COEFFICIENTS OF ALGEBRAIC FUNCTIONS: FORMULAE AND ASYMPTOTICS

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ABSTRACT. This paper studies the coefficients of algebraic functions. First, we recall the too-less-known fact that these coefficients f_n always a closed form. Then, we study their asymptotics, known to be of the type $f_n \sim CA^n n^\alpha$. When the function is a power series associated to a context-free grammar, we solve a folklore conjecture: the appearing critical exponents α belong to a subset of dyadic numbers, and we initiate the study the set of possible values for A. We extend what Philippe Flajolet called the Drmota–Lalley–Woods theorem (which is assuring $\alpha = -3/2$ as soon as a "dependency graph" associated to the algebraic system defining the function is strongly connected): We fully characterize the possible singular behaviors in the non-strongly connected case. As a corollary, it shows that certain lattice paths and planar maps can not be generated by a context-free grammar (i.e., their generating function is not $\mathbb{N}$-algebraic). We give examples of Gaussian limit laws (beyond the case of the Drmota–Lalley–Woods theorem), and examples of non Gaussian limit laws. We then extend our work to systems involving non-polynomial entire functions (non-strongly connected systems, fixed points of entire function with positive coefficients). We end by discussing few algorithmic aspects.

RÉSUMÉ. Cet article a pour héros les coefficients des fonctions algébriques. Après avoir rappelé le fait trop peu connu que ces coefficients f_n admettent toujours une forme close, nous étudions leur asymptotique $f_n \sim CA^n n^\alpha$. Lorsque la fonction algébrique est la série génératrice d'une grammaire non-contextuelle, nous résolvons une vieille conjecture du folklore : les exposants critiques α sont restreints à un sous-ensemble des nombres dyadiques, et nous amorçons l'étude de l'ensemble des valeurs possibles pour A. Nous étendons ce que Philippe Flajolet appelait le théorème de Drmota–Lalley–Woods (qui affirme que $\alpha = -3/2$ dès lors qu'un "graphe de dépendance" associé au système algébrique est fortement connexe) : nous caractérisons complètement les exposants critiques dans le cas non fortement connexe. Un corolaire immédiat est que certaines marches et cartes planaires ne peuvent pas être engendrées par une grammaire non-contextuelle non ambigüe (*i. e.*, leur série génératrice n'est pas $\mathbb{N}$-algébrique). Nous donnons un critère pour l'obtention d'une loi limite gaussienne (cas non couvert par le théorème de Drmota–Lalley–Woods), et des exemples de lois *non* gaussiennes. Nous étendons nos résultats aux systèmes d'équations de degré infini (systèmes non fortement connexes impliquant des points fixes de fonctions entières à coefficients positifs). Nous terminons par la discussion de quelques aspects algorithmiques.

1. INTRODUCTION

The theory of context-free grammars and its relationship with combinatorics was initiated by the article of Noam Chomsky and Marcel-Paul Schützenberger in 1963 [27], where it is shown that the generating function of number of words generated by a non ambiguous context-free grammar is algebraic.

Since then, there has been much use of context-free grammars in combinatorics, several chapters of the Flajolet & Sedgewick book "Analytic Combinatorics" [47] are dedicated to what they called the "symbolic method" (which is in large parts isomorphic to the Joyal theory of species [53, 13], and when restricted to context-free grammar, it is sometimes called the "DVS methodology", for Delest–Viennot–Schützenberger, as the Bordeaux combinatorics school indeed made a deep use of it, e.g. for enumeration of polyominoes [34], lattice paths [42]). They also allow to enumerate trees [49, 63], avoiding-pattern permutations [57, 62, 2]. some type of planar maps/triangulations/Apollonian networks [15] non crossing configurations, dissections of polygons [45] (a result going back to Euler in 1763, one of the founding problems of analytic combinatorics!). Links between asymptotics of algebraic functions and (inherent) ambiguity of context-free languages was studied in [55, 44],

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and for prefix of infinite words in [3]. Growth rates are studied in [24, 25], in link with asymptotics of random walks [80, 26, 50, 58, 59]. Application in bio-informatics or patterns in RNA are given in [67, 79, 36, 29]. Its key rôle for uniform random generation is illustrated by [43, 33, 23, 37]. More links with monadic second-order logic, tiling problems and vector addition systems appears in [65, 81].

Many algebraic functions also pop ups in combinatorics [19]. Quite often, they come as "diagonal" of rational functions [10, 12], or as solution of functional equations (solvable by the kernel method and its variants [20, 6]), and the interplay with their asymptotics is crucial for analysis of lattice paths [7], walks with an infinite set of jumps [4, 11] (which are thus not coded by a grammar on a finite alphabet), or planar maps [8].

Plan of this article:

- In Section 2, we give few definitions, mostly illustrating the link between context-free grammars, solutions of positive algebraic systems and $\mathbb{N}$-algebraic functions.
- In Section 3, we survey some closure properties of algebraic functions and give a closed form for their coefficients.
- Section 4, we state and prove our main theorem on the possible asymptotics of algebraic functions (associated to a context-free grammar with positive weights).
- Section 5, we prove that the associated limit laws are Gaussian for a broad variety of cases.
- We end with a conclusion pinpointing some extensions (algorithmic considerations, extension to infinite systems, or systems involving entire functions).

2. Definitions: N-algebraic functions, context-free grammars and pushdown automata

For the notions of automata, pushdown automata, context-free grammars, we refer to the first three chapters of [78] (by Perrin on finite automata, by Berstel and Boasson on context-free languages, by Salomaa [71] on formal languages and power series) or to the more recent survey [66] in [40]. Another excellent compendium on the subject is the handbook of formal languages [70] and the Lothaire trilogy [61].

An $\mathbb{S}$-algebraic function is a function $y_1(z)$ solution of a system[1]:

$$\begin{cases} y_1 = P_1(z, y_1, \dots, y_d) \\ \vdots \\ y_d = P_d(z, y_1, \dots, y_d) \end{cases} \tag{1}$$

where each polynomial P_i has coefficients in any set $\mathbb{S}$ (in this article, we consider $\mathbb{S} = \mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}_+$, or $\mathbb{C}$). We restrict (with no loss of generality) to systems satisfying: each P_i is involving at least one y_j, the coefficient of y_i in P_i is not 1, and there is at least one $P_i(z, 0, \dots, 0)$ for which the coefficient of z is not 0. Such systems are called "proper" (or "well defined" or "well founded" or "well posed"), and correspond to context-free grammars for which one has no "infinite chain rules" (no epsilon production, no monic production). On the set of power series, $d(F(z), G(z)) := 2^{-\mathrm{val}(F(z)-G(z))}$ is an ultrametric distance, this distance extends to vectors of functions, and allows to apply the Banach fixed-point theorem: it implies existence and uniqueness of a solution of the system as a d-tuple of power series $(y_1, \dots, y_d)$ (and they are analytic functions in 0, as we already know that they are algebraic by nature). A current mistake is to forget that there exist situations for which the system (1) can admit several solutions as power series for y_1 (*nota bene:* there is no contradiction with our previous claim, which is considering *tuples*). By elimination theory (resultant or Gröbner bases), $\mathbb{S}$-algebraic functions are algebraic functions.

