell u^m]B^{(+1)}(t,u) = \binom{l+2m+1}{l,m,m+1}, \tag{6}$$

$$\overline{B}_{\ell,m} = [t^\ell u^m]\overline{B}(t,u) = B_{\ell-1,m} + B^{(+1)}_{\ell,m-1} = \frac{l+m}{l+2m}\binom{l+2m}{l,m,m}. \tag{7}$$

Using the above, we can now finally compute relations for generating functions of proper classes of mobiles. We define the following series, where t corresponds to the number of white vertices, z to the number of edges, and y_i, $i \geq 1$, to the number of black vertices of degree i:

- $L_\triangledown(t, z, y_1, y_2, \ldots)$ is the series counting rooted mobiles that are rooted at a black vertex and where an additional edge is attached to the black vertex.
- $L_\circ(t, z, y_1, y_2, \ldots)$ is the series counting rooted mobiles that are rooted at a univalent white vertex, which is not counted in the series.

- $R(t, z, y_1, y_2, \ldots)$ is the series counting rooted mobiles that are rooted at a white vertes and where an additional edge is attached to the root vertex.

Similarly to the above we obtain the following equations for the generating functions of mobiles and rooted maps.

Proposition 2. *Let $L_\triangledown = L_\triangledown(t, z, y_1, y_2, \ldots)$, $L_\circ = L_\circ(t, z, y_1, y_2, \ldots)$, and $R = R(t, z, y_1, y_2, \ldots)$ be the solutions of the equation*

$$\begin{aligned} L_\triangledown &= z \sum_{\ell,m} y_{2m+\ell+1} B_{\ell,m} L_\triangledown^\ell R^m, \\ L_\circ &= z \sum_{\ell,m} y_{\ell+2m+2} B_{\ell,m}^{(+1)} L_\triangledown^\ell R^m, \\ R &= \frac{tz}{1 - L_\circ}, \end{aligned} \tag{8}$$

and let $T = T(t, z, y_1, y_2, \ldots)$ be given by

$$T = 1 + \sum_{\ell,m} y_{2m+\ell} \overline{B}_{\ell,m} L_\triangledown^\ell R^m, \tag{9}$$

where the numbers $B_{\ell,m}$, $B_{\ell,m}^{(+1)}$, and $\overline{B}_{\ell,m}$ are given by (5)–(7). Then the generating function $M = M(t, z, y_1, y_2, \ldots)$ of rooted maps satisfies

$$\frac{\partial M}{\partial t} = R/z - t + T, \tag{10}$$

where the variable t corresponds to the number of vertices, z to the number of edges, and y_i, $i \geq 1$, to the number of faces of valency i.

Proof: The system (8) is just a rephrasement of the recursive structure of rooted mobiles. Note that the numbers $B_{\ell,m}$ and $B_{\ell,m}^{(+1)}$ are used to count the number of ways to circumscribe a specific black vertex and considering white vertices, black vertices and "legs" as steps -1, 0 and $+1$. The generating function T given in (9) is then the generating function of rooted mobiles where the root vertex is black.

Finally, the equation (10) follows from Theorem 2 since $R/z - t$ corresponds to rooted mobiles with at least one black vertex where the root vertex is white and T corresponds to rooted mobiles where the root vertex is black.

Remark 2. Note that Proposition 1 is a special case of Proposition 2. We just have to restrict to the terms corresponding to $\ell = 0$ since bipartite mobiles have no black–black edges. In particular, the series for $L_\triangledown$ is not needed any more and second and third equation from (8) can be easily used to eliminate $L_\circ$ in order to recover the equation (3).

3 Asymptotic Enumeration

In this section we prove the asymptotic expansion (1). It turns out that it is much easier to start with bipartite maps. Actually, the bipartite case has been already treated by Bender and Canfield [2]. However, we apply a slightly different approach, which will then be extended to cover the general case as well the central limit theorem.

3.1 Bipartite maps

Let D be a non-empty subset of even positive integers different from $\{2\}$. Then by Proposition 1 the counting problem reduces to the discussion of the solutions $R_D = R_D(z,t)$ of the functional equation

$$R_D = tz + z \sum_{2i \in D} \binom{2i-1}{i} R_D^i \tag{11}$$

and the generating function $M_D(z,t)$ that satisfies the relation

$$\frac{\partial M_D}{\partial t} = 2\left(R_D/z - t\right). \tag{12}$$

Let $d = \gcd\{i : 2i \in D\}$. Then for combinatorial reasons it follows that there only exist maps with n edges for n that are divisible by d. This is reflected by the fact that the equation (11) can we rewritten in the form

$$\tilde{R} = t + \sum_{2i \in D} \binom{2i-1}{i} z^{i/d} \tilde{R}^i, \tag{13}$$

where we have substituted $R_D(z,t) = z\tilde{R}(z^d,t)$. (Recall that we finally work with R_D/z.)

Lemma 1. *There exists an analytic function $\rho(t)$ with $\rho(1) > 0$ and $\rho'(1) \neq 0$ that is defined in a neighborhood of $t = 1$, and there exist analytic functions $g(z,t)$, $h(z,t)$ with $h(\rho,1) > 0$ that are defined in a neighborhood of $z = \rho(1)$ and $z = 1$ such that the unique solution $R_D = R_D(z,t)$ of the equation (11) that is analytic at $z = 0$ and $t = 0$ can be represented as*

$$R_D = g(z,t) - h(t,z)\sqrt{1 - \frac{z}{\rho(t)}}. \tag{14}$$

Furthermore, the values $z = \rho(t)e(2\pi ij/d)$, $j \in \{0,1,\ldots,d-1\}$, are the only singularities of the function $z \mapsto R_D(z,t)$ on the disc $|z| \leq \rho(t)$, and there exists an analytic continuation of R_D to the range $|z| < |\rho(t)| + \eta$, $\arg(z - \rho(t)e(2\pi ij/d)) \neq 0$, $j \in \{0,1,\ldots,d-1\}$.

Proof: See Appendix.

It is now relatively easy to obtain similar properties for $M_D(z,t)$.

Lemma 2. *The function $M = M_D(z,t)$ that is given by (12) has the representation*

$$M_D = g_2(z,t) + h_2(t,z)\left(1 - \frac{z}{\rho(t)}\right)^{3/2} \tag{15}$$

in a neighborhood of $z = \rho(1)$ and $z = 1$, where the functions $g_2(z,t)$, $h_2(z,t)$ are analytic in a neighborhood of $z = \rho(1)$ and $z = 1$ and we have $h_2(\rho,1) > 0$. Furthermore, the values $z = \rho(t)e(2\pi ij/d)$, $j \in \{0,1,\ldots,d-1\}$, are the only singularities of the function $z \mapsto M_D(z,t)$ on the disc $|z| \leq \rho(t)$, and there exists an analytic continuation of M_D to the range $|z| < |\rho(t)| + \eta$, $\arg(z - \rho(t)e(2\pi ij/d)) \neq 0$, $j \in \{0,1,\ldots,d-1\}$.

Proof: This is a direct application of [8, Lemma 2.27].

In particular it follows that $M_D(z,1)$ has the singular representation

$$M_D = g_2(z,1) + h_2(z,1)\left(1-\frac{z}{\rho}\right)^{3/2}$$

around $z=\rho$. The singular representation are of the same kind around $z=\rho e(2\pi ij/d)$, $j\in\{1,\ldots,d-1\}$ and we have the analytic continuation property. Hence it follows by usual singularity analysis (see for example [8, Corollary 2.15]) that there exists a constant $c_D>0$ such that

$$[z^n]M_D(z,1)\sim c_D n^{-5/2}\rho^{-n}, \qquad n\equiv 0 \bmod d,$$

which completes the proof of the asymptotic expansion in the bipartite case.

3.2 General Maps

We now suppose that D contains at least one odd number. It is easy to observe that in this case we have $[z^n]M_D(z,1)>0$ for $n\ge n_0$ (for some n_0) so that we do not have to deals with several singularities.

By Proposition 2 we have to consider the system of equations for $L_{\triangledown,D}=L_{\triangledown,D}(z,t)$, $L_{\circ,D}=L_{\circ,D}(z,t)$, $R_D=R_D(z,t)$:

$$\begin{aligned} L_{\triangledown,D} &= z\sum_{i\in D}\sum_m B_{i-2m-1,m}L_{\triangledown,D}^{i-2m-1}R_D^m,\\ L_{\circ,D} &= z\sum_{i\in D}\sum_m B^{(+1)}_{i-2m-2,m}L_{\triangledown,D}^{i-2m-2}R_D^m,\\ R_D &= \frac{tz}{1-L_{\circ,D}}, \end{aligned} \tag{16}$$

and also the function

$$T_D = T_D(z,t) = 1+\sum_{i\in D}\sum_m \overline{B}_{i-2m,m}L_{\triangledown,D}^{i-2m-i}R_D^m.$$

Lemma 3. *There exists an analytic function $\rho(t)$ with $\rho(1)>0$ and $\rho'(1)\neq 0$ that is defined in a neighborhood of $t=1$, and there exist analytic functions $g(z,t)$, $h(z,t)$ with $h(\rho,1)>0$ that are defined in a neighborhood of $z=\rho(1)$ and $z=1$ such that*

$$R_D/z - t + T_D = g(z,t) - h(t,z)\sqrt{1-\frac{z}{\rho(t)}}. \tag{17}$$

Furthermore, the values $z=\rho(t)$ is the only singularitiy of the function $z\mapsto R_D/z-t+T_D$ on the disc $|z|\le\rho(t)$, and there exists an analytic continuation of R_D to the range $|z|<|\rho(t)|+\eta$, $\arg(z-\rho(t))\neq 0$.

Proof: See Appendix.

Lemma 3 shows that we are precisely in the same situation as in the bipartite case (actually, it is slightly easier since there is only one singularity on the circle $|z|=\rho(t)$). Hence we immediately get the same property for M_D as stated in Lemma 2 and consequently the proposed asymptotic expansion (1).

4 Central Limit Theorem for bipartite maps

Based on this previous result, we now extend our analysis to obtain a central limit theorem. Actually, this is immediate if the set D is finite, whereas the infinite case needs much more care.

Let D be a non-empty subset of even positive integers different from $\{2\}$. Then by Proposition 1 the generating functions $R_D = R_D(z,t,(x_{2i})_{2i\in D})$ and $M_D = M_D(z,t,(x_{2i})_{2i\in D})$ satisfy the equations

$$R_D = tz + z\sum_{2i\in D} x_{2i}\binom{2i-1}{i}R_D^i \tag{18}$$

and

$$\frac{\partial M_D}{\partial t} = 2\left(R_D/z - t\right). \tag{19}$$

If D is finite, then the number of variables is finite, too, and we can apply [8, Theorem 2.33] to obtain a representation of R_D of the form

$$R_D = g(z,t,(x_{2i})_{2i\in D}) - h(t,z,(x_{2i})_{2i\in D})\sqrt{1-\frac{z}{\rho(t,(x_{2i})_{2i\in D})}}, \tag{20}$$

a proper extension of the transfer lemma [8, Lemma 2.27] (where the variables x_{2i} are considered as additional parameters) leads to fdsakihfs
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wdhajlfhd

$$M_D = g_2(z,t,(x_{2i})_{2i\in D}) + h_2(t,z,(x_{2i})_{2i\in D})\left(1-\frac{z}{\rho(t,(x_{2i})_{2i\in D})}\right)^{3/2}, \tag{21}$$

and finally [8, Theorem 2.25] implies a multivariate central limit theorem for the random vector $\mathbf{X}_n = (X_n^{(2i)})_{2i\in D}$ of the proposed form.

Thus, we just have to concentrate on the infinite case. Actually, we proceed there in a similar way, however, we have to take care of infinitely many variables. There is no real problem to derive the same kind of representation (20) and (21) if D is infinite. Everything works in the same way as in the finite case, we just have to assume that the variables x_i are uniformly bounded. And of course we have to use a proper notion of analyticity in infinitely many variables. We only have to apply the functional analytic extension of the above cited theorems that are given in [9]. Moreover, in order to obtain a proper central limit theorem we need a proper adaption of [9, Theorem 3]. In this theorem we have also a single equation $y = F(z,(x_i)_{i\in I}, y)$ for a generating function $y = y(z,(x_i)_{i\in I})$ that encodes the distribution of a random vector $(X_n^{(i)})_{i\in I}$ in the form

$$y = \sum_n y_n \left(\mathbb{E}\prod_{i\in I} x_i^{X_n^{(i)}}\right) z^n,$$

where $X_n^{(i)} = 0$ for $i > cn$ (for some constant $c > 0$) which also implies that all appearing potentially infinite products are in fact finite. (In our case this is satisfied since there is no vertex of degree larger than n if we have n edges.) As we see from the proof of [9, Theorem 3], the essential part is to provide tightness of the involved normalized random vector, and tightness can be checked with the help of moment conditions. It is clear that asymptotics of moments for $X_n^{(i)}$ can be calculated with the help of derivatives of F, for example $\mathbb{E}X_n^{(i)} = F_{x_i}/(\rho F_z) \cdot n + O(1)$. This follows from the fact all information on the asymptotic behavior of the moments is *encoded* in the derivatives of the singularity $\rho(z,(x_i)_{i\in I})$ and by implicit differentiation these derivatives relate to derivatives of F. More precisely, [9, Theorem 3] says that the following conditions are sufficient to deduce tightness of the normalized random vector:

$$\sum_{i\in I} F_{x_i} < \infty, \quad \sum_{i\in I} F_{yx_i}^2 < \infty, \sum_{i\in I} F_{x_ix_i} < \infty,$$

$$F_{zx_i} = o(1), \quad F_{zx_ix_i} = o(1), \quad F_{yyx_i} = o(1), \quad F_{yyx_ix_i} = o(1),$$

$$F_{zzx_i} = O(1), \quad F_{zyx_i} = O(1), \quad F_{zyyx_i} = O(1), \quad F_{yyyx_i} = O(1), \quad (i \to \infty),$$

where all derivatives are evaluated at $(\rho, y(\rho), (1)_{i\in I})$.

The situation is slightly different in our case since we have to work with M_D instead of R_D. However, the only real difference between R_D and M_D is that the critical exponent in the singular representations (20) and (21) are different, but the behavior of the singularity $\rho(z,t,(x_i)_{i\in I})$ is precisely the same. Note that after the integration step we can set $t = 1$. Now tightness for the normalized random vector that is encoded in the function M_D follows in the same way as for R_D. And since the singularity $\rho(z,1,(x_i)_{i\in I})$ is the same, we get precisely the same conditions as in the case of [9, Theorem 3].

This means that we just have to check the above conditions applied to

$$F = F(z,(x_{2i})_{2i\in D}, y) = z + z\sum_{2i\in D} x_{2i}\binom{2i-1}{i} y^i,$$

where all derivatives are evaluated at $z = \rho$, $x_{2i} = 1$, and $y = R_D(\rho) < 1/4$. However, they are trivially satisfied since

$$\sum_{i\geq 1}\binom{2i-1}{i} i^K y^i < \infty$$

for all $K > 0$ and for positive real $y < 1/4$.

Remark 3. As stated in Theorem 1, the results and methods extend to the general case as well. The main idea is to reduce the (positive strongly connected) system of two equations (16) to a single functional equation, by applying [8, Theorem 3]. A more detailed proof is provided in the Appendix.

5 Maps of Higher Genus

The bijection used in Section 2 relies solely on the orientability of the surface on which the maps are embedded. Therefore it can easily be extended to maps of higher genus, i.e., embedded on an orientable surface of genus $g \in \mathbb{Z}_{>0}$ (while planar maps correspond to maps of genus 0). The main difference lies in the fact that the corresponding mobiles are no longer trees but rather *one-faced* maps of higher genus, while the other properties still hold.

However, due to the apparition of cycles in the underlying structure of mobiles, another difficulty arises. Indeed, in the original bijection, vertices and edges in mobiles could carry labels (related to the geodesic distance in the original map), subject to local constraints. In our setting, the legs actually encode the *local* variations of these labels, which are thus implicit. Local constraints on labels are naturally translated into

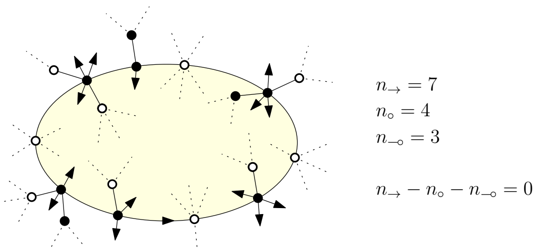

Fig. 1: An oriented cycle in a g-mobile and the constraint on its left (colored area). Notice that a similar constraint holds on its right, but is necessarily satisfied thanks to the properties of a g-mobile.

local constraints on the number of legs. But the labels have to remain consistent along each cycle of the mobiles, which gives rise to non-local constraints on the repartition of legs.

In order to deal with these additional constraints, and to be able to control the degrees of the vertices at the same time, we will now use a hybrid formulation of mobiles, carrying both labels and legs. As before, we will focus on the simpler case of mobiles coming from bipartite maps.

5.1 g-Mobiles

Definition 3. Given $g \in \mathbb{Z}_{\geq 0}$, a g-mobile is a one-faced map of genus g – embedded on the g-torus – such that there are two kinds of vertices (black and white), edges only occur as black–black edges or black–white edges, and black vertices additionally have so-called "legs" attached to them (which are not considered edges), whose number equals the number of white neighbor vertices.

Furthermore, for each cycle c of the g-mobile, let $n_{\circ}$, $n_{\to}$ and $n_{\multimap}$ respectively be the numbers of white vertices on c, of legs dangling to the left of c and of white neighbours to the left of c. One has the following constraint (see Figure 5.1):

$$n_{\to} = n_{\circ} + n_{\multimap} \tag{22}$$

The *degree* of a black vertex is the number of half-edges plus the number of legs that are attached to it. A *bipartite* g-mobile is a g-mobile without black–black edges. A g-mobile is called *rooted* if an edge is distinguished and oriented.
Notice that a 0-mobile is simply a mobile as described in Definition 1.

Theorem 3. *Given $g \geq 0$, there is a bijection between g-mobiles that contain at least one black vertex and pointed maps of genus g, where white vertices in the mobile correspond to non-pointed vertices in the equivalent map, black vertices correspond to faces of the map, and the degrees of the black vertices correspond to the face valencies. This bijection induces a bijection on the edge sets so that the number of edges is the same. (Only the pointed vertex of the map has no counterpart.)*

Similarly, rooted g-mobiles that contain at least one black vertex are in bijection to rooted and vertex-pointed maps of genus g.

Proof: This generalization of the bijection to higher genus was first given in [6] for quadrangulations and [4] for Eulerian maps, from which we will exploit many ideas in the present section.

5.2 Schemes of g-Mobile

g-Mobiles are not as easily decomposed as planar mobiles, due to the existence of cycles. However, they still exhibit a rather simple structure, based on *scheme* extraction.

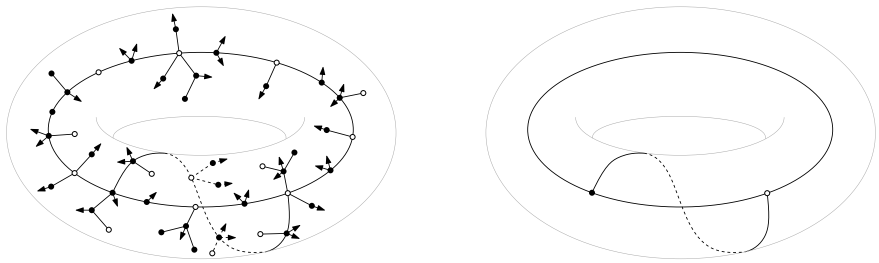

Fig. 2: A 1-mobile on the torus and its scheme.

The *g-scheme* (or simply the *scheme*) of a g-mobile is what remains when we apply the following operations (see Figure 2): first remove all legs, then remove iteratively all vertices of degree 1 and finally replace any maximal path of degree-2-vertices by a single edge.

Once these operations are performed, the remaining object is still a one-faced map of genus g, with black and white vertices (white–white edges can now occur), where the vertices have minimum degree 3.

To count g-mobiles, one key ingredient is the fact that there is only a finite number of schemes of a given genus. Indeed, let d_i be the number of degree i vertices of a g-scheme:

$$\sum_{k\geq 3}(i-2)d_i = \sum_{k\geq 3} id_i - 2\sum_{k\geq 3} d_i = 2(\#\text{edges} - \#\text{vertices}) = 4g-2.$$

The number of vertices (respectively edges) is then bounded by $4g-2$ (respectively $6g-3$), where this bound is reached for cubic schemes (see an example in Figure 2).

To recover a proper g-mobile from a given g-scheme, one would have to insert a suitable planar mobile into each corner of the scheme and to substitute each edge with some kind of path of planar mobiles. Unfortunately, this cannot be done independently: Around each black vertex, the total number of legs in every corner must equal the number of white neighbors, and around each cycle, (22) must hold.

In order to make these constraints more transparent, we will equip schemes with labels on white vertices and black corners. Now, when trying to reconstruct a g-mobile from a scheme, one has to ensure that the local variations are consistent with the global labelling. To be precise, the label variations are encoded as follows (see Figure 3):

- Around a black vertex of degree d, let $(l_1,\ldots,l_d)$ be the labels of its corners read in clockwise order:

$$\forall i, l_{i+1}-l_i = \begin{cases} +1 & \text{if there is a leg between the two corresponding corners,} \\ 0 & \text{if there is a black neighbor,} \\ -1 & \text{if there is a white neighbor.} \end{cases}$$

- Along the left side of an oriented cycle, the label decreases by 1 after a white vertex or when encountering a white neighbor and increases by 1 when encountering a leg.

The above statements hold for general – as well as bipartite – mobiles. In the following, we will only consider bipartite mobiles, as they are much easier to decompose.

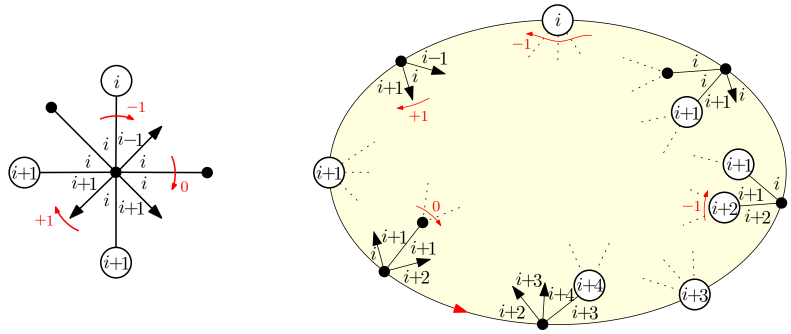

Fig. 3: The variations of labels around a black vertex and along an oriented cycle.

5.3 Reconstruction of Bipartite Maps of Genus g

In the following, it will be convenient to work with rooted schemes. One can then define a canonical labelling and orientation for each edge of a rooted scheme. An edge e now has an origin e_- and an endpoint e_+. The k corners around a vertex of degree k are clockwisely ordered and denoted by $c_1, \ldots, c_k$.

Given a scheme S, let $V_\circ, V_\bullet, C_\circ, C_\bullet$ be respectively the sets of white and black vertices and of white and black corners. A *labelled scheme* $(S, (l_c)_{c \in V_\circ \cup C_\bullet})$ is a pair consisting of a scheme S and a labelling on white vertices and black corners, with $l_c \geq 0$ for all c. Labellings are considered up to translation, as they will not affect local variations. For $e \in E_S$, an edge of S, we associate a label to each extremity l_{e_-}, l_{e_+}. If an extremity is a white vertex of label l, its label is l. If the extremity is a black vertex, its label is the same as the next clockwise corner of the black vertex.

Let a *doubly-rooted planar mobile* be a rooted (on a black or white vertex) planar mobile with a secondary root (also black or white). These two roots are the extremities of a path $(v_1, \ldots, v_k)$. The *increment* of the doubly-rooted mobile is then defined as $n_\rightarrow - n_\circ - n_{-\circ}$, which is not necessarily 0, as the path is not a cycle.

Similarly as in [4], we present a non-deterministic algorithm to reconstruct a g-mobile:

Algorithm.

(1) *Choose a labelled scheme of genus* g $(S, (l_c)_{c \in V_\circ \cup C_\bullet})$.

(2) $\forall v \in V_\bullet$, *choose a sequence of non-negative integers* $(i_k)_{1 \leq k \leq deg(v)}$, *then attach* i_k *planar mobiles and* $i_k + l_{c_{k+1}} - l_{c_k} + 1$ *legs to* c_k *(the* k^{th} *corner of* v*).*

(3) $\forall e \in S$, *replace* e *by a doubly-rooted mobile of increment* $incr(e) = l_{e_+} - l_{e_-} + \begin{cases} +1 & \text{if } e_- \text{ is white,} \\ -1 & \text{if } e_- \text{ is black.} \end{cases}$

(4) *On each white corner of* S, *insert a planar mobile.*

(5) *Distinguish and orient an edge as the root.*

Proposition 3. *Given* $g > 0$, *the algorithm generates each rooted bipartite* g*-mobile whose scheme has* k *edges in exactly* $2k$ *ways.*

Proof: One can easily see that the obtained object is indeed bipartite. Attaching planar mobiles and legs added at step (2) in a corner c_k create new corners, such that:

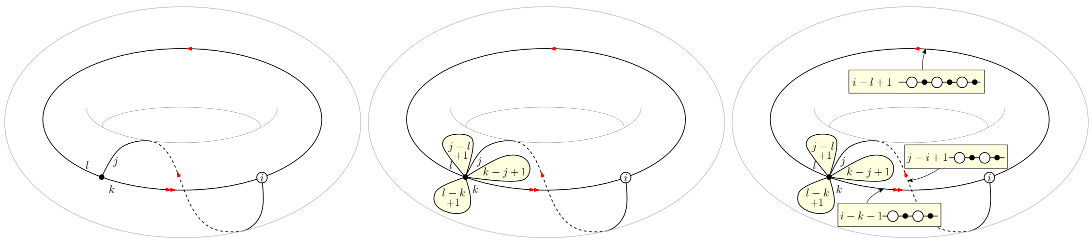

Fig. 4: Steps (1)–(3) of the algorithm.

- The first carries the same label l_{c_k} as c_k, and
- the last carries the label $l_{c_k} + (i_k + l_{c_{k+1}} - l_{c_k} + 1) - i_k = l_{c_{k+1}} + 1$.

The next corner should then be labelled $(l_{c_{k+1}} + 1) - 1 = l_{c_{k+1}}$, due to the next white neighbor, which is precisely what we want.

In the same fashion, at step (3), a simple counting shows that each edge is replaced by a path such that the labels along it evolve according to the scheme labelling.

We thus obtain a well-formed rooted bipartite g-mobile, with a secondary root on its scheme. Since the first root destroys all symmetries, there are exactly $2k$ choices for the secondary root, which would give the same rooted g-mobile.

5.4 g-Mobile Counting

A doubly-rooted bipartite planar mobile can be decomposed along a sequence of elementary cells forming the path between its two roots. Its increment is simply the sum of the increments of its cells.

Definition 4. An *elementary cell* is a half-edge connected to a black vertex itself connected to a white vertex with a dangling half-edge. The white vertex has a sequence of black-rooted mobiles attached on each side. The black vertex has $j \geq 0$ legs and $k \geq 0$ white-rooted mobiles on its left, $l \geq 0$ and $k + l - j + 2$ legs on its right, and its degree is $2(k + l + 1)$. The *increment* of the cell is then $k - j - 1$.

The generating series $P := P(t, (x_{2i}), z, s)$ of a cell, where s marks the increment, is:

$$P(t, (x_{2i}), z, s) = \frac{z^2 R^2}{t} \sum_{j,k,l \geq 0} \binom{j+k}{j} \binom{k+l-j+2}{l} s^{k-j-1} x_{2(k+l+1)} R^{k+l} = \frac{z^2 R^2}{st} \widehat{P}$$

The generating series $S := S(t, (x_{2i}), z, s)$ of a doubly-rooted mobile depends on the color of its roots (u, v):

$$S_{(u,v)}(t, (x_{2i}), z, s) = \begin{cases} \frac{1}{1-P} & \text{if } (u, v) = (\circ, \bullet) \text{ or } (\bullet, \circ), \\ \frac{z\widehat{P}}{1-P} & \text{if } (u, v) = (\circ, \circ), \\ \frac{zR^2}{st(1-P)} & \text{if } (u, v) = (\bullet, \bullet). \end{cases}$$

We can now express the generating series $R_S := R_S(t, (x_{2i}), z)$ of rooted bipartite g-mobiles with scheme S:

$$R_S(t,(x_{2i}),z) = 2\frac{z\partial}{\partial z}\frac{1}{2|E|}z^{|E|}t^{|V_\circ|}\left(\frac{R}{zt}\right)^{|C_\circ|} \bullet$$

$$\bullet \sum_{(l_c)\text{ labelling}}\left[\prod_{e\in E}[s^{incr(e)}]S_{(e_-,e_+)}\prod_{v\in V_\bullet}\sum_{i_1,\dots,i_{\deg(v)}\geq 0}\left(\prod_{k=1}^{\deg(v)}\binom{2i_k+l_{c_{k+1}}-l_{c_k}+1}{i_k}\right)x_{2(\deg(v)+\sum i_k)}\right] \tag{23}$$

Proposition 4. *The generating series $M_D^{(g)} := M_D^{(g)}(t,(x_{2i}),z)$ for the family of rooted bipartite maps of genus g, where the vertex degrees belong to D, satisfies the relation:*

$$\frac{\partial M_D^{(g)}}{\partial t} = \frac{2}{z}\sum_{\substack{S\text{ scheme}\\ \text{of genus } g}} R_S(t,(x_{2i}\mathbb{1}_{\{2i\in D\}}),z) \tag{24}$$

Proof: It follows directly from Theorem 3 and Equation (23).

6 Conclusion

Theorem 1 confirms the existence of a universal behaviour of planar maps. The asymptotics (with exponent $-5/2$) and this central limit theorem for the expected number of vertices of a given degree are believed to hold for any "reasonable" family of maps. It has also been shown in [6, 4] that a similar phenomenom occurs for maps of higher genus: the generating series of several families (quadrangulations, general and Eulerian maps) of genus g exhibit the same asymptotic exponent $5g/2-5/2$.

The expression obtained in Section 5 needs to be properly studied in order to obtain an asymptotic expansion. It refines previous results by controlling the degree of each vertex in the corresponding map.

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A Proof of Proposition 1

Let $R = R(t, z, x_1, x_2, \ldots)$ be the generating function of rooted bipartite mobiles, where the root vertex is white and where we additionally attach a *planted edge* to the (white) root vertex next to the root edge (for example, in counterclockwise order). The variable t corresponds to the number of white vertices, z to the number of edges, and x_{2i}, $i \ge 1$, to the number of black vertices of degree $2i$.

Since rooted mobiles can be considered as ordered rooted trees (which means that the neighboring vertices of the root vertex are linearly ordered and the subtrees rooted at these neighboring vertices are again ordered trees) we can describe them recursively. This leads directly to a functional equation for R of the form

$$R = \frac{tz}{1 - z\sum_{i\ge 1} x_{2i}\binom{2i-1}{i}R^{i-1}}$$

which is apparently the same as (3). Note that the factor $\binom{2i-1}{i}$ is precisely the number of ways of grouping i legs and $i-1$ edges around a black vertex (of degree $2i$; one edge is already there).

Hence, the generating function of rooted mobiles that are rooted by a white vertex is given by R/z. Since we have to discount the mobile that consists just of one (white) vertex, the generating function of rooted mobiles that are rooted at a white vertex and contain at least two vertices is given by

$$R/z - t = \sum_{i\ge 1} x_{2i}\binom{2i-1}{i}R^i. \tag{25}$$

We now observe that the right hand side of (25) is precisely the generating function of rooted mobiles that are rooted at a black vertex (and contain at least two vertices). Summing up, the generating function of bipartite rooted mobiles (with at least two vertices) is given by

$$2(R/z - t).$$

Finally, if M denotes the generating function of bipartite rooted maps (with at least two vertices) then $\frac{\partial M}{\partial t}$ corresponds to rooted maps, where a non-root vertex is pointed (and discounted). Thus, by Theorem 2 we obtain (4).

B Proof of Lemma 1

From general theory (see [8, Theorem 2.21]) we know that an equation of the form $R = F(z,t,R)$, where F is a power series with non-negative coefficients, has – usually – a square-root singularity of the form (14). We only have to assume that the function $R \to F(z,t,R)$ is neither constant nor a linear polynomial and that there exist solutions $\rho > 0$, $R_0 > 0$ of the system of equations

$$R_0 = F(\rho, 1, R_0), \qquad 1 = F_R(\rho, 1, R_0),$$

which are inside the range of convergence of F. Furthermore, we have to assume that $F_z(\rho, 1, R_0) > 0$ and $F_{RR}(\rho, 1, R_0) > 0$ to ensure that (14) holds not only for $t = 1$ but in a neighborhood of $t = 1$, and the condition $F_t(\rho, 1, R_0) > 0$ ensures that $\rho'(1) \neq 0$.

This means that in our case we have to deal with the system of equations

$$R_0 = \rho + \rho \sum_{2i \in D} \binom{2i-1}{i} R_0^i, \quad 1 = \rho \sum_{2i \in D} i \binom{2i-1}{i} R_0^{i-1},$$

or (after eliminating ρ) with the equation

$$\sum_{2i \in D} i \binom{2i-1}{i} R_0^i = 1 + \sum_{2i \in D} \binom{2i-1}{i} R_0^i$$

which we can rewrite to

$$\sum_{2i \in D} (i-1) \binom{2i-1}{i} R_0^i = 1. \tag{26}$$

It is clear that (26) has a unique positive solution if D is finite. (We also recall that all $i \geq 1$, since $2i$ has to be positive.) If D is infinite, we have to be more precise. Actually, we will show that (26) has a unique positive solution $R_0 < 1/4$. This follows from the fact that

$$(i-1)\binom{2i-1}{i} \sim \frac{4^i \sqrt{i}}{2\sqrt{\pi}}.$$

Thus, if D is infinite, it follows that the power series

$$x \mapsto H(x) = \sum_{2i \in D} (i-1) \binom{2i-1}{i} x^i$$

has radius of convergence $1/4$ and we also have $H(x) \to \infty$ as $x \to 1/4-$ since each non-zero term satisfies

$$\lim_{x \to 1/4} (i-1) \binom{2i-1}{i} x^i \sim \frac{\sqrt{i}}{2\sqrt{\pi}},$$

which is unbounded for $i \to \infty$. Finally, we set $\rho = \left(\sum_{2i\in D} i\binom{2i-1}{i} R_0^{i-1}\right)^{-1}$.

It is clear that $F_z(\rho,1,R_0) > 0$, $F_{RR}(\rho,1,R_0) > 0$, and $F_t(\rho,1,R_0) > 0$. Hence we obtain the representation (14) in a neighborhood of $z = \rho = \rho(1)$ and $t = 1$.

Next, let us discuss the analytic continuation property. If $d = \gcd\{i : 2i \in D\} = 1$ then it follows from the equation (11) that the coefficients $[z^n]R_D(z,1)$ are positive for $n \ge n_0$ (for some n_0). Consequently [8, Theorem 2.21] (see also [8, Theorem 2.16]) implies that there is an analytic continuation to the region $|z| < |\rho(t)| + \eta$, $\arg(z - \rho(t)) \ne 0$. If $d > 1$, then we can first reduce equation (11) to a an equation (13) for the function $\tilde{R}$ that is given by $R_D(z,t) = z\tilde{R}(z^d,t)$. We now apply the above method to this equation and obtain corresponding properties for $\tilde{R}$. Of course, these properties directly translate to R_D, and we are done.

C Proof of Lemma 3

It is convenient to reduce the number of equations. If we substitute the second equation of (16) for $L_{\circ,D}$ into the third one and multiply with the denominator, we obtain the equivalent system

$$L_{\triangledown,D} = z\sum_{i\in D} y_i \sum_m B_{2m-i-1,m} L_{\triangledown,D}^{2m-i-1} R_D^m,$$
$$R_D = zt + z\sum_{i\in D} y_i \sum_m B^{(+1)}_{2m-i-2,m} L_{\triangledown,D}^{2m-i-2} R_D^{m+1}.$$

This is a strongly connected system of two equations of the form $L_{\triangledown,D} = F(z,t,L_{\triangledown,D},R_D)$, $R_D = G(z,t,L_{\triangledown,D},R_D)$, where F and G are power series with non-negative coefficients. It is known that such a system of equations has in principle the same analytic properties (including the singular behavior of its solutions) as a single equation, see [8, Theorem 2.33]; however, we have to be sure that the regions of convergence of F and G are large enough.

In particular, if D is finite, then we have a positive algebraic system and we are done, see [1].

In the infinite case we have to argue in a different way. First of all, it is clear from the explicit solutions of $E = E(t,u) = ((1-t-\sqrt{(1-t)^2-4u})/(2u)$ and $B = B(t,u) = 1/\sqrt{(1-t)^2-4u}$ that F and G (and all their derivatives with respect to $L_{\triangledown,D}$ and R_D) are certainly convergent if $2|L_{\triangledown,D}| - |L_{\triangledown,D}|^2 + 2|R_D| < 1$. On the other hand, it follows similarly to the bipartite case that the derivatives of F and G are divergent if $L_{\triangledown,D} > 0$, $R_D > 0$, and $2L_{\triangledown,D} - L_{\triangledown,D}^2 + 2R_D = 1$. To see this we consider the function

$$B(t/s, us^2) = \frac{1}{1 - 2t/s - t^2/s^2 - 4uw^2} = \sum_{\ell,m} B_{\ell,m} s^{2m-\ell} t^\ell u^m = \sum_i s^i \sum_m B_{2m-i,m} t^{2m-i} u^m.$$

By singularity analysis it follows (for $t,u > 0$) that

$$\sum_m B_{2m-i,m} t^{2m-i} u^m \sim c\, i^{-1/2} h(t,u)^{-i},$$

where $c > 0$ and $h = h(t,u) > 0$ satisfy the equation $1 - 2t/h - t^2/h^2 - 4uh^2 = 0$. Similarly, we can consider derivatives of F which correspond, for example, to sums of the form

$$\sum_m B_{2m-i,m} m t^{2m-i} u^m \sim c' i^{1/2} h(t,u)^{-i}.$$

In particular, if $h(t,u)=1$ (which is the case if $2t-t^2-4u=1$), then this term diverges for $i\to\infty$. Thus, the derivatives of F and G diverge if $L_{\triangledown,D}>0$, $R_D>0$, and $2L_{\triangledown,D}-L_{\triangledown,D}^2+2R_D=1$.

In order to determine the singularity of the system $L_{\triangledown,D}=F(z,t,L_{\triangledown,D},R_D)$, $R_D=G(z,t,L_{\triangledown,D},R_D)$ we have to find positive solutions of L_0,R_0,ρ of the system

$$
\begin{aligned}
L_0 &= F(\rho,1,L_0,R_0),\\
R_0 &= G(\rho,1,L_0,R_0),\\
1 &= \frac{G_{L_{\triangledown,D}}F_{R_D}}{1-F_{L_{\triangledown,D}}}+G_{R_D}.
\end{aligned}
$$

We do this in the following way. Starting with $\rho=0$, we increase ρ and solve the first two equations to get $L_0=L_0(\rho)$, $R_0=R_0(\rho)$ till the third equation is satisfied. (Note that for $\rho=0$, the right-hand side 0 and, thus, smaller than 1.) As long as the right-hand side of the third equation is smaller than 1, it follows from the implicit function theorem that there is a local analytic continuation of the solutions $L_0=L_0(\rho)$, $R_0=R_0(\rho)$. Furthermore, since $L_0>0$ and $R_0>0$, we have to be in the region of convergence of the derivatives of F and G, that is, $2L_0-L_0^2+2R_0<1$. From this it also follows that the solutions $L_0=L_0(\rho)$, $R_0=R_0(\rho)$ naturally extend to a point where the right-hand side of the third equation equals 1, so that the above system has a solution (ρ,L_0,R_0). Of course, at this point the derivatives of F and G have to be finite, which implies that (ρ,L_0,R_0) lies inside the region of convergence of F and G.

This finally shows that all assumptions of [8, Theorem 2.33] are satisfied. Thus, singular representation of type (17) and the analytic continuation propery follow for the functions $L_{\triangledown,D}=L_{\triangledown,D}(z,t)$, $L_{\circ,D}=L_{\circ,D}(z,t)$, $R_D=R_D(z,t)$. Hence, the same kind of properties follows for $T_D=T_D(z,t)$ and consequently also for $R_D/z-t+T_D$.

D Central Limit Theorem for General Maps

We now assume that D contains at least one odd number. By Proposition 2 (and by the applying the same elimination procedure as in the proof of Lemma 3) we have to consider the system of equations

$$
\begin{aligned}
L_{\triangledown,D} &= z\sum_{i\in D} y_i \sum_m B_{2m-i-1,m} L_{\triangledown,D}^{2m-i-1} R_D^m,\\
R_D &= zt + z\sum_{i\in D} y_i \sum_m B_{2m-i-2,m}^{(+1)} L_{\triangledown,D}^{2m-i-2} R_D^{m+1},
\end{aligned}
$$

for the generating functions $L_{\triangledown,D}=L_{\triangledown,D}(z,t,(y_i)_{i\in D})$ and $R_D=R_D(z,t,(y_i)_{i\in D})$, the generating function

$$
T_D = T_D(z,t,(y_i)_{i\in D}) = 1+\sum_{i\in D} y_i \sum_m \overline{B}_{2m-i,m} L_{\triangledown,D}^{2m-i} R_D^m
$$

and finally the generating function $M_D=M_D(z,t,(y_i)_{i\in D})$ that satisfies the relation

$$
\frac{\partial M_D}{\partial t} = R_D/z - t + T_D.
$$

Again, if D is finite, we can proceed as in the bipartite case by applying [8, Theorem 2.33, Lemma 2.27, and Theorem 2.25] which implies the proposed central limit theorem.

If D is infinite, we argue in a similar way as in the bipartite case. The only difference is that we are not starting with one equation but with a system of two equations that have the (general) form

$$L = F(z, t, (y_i)_{i\in D}, L, R), \quad R = G(z, t, (y_i)_{i\in D}, L, R).$$

Nevertheless, it is possible to reduce two equations of this form to a single one. The proof of [8, Theorem 2.33] shows that there are no analytic problems since we have a positive and strongly connected system. We use the first equation to obtain an implicit function solution $f = f(z, t, (y_i)_{i\in D}, r)$ that satisfies

$$f = F(z, t, (y_i)_{i\in D}, f, r).$$

Then we substitute f for L in the second equation and arrive at a single functional equation

$$R = G(z, t, (y_i)_{i\in D}, f(z, t, (y_i)_{i\in D}, R), R)$$

for $R = R_D(z, t, (y_i)_{i\in D})$. Note that the proof of [8, Theorem 2.33] assures that f is analytic although L and R get singular. Hence by setting

$$H(z, t, (y_i)_{i\in D}, r) = G(z, t, (y_i)_{i\in D}, f(z, t, (y_i)_{i\in D}, r), r)$$

we obtain a single equation $R = H(z, t, (y_i)_{i\in D}, R)$ for $R = R_D$ and we can apply the same method as in the bipartite case. Of course, the calculations get more involved. For example, we have

$$H_{y_i} = G_{y_i} + \frac{G_L F_{y_i}}{1 - F_L},$$

where

$$\begin{aligned}
F_L &= \rho \sum_{i\in D} \sum_m (2m - i - 1) B_{2m-i-1,m} L_0^{2m-i-2} R_0^m, \\
F_{y_i} &= \rho \sum_m B_{2m-i-1,m} L_0^{2m-i-1} R_0^m, \\
G_L &= \rho \sum_{i\in D} \sum_m (2m - i - 2) B^{(+1)}_{2m-i-1,m} L_0^{2m-i-3} R_0^m, \\
G_{y_i} &= \rho \sum_m B^{(+1)}_{2m-i-2,m} L_0^{2m-2-i} R_0^m.
\end{aligned}$$

From the proof of Lemma 3 we already know that $2L_0 - L_0^2 + 4R_0 < 1$, which implies that

$$\sum_{i\geq 1} \sum_m m^K (2m - i - 1) B_{2m-i-1,m} L_0^{2m-i-2} R_0^m < \infty$$

for all $K > 0$. Furthermore, we have $F_L < 1$ and $G_R < 1$. Hence it follows that

$$\sum_{i\in D} H_{y_i} < \infty.$$

In the same way, we can handle the other conditions which completes the proof of Theorem 1.