

# Singularities for Systems of Functional Equations

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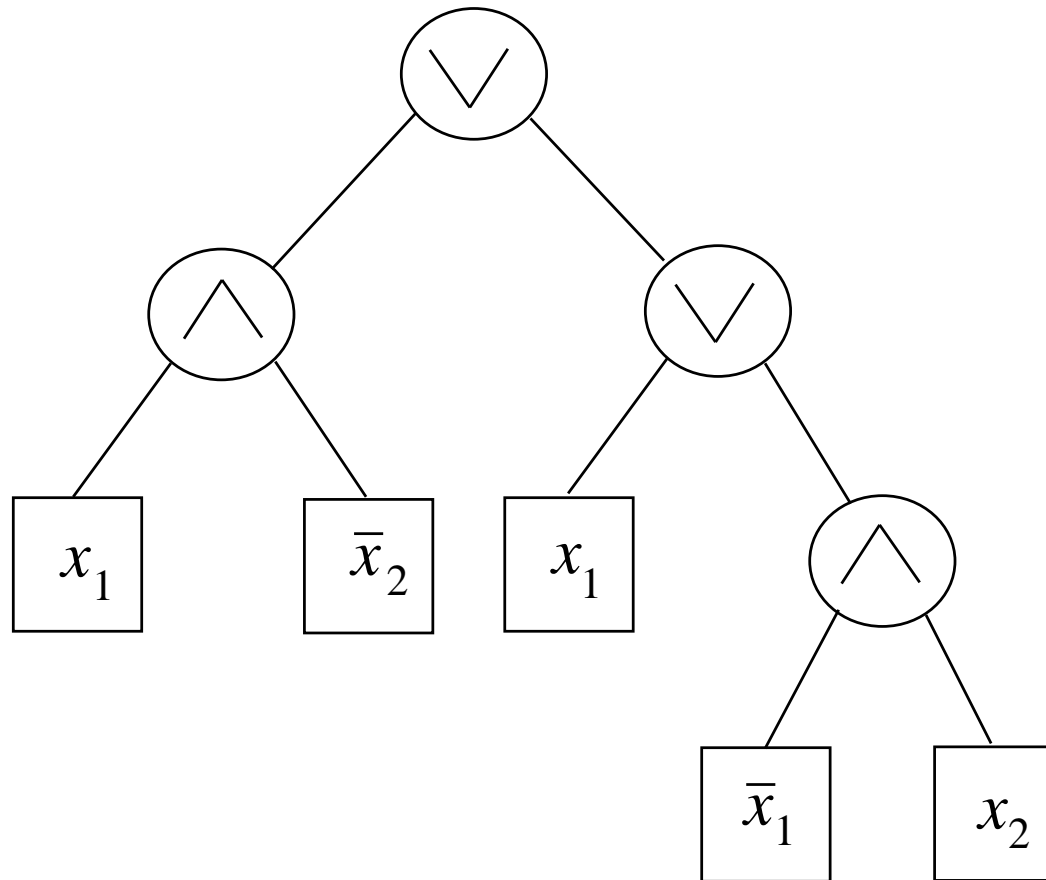
# Contents

1. Boolean Functions and AND/OR-Trees
2. Generating Functions
3. Systems of Functional Equations
4. General Dependency Graphs
5. Infinite Systems

# Boolean Functions and AND/OR-Trees

**Boolean formula.**  $F = (x_1 \wedge \bar{x}_2) \vee (x_1 \vee (\bar{x}_1 \wedge x_2))$

**Boolean function.**  $f = x_1 \vee x_2$



# Boolean Functions and AND/OR-Trees

$B_m$  ... **binary trees** with  $m$  internal nodes:  $b_m = |B_m| = \frac{1}{m} \binom{2m}{m}$

$T_m$  ... **Boolean AND/OR-formulas** with  $n$  literals (+ their negations)

$$t_m = |T_m| = b_m 2^m (2n)^{m+1} = \frac{2n}{m+1} (4n)^m \binom{2m}{m}.$$

$f$  ... **Boolean function** in  $x_1, \dots, x_m$

$$P_m(f) = \frac{\#\{F \in T_m : F \text{ represents } f\}}{t_m}$$

$$P(f) = \lim_{m \rightarrow \infty} P_m(f)$$

# Binary Trees

Generating Function.  $b(z) = \sum_{m \geq 0} b_m z^m$

$$b(z) = 1 + z b(z)^2$$

$$b(z) = \frac{1 - \sqrt{1 - 4z}}{2z} = 2 - 2\sqrt{1 - 4z} + \dots$$

$$b_m = [z^m]b(z) = \frac{1}{m} \binom{2m}{m} \sim \frac{4^m}{\sqrt{\pi} m^{3/2}}$$

# Binary Trees

Generating Function.  $t(z) = \sum_{m \geq 0} t_m z^m$

$$t(z) = 2n + 2z t(z)^2$$

$$t(z) = \frac{1 - \sqrt{1 - 16nz}}{4z} = 4n - 4n\sqrt{1 - 16nz} + \dots$$

$$t_m = [z^m]t(z) = 2^m (2n)^{m+1} \frac{1}{m} \binom{2m}{m} \sim 2n \frac{(16n)^m}{\sqrt{\pi} m^{3/2}}$$