We give now few trivial/folklore results: $\mathbb{N}$-algebraic functions correspond to generating function of context-free grammar (this is often called the Chomsky–Schützenberger theorem), or, equivalently, pushdown automata (via e.g. a Greibach normal form). $\mathbb{Z}$-algebraic functions have no natural simple combinatorial structures associated to them, but they are the difference of two $\mathbb{N}$-algebraic functions (as can be seen by introducing new unknowns splitting in two the previous ones, and writing the system involving positive coefficients on one

[1] In this article, we will often summarize the system (1) via the convenient short notation $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$, where bold fonts are used for vectors.

side, and negative coefficients on the other side). They also play an important rôle as any algebraic generating with integer coefficients can be considered as a $\mathbb{Z}$-algebraic function: this is a priori no evident, as an algebraic function is not defined by an equation of the type $y_1 = P(z, y_1, \dots)$ but by a more general type of equation, namely $P(z, y(z)) = 0$ for P a polynomial $\in \mathbb{Z}[z, y]$.

An $\mathbb{N}$-rational function is a function solution of a system (1) where each polynomial P_i has coefficients in $\mathbb{N}$ and total degree 1. Such functions correspond to generating functions of regular expressions or, equivalently, automata (a result often attributed to Kleene [56]).

3. Closed form for coefficient of algebraic function

A first natural question is how can we compute the n-th coefficient f_n of an algebraic power series? The fastest way is relying on the theory of D-finite functions. A function $F(z)$ is D-finite if it satisfies a differential equation with coefficients which are polynomials in z; equivalently, its coefficients f_n satisfy a linear recurrence with coefficients which are polynomials in n. They are numerous algorithms to deal with this important class of functions, which includes a lot of special functions from physics, number theory and also combinatorics [75]. In combinatorics, Comtet [31, 32] popularized the fact that algebraic functions are D-finite. It is amusing that this is in fact an old theorem rediscovered many times, by Tannery [77], Cockle and Harley [30, 51] in their method for solving quintic equations via ${}_4F_3$ hypergeometric functions. Last but not least, this theorem can also be found in an unpublished manuscript of Abel [1, p. 287]!

The world of D-finite functions offers numerous closure properties, let us mention some of them related to algebraic functions (due respectively to Harris & Sibuya [52], Singer [73] for the two next ones, Jungen [54], Denef and Lipshitz [35], Furstenberg [48], Schwarz [72] and Beukers & Heckman [14]):

- f and $1/f$ are simultaneously D-finite if and only if f'/f is algebraic.
- f and $\exp(\int f)$ are simultaneously D-finite if and only if f'/f is algebraic.
- Let g be algebraic of genus ≥ 1, then f and $g(f)$ are simultaneously D-finite if and only if f is algebraic.
- The Hadamard product of a rational and an algebraic function is algebraic.
- Each algebraic function is the diagonal of a bivariate rational function.
- In finite fields, Hadamard products of algebraic functions are algebraic.
- The set of generalized hypergeometric functions ${}_nF_{n-1}$ which are algebraic is well identified.

The linear recurrence satisfied by f_n allows to compute in linear time all the coefficients $f_0, \dots, f_n$, more precisely, it is proved in [16] that there exists an algorithm of complexity $O(nd^2 \ln d)$, where d is the degree of the function. If one just wants the n-th coefficient f_n, it is possible to get it in $O(\sqrt{n})$ operations [28]. Many of these features (and few others related to random generation and context-free grammars, and corresponding asymptotics) are implemented in the "Algolib" library, a Maple package developed by Flajolet, Salvy, Zimmermann, Chyzak, Mishna, Mezzarobba, ... (see `http://algo.inria.fr/libraries/`).

A less known fact (however mentioned in VII.34 p. 495 in [47]) is that these coefficients admit a closed form expression as a finite linear combination of weighted multinomial numbers. The multinomial number is the number of ways to divide m objects into d groups, of cardinality $m_1, \dots, m_d$ (with $m_1 + \dots + m_d = m$):

$$[{u_1}^{m_1} \dots {u_d}^{m_d}](u_1 + \dots + u_d)^m = \binom{m}{m_1, \dots, m_k} = \frac{m!}{m_1! \dots m_d!}\,.$$

More precisely, one has the following theorem:

Theorem 1 (The Flajolet–Soria formula for coefficients of algebraic function)**.** *Let $P(z, y)$ be a bivariate polynomial such that $P(0,0) = 0$, $P'_y(0,0) = 0$ and $P(z, 0) \neq 0$. Consider the algebraic function implicitly defined by $f(z) = P(z, f(z))$. Then, the Taylor coefficients of $f(z)$ are given by the following finite sum*

$$f_n = \sum_{m \geq 1} \frac{1}{m} [z^n y^{m-1}] P^m(z, y). \tag{2}$$

Accordingly, applying the multinomial theorem on $P(z,y)=\sum a_i z^{b_i} y^{c_i}$ *leads to*

$$f_n=\sum_{m\geq 1}\frac{1}{m}\sum_{\substack{m_1+\cdots+m_d=m\\ b_1m_1+\cdots+b_dm_d=n\\ c_1m_1+\cdots+c_dm_d=m-1}}\binom{m}{m_1,\ldots,m_d}{a_1}^{m_1}\ldots{a_d}^{m_d}\,. \tag{3}$$

The proof considers $y=P(z,y)$ as the perturbation at $u=1$ of the equation $y=uP(z,y)$, and then applying the Lagrange inversion formula (considering u as the main variable, and z as a fixed parameter) leads to the theorem.

This Flajolet–Soria formula was first published in the habilitation thesis of Michèle Soria in 1990, and then in 1998 in the proceedings of the Algorithms Seminar, it has also been found by Gessel (as published in 1999 in the exercise 5.39 p.148 of [75]), and it was finally also rediscovered in 2009 by Sokal [74].

4. Asymptotics for coefficients of algebraic function

The theory of Puiseux expansions or the theory of G-functions implies that the critical exponents are pure rational numbers for pure algebraic functions. Pure algebraic means algebraic but not rational, pure rational means rational but integer.

The following proposition shows that all rational numbers are reached:

Theorem 2. *For any rational number* a/b ($\notin\mathbb{N}$)*, there exists an algebraic power series with positive integer coefficients which has exactly the critical exponent* a/b.

Proof. First consider $F(z):=\frac{1-(1-a^2z)^{1/a}}{z}$, where a is any positive or negative integer. Accordingly, its coefficients are given by $f_n=\binom{1/a}{n+1}a^{2n+1}(-1)^n$. We are not aware of any trivial proof showing that these numbers f_n are integers. One possibility is to compare the p-adic valuation of $(n+2)!$ via the De Polignac/Legendre's formula with the one of $a^n\Pi_{k=0}^n(ak+a-1)$, as they satisfy $f_{n+1}=a(an+a-1)f_n/(n+2)$. Another nicer way, because it conveys more combinatorial insights is to use a link with a variant of Stirling numbers, as studied in [60]. To get get our claim, it is easy to see that $G(z):=((zF-1)^b+1)$ minus few of its first coefficients is also a power series with positive integer coefficients and have the critical behavior a/b: $G(z)\sim 2a^2-a(-a^2)^{a/b}(z-1/a^2)^{a/b}$ for $z\sim 1/a^2$. □

One may then wonder if there is something stronger. For example, is it the case that for any radius of convergence, any critical exponent is possible? It happens not to be the case, as can be seen via a result of Fatou: a power series with integer coefficients and radius of convergence 1 is either rational or transcendental (in fact the transcendental case is necessarily involving a natural boundary, this was a conjecture of Pólya proved by Carlson).

4.1. Well posed systems of functional equations.

We will only consider *positive well posed systems of functional equations* $\mathbf{y}=\mathbf{P}(z,\mathbf{y})$ that have the property that they have (unique) power series solutions $y_j=f_j(z)$ with non-negative coefficients.

In particular this means that $\mathbf{P}$ has non-negative coefficients. Furthermore, if $z=0$ there exists a (unique) non-negative vector $\mathbf{y}_0=\mathbf{f}(0)=(f_j(0))$ with $\mathbf{y}_0=\mathbf{P}(0,\mathbf{y}_0)$ that should be obtained iteratively by the recurrence $\mathbf{y}_{0,0}=\mathbf{0}$ and $\mathbf{y}_{0,k+1}=\mathbf{P}(0,\mathbf{y}_{0,k})$ (for $k\geq 0$). As the 1-dimensional example $y=P(z,y)$ with $P(z,y)=1+z+y^2$ shows this need not be the case even if $P_y(0,0)=0<1$.

If there is a non-negative vector $\mathbf{y}_0$ with $\mathbf{y}_0=\mathbf{P}(0,\mathbf{y}_0)$ then is recommendable to set $\tilde{\mathbf{y}}=\mathbf{y}+\mathbf{y_0}$ so that we obtain a system for $\tilde{\mathbf{y}}$ of the form $\tilde{\mathbf{y}}=\mathbf{G}(z,\tilde{\mathbf{y}})$ with $\mathbf{G}(z,\tilde{\mathbf{y}})=\mathbf{P}(z,\tilde{\mathbf{y}}+\mathbf{y}_0)-\mathbf{y}_0$. Since $\mathbf{P}$ has non-negative coefficients, the same holds for $\mathbf{G}$. Consequently it is no loss of generality to assume that we have a system $\mathbf{y}=\mathbf{P}(z,\mathbf{y})$ with $\mathbf{P}(0,\mathbf{0})=\mathbf{0}$.

The second property that we should have is that we the the iteration $\mathbf{f}_0(z)=\mathbf{0}$, $\mathbf{f}_{k+1}(z)=\mathbf{P}(z,\mathbf{f}_k(z))$ (for $k\geq 0$) should converge to the solution $\mathbf{f}(z)$. This can be seen either from the formal point of view or from the analytic point of view (in the latter case we only require this property for sufficiently small $z\neq 0$). If we require convergence in the formal sense (that is, for every n the n-coefficient of $\mathbf{y}_k(z)$ should be constant for sufficiently large k) then we have to require that the Jacobian matrix $\mathbf{P}_{\mathbf{y}}(0, \mathbf{0})$ is nil-potent (compare with [68]). This is certainly satisfied if $\mathbf{P}$ is of the form $\mathbf{P}(z, \mathbf{y}) = z\mathbf{H}(z, \mathbf{y})$, where the Jacobian is identically zero. Actually, if the Jacobian is nil-potent of order k then we can replace the original equation by the k-th iterated equation $\mathbf{y} = \mathbf{P}^k(z, \mathbf{y})$, where the Jacobian is identically to zero. From the analytic point of view we just have to require that the spectral radius $r(\mathbf{P}_{\mathbf{y}}(0, \mathbf{0})$ satisfies $r(\mathbf{P}_{\mathbf{y}}(0, \mathbf{0})) < 1$. In this case the iteration is a contraction (if z is sufficiently small). Finally we note that by construction the coefficients of $\mathbf{f}_k(z)$ are non-negative. Hence, the same holds for the limit $\mathbf{f}(z)$.

In order simplify the statement of Theorem 3 we also assume that none of the solution functions $f_j(z)$ are identically to zero or just polynomials. For example, the system $y_1 = z + z^2y_2^2, y_2 = z + z^2$ has just a polynomial solution. Actually it is easy to detect (even algorithmically) whether some of the functions $f_j(z)$ are polynomials or not. And it is clear that in the case, where polynomial solutions appear we can eliminate them easily by replacing them by their polynomial representation. Hence, without loss of generality we can assume that all functions $f_j(z)$ are *real power series*.

In what follows, a system of positive system of equations will be called *well posed* if $\mathbf{P}(0, \mathbf{0}) = \mathbf{0}$, if the Jacobian $\mathbf{P}_{\mathbf{y}}(0, \mathbf{0})$ has spectral radius is smaller than 1, and if all functions $f_j(z)$ are no polynomials.

4.2. Main result: dyadic critical exponents for $\mathbb{N}$-algebraic function.

Theorem 3. *Let $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ be a well posed positive polynomial system of functional equations.*

Then the solutions $f_j(z)$ have positive and finite radii of convergence ρ_j. Furthermore, the singular behavior of $f_j(z)$ around ρ_j is either of algebraic type

$$f_j(z) = f_j(\rho_j) + c_j(1 - z/\rho_j)^{2^{-k_j}} + c_j'(1 - z/\rho_j)^{2 \cdot 2^{-k_j}} + \cdots, \tag{4}$$

where $c_j \neq 0$ and where k_j is a positive integer or of type

$$f_j(z) = \frac{d_j}{(1 - z/\rho_j)^{m_j 2^{-k_j}}} + \frac{d_j'}{(1 - z/\rho_j)^{(m_j-1)2^{-k_j}}} + \cdots, \tag{5}$$

where $d_j \neq 0$, m_j are positive integers and k_j are non-negative integers.

The following example show that all the cases mentionned in Theorem 3 are indeed appearing:

Example 1. The system of equations $y_1 = z(y_2 + y_1^2), y_2 = z(y_3 + y_2^2), y_3 = z(1 + y_3^2)$ has the following (explicit) solution

$$f_1(z) = \frac{1 - (1 - 2z)^{1/8}\sqrt{2z\sqrt{2z\sqrt{1 + 2z} + \sqrt{1 - 2z}} + (1 - 2z)^{3/4}}}{2z}$$
$$f_2(z) = \frac{1 - (1 - 2z)^{1/4}\sqrt{2z\sqrt{1 + 2z} + \sqrt{1 - 2z}}}{2z}$$
$$f_3(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z}$$

Here $f_1(z)$ has dominant singularity $(1 - 2z)^{1/8}$ and it is clear that this example can be generalized: indeed, consider the system $y_i = z(y_{i+1} + y_i^2)$ for $i = 1, \ldots, m - 1$, and $y_k = z(1 + y_k^2)$, it leads to behavior $(1 - 2z)^{2^{-k}}$ for each $k \geq 1$. Now, taking the system of equations $y = z(y_0^m + y), y_0 = z(1 + 2y_0y_1)$ leads to a behavior $(1 - 2z)^{-m2^{-k}}$ for each $m \geq 1$ and $k \geq 0$. See also [76] for another explicit combinatorial structure (a familly of colored tree related to a critical composition) exhibiting all these critical exponents.

Example 2. Similarly, many families of planar maps can not be generated by a non ambiguous context-free grammar, because of their critical exponent $-5/2$ [8].

The tables of lattice paths in the quarter plane [21] and their asymptotics (see e.g. the table page 13 of [17] where some of the connection constants are guessed, but all the critical exponents are proved, and this is enough for our point) allow to prove that many sets of jumps are giving a non algebraic generating function, as they lead to a critical exponent which is a negative integer. For many of the remaining asymptotics compatible with algebraicity, it is possible to use Ogden's pumping lemma for context-free languages, to prove that these walks can be not generated by a context-free grammar. One very neat exemple are Gessel walks (their algebraicity were a nice suprise [18]), where the hypergeometric formula for their coefficients leads to an asymptotic in $4^/n^{2/3}$ not compatible with $\mathbb{N}$-algebraicity.[2]

The critical exponents $-3/4, -1/4, 1/4$ which appear for walks on the slit plane [22] and other lattice paths questions [18] is compatible with $\mathbb{N}$-algebraicity, this for sure by no way a proof that are indeed $\mathbb{N}$-algebraic (typically, they are not), and to get a constructive method to solve this question (input: a polynomial equation, output: a context-free specification, whenever it exists) is a challenging task.

4.3. **Auxiliary Results.** A main ingredient of the proof of Theorem 3 is the analysis of the *dependency graph* $G_{\mathbf{P}}$ of the system $y_j = P_j(z, y_1, \ldots, y_K)$, $1 \le j \le K$. The vertex set is $\{1, \ldots, K\}$ and there is a directed edge from i to j if P_j depends on y_i. (See subsection 4.4 for an example.) If the dependency graph is strongly connected then we are in very special case of Theorem 3 and here the result is already known (see [38]). Informally this means that there is no sub-system that can be solved before the whole system.

Actually there are two different situations.

Lemma 1. *Let $\mathbf{y} = \mathbf{A}(z)\mathbf{y} + \mathbf{B}(z)$ a positive affine and well posed system of equations, where the dependency graph is strongly connected. Then the functions $f_j(z)$ have a joint polar singularity ρ or order one as the dominant singularity:*

$$f_j(z) = \frac{c_j(z)}{1 - z/\rho},$$

where $c_j(z)$ is non-zero and analytic at $z = \rho$.

Lemma 2. *Let $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ a positive and well posed polynomial system of equations that is not affine and where the dependency graph is strongly connected. Then the functions $f_j(z)$ have a joint square-root singularity ρ as the dominant singularity, that is, they can be locally represented as*

$$f_j(z) = g_j(z) - h_j(z)\sqrt{1 - \frac{z}{\rho}},$$

where $g_j(z)$ and $h_j(z)$ are non-zero and analytic at $z = \rho$.

In the proof of Theorem 3 we will use in fact extended version of Lemma 1 and 2, where we introduce additional (polynomial) parameters, that is, we consider systems of functional equations of the form $\mathbf{y} = \mathbf{P}(z, \mathbf{y}, \mathbf{u})$, where $\mathbf{P}$ is now a polynomial in $z, \mathbf{y}, \mathbf{u}$ with non-negative coefficients and where the dependency graph (with respect to $\mathbf{y}$) is strongly connected. We also assume that $\mathbf{u}$ is strictly positive such that the spectral radius of the Jacobian $\mathbf{P}_{\mathbf{y}}(0, \mathbf{0}, \mathbf{u})$ is smaller than 1. Hence, we can consider the solution that we denote by $\mathbf{f}(z, \mathbf{u})$.

If we are in the affine setting ($\mathbf{y} = \mathbf{A}(z, \mathbf{u})\mathbf{y} + \mathbf{B}(z, \mathbf{u})$) it follows that $\mathbf{f}(z, \mathbf{u})$ has a polar singularity:

$$f_j(z, \mathbf{u}) = \frac{c_j(z, \mathbf{u})}{1 - z/\rho(\mathbf{u})}, \tag{6}$$

where the functions $\rho(\mathbf{u})$ and $c_j(z, \mathbf{u})$ are non-zero and analytic (see Lemma 3). Please observe that we have to distinguish two cases. If $\mathbf{A}(z, \mathbf{u}) = \mathbf{A}(z)$ does not depend on $\mathbf{u}$ then $\rho(\mathbf{u}) = \rho$ is constant and the dependence from $\mathbf{u}$ just comes from $\mathbf{B}(z, \mathbf{u})$. Of course, if $\mathbf{A}(z, \mathbf{u})$ depends on $\mathbf{u}$ then $\rho(\mathbf{u})$ is not constant. More precisely it depends exactly on those parameters that appear in $\mathbf{A}(z, \mathbf{u})$.

Similarly in the non-affine setting we obtain representations of the form

$$f_j(z, \mathbf{u}) = g_j(z, \mathbf{u}) - h_j(z, \mathbf{u})\sqrt{1 - \frac{z}{\rho(\mathbf{u})}}, \tag{7}$$

[2] The fact that critical exponents involving 1/3 were not possible was an informal conjecture in the communauty for years. We thank Philippe Flajolet, Mireille Bousquet-Mélou and Gilles Schaeffer, who encouraged us to work on this question.

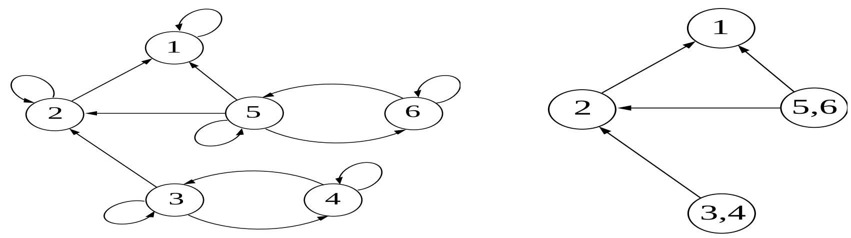

FIGURE 1. Dependency graph $G_{\mathbf{P}}$ and reduced dependency graph $\widetilde{G_{\mathbf{P}}}$

where the functions $\rho(\mathbf{u})$, $g_j(z,\mathbf{u})$, and $h_j(z,\mathbf{u})$ are non-zero and analytic. In this case $\rho(\mathbf{u})$ is always non-constant and depends on all parameters (see Lemma 3).

Actually we have to be careful with the property that $\rho(\mathbf{u})$ is analytic. By inspecting the proofs of Lemma 1 and 2 it immediately follows that $\rho(\mathbf{u})$ exists but analyticity is not immediate. For notational convenience we will denote by D_0 the set of positive real vectors $\mathbf{u}$, for which $r(\mathbf{P_y}(0,\mathbf{0},\mathbf{u}))<1$.

Lemma 3. *The function $\rho(\mathbf{u})$ that appears in the representations (6) and (7) is analytic in a proper complex neighborhood of D_0. Moreover, if $\mathbf{u}\in D_0$ is real and increasing then $\rho(\mathbf{u})$ tends to 0 when $\mathbf{u}$ approaches the boundary of D_0.*

4.4. **Proof of our Theorem 3 on dyadic critical exponents.** We fix some notation. Let $G_{\mathbf{P}}$ denote the dependency graph of the system and $\widetilde{G_{\mathbf{P}}}$ the reduced dependency graph. Its vertices are the strongly connected components $C_1,\dots C_L$ of $G_{\mathbf{P}}$. For example, for the system $y_1 = P_1(z,y_1,y_2,y_5), y_2 = P_2(z,y_2,y_3,y_5), y_3 = P_3(z,y_3,y_4), y_4 = P_4(z,y_3,y_4), y_5 = P_5(z,y_5,y_6), y_6 = P_6(z,y_5,y_6)$. we get the dependency graph $G_{\mathbf{P}}$ that is depicted in Figure 1. We can also *reduce* the dependency graph to its components. Let $\mathbf{y}_1,\dots,\mathbf{y}_L$ denote the system of vector functions corresponding to the components $C_1,\dots C_L$ and let $\mathbf{u}_1,\dots,\mathbf{u}_L$ denote the input vectors related to these components. In the above example we have $C_1=\{1\}$, $C_2=\{2\}$, $C_3=\{3,4\}$, $C_4=\{5,6\}$, $\mathbf{y}_1=y_1$, $\mathbf{y}_2=y_2$, $\mathbf{y}_3=(y_3,y_4)$, $\mathbf{y}_4=(y_5,y_6)$, and $\mathbf{u}_1=(y_2,y_5)$, $\mathbf{u}_2=(y_3,y_5)$, $\mathbf{u}_3=\emptyset$, $\mathbf{u}_4=\emptyset$.

Finally, for each component C_ℓ we define the set D_ℓ of real vectors $\mathbf{u}_\ell$ for which the spectral radius of the Jacobian of ℓ-th subsystem evaluated at $z=0$, $\mathbf{y}_\ell=\mathbf{0}$ is smaller than 1.

The first step we for each strongly connected component C_ℓ we solve the corresponding subsystem in the variables z and $\mathbf{u}_\ell$ and obtain solutions $\mathbf{f}(z,\mathbf{u}_\ell)$, $1\le\ell\le L$. In our example these are the functions $\mathbf{f}_1(z,\mathbf{u}_1)=f_1(z,y_2,y_5)$, $\mathbf{f}_2(z,\mathbf{u}_2)=f_2(z,y_3,y_5)$, $\mathbf{f}_3(z,\mathbf{u}_3)=(f_3(z),f_4(z))$, $\mathbf{f}_4(z,\mathbf{u}_4)=(f_5(z),f_6(z))$.

Since the dependency graph $\widetilde{G_{\mathbf{P}}}$ is acyclic there are components $C_{\ell_1},\dots,C_{\ell_m}$ with no input, that is, they corresponding functions $\mathbf{f}_{\ell_1}(z),\dots,\mathbf{f}_{\ell_m}(z)$ can be computed without any further information. By Lemma 1 and 2 they either have a polar singularity or a square-root singularity, that is, they are are precisely of the types that are stated in Theorem 3.

Now we proceed inductively. We consider a strongly connected component C_ℓ with the function $\mathbf{f}_\ell(z,\mathbf{u}_\ell)$ and assume that all the functions $f_j(z)$ that are contained in $\mathbf{u}_\ell$ are already known and and that their leading singularities of the two types stated in Theorem 3.

By the discussion following Lemma 1 and 2 it follows that functions contained in $\mathbf{f}_\ell(z,\mathbf{u}_\ell)$ have either a common polar singularity or a common square-root singularity $\rho(\mathbf{u}_\ell)$.

We distinguish between three cases:

(1) First let us assume that $\mathbf{f}_\ell(z,\mathbf{u}_\ell)$ comes from an affine system and, thus, has a polar singularity. Since all functions contained in $\mathbf{f}_\ell(z,\mathbf{u}_\ell)$ have the same form we just consider one of these functions and denote it by $f(z,\mathbf{u}_j)$:

$$f(z,\mathbf{u}_\ell)=\frac{c(z,\mathbf{u}_\ell)}{1-z/\rho(\mathbf{u}_\ell)}. \tag{8}$$

If $\rho(\mathbf{u}_\ell) = \rho'$ is constant then the only dependence from $\mathbf{u}_\ell$ comes from the numerator $c(z, \mathbf{u}_\ell)$. Since this solution comes from an affine system, $c(z, \mathbf{u}_\ell)$ is just a linear combination of the polynomials of $\mathbf{B}(z, \mathbf{u}_\ell)$ with coefficient functions that depend only on z (this follows from the expansion of $(\mathbf{I}-\mathbf{A}(z))^{-1}\mathbf{B}(z, \mathbf{u}_\ell)$). Furthermore, since $f(z, \mathbf{u}_\ell)$ is (in principle) a power series in z and $\mathbf{u}_j$ with non-negative coefficients the coefficients of this polynomial (if z is some positive real number) have to be non-negative, too.

When we substitute $\mathbf{u}_\ell$ by the the functions $f_j(z)$ that correspond to $\mathbf{u}_j$ then we obtain the functions $f(z)$ that correspond to the component C_ℓ. We have to consider the following cases:

(1.1) The dominating singularities ρ_j of the functions $f_j(z)$ are larger than ρ': In this case the resulting dominating singularity ρ_ℓ is ρ' and we just get a polar singularity for $f(z)$.

(1.2) At least one of the dominating singularities ρ_j of the functions $f_j(z)$ is smaller than ρ':
Let ρ'' denote the smallest of these singularities. If all of the functions $f_j(z)$ with $\rho_j = \rho''$ have a singular behavior of the form (4) then we just make a local expansion and of $c(z, \mathbf{u}_\ell)$ at the corresponding points $f_j(\rho'')$ (for u_j) and observe again an expansion of this form. Note the largest appearing k_j reappears in the expansion of $f(z)$.
Second, if at least one of the functions $f_j(z)$ with $\rho_j = \rho''$ is of type (5) then we use the property that $c(z, \mathbf{u}_\ell)$ is just a polynomial in $\mathbf{u}_j$ (with non-negative coefficients). It is clear that the leading singular behavior comes from these functions, actually they have to be multiplied and added. However, since functions of the type (5) are closed under multiplication and addition this gives again a function of type (5). Note that the appearing coefficient functions that depend just on z have to expanded at ρ'', too, and do not disturb the overall structure.

(1.3) The smallest dominating singularities ρ_j of the functions $f_j(z)$ equals ρ':
Here we can argue similarly to the previous case. If all of the functions $f_j(z)$ with $\rho_j = \rho'$ have a singular behavior of the form (4) then we perform a local expansion in the numerator. Let $\tilde{k}$ be the largest k_j that appears. Then we interpret the polar singularity $(1 - z/\rho')^{-1}$ as $(1 - z/\rho')^{-m2^{-\tilde{k}}}$ with $m = 2^{\tilde{k}}$ and obtain a singular expansion of the form (5).
If at least one of the functions $f_j(z)$ with $\rho_j = \rho'$ is of type (5) then we use the polynomial structure of the numerator as above and obtain an expansion of the form (5). By combining this with the factor $(1 - z/\rho')^{-1}$ we finally obtain an expansion of the form (5) for $f(z)$, too.

(2) Second let us (again) assume that $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ comes from an affine system (and has a polar singularity) of the form (8), however, we now assume that $\rho(\mathbf{u}_\ell)$ is not constant but depends on some of the u_j (not necessarily on all of them).

In this case we study first the behavior of the denominator when $\mathbf{u}_j$ is substituted by the corresponding functions $f_j(z)$. For the sake of simplicity we will work with the difference $\rho(\mathbf{u}_j) - z$. Of course this is equivalent to the discussion of the denominator $1 - z/\rho(\mathbf{u}_j)$, since the factor $\rho(\mathbf{u}_j)$ can be also put to the numerator. Finally let J'_ℓ denote the set of indices of functions u_j for which the function $\rho(\mathbf{u}_\ell)$ really depends on.

Let ρ' denote the smallest radius of convergence of the functions $f_j(z)$, $j \in J'_\ell$. Then we consider the difference $\delta(z) = \rho((f_j(z))_{j\in J'_\ell}) - z$. We have to consider the following cases for the denominator:

(2.1) $\delta(\rho'') = 0$ for some $\rho'' < \rho'$ such that $(f_j(\rho''))_{j\in J'_\ell} \in D_\ell$:
First we note that $\delta(z)$ has at most one positive zero since $\rho((f_j(z))_{j\in J'_\ell})$ is decreasing and z is increasing. Furthermore the derivative satisfies $\delta'(\rho'') > 0$. Consequently we have a simple zero ρ'' in the denominator.

(2.2) We have $\delta(\rho') = 0$ such that $(f_j(\rho'))_{j\in J'_\ell} \in D_\ell$:
In this case all functions $f_j(z)$, $j \in J'_\ell$, with $\rho_j = \rho'$ have to be of type (4). Consequently $\delta(z)$ behaves like

$$c(1 - z/\rho')^{2^{-\tilde{k}}} + \dots,$$

where $c > 0$ and $\tilde{k}$ is the largest appearing k_j (among those functions $f_j(z)$ with $\rho_j = \rho'$).

(2.3) We have $\delta(\rho') > 0$ such that $(f_j(\rho'))_{j\in J'_\ell} \in D_\ell$:
In this case all functions $f_j(z)$, $j \in J'_\ell$, with $\rho_j = \rho'$ have to be (again) of type (4). Consequently $\delta(z)$ behaves like

$$c_0 - c_1(1 - z/\rho')^{2^{-\tilde{k}}} + \dots,$$

where $c_0 > 0$ and $c_1 > 0$ and $\tilde{k}$ is the largest appearing k_j (among those functions $f_j(z)$ with $\rho_j = \rho'$). Hence, $1/\delta(z)$ is of type (4).

Note that there are no other cases. This follows from the fact that $\rho(\mathbf{u}_\ell) \to 0$ if $\mathbf{u}_\ell$ approaches the boundary of D_ℓ. This means that if we trace the function $z \to \delta(z)$ for $z > 0$ then we either meet a singularity of $\delta(z)$ or we pass a zero of $\delta(z)$ before $(f_j(z))_{j\in J'_\ell})$ leaves D_ℓ.

Finally we have to discuss the numerator (as in the above case). Note that there might occur u_j with $j \notin J'_\ell$, so that more functions $f_j(z)$ than in the denominator are involved. Nevertheless in all possible cases we can combine the expansions of the numerator and and denominator and obtain for $f(z)$ either type (4) or type (5).

(3) Finally, let us assume that $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ comes from a non-affine system and, thus, has a square-root singularity. Again, since all functions contained in $\mathbf{f}_\ell(z, \mathbf{u}_\ell)$ have the same form we just consider one of these functions and denote it by $f(z, \mathbf{u}_j)$:

$$f(z, \mathbf{u}_\ell) = g(z, \mathbf{u}_\ell) - h(z, \mathbf{u}_\ell)\sqrt{1 - \frac{z}{\rho(\mathbf{u}_\ell)}}. \tag{9}$$

In this case $\rho(\mathbf{u}_\ell)$ depends on all components of $\mathbf{u}_\ell$ which makes the analysis slightly more easy. As above we will study the behavior of the square-root $\sqrt{\rho(\mathbf{u}_\ell) - z}$ instead of $\sqrt{1 - z/\rho(\mathbf{u}_\ell)}$ since the non-zero factor $\sqrt{\rho(\mathbf{u}_\ell)}$ can be put to $h(z, \mathbf{u}_\ell)$.

Let ρ' denote the smallest radius of convergence of the functions $f_j(z)$ that correspond to $\mathbf{u}_\ell$. Here we have to consider the following cases:

(3.1) $\delta(\rho'') = 0$ for some $\rho'' < \rho'$ such that $(f_j(\rho'')) \in D_\ell$:
This means that $\rho((f_j(z)) - z$ has a simple zero. By the Weierstrass preparation theorem we can, thus, represent this function as

$$\rho((f_j(z)) - z = (\rho'' - z)H(z),$$

where $H(z)$ is non-zero and analytic at ρ''. Consequently

$$\sqrt{\rho((f_j(z)) - z} = \sqrt{\rho'' - z}\sqrt{H(z)}$$

and we observe that $f(z)$ has a (simple) square-root singularity.

(3.2) We have $\delta(\rho') = 0$ such that $(f_j(\rho')) \in D_\ell$:
In this case all functions $f_j(z)$ with $\rho_j = \rho'$ have to be of type (4). Hence the square-root of $\delta(z)$ behaves as

$$\sqrt{c(1 - z/\rho')^{2^{-\tilde{k}}} + \dots} = \sqrt{c}(1 - z/\rho')^{2^{-\tilde{k}-1}} + \dots,$$

where the corresponding k equals the largest appearing k_j plus 1. Thus, $f(z)$ is of type (4).

(3.3) We have $\delta(\rho') > 0$ such that $(f_j(\rho'))_{j\in J'_\ell} \in D_\ell$:
In this case all functions $f_j(z)$, with $\rho_j = \rho'$ have to be (again) of type (4). Consequently the square-root of $\delta(z)$ behaves like

$$\sqrt{c_0 - c_1(1 - z/\rho')^{2^{-\tilde{k}}} + \dots} = \sqrt{c_0}\left(1 - \frac{c_1}{2c_0}(1 - z/\rho')^{2^{-\tilde{k}}} + \dots\right),$$

where $c_0 > 0$ and $c_1 > 0$ and $\tilde{k}$ is the largest appearing k_j (among those functions $f_j(z)$ with $\rho_j = \rho'$). Hence, $f(z)$ is of type (4).

This completes the induction proof of Theorem 3.

4.5. **Periodicities.** When we are interested in the asymptotic properties of the coefficients of a function $f(z)$ that is (part of the) solution of a positive system of algebraic equations we need the structure of all singularities z with modulus $|z| = \rho$, where ρ denotes the radius of convergence.

We will call a function $f(z)$ that is solution of a positive system of algebraic equations **strongly aperiodic** if $z = \rho$ is the only singularity on the cycle $|z| = \rho$ and **aperiodic** if the coefficients have an asymptotic expansion of the form $[z^n]\, f(z) \sim cn^\alpha \rho^{-n}$ for some constants $c > 0$ (and a proper dyadic number α).

Similarly we call such a function $f(z)$ **strongly periodic** with period $m > 1$ if the only singularities on the cycle $|z| = \rho$ are of the form $z = \rho e^{2\pi ij/m}$, $j = 0, 1, \ldots, m-1$. Finally we call a function $f(z)$ **periodic** with period $m > 1$ if $f(z)$ can be represented as $f(z) = \sum_{j=0}^{m} z^j f_j(z^m)$ such that all functions $f_j(z)$ are either polynomials or aperiodic function, where at least one of these functions is aperiodic.

Since algebraic functions have only algebraic singularities (and can be analytically continued to a region that contains the circle of convergence), it follows from the transfer principles of Flajolet and Odlyzko [46] that every strongly aperiodic function is aperiodic and every strongly periodic function (with period m) is periodic (with period m).The main purpose of this section is to provide the following property.

Theorem 4. *Every function $f(z)$ that is solution of a well posed positive polynomial system of equations is either strongly aperiodic or strongly periodic (with some period $m > 1$). Furthermore, the singularity $z = \rho$ dominates all other singularities on the cycle $|z| = \rho$ in the periodic case.*

In particular this implies the following asymptotic relations for the coefficients of solutions of positive polynomial systems. We just have to apply the transfer principle of Flajolet and Odlyzko [46].

Theorem 5. *Suppose that $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ is positive polynomial system of equations that has solution $\mathbf{f}(z)$. Furthermore let $f(z)$ be given by*

$$f(z) = \sum_{n\ge 0} f_n z^n = G(z, \mathbf{f}(z)),$$

where G is a polynomial function with non-negative coefficients.

Then there exists an integer $m \ge 1$ such that for all $j = 0, 1, \ldots, m-1$ we either have $f_{j+mn} = 0$ for almost all $n \ge n_{0,j}$ or

$$f_{j+mn} \sim c_j n^{\alpha_j} \rho_j^{-n/m} \qquad (n \to \infty),$$

where $c_j > 0$, $\rho_j > 0$, and α_j is either of the form $\alpha_j = -2^{-k_j} - 1$ for some integer $k_j \ge 1$ or of the form $\alpha_j = m_j/2^{k_j} - 1$ for some integers $k_j \ge 0$ and $m_j \ge 1$.

The proof of Theorem 4 runs along similar lines as the proof of Theorem 3, that is, we partition the dependency graph into strongly connected components and solve the system step by step. The core of the problem is to characterize the singularities on the cycle of convergence of a system of functions that correspond to a strongly connected dependency graph. Actually the singularities are situated at $\rho e^{2\pi ij/m}$ for some integers j and m.

4.6. **Possible radius of convergence of $\mathbb{Q}_+$ and $\mathbb{N}$-algebraic functions.** In this section we shortly discuss the radius of convergence ρ that can appear in an algebraic system with positive rational coefficients. Pringsheim's theorem and resultant theory [47] imply that ρ has to be a positive algebraic number, however, it is not immediate whether all positive algebraic numbers actually appear.

Conjecture. *Let $R_{\mathbb{S}}$ be the set of possible radius of $\mathbb{S}$-algebraic functions. Then $R_{\mathbb{Q}_+}$ is the set of positive algebraic numbers and $R_{\mathbb{N}}$ is the set of positive algebraic numbers smaller than 1.*

In what follows we present some properties of these algebraic numbers, as a first step towards a proof of the above conjecture.

Theorem 6. *The set $R_{\mathbb{Q}_+}$ has the following properties.*

(1) *All positive roots of equations of the form $p(z) = 1$, where $p(z)$ is a polynomial with non-negative rational coefficients, are contained in $R_{\mathbb{Q}_+}$, in particular all rational numbers and all roots of rational numbers.*

(2) *If ρ_1 and ρ_2 are radii of convergence of a $\mathbb{Q}_+$-rational and a $\mathbb{Q}_+$-algebraic function, and if at least one of these functions is aperiodic, then $\rho_1\rho_2 \in R_{\mathbb{Q}_+}$.*
(3) *All positive quadratic irrational numbers are contained in $R_{\mathbb{Q}_+}$.*

The proof of the first property is immediate. The proof of the second property relies on the fact that the Hadamard product of a rational and an algebraic languages can be described with the help of a pushdown automaton. Finally, the third property can be deduced from the second one. This theorem extends to $R_{\mathbb{N}}$ (by adding the constraint $\rho \leq 1$).

5. Limit laws

5.1. **The classical Drmota–Lalley-Woods theorem.** In several applications in combinatorics, we are not only interested in a univariate situation, where z is the *counting variable* but we are interested, too, in a second parameter that we *count* with the help of another variable (say v). Hence we are led to consider systems of equations of the form $\mathbf{y} = \mathbf{P}(z, \mathbf{y}, v)$. Of course, if we set $v = 1$, we come back to the *original* counting problem. The next theorem shows that the limiting distribution of the additional parameter is always Gaussian if the system is strongly connected (see [38]).

Theorem 7. *Suppose that $\mathbf{y} = \mathbf{P}(z, \mathbf{y}, v)$ is a strongly connected and positive well posed entire or polynomial system of equations that depends on v and has solution $\mathbf{f}(z, v)$ that exist in a neighborhood of $v = 1$. Furthermore let $f(z, v)$ be given by*

$$f(z, v) = \sum_{n\geq 0} f_n(v)z^n = G(z, \mathbf{f}(z, v), v),$$

where G is an entire or polynomial function with non-negative coefficients that depends on $\mathbf{y}$ and suppose that $f_n(v) \neq 0$ for all $n \geq n_0$ (for some $n_0 \geq 0$).

Let X_n be a random variable which distribution is defined by

$$\mathbb{E}[v^{X_n}] = \frac{f_n(v)}{f_n(1)}.$$

Then X_n has a Gaussian limiting distribution. More precisely, we have $\mathbb{E}[X_n] = \mu n + O(1)$ and $\mathbb{V}\mathrm{ar}[X_n] = \sigma^2 n + O(1)$ for constants $\mu > 0$ and $\sigma^2 \geq 0$ and

$$\frac{1}{\sqrt{n}} (X_n - \mathbb{E}[X_n]) \to N(0, \sigma^2).$$

5.2. **More Gaussian examples, beyond the Drmota–Lalley–Woods case.** If the system of equations is not strongly connected that we can still define a random variable X_n, however, it is not necessarily Gaussian as we will see in the next section. Nevertheless, it is possible to state sufficient conditions, where a Gaussian limiting distribution is present.

Theorem 8. *Let $\mathbf{y} = \mathbf{P}(z, \mathbf{y}, v)$ be a system of equations as in Theorem 7 with the only difference that it is not strongly connected. Furthermore we assume that the function $f(x, 1)$ is strongly aperiodic.*

For every strongly connected component C_ℓ of the dependency graphs $G_{\mathbf{P}}$ let ρ_ℓ denote the radius of convergence of those functions $f_j(z, 1)$ that correspond to C_ℓ.

If all ρ_ℓ are different then X_n (that is defined as in Theorem 7) has a Gaussian limiting distribution.

5.3. **Non-Gaussian limit laws.** This section illustrates the wide variety of distributions followed by a parameter in a non strongly-connected grammar.

Theorem 9 (Diversity of possible limit laws for context-free systems)**.** *Let X_n be the number of occurrences of any given pattern (this pattern could be a given letter!) in a word of length n, generated by a grammar (or even by a simpler model of Markov chain, with an alphabet of 2 letters, each letter having an integer weight). Then X_n can follow "any limit law", in the sense that there exist some patterns and some grammars for which the limit curve (for large n) of $(k, Prob(X_n = k))$ can, once rescaled, be arbitrary near from any càdlàg curve.*

Proof. This a consequence of the fact that one can get any piecewise-affine function, as proved in [5], and so by the Weierstrass theorem, one gets any continuous (or càdlàg) distribution. Due to the (possible) periodic behavior of the coefficients of the solution functions there is also a (possible) periodic behavior of the limiting distribution, that is, for every fixed residue class $\bmod m$ we get different laws. Putting these finitely many limit laws into one figure this leads to a multivalued curve, as illustrated below. □

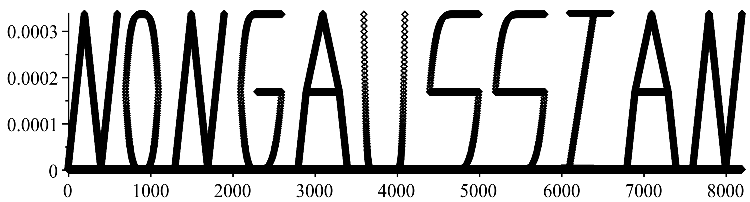

FIGURE 2. This figures, taken from [5], gives the distribution of the letter "b" in words of a language generated by an ad-hoc regular expression of few lines. This distribution is converging towards a curve, "NONGAUSSIAN". Note that this curve is, *at the limit*, a curve of a *multivalued* functional (as can be seen in the O, G, A, S, I letters), however we achieve it for *finite length words* via a *single valued* function, by interlacing two sequences mod 2. This figure illustrates the huge diversity of possible limit laws, even for the distribution of a single letter.

6. CONCLUSION

Now that we have a better picture of the behavior of algebraic coefficients, several extensions are possible and in the full version of this article, we will say more on

- **Algorithmic aspects:** In order to automatize the asymptotics, one has to follow the right branch of the algebraic equations, this is doable by a disjunction of cases following the proof of our main theorem, coupled with an inspection of the associated spectral radii, this leads to a more "algebraic" approach suitable for computer algebra, shortcutting some numerical methods like e.g. the Flajolet–Salvy ACA (analytic continuation of algebraic) algorithm [47]. Giving an algorithm to decide in a constructive way if a function is $\mathbb{N}$-algebraic would be nice. (This is doable for $\mathbb{N}$-rational functions). With respect to the Pisot problem (i.e., deciding if one, or an infinite number of f_n are zeroes), finding the best equivalent for $\mathbb{N}$-algebraic functions of the Skolem–Lerch–Mahler theorem for $\mathbb{N}$-rational functions is also a nice question. The binomial formula of Section 3 leads to many identities, it is not always easy to predict when the nested sums can simplified, this has as also some links with diagonals of rational generating functions.
- **Extension to entire functions system:** Most parts of the analysis of positive polynomial systems of equations also works for positive entire systems, however, one quickly gets "any possible asymptotic behavior" as illustrated by the system of equations $y_1 = z(e^{y_2} + y_1), y_2 = z(1 + 2y_2y_3), y_3 = z(1 + y_3^2)$, as it has the following explicit solution $f_1(z) = \frac{z}{1-z} \exp\left(\frac{z}{\sqrt{1-4z^2}}\right)$, which exhibits a non-algebraic behavior. However, adding the constraints $\frac{\partial^2 P_j}{\partial y_j^2} \neq 0$ or if P_j is affine in y_j leads to the same conclusion as Theorem 3, with a smaller set of possible critical exponents (now, all $m_j = 1$).
- **Extension to infinite systems:** If one considers systems having an infinite (but countable) number of unknowns $y_i(z)$, it is proved in [64] that strongly connected systems also lead to a square-root behavior. The fact that the limit law is Gaussian (as soon as a Jacobian operator associated to the system is compact) is proved in [39]. When the conditions of strong connectivity or of compactness are dropped, a huge diversity of behavior appears, but it is however possible to give interesting subclasses having a regular behaviors.

- **Extension to attributed grammars:** Attribute grammars were introduced by Knuth. Many interesting parameters (like internal paths length in trees or area below lattice paths [9, 41, 69]) are well captured by such grammars. They lead to statistics with a mean which is no more linear. For a large class of strongly connected positive systems (with a Jacobian condition), it leads to the Airy function, and it is expected that it is also the case for a class of functional equations with non positive coefficients.

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