On a mixed Littlewood conjecture in Diophantine approximation

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Abstract. *In a recent paper, de Mathan and Teulié asked whether* $\liminf_{q\to+\infty} q \cdot \|q\alpha\| \cdot |q|_p = 0$ *holds for every badly approximable real number* α *and every prime number* p*. We establish that if the sequence of partial quotients of a real number* α *satisfies a simple, combinatorial condition, then their conjecture is true for the pair* (α, p) *with* p *an arbitrary prime.*

1. Introduction

A famous open problem in simultaneous Diophantine approximation is the Littlewood conjecture [9]. It claims that, for every given pair (α, β) of real numbers, we have

$$\liminf_{q\to+\infty} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0, \tag{1}$$

where $\|\cdot\|$ denotes the distance to the nearest integer. The first significant contribution on this question goes back to Cassels and Swinnerton-Dyer [3] who showed that (1) holds when α and β belong to the same cubic field. Further explicit examples of pairs (α, β) of real numbers satisfying (1) have been given in [10,1]. Despite some recent remarkable progress [13,5] the Littlewood conjecture remains an open problem.

Recently, de Mathan and Teulié [12] proposed a 'mixed Littlewood conjecture' that can be stated as follows. Let $\mathcal{D} = (d_k)_{k\in\mathbf{Z}}$ be a sequence of integers greater than or equal to 2. Set $e_0 = 1$ and, for any $n \geq 1$,

$$e_n = \prod_{0<k\leq n} d_k.$$

For $q \in \mathbf{Z}$, set

$$w_{\mathcal{D}}(q) = \sup\{n \in \mathbf{N} : q \in e_n\mathbf{Z}\}$$

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and

$$|q|_{\mathcal{D}} = 1/e_{w_{\mathcal{D}}(q)} = \inf\{1/e_n : q \in e_n\mathbf{Z}\}.$$

When $\mathcal{D}$ is the constant sequence equal to p, where p is a prime number, then $|\cdot|_{\mathcal{D}}$ is the usual p-adic value $|\cdot|_p$, normalized by $|p|_p = p^{-1}$. In analogy with the Littlewood conjecture, de Mathan and Teulié asked whether

$$\liminf_{q\to+\infty} q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} = 0 \tag{2}$$

holds for every real number α. They proved that (2) holds for every quadratic irrationality α when the sequence $\mathcal{D}$ is bounded.

In the present paper, we are focusing on the particular case when $\mathcal{D}$ is the constant sequence equal to a prime number. Thus, we investigate the following conjecture.

Mixed Littlewood Conjecture. *For every real number α and every prime number p, we have*

$$\liminf_{q\to+\infty} q \cdot \|q\alpha\| \cdot |q|_p = 0. \tag{3}$$

Obviously, the above conjecture holds if α is rational or has unbounded partial quotients. Thus, we only consider the case when α is an element of the set $\boldsymbol{Bad}$ of badly approximable numbers, where

$$\boldsymbol{Bad} = \{\alpha \in \mathbf{R} : \inf_{q\geq 1} q \cdot \|q\alpha\| > 0\}.$$

We are concerned with the following question:

Problem 1. Is there any α in $\boldsymbol{Bad}$, which is irrational and not quadratic, such that, for any prime number p, the pair (α, p) satisfies (3)?

As briefly outlined on page 231 of [12], the answer to Problem 1 is positive when α lies in a subset of $\boldsymbol{Bad}$ with Hausdorff dimension 1; see also [6] for a stronger result. Nevertheless, these approaches do not provide any new explicit examples of pairs (α, p) satisfying (3) with α in $\boldsymbol{Bad}$. The purpose of the present note is precisely to construct explicitly uncountably many real numbers α in $\boldsymbol{Bad}$ such that the pair (α, p) satisfies (3) for any prime number p.