# Boolean Functions and AND/OR-Trees

## Lemma

$f$  ... Boolean function in  $x_1, \dots, x_n$

$$t_m(f) = \#\{F \in T_m : F \text{ represents } f\}, \quad t_f(z) = \sum_{m \geq 0} t_m(f) z^m$$

Suppose that  $t_f(z)$  has radius of convergence  $1/(16n)$  and a local expansion in terms of  $\sqrt{1 - 16nz}$  of the form

$$t_f(z) = \alpha_f - \beta_f \sqrt{1 - 16nz} + \dots$$

[+ some technical conditions] then

$$P(f) = \lim_{m \rightarrow \infty} \frac{t_m(f)}{t_m} = \frac{\beta_f}{4n}.$$

# Boolean Functions and AND/OR-Trees

**Lemma** [Chauvin+Flajolet+Gardy+Gittenberger]

$f$  ... Boolean function in  $x_1, \dots, x_n$

$$t_f(z) = \mathbf{1}_{[f \text{ literal}]} + z \sum_{g,h:g \vee h=f} t_g(z)t_h(z) + z \sum_{g,h:g \wedge h=f} t_g(z)t_h(z)$$

**Remark.** This is a system of  $2^{2^n}$  equations.



# Boolean Functions and AND/OR-Trees

**Example.**  $n = 1$ : *True*, *False*,  $x_1$ ,  $\bar{x}_1$

$$t_{True}(z) = 2zt_{x_1}t_{\bar{x}_1} + 2zt_{True}(z)t(z)$$

$$t_{False}(z) = 2zt_{x_1}t_{\bar{x}_1} + 2zt_{False}(z)t(z)$$

$$t_{x_1}(z) = 1 + 2zt_{x_1}(z)^2 + 2zt_{x_1}(z)t_{True}(z) + 2zt_{x_1}(z)t_{False}(z)$$

$$t_{\bar{x}_1}(z) = 1 + 2zt_{\bar{x}_1}(z)^2 + 2zt_{\bar{x}_1}(z)t_{True}(z) + 2zt_{\bar{x}_1}(z)t_{False}(z)$$

[ $t(z)$  abbreviates  $t(z) = t_{True}(z) + t_{False}(z) + t_{x_1}(z) + t_{\bar{x}_1}(z)$ ]

This system can be solved explicitly. For example, one gets:

$$\begin{aligned} t_{x_1}(z) &= -\frac{1}{8z} \left( 1 + \sqrt{1 - 16z} - \sqrt{2 + 16z + 2\sqrt{1 - 16z}} \right) \\ &= 2(\sqrt{3} - 1) + 2(1/\sqrt{3} - 1)\sqrt{1 - 16z} + \dots \end{aligned}$$

# Boolean Functions and AND/OR-Trees

**Example.**  $n = 1$ : *True*, *False*,  $x_1$ ,  $\bar{x}_1$

$$P(\textit{True}) = P(\textit{False}) = \frac{1}{2\sqrt{3}} = 0.28867513\dots$$

$$P(x_1) = P(\bar{x}_1) = \frac{\sqrt{3} - 1}{2\sqrt{3}} = 0.21132486\dots$$

# A Single Functional Equation

**Theorem** (Bender, Canfield, Meir & Moon)

Suppose that  $y(z)$  satisfies  $y(z) = \Phi(z, y(z))$ , where  $\Phi(z, y)$  has a power series expansion at  $(0, 0)$  with non-negative coefficients,  $\Phi_{yy}(z, y) \neq 0$ , and  $\Phi_z(z, y) \neq 0$

Let  $z_0 > 0$ ,  $y_0 > 0$  (inside the region of convergence) satisfy the system of equations:

$$y_0 = \Phi(z_0, y_0), \quad 1 = \Phi_y(z_0, y_0).$$

Then there exists analytic function  $g(z)$  and  $h(z)$  such that locally

$$y(z) = g(z) - h(z) \sqrt{1 - \frac{z}{z_0}},$$

where  $g(z_0) = y_0$  and  $h(z_0) \neq 0$ .

# A Single Functional Equation

**Example.**  $y(z) = 1 + zy(z)^2$ ,  $\Phi(z, y) = 1 + zy^2$ .

$$y_0 = 1 + z_0 y_0^2, \quad 1 = 2z_0 y_0$$

$$z_0 = \frac{1}{4}, \quad y_0 = 2$$

$$g(z_0) = 2, \quad h(z_0) = 2$$

$$y(z) = 2 - 2\sqrt{1 - 4z} + \dots$$

# A Single Functional Equation

The case  $\Phi_{yy}(z, y) = 0$ .

$$y = \Phi(z, 0) + \Phi_y(z, 0)y$$

$$y(z) = \frac{\Phi(z, 0)}{1 - \Phi_y(z, 0)}$$

$$1 = \Phi_y(z_0, 0) \implies 1 - \Phi_y(z, 0) = K(z)(1 - z/z_0)$$

$$y(z) = \frac{\Phi(z, 0)}{K(z)(1 - z/z_0)}$$

$\implies$  **Polar singularity**

# A Single Functional Equation

## Idea of the Proof.

Set  $F(z, y) = \Phi(z, y) - y$ . Then we have

$$F(z_0, y_0) = 0$$

$$F_y(z_0, y_0) = 0$$

$$F_z(z_0, y_0) \neq 0$$

$$F_{yy}(z_0, y_0) \neq 0.$$

Weierstrass preparation theorem implies that there exist analytic functions  $H(z, y)$ ,  $p(z)$ ,  $q(z)$  with  $H(z_0, y_0) \neq 0$ ,  $p(z_0) = q(z_0) = 0$  and

$$F(z, y) = H(z, y) \left( (y - y_0)^2 + p(z)(y - y_0) + q(z) \right).$$

# A Single Functional Equation

$$F(z, y) = 0 \iff (y - y_0)^2 + p(z)(y - y_0) + q(z) = 0.$$

Consequently

$$\begin{aligned} y(z) &= y_0 - \frac{p(z)}{2} \pm \sqrt{\frac{p(z)^2}{4} - q(z)} \\ &= \boxed{g(z) - h(z) \sqrt{1 - \frac{z}{z_0}}}, \end{aligned}$$

where we write

$$\frac{p(z)^2}{4} - q(z) = K(z)(z - z_0)$$

which is again granted by the Weierstrass preparation theorem and we set

$$g(z) = y_0 - \frac{p(z)}{2} \quad \text{and} \quad h(z) = \sqrt{-K(z)z_0}.$$

# A Single Functional Equation

**Variation of the Theorem.**  $u$  denotes an **additional parameter**.

Suppose that  $y(z; u)$  satisfies  $y(z; u) = \Phi(z, y(z; u); u)$ , where  $\Phi(z, y; u)$  has a power series expansion at  $(0, 0)$  with non-negative coefficients,  $\Phi_{yy}(z, y; u) \neq 0$ , and  $\Phi_z(z, y; u) \neq 0$

Let  $z_0(u) > 0$ ,  $y_0(u) > 0$  (inside the region of convergence) satisfy the system of equations:

$$y_0(u) = \Phi(z_0(u), y_0(u)), \quad 1 = \Phi_y(z_0(u), y_0(u)).$$

Then there exists analytic function  $g(z; u), h(z; u)$  such that locally

$$y(z; u) = g(z; u) - h(z; u) \sqrt{1 - \frac{z}{z_0(u)}},$$

where  $g(z_0; u) = y_0(u)$  and  $h(z_0(u)) \neq 0$ .



# Systems of functional equations

## Positive System.

Suppose, that several generating functions  $y_1(z), \dots, y_r(z)$  satisfy a **system of equations**

$$y_j(z) = \Phi_j(z, y_1(z), \dots, y_r(z)),$$

where  $\Phi_j(z, y_1, \dots, y_r)$  has as a power series expansion at  $(0, 0)$  in  $y_1, \dots, y_r$ . If these coefficients are **non-negative coefficients** (for all  $j$ ) then we call it **positive system**.

# Systems of functional equations

## Dependency Graph.

$$y_1 = \Phi_1(z, y_1, y_2, y_5)$$

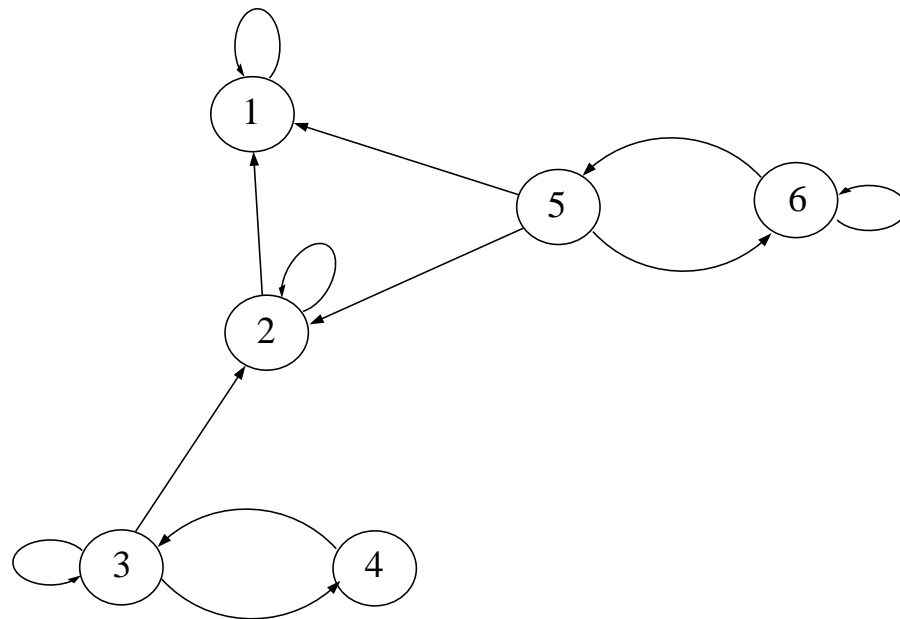
$$y_2 = \Phi_2(z, y_2, y_3, y_5)$$

$$y_3 = \Phi_3(z, y_3, y_4)$$

$$y_4 = \Phi_4(z, y_3)$$

$$y_5 = \Phi_5(z, y_6)$$

$$y_6 = \Phi_6(z, y_5, y_6)$$



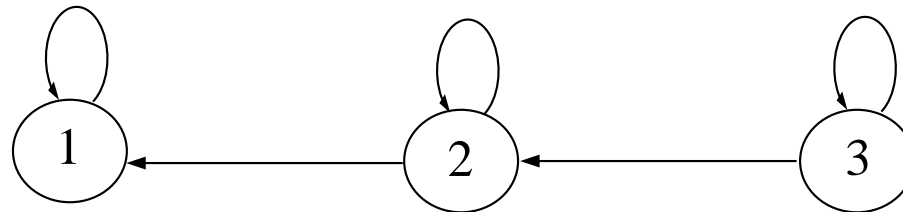
# Systems of functional equations

## Example

$$y_1 = z(e^{y_2} + y_1)$$

$$y_2 = z(1 + 2y_2y_3)$$

$$y_3 = z(1 + y_3^2)$$



$$y_1(z) = \frac{z}{1-z} \exp\left(\frac{z}{\sqrt{1-4z^2}}\right)$$

$$y_2(z) = \frac{z}{\sqrt{1-4z^2}}$$

$$y_3(z) = \frac{1 - \sqrt{1-4z^2}}{2z}$$

# Systems of functional equations

**Dependency graph:**  $G_{\Phi} = (V, E)$

$V$  ... vertex set =  $\{y_1, y_2, \dots, y_r\}$

$E$  ... (directed) edge set:

$$(y_i, y_j) \in E : \iff y_j(z) \text{ depends on } y_i(z)$$

$$\iff \Phi_j \text{ depends on } y_i$$

$$\iff \frac{\partial \Phi_j}{\partial y_i} \neq 0.$$

**Stongly connected dependency graphs.**

$$G_{\Phi} \text{ is strongly connected} \iff \Phi_{\mathbf{y}} := \left( \frac{\partial \Phi_j}{\partial y_i} \right) \text{ irreducible}$$

$\iff$  no subsystem can be solved  
before the whole system

# Systems of functional equations

A digraph  $G$  is **strongly connected** if each pair of vertices  $(v_1, v_2)$  is connected by a (directed) path.

A **positive** matrix  $A = (a_{i,j})$ , i.e.  $a_{i,j} \geq 0$ , is **irreducible** if for every pair of indices  $(i_1, i_2)$  there exists an integer  $m$  such that  $a_{i_1, i_2}^{(m)} > 0$ , where  $A^m = (a_{i,j}^{(m)})$ .

A digraph  $G$  is **strongly connected** if and only if its adjacency matrix  $A(G)$  is **irreducible**.

# Systems of functional equations

## Perron-Frobenius theory.

Every **positive irreducible** matrix  $A = (a_{i,j})$  has a real positive eigenvalue  $r(A)$  with the property that all other eigenvalues have modulus  $\leq r(A)$ . Furthermore,  $r(A)$  is a **simple eigenvalue**.

If  $B < A$ , that is  $b_{i,j} \leq a_{i,j}$  for all pairs  $(i,j)$  but  $A \neq B$ , then  $r(B) < r(A)$ .

In particular if  $B$  is a **submatrix** of  $A$ , then we also have  $r(B) < r(A)$ .

# Systems of functional equations

**Theorem** [D., Lalley, Woods]

Suppose that  $y = \Phi(z, y)$  is a **positive** and **non-linear** system.  
Suppose further, that the **dependency graph** of the system  
 $y = \Phi(z, y)$  is **strongly connected**.

Let  $z_0 > 0$ ,  $y_0 = (y_{0,0}, \dots, y_{r,0}) > 0$  (inside the region of convergence)  
satisfy the system of equations:  $(\Phi = (\Phi_1, \dots, \Phi_r))$

$$y_0 = \Phi(z_0, y_0), \quad 0 = \det(\mathbf{I} - \Phi_y(z_0, y_0))$$

such that all eigenvalues of  $\Phi_y(z_0, y_0)$  have modulus  $\leq 1$ .

Then there exists analytic function  $g_j(z), h_j(z) \neq 0$  such that locally

$$y_j(z) = g_j(z) - h_j(z) \sqrt{1 - \frac{z}{z_0}}.$$

# Systems of functional equations

Example.

$$t_{True}(z) = 2zt_{x_1}t_{\bar{x}_1} + 2zt_{True}(z)t(z)$$

$$t_{False}(z) = 2zt_{x_1}t_{\bar{x}_1} + 2zt_{False}(z)t(z)$$

$$t_{x_1}(z) = 1 + 2zt_{x_1}(z)^2 + 2zt_{x_1}(z)t_{True}(z) + 2zt_{x_1}(z)t_{False}(z)$$

$$t_{\bar{x}_1}(z) = 1 + 2zt_{\bar{x}_1}(z)^2 + 2zt_{\bar{x}_1}(z)t_{True}(z) + 2zt_{\bar{x}_1}(z)t_{False}(z)$$

[ $t(z)$  abbreviates  $t(z) = t_{True}(z) + t_{False}(z) + t_{x_1}(z)$ ]



# Systems of functional equations

## Linear Systems.

$$y = \Phi(z, \mathbf{0}) + \Phi_y(z, \mathbf{0}) \cdot y$$

$$y(z) = (\mathbf{I} - \Phi_y(z, \mathbf{0}))^{-1} \Phi(z, \mathbf{0}).$$

If  $\Phi_y(z_0, \mathbf{0})$  is **irreducible**, has eigenvalue 1 and all other eigenvalues have modulus  $\leq 1$  then

$$\det(\mathbf{I} - \Phi_y(z, \mathbf{0})) = (1 - z/z_0)K(z)$$

and consequently all functions  $y_j(z)$  have a **polar singularity** of order 1 at  $z = z_0$ .

**Conclusion.** In a **positive irreducible system** we have either a common **polar singularity** or a **squareroot singularity**.

# Systems of functional equations

**Idea of the proof** (reduction to a single equation)

$$y = (y_1, \dots, y_r) = (y_1, \bar{y}), \quad \Phi = (\Phi_1, \dots, \Phi_r) = (\Phi_1, \bar{\Phi})$$

$$y = \Phi(y, z) \quad \Longleftrightarrow \quad \begin{aligned} y_1 &= \Phi_1(y_1, \bar{y}, z), \\ \bar{y} &= \bar{\Phi}(y_1, \bar{y}, z) \end{aligned}$$

The second system has dominant eigenvalue  $< 1$

$\implies \bar{y} = \bar{y}(z, \boxed{y_1})$  is **analytic**

Insertion of this analytic solution into the first equation:

$$\boxed{y_1 = \Phi_1((y_1, \bar{y}(z, y_1)), z) = G(y_1, z)}$$

leads to **single equation**.

# Systems of functional equations

**Existence of limiting probabilities for Boolean functions.**

[Chauvin+Flajolet+Gardy+Gittenberger]

The system

$$t_f(z) = \mathbf{1}_{[f \text{ literal}]} + z \sum_{g,h:g \vee h=f} t_g(z)t_h(z) + z \sum_{g,h:g \wedge h=f} t_g(z)t_h(z)$$

has a strongly connected dependency graph. The common radius of convergence is  $z_0 = 1/(16n)$ . Consequently we have

$$t_f(z) = \alpha_f - \beta_f \sqrt{1 - 16nz} + \dots$$

and the limiting probabilities  $P(f) = \beta_f/(4n)$  exist.

# General Dependency Graphs

## Dependency Graph and Reduced Dependency Graph

$$y_1 = \Phi_1(z, y_1, y_2, y_5)$$

$$y_2 = \Phi_2(z, y_2, y_3, y_5)$$

$$y_3 = \Phi_3(z, y_3, y_4)$$

$$y_4 = \Phi_4(z, y_3)$$

$$y_5 = \Phi_5(z, y_6)$$

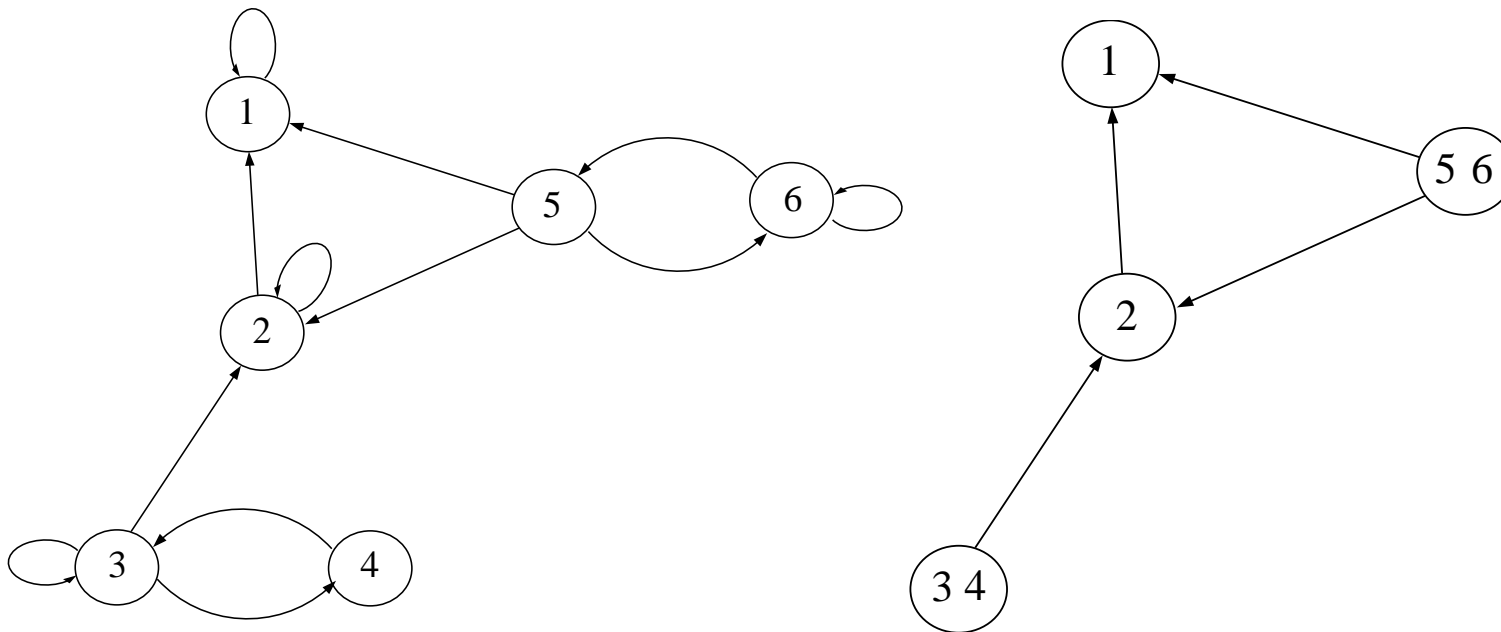
$$y_6 = \Phi_6(z, y_5, y_6)$$

$$y_1 = \Phi_1(z, y_1, y_2, (y_5, y_6))$$

$$y_2 = \Phi_2(z, y_2, (y_3, y_4), (y_5, y_6))$$

$$(y_3, y_4) = (\Phi_3, \Phi_4)(z, y_3, y_4)$$

$$(y_5, y_6) = (\Phi_5, \Phi_6)(z, y_5, y_6)$$



# General Dependency Graphs

## Theorem 1

Suppose that  $y = \Phi(z, y)$  is a **positive** system of entire functions such that there is a unique solution  $(y_1(z), \dots, y_r(z))$  that is analytic at  $z = 0$ .

Then all functions  $y_j(z)$  have **non-negative coefficients** and a **finite radius of convergence**  $\rho_j$ .

(A) If  $\frac{\partial^2 \Phi_j}{\partial y_j^2} \neq 0$  (for all  $j$ ) then for every  $j$  there exists an integer  $k_j \geq 1$  such that locally

$$y_j(z) = a_{0,j} + a_{1,j}(1 - z/\rho_j)^{1/2^{k_j}} + a_{2,j}(1 - z/\rho_j)^{2/2^{k_j}} + \dots.$$

# General Dependency Graphs

## Theorem 1 (cont.)

**(B)** If we just have the condition that for all pairs  $(i, j)$  with  $\frac{\partial \Phi_j}{\partial y_i} \neq 0$

there exists  $k$  with  $\frac{\partial^2 \Phi_j}{\partial y_i y_k} \neq 0$  then for every  $j$  we either have

$$y_j(z) = a_{0,j} + a_{1,j}(1 - z/\rho_j)^{2^{-k_j}} + a_{2,j}(1 - z/\rho_j)^{2 \cdot 2^{-k_j}} + \dots$$

for an integer  $k_j \geq 1$  or

$$y_j(z) = \frac{a_{-1,j}}{(1 - z/\rho_j)^{2^{-k_j}}} + a_{0,j} + a_{1,j}(1 - z/\rho_j)^{2^{-k_j}} + \dots.$$

for an integer  $k_j \geq 0$ .

# General Dependency Graphs

(Counter-)Example.

$$y_1 = z(e^{y_2} + y_1)$$

$$y_2 = z(1 + 2y_2y_3)$$

$$y_3 = z(1 + y_3^2)$$

$$y_1(z) = \frac{z}{1-z} \exp\left(\frac{z}{\sqrt{1-4z^2}}\right)$$

$$y_2(z) = \frac{z}{\sqrt{1-4z^2}}$$

$$y_3(z) = \frac{1 - \sqrt{1-4z^2}}{2z}$$

- $\Phi_3 = z(1 + y_3^2)$  ... satisfies (A)
- $(\Phi_2, \Phi_3) = (z(1 + 2y_2y_3), z(1 + y_3^2))$  ... satisfies (B)
- $(\Phi_1, \Phi_2, \Phi_3) = (z(e^{y_2} + y_1), \dots)$  **does not satisfy (B)**

# General Dependency Graphs

## Theorem 2

Suppose that  $y = \Phi(z, y)$  is a **positive polynomial** system of functions such that there is a unique solution  $(y_1(z), \dots, y_r(z))$  that is analytic at  $z = 0$ .

Then all functions  $y_j(z)$  have **non-negative coefficients** and a **finite radius of convergence**  $\rho_j$  and for every  $j$  we either have

$$y_j(z) = a_{0,j} + a_{1,j}(1 - z/\rho_j)^{2^{-k_j}} + a_{2,j}(1 - z/\rho_j)^{2 \cdot 2^{-k_j}} + \dots$$

for an integer  $k_j \geq 1$  or

$$y_j(z) = \frac{a_{-m_j,j}}{(1 - z/\rho_j)^{m_j 2^{-k_j}}} + \frac{a_{-m_j+1,j}}{(1 - z/\rho_j)^{(m_j-1) 2^{-k_j}}} + \dots$$

for integer  $k_j \geq 0$  and  $m_j \geq 1$ .



# General Dependency Graphs

Example.

$$y_1 = z(y_2^3 + y_1)$$

$$y_2 = z(1 + 2y_2y_3)$$

$$y_3 = z(1 + y_3^2)$$

$$y_1(z) = \frac{z}{1-z} \left( \frac{z}{\sqrt{1-4z^2}} \right)^3$$

$$y_2(z) = \frac{z}{\sqrt{1-4z^2}}$$

$$y_3(z) = \frac{1 - \sqrt{1-4z^2}}{2z}$$

$y_1(x)$  has dominant singularity  $(1 - 2z)^{-3/2}$ .

# General Dependency Graphs

Example.

$$y_1 = z(y_2 + y_1^2)$$

$$y_2 = z(y_3 + y_2^2)$$

$$y_3 = z(1 + y_3^2)$$

$$y_1(z) = \frac{1 - (1 - 2z)^{1/8} \sqrt{2z \sqrt{2z \sqrt{1 + 2z} + \sqrt{1 - 2z}} + (1 - 2z)^{3/4}}}{2z}$$

$$y_2(z) = \frac{1 - (1 - 2z)^{1/4} \sqrt{2z \sqrt{1 + 2z} + \sqrt{1 - 2z}}}{2z}$$

$$y_3(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z}$$

$y_1(x)$  has dominant singularity  $(1 - 2z)^{1/8}$ .

# General Dependency Graphs

Two equations for case (A)

$$y_1 = \Phi_1(z, y_1, y_2)$$

$$y_2 = \Phi_2(z, y_2)$$

$$\implies y_2(z) = g_2(z) - h_2(z)\sqrt{1 - z/\rho_2},$$

$$y_1(z, y_2) = g_1(z, y_2) - h_1(z, y_2)\sqrt{1 - z/\rho(y_2)}$$

$$\implies y_1(z) = y_1(z, y_2(z))$$

$$= g_1(z, y_2(z)) - h_1(z, y_2(z))\sqrt{1 - z/\rho(y_2(z))}$$

$$= g_1(z, y_2(z)) - h_1(z, y_2(z))\rho(y_2(z))^{-1/2}\sqrt{\rho(y_2(z)) - z}$$

**3 cases:** (1)  $\rho(y_2(\rho_2)) > \rho_2$    (2)  $\rho(y_2(\rho_2)) = \rho_2$    (3)  $\rho(y_2(\rho_2)) < \rho_2$

# General Dependency Graphs

Case (1).  $\rho(y_2(\rho_2)) > \rho_2$

$$\begin{aligned} g_1(z, y_2(z)) &= g_1\left(z, g_2(z) - h_2(z)\sqrt{1 - z/\rho_2}\right) \\ &= g_1(\rho_2, g_2(\rho_2)) - g_{1,y}(\rho_2, g_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2} + \dots \end{aligned}$$

$$h_1(z, y_2(z)) = h_1(\rho_2, g_2(\rho_2)) - h_{1,y}(\rho_2, g_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2} + \dots$$

$$\rho(y_2(z)) - z = \rho(y_2(\rho_2)) - \rho_2 - \rho'(y_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2} + \dots$$

$$\begin{aligned} \sqrt{\rho(y_2(z)) - z} &= \sqrt{\rho(y_2(\rho_2)) - \rho_2 - \rho'(y_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2} + \dots} \\ &= \sqrt{\rho(y_2(\rho_2)) - \rho_2} - c_1\sqrt{1 - z/\rho_2} + \dots \end{aligned}$$

$$\begin{aligned} \implies y_1(z) &= g_1(z, y_2(z)) - h_1(z, y_2(z))\rho(y_2(z))^{-1/2}\sqrt{\rho(y_2(z)) - z} \\ &= c_0 - c_1\sqrt{1 - z/\rho_2} + \dots \end{aligned}$$

# General Dependency Graphs

Case (2).  $\boxed{\rho(y_2(\rho_2)) = \rho_2}$

$$\begin{aligned}\rho(y_2(z)) - z &= \rho(y_2(\rho_2)) - \rho_2 - \rho'(y_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2} + \dots \\ &= c'_1\sqrt{1 - z/\rho_2} + \dots\end{aligned}$$

$$\begin{aligned}\sqrt{\rho(y_2(z)) - z} &= \sqrt{c'_1\sqrt{1 - z/\rho_2} + \dots} \\ &= \sqrt{c'_1}(1 - z/\rho_2)^{1/4} + c'_2(1 - z/\rho_2)^{3/4} + \dots\end{aligned}$$

$$\begin{aligned}\implies y_1(z) &= g_1(z, y_2(z)) - h_1(z, y_2(z))\rho(y_2(z))^{-1/2}\sqrt{\rho(y_2(z)) - z} \\ &= c_0 + c_1(1 - z/\rho_2)^{1/4} + c_2\sqrt{1 - z/\rho_2} + \dots\end{aligned}$$

# General Dependency Graphs

Case (3).  $\rho(y_2(\rho_2)) < \rho_2$

There exists  $\rho_1 < \rho_2$  with  $\rho(y_2(\rho_1)) = \rho_1$ :

$$\begin{aligned}\rho(y_2(z)) - z &= \rho(y_2(\rho_1)) - \rho_1 + \rho'(y_2(\rho_1))y_2'(\rho_1)(z - \rho_1) \\ &= c_1''(\rho_1 - z) + \dots\end{aligned}$$

$$\begin{aligned}\sqrt{\rho(y_2(z)) - z} &= \sqrt{c_1''}\sqrt{\rho_1 - z} + \dots \\ &= \sqrt{c_1''\rho_1}\sqrt{1 - z/\rho_1} + \dots\end{aligned}$$

$$\begin{aligned}\implies y_1(z) &= g_1(z, y_2(z)) - h_1(z, y_2(z))\rho(y_2(z))^{-1/2}\sqrt{\rho(y_2(z)) - z} \\ &= c_0 - c_1\sqrt{1 - z/\rho_1} + \dots\end{aligned}$$

with  $\rho_1 < \rho_2$ .

**Remark.** It is important that  $\lim_{u \rightarrow \infty} \rho(u) = 0$ . This is assured by conditions (A) or (B).

# Infinite Systems

Infinite linear systems.  $y = A(z)y + b(z) \implies \boxed{y(z) = (I - A(z))^{-1}b(z)}$

Example.

$$y_1 = 1 + zy_2$$

$$y_j = z(y_{j-1} + y_{j+1})$$

$$\implies \boxed{y_j(z) = \frac{1}{z} \left( \frac{1 - \sqrt{1 - 4z^2}}{2z} \right)^j}$$

$$A = \begin{pmatrix} 0 & z & 0 & 0 & \cdots \\ z & 0 & z & 0 & \cdots \\ 0 & z & 0 & z & \\ 0 & 0 & z & \cdots & \cdots \\ \vdots & \vdots & \vdots & & \cdots \end{pmatrix}$$

# Infinite Systems

Example.

$$y_0 = 1 + z^2 y_0 + z y_1$$

$$y_1 = z(1 + z)y_0 + z y_2$$

$$y_j = z(y_{j-1} + y_{j+1})$$

$$\Rightarrow y_j(z) = \frac{2}{\sqrt{1-2z}(\sqrt{1+2z} - \sqrt{1-2z})} \left( \frac{1 - \sqrt{1-4z^2}}{2z} \right)^{j+1} \quad (j \geq 1)$$

$$A = \begin{pmatrix} z^2 & z & 0 & 0 & \cdots \\ z(1+z) & 0 & z & 0 & \cdots \\ 0 & z & 0 & z & \\ 0 & 0 & z & \ddots & \ddots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$



# Infinite Systems

**Compact operator**  $A(z)$ .  $y = A(z)y + b(z)$

$A(z)$  ... irreducible (and compact in a proper  $\ell^p$ -space)

$r(A(z))$  ... spectral radius of  $A(z)$ .

$r(A(z_0)) = 1 \implies$  resolvent  $(x\mathbf{I} - \mathbf{A}(z_0))^{-1}$  has a simple pole

$\implies y(z) = (\mathbf{I} - A(z))^{-1}b(z)$  has a **simple pole** at  $z = z_0$ .

This is the same situation as in the finite dimensional case

# Infinite Systems

**Theorem** [Lalley, Morgenbesser]

Suppose that  $y = (y_j)_{j \geq 1} = \Phi(z, y)$  is a **positive, non-linear, infinite** and **irreducible** system such that  $\Phi_y(z, y)$  is **compact**.

Let  $z_0 > 0$ ,  $y_0 = (y_{0,0}, \dots, y_{r,0}) > 0$  (inside the region of convergence) satisfy the system of equations:  $(\Phi = (\Phi_1, \dots, \Phi_r))$

$$\boxed{y_0 = \Phi(z_0, y_0), \quad r(\Phi_y(z_0, y_0)) = 1.}$$

Then there exists analytic function  $g_j(z), h_j(z) \neq 0$  such that locally

$$\boxed{y_j(z) = g_j(z) - h_j(z) \sqrt{1 - \frac{z}{z_0}}.}$$

with  $g_j(z_0) = (y_0)_j$  and  $h_j(z_0) \neq 0$ .

# Infinite Systems

A linear operator  $A$  is **compact** if the image of a bounded set is relative compact.

[**Informally**, an infinite matrix  $A$  is compact if it can be **well approximated** by finite dimensional matrices.]

An infinite matrix  $A = (a_{i,j})$  is **irreducible** if for every pair of indices  $(i_1, i_2)$  there exists an integer  $m$  such that  $a_{i_1, i_2}^{(m)} > 0$ , where  $A^m = (a_{i,j}^{(m)})$ .

An **infinite, irreducible, positive and compact** matrix  $A = (a_{i,j})$  has a dominant positive real eigenvalue  $r(A)$  (the **spectral radius**) that is **isolated** and **simple**.

# Infinite Systems

## Lemma

$A = (a_{i,j})_{i,j \geq 1}$  ... positive, irreducible, compact

$B = (a_{i+1,j+1})_{i,j \geq 1}$  [i.e., first column and row are deleted]

$$\implies \boxed{r(B) < r(A)}$$

**Remark.** With the help of this property the proof is precisely the same as in the finite dimensional case. It is possible to reduce the infinite system to a single equation.