We further extend the problem posed by de Mathan and Teulié, by considering an 'inhomogeneous mixed Littlewood conjecture'. We ask whether

$$\liminf_{q\to+\infty} q \cdot \|q\alpha\| \cdot |q - y|_p = 0 \tag{4}$$

holds for any real number α, any prime number p and any y in $\mathbf{Z}_p$, the ring of p-adic integers. It turns out that our methods allow us to establish (4) for a wide class of real numbers α in $\boldsymbol{Bad}$.

Our proofs heavily depend on p-adic analysis, and our key tool is the p-adic logarithm function. It is not clear to us whether our Theorem 1 has a real analogue, or an analogue for formal power series over a finite field.

2. Results

Our main result shows that (3) holds for any real number α whose sequence of partial quotients is, in some sense, quasi-periodic.

Theorem 1. *Let α be in **Bad** and write*

$$\alpha = [a_0; a_1, a_2, \ldots].$$

Let $T \geq 1$ be an integer and $b_1, \ldots, b_T$ be positive integers. If there exist two sequences $(m_k)_{k\geq 1}$ and $(h_k)_{k\geq 1}$ of positive integers with $(h_k)_{k\geq 1}$ being unbounded and

$$a_{m_k+j+nT} = b_j, \quad \text{for every } j = 1, \ldots, T \text{ and every } n = 0, \ldots, h_k - 1,$$

then we have

$$\liminf_{q\to+\infty} q \cdot \|q\alpha\| \cdot |q - y|_p = 0. \tag{5}$$

for every prime number p and every $y \in \mathbf{Z}_p$. If, furthermore, there exists a constant C such that

$$m_k \leq Ch_k, \quad \text{for } k \geq 1, \tag{6}$$

then we have

$$\liminf_{q\to+\infty} q \cdot \log q \cdot \|q\alpha\| \cdot |q - y|_p < +\infty, \tag{7}$$

for every prime number p and every $y \in \mathbf{Z}_p$.

It is worth rephrasing the assumption of Theorem 1 by using the terminology from combinatorics on words. Let $\alpha = [a_0; a_1, a_2, \ldots]$ be in ***Bad*** and view its sequence of partial quotients as the infinite word $\mathbf{a} = a_0a_1a_2\ldots$ on the alphabet $\mathcal{A} = \{1, \ldots, M\}$, where M is an upper bound for the a_i's. Theorem 1 asserts that, if there exists a finite, non-empty word B on the alphabet $\mathcal{A}$ such that, for every $k \geq 1$, the concatenation of k copies of B occurs in the word $\mathbf{a}$, then (5) holds for every prime number p and every y in $\mathbf{Z}_p$. Consequently, Theorem 1 provides an uncountable, explicit class of badly approximable real numbers for which the mixed Littlewood conjecture, and even the inhomogeneous mixed Littlewood conjecture, is true.

We further mention that, if the real number α satisfies the assumption of Theorem 1 with sequences $(m_k)_{k\geq 1}$ and $(h_k)_{k\geq 1}$ such that (6) holds, then α is either a quadratic irrationality, or a transcendental number. This follows from Theorem 3.2 of [2].

We display an immediate consequence of the proof of Theorem 1.

Theorem 2. *Let α be a quadratic real number. Let p be a prime number. For any y in $\mathbf{Z}_p$, there exist a positive constant $c(\alpha, p, y)$, depending only on α, p and y, and arbitrarily large positive integers q with*

$$\|q\alpha\| \leq c(\alpha, p, y)/q$$

and

$$|q - y|_p \leq c(\alpha, p, y)/\log q. \tag{8}$$

In particular, we have

$$\liminf_{q\to+\infty} q\cdot\log q\cdot\|q\alpha\|\cdot|q-y|_p<+\infty. \tag{7}$$

The case $y=0$ was already established in Théorème 2.1 from [12]. Notice that their result actually covers the more general case of a bounded sequence $\mathcal{D}$ of positive integers. In view of [11], we cannot replace $\log q$ in (8) by $(\log q)^\lambda$ with λ very large.

Theorem 2 also holds for an arbitrary bounded sequence $\mathcal{D}$ when y is an integer.

Using some of the ideas occurring in the proof of Theorem 1, we get another uncountable, explicit class of badly approximable real numbers for which the mixed Littlewood conjecture, and even the inhomogeneous mixed Littlewood conjecture, is true.

Theorem 3. *Let $\alpha=[a_0;a_1,a_2,\ldots]$ be in **Bad**. If for every integer $h\geq 1$ there exists an integer T such that*

$$a_{j+nT}=a_j,\quad \text{for every } j=1,\ldots,T \text{ and every } n=0,\ldots,h,$$

then, for any prime number p and any integer y we have

$$\liminf_{q\to+\infty} q\cdot\|q\alpha\|\cdot|q-y|_p<+\infty.$$

More generally, Theorem 3 holds for an arbitrary (bounded) sequence $\mathcal{D}$. Its analogue in the function fields case can also easily be established.

Using arguments from [13], we get that (4) holds for 'many' real numbers α with bounded partial quotients.

Theorem 4. *The set of real numbers α with bounded partial quotients for which*

$$\liminf_{q\to+\infty} q\cdot\|q\alpha\|\cdot|q-y|_p=0,$$

for every prime number p and for every y in $\mathbf{Z}$, has Hausdorff dimension 1.

A stronger result holds when $y=0$. Namely, Einsiedler and Kleinbock [6] established that the set of α for which (3) does not hold has Hausdorff dimension 0. Presumably, their approach could be modified to get an analogous result for (4), but this is not clear to us.

In view of Theorem 1, we would like to address the following problem.

Problem 2. Let ε be a positive real number. Find a real number α in ***Bad***, a prime number p, and a rational integer y such that

$$\liminf_{q\to+\infty} q^{1+\varepsilon}\cdot\|q\alpha\|\cdot|q-y|_p<+\infty. \tag{9}$$

It follows from the p-adic analogue of the Schmidt Subspace Theorem, established by Schlickewei [15], that any real number α satisfying (9) must be transcendental. Apparently, our construction does not allow us to tackle Problem 2. Nevertheless, by using the Folding Lemma recalled in Section 6, it is possible to give a positive answer to Problem 2.

Theorem 5. *For any prime number p, there exists a real number α in* ***Bad*** *such that*

$$\liminf_{q\to+\infty} q^2 \cdot \|q\alpha\| \cdot |q|_p \leq 1.$$

Since (9) cannot hold with $\varepsilon > 1$ and α in ***Bad***, Theorem 5 is best possible.
We end this section by a third question.

Problem 3. Given α in ***Bad***, is there a prime number p such that (3) holds for the pair (α, p)?

Apparently, there is no contribution towards Problem 3, which seems to be quite difficult.

3. Proof of Theorem 1

– Preliminaries to the proof.

Let $(q_n)_{n\geq -1}$ be the sequence of the denominators of the convergents of α. These numbers satisfy the recurrence relation

$$q_n = a_n q_{n-1} + q_{n-2}$$

with $q_0 = 1$ and $q_{-1} = 0$. For $n \geq 0$, set

$$Q_n = \begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix}.$$

We can write

$$Q_n = P_n Q_{n-1}$$

where P_n is the matrix

$$P_n = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

For $j = 1, \ldots, T$, set

$$M_j = \begin{pmatrix} b_j & 1 \\ 1 & 0 \end{pmatrix}$$

and $M = M_T \ldots M_1$. Replacing if necessary $b_1, \ldots, b_T$ by $b_1, \ldots, b_T, b_1, \ldots, b_T$, we may replace T by $2T$, and thus we can suppose $\det M = 1$. The matrix M is diagonalizable and its eigenvalues are quadratic units $\omega > 1$ and $1/\omega$. We have $N_{\mathbf{Q}(\omega)/\mathbf{Q}}\omega = 1$. Set $a = Tr_{\mathbf{Q}(\omega)/\mathbf{Q}}\omega$. Then $a > 2$ is a positive integer, and ω is a root of the polynomial $X^2 - aX + 1$. We have $Q_{m_k+nT} = MQ_{m_k+(n-1)T}$ for every $1 \leq n \leq h_k$, hence $Q_{m_k+nT} = M^n Q_{m_k}$. The Cayley–Hamilton theorem implies that, for $2 \leq n \leq h_k$, we have

$$Q_{m_k+nT} = M^2 Q_{m_k+(n-2)T} = (aM - I)Q_{m_k+(n-2)T} = aQ_{m_k+(n-1)T} - Q_{m_k+(n-2)T}.$$

Hence the sequence $(q_{m_k+nT})_{n\geq 0}$ satisfies

$$q_{m_k+nT} = aq_{m_k+(n-1)T} - q_{m_k+(n-2)T} \tag{10}$$

when $2 \leq n \leq h_k$.

Note that we have $Q_{m_k+1+nT} = M'Q_{m_k+1+(n-1)T}$ for $0 < n < h_k$, where $M' = M_1 M_T \dots M_2 = M_1 M M_1^{-1}$. Hence, M' has the same characteristic polynomial as M, and the sequence $(q_{m_k+1+nT})_{n\geq 0}$ also satisfies the recurrence

$$q_{m_k+1+nT} = aq_{m_k+1+(n-1)T} - q_{m_k+1+(n-2)T} \tag{10}$$

when $2 \leq n < h_k$. Consequently, we may replace m_k by $m_k + 1$. Since q_{m_k} and q_{m_k+1} are coprime, we can, without any loss of generality, suppose that p does not divide q_{m_k}.

We have

$$q_{m_k+nT} = A_k\omega^n + B_k\omega^{-n}, \tag{11}$$

for $0 \leq n \leq h_k$, where A_k and B_k are given by (11) with $n = 0$ and $n = 1$:

$$A_k = \frac{\omega q_{m_k+T} - q_{m_k}}{\omega^2 - 1} \tag{12}$$

and

$$B_k = -\omega\frac{q_{m_k+T} - \omega q_{m_k}}{\omega^2 - 1}. \tag{13}$$

We also denote by ω a zero of the polynomial $X^2 - aX + 1$ in the algebraic closure of $\mathbf{Q}_p$. We still denote by $|\cdot|_p$ the p-adic value extended to this field. As ω is a unit, we have $|\omega|_p = 1$. The formulæ (12) and (13) hold in $\mathbf{Q}_p$. Since ω and ω^{-1} are conjugate numbers in $\mathbf{Q}(\omega)$, and so are A_k and B_k, we can change A_k into B_k by changing the embedding of ω in $\mathbf{Q}_p(\omega)$ — it is convenient that this embedding depend upon k. We thus can suppose that $|A_k|_p \geq |B_k|_p$ (i.e., $|q_{m_k+T} - \omega q_{m_k}|_p \leq |\omega q_{m_k+T} - q_{m_k}|_p$). Since we suppose that $|A_k + B_k|_p = |q_{m_k}|_p = 1$, we then have $1 \leq |A_k|_p \leq 1/|\omega^2 - 1|_p$.

The field $\mathbf{Q}_p(\omega)$ is complete. The ball

$$G = \{x \in \mathbf{Q}_p(\omega) : |x - 1|_p < p^{-1/(p-1)}\}$$

is a subgroup of finite index in the multiplicative group $\{x \in \mathbf{Q}_p(\omega) : |x|_p = 1\}$. Hence, replacing again T by ℓT, therefore ω by ω^ℓ, where ℓ is a suitable positive integer, we may also suppose that $|\omega - 1|_p < p^{-1/(p-1)}$.

In the sequel, we shall make use of the p-adic logarithm function, which is defined on the multiplicative group $\{x \in \mathbf{Q}_p(\omega) : |x - 1|_p < 1\}$ in $\mathbf{Q}_p(\omega)$ by

$$\log x = \sum_{n=1}^{+\infty} (-1)^{n-1}(x-1)^n/n.$$

We have

$$\log xy = \log x + \log y,$$

for x, y in $\{x \in \mathbf{Q}_p(\omega) : |x-1|_p < 1\}$, and, for every x, y in G,

$$|\log x|_p = |x-1|_p,$$

and

$$|\log x - \log y|_p = \left|\log \frac{x}{y}\right|_p = \left|\frac{x}{y} - 1\right|_p = |x-y|_p.$$

The constants implied in the symbols $\ll$ and $\asymp$ occurring below will only depend upon $b_1, \ldots, b_T$ and on p.

The proofs in the cases $y = 0$ and $y \neq 0$ are rather different. First, we deal with the case $y = 0$, which is slightly more difficult.

– *An auxiliary result for the case $y = 0$.*

Keep the above notations and set either

$$\varepsilon_k = |\omega^2 - 1|_p \cdot |q_{m_k+T} - \omega q_{m_k}|_p$$

or

$$\varepsilon_k = p^{-1/2} \cdot |\omega^2 - 1|_p \cdot |q_{m_k+T} - \omega q_{m_k}|_p,$$

in such a way that $\varepsilon_k < 1$ is an integral power of p.

Lemma 1. *There exist integers x_k and y_k, with $|y_k|_p = 1$, such that*

$$|x_k q_{m_k} + y_k q_{m_k+T}|_p \leq \varepsilon_k \tag{14}$$

and

$$\max\{|x_k|, |y_k|\} \ll \varepsilon_k^{-1/2}. \tag{15}$$

Furthermore, we have

$$|x_k q_{m_k} + y_k q_{m_k+T}|_p \leq |\omega^2 - 1|_p |x_k + y_k \omega|_p \ll (\max\{|x_k|, |y_k|\})^{-2}. \tag{16}$$

In course of the proof of Lemma 1, we need the following version of Liouville's Lemma.

Lemma 2. *For any integers X and Y, not both zero, we have*

$$|X + Y\omega|_p \gg (\max\{|X|, |Y|\})^{-2}.$$

Proof. We have

$$|N_{\mathbf{Q}(\omega)/\mathbf{Q}}(X + Y\omega)| \ll \max\{|X|, |Y|\}^2$$

and

$$|N_{\mathbf{Q}(\omega)/\mathbf{Q}}(X + Y\omega)|_p = |(X + Y\omega)(X + Y/\omega)|_p \leq |X + Y\omega|_p.$$

As $N_{\mathbf{Q}(\omega)/\mathbf{Q}}(X + Y\omega)$ is a non zero integer, we get

$$|N_{\mathbf{Q}(\omega)/\mathbf{Q}}(X + Y\omega)| \cdot |(N_{\mathbf{Q}(\omega)/\mathbf{Q}}(X + Y\omega)|_p \geq 1.$$

It then follows that $|X+Y\omega|_p \max\{|X|,|Y|\}^2 \gg 1$, as claimed. □

Proof of Lemma 1. Let s be a positive integer. By the pigeonhole principle, there exist integers $x_{k,s}$ and $y_{k,s}$, not both zero, such that

$$|x_{k,s}q_{m_k} + y_{k,s}q_{m_k+T}|_p \le \varepsilon_k p^{-s} \tag{17}$$

and

$$\max\{|x_{k,s}|, |y_{k,s}|\} \le \varepsilon_k^{-1/2} p^{s/2}. \tag{18}$$

First, we prove that there exists a non-negative integer S, depending only on ω, such that, if $s > S$, then $y_{k,s}$ cannot be divisible by p^s. Indeed, let σ be a positive integer, with $0 \le \sigma \le s$, such that $y_{k,s}$ is divisible by p^σ. Then, $x_{k,s}$ as well is divisible by p^σ. Indeed, we have $|y_{k,s}q_{m_k+T}|_p \le p^{-\sigma}$ and $|x_{k,s}q_{m_k} + y_{k,s}q_{m_k+T}|_p \le p^{-\sigma}$, hence $|x_{k,s}q_{m_k}|_p \le p^{-\sigma}$. Since we have assumed that $|q_{m_k}|_p = 1$, we get $|x_{k,s}|_p \le p^{-\sigma}$. Setting $x'_{k,s} = p^{-\sigma}x_{k,s}$ and $y'_{k,s} = p^{-\sigma}y_{k,s}$, we deduce from (17) that we have

$$\left| x'_{k,s} + y'_{k,s}\frac{q_{m_k+T}}{q_{m_k}} \right|_p \le \varepsilon_k p^{-s+\sigma} \tag{19}$$

and from (18) that

$$\max\{|x'_{k,s}|, |y'_{k,s}|\} \le \varepsilon_k^{-1/2} p^{s/2-\sigma}. \tag{20}$$

Hence, writing

$$x'_{k,s} + y'_{k,s}\omega = x'_{k,s} + y'_{k,s}\frac{q_{m_k+T}}{q_{m_k}} + y'_{k,s}\left(\omega - \frac{q_{m_k+T}}{q_{m_k}}\right), \tag{21}$$

and noticing that (19) implies that

$$\left| x'_{k,s} + y'_{k,s}\frac{q_{m_k+T}}{q_{m_k}} \right|_p < |q_{m_k+T} - \omega q_{m_k}|_p,$$

we get

$$|x'_{k,s} + y'_{k,s}\omega|_p \le |q_{m_k+T} - \omega q_{m_k}|_p \ll \varepsilon_k.$$

Then by (20)

$$|x'_{k,s} + y'_{k,s}\omega|_p \max\{|x'_{k,s}|, |y'_{k,s}|\}^2 \ll p^{s-2\sigma}.$$

If $\sigma = s$, we thus get

$$|x'_{k,s} + y'_{k,s}\omega|_p \max\{|x'_{k,s}|, |y'_{k,s}|\}^2 \le p^{-s}$$

and it follows from Lemma 2 that we must have $s \ll 1$, i.e., $s \le S$, where $S \ge 0$ is an integral constant, depending only upon ω. Thus, if $s > S$, then $y_{k,s}$ cannot be divisible

by p^s. Take $s = S + 1$. If we define σ by $|y_{k,s}|_p = p^{-\sigma}$, we then have $\sigma < s$. We define $x_k = p^{-\sigma} x_{k,s}$, and $y_k = p^{-\sigma} y_{k,s}$, so that

$$|y_k|_p = 1. \tag{22}$$

By (19) and (20), as $s = S + 1$, we have

$$|x_k q_{m_k} + y_k q_{m_k+T}|_p \le \varepsilon_k \tag{14}$$

and

$$\max\{|x_k|, |y_k|\} \ll \varepsilon_k^{-1/2}. \tag{15}$$

To conclude, let us show that the pair of integers (x_k, y_k) which we have constructed satisfies (16). Indeed, as $\varepsilon_k < |q_{m_k+T} - \omega q_{m_k}|_p$, we deduce from (22), (14) and (21) that

$$|x_k + y_k\omega|_p = |q_{m_k+T} - \omega q_{m_k}|_p.$$

Accordingly, conditions (14) and (15) lead to Lemma 1. □

– *Completion of the proof for the case* $y = 0$.

For any $k \ge 1$, let x_k and y_k be the integers given by Lemma 1. For any $n \ge 0$, set

$$Q_{k,n} = x_k q_{m_k+nT} + y_k q_{m_k+(n+1)T}. \tag{23}$$

Our aim is to prove that there exists an integer $n(k)$ at most equal to $h_k - 1$ such that $Q_{k,n(k)} \ne 0$, and

$$\inf_{k\ge1} |Q_{k,n(k)}| \cdot |Q_{k,n(k)}|_p \cdot \|Q_{k,n(k)}\alpha\| = 0.$$

First, we note that, for every $0 \le n < h_k$, we have

$$|Q_{k,n}| \ll \max\{|x_k|, |y_k|\}\, q_{m_k+nT}$$

and

$$\|Q_{k,n}\alpha\| \ll \max(|x_k|, |y_k|)/q_{m_k+nT}.$$

We thus get

$$|Q_{k,n}| \cdot \|Q_{k,n}\alpha\| \ll \max\{|x_k|, |y_k|\}^2. \tag{24}$$

Further, with A_k and B_k being defined as in (12) and (13), we can write

$$Q_{k,n} = C_k\omega^n + D_k\omega^{-n}$$

where

$$C_k = A_k(x_k + y_k\omega)$$

and

$$D_k = B_k(x_k + y_k\omega^{-1}).$$

Since $|A_k|_p \geq 1$, we deduce from Lemma 1 that

$$|C_k + D_k|_p = |x_k q_{m_k} + y_k q_{m_k+T}|_p \leq |\omega^2 - 1|_p \cdot |x_k + y_k\omega|_p \leq |\omega^2 - 1|_p \cdot |C_k|_p.$$

Hence, we have $|C_k + D_k|_p < |C_k|_p$, and thus $|C_k|_p = |D_k|_p$. We then get

$$|1 + D_k/C_k|_p \leq |\omega^2 - 1|_p < p^{-1/(p-1)}. \tag{25}$$

Write

$$Q_{k,n} = C_k\omega^{-n}(D_k/C_k + \omega^{2n}).$$

Since we have $|C_k|_p \ll |x_k + y_k\omega|_p \ll (\max\{|x_k|, |y_k|\})^{-2}$, we thus get

$$|Q_{k,n}|_p \ll (\max\{|x_k|, |y_k|\})^{-2}|D_k/C_k + \omega^{2n}|_p$$

and, by (24),

$$|Q_{k,n}| \cdot |Q_{k,n}|_p \cdot \|Q_{k,n}\alpha\| \ll |D_k/C_k + \omega^{2n}|_p. \tag{26}$$

Now, inequality (25) enables us to use the p-adic logarithm in the domain G where the equality $|\log x - \log y|_p = |x - y|_p$ holds. As $|D_k/C_k + \omega^{2n}|_p = |\log(-D_k/C_k) - 2n\log\omega|_p$, we may also write (26) as

$$|Q_{k,n}| \cdot |Q_{k,n}|_p \cdot \|Q_{k,n}\alpha\| \ll \left|\frac{\log(-D_k/C_k)}{2\log\omega} - n\right|_p.$$

Now let us show that $(\log(-D_k/C_k))/(2\log\omega)$ lies in $\mathbf{Q}_p$. This is trivial if $\omega \in \mathbf{Q}_p$. In the case where $\omega \notin \mathbf{Q}_p$, there exists a unique $\mathbf{Q}_p$-automorphism σ of $\mathbf{Q}_p(\omega)$, different from the identity. We have $\sigma(\omega) = 1/\omega$, and σ is isometrical. We have $\sigma(\log\omega) = \log(\sigma(\omega)) = -\log\omega$, and $\sigma(-D_k/C_k) = -C_k/D_k$, since $\sigma(C_k) = D_k$. We thus have

$$\sigma\big(\log(-D_k/C_k)\big) = \log(-C_k/D_k) = -\log(-D_k/C_k).$$

Hence,

$$\sigma\left(\frac{\log(-D_k/C_k)}{2\log\omega}\right) = \frac{\log(-D_k/C_k)}{2\log\omega},$$

and the number $\log(-D_k/C_k)/(2\log\omega)$ of $\mathbf{Q}_p(\omega)$ lies in $\mathbf{Q}_p$. Further, this number lies in $\mathbf{Z}_p$, since

$$|\log(-D_k/C_k)|_p = |D_k/C_k + 1|_p \leq |\omega^2 - 1|_p = |2\log\omega|_p,$$

by (25). If t_k is a positive integer with

$$\frac{1}{2p}h_k < p^{t_k} \leq \frac{1}{2}h_k,$$

then there exists an integer n, with $0 \leq n < p^{t_k}$, such that

$$\left|\frac{\log(-D_k/C_k)}{2\log\omega} - n\right|_p \leq p^{-t_k}.$$

Replacing if necessary n by $n + p^{t_k}$, we may ensure that $\log(-D_k/C_k) \neq 2n \log \omega$, hence, $D_k/C_k + \omega^{2n} \neq 0$. Taking either $n(k) = n$ or $n(k) = n + p^{t_k}$, we have thus constructed an integer $n(k)$ satisfying $0 \leq n(k) < h_k$, $Q_{k,n(k)} \neq 0$ and

$$|Q_{k,n(k)}| \cdot |Q_{k,n(k)}|_p \cdot \|Q_{k,n(k)}\alpha\| \ll p^{-t_k} \ll \frac{1}{h_k}. \tag{27}$$

Since the sequence $(h_k)_{k\geq 1}$ is unbounded, we have

$$\inf_{k\geq 1} |Q_{k,n(k)}| \cdot |Q_{k,n(k)}|_p \cdot \|Q_{k,n(k)}\alpha\| = 0.$$

It remains for us to prove that the estimate (7) holds when (6) is satisfied. Note that as the partial quotients of α are bounded, we have by (23):

$$\log |Q_{k,n(k)}| \ll \log \max\{|x_k|, |y_k|\} + m_k + n(k),$$

hence,

$$\log |Q_{k,n(k)}| \ll \log \max\{|x_k|, |y_k|\} + m_k + h_k.$$

Now, by (15) and Lemma 2, we have

$$\max\{|x_k|, |y_k|\} \ll |q_{m_k+T} - \omega q_{m_k}|_p^{-1/2} \ll q_{m_k}.$$

Thus, we get

$$\log \max\{|x_k|, |y_k|\} \ll m_k$$

and

$$\log |Q_{k,n(k)}| \ll m_k + h_k.$$

Accordingly, if (6) is satisfied, then we have $\log |Q_{k,n(k)}| \ll h_k$, and (7) follows from (27).

– *The case* $y \neq 0$.

We shall prove that for any non-zero y in $\mathbf{Z}_p$, there exists an infinite set $\mathcal{Q}$ of positive integers Q satisfying

$$Q \cdot \|Q\alpha\| \ll 1 \tag{28}$$

and

$$\liminf_{\mathcal{Q}\ni Q\to+\infty} |Q - y|_p = 0. \tag{29}$$

Note that if (29) is true for y, then it is true for My, where M is any positive integer. Indeed, we may replace Q by MQ, while preserving (28). Then it is enough to prove (29) when $|y|_p = 1$. Actually we shall construct a set of positive integers Q, with (28), such that the p-adic topological closure of this set contains the unit circumference $|y - 1|_p = 1$. For this purpose, it is enough to construct a set of Q, with (28), whose p-adic closure contains the ball $|y - 1|_p \leq p^{-\lambda}$, for some positive integer λ. Indeed, replacing then the set of Q by the μQ, where μ runs among the integers $0 < \mu < p^\lambda$ with $|\mu|_p = 1$, the p-adic closure of this set will contain the unit circumference $|y - 1|_p = 1$.

First, suppose that $p \neq 2$. We take the sequences $(m_k)_{k\geq 1}$ and $(h_k)_{k\geq 1}$ as in the statement of the theorem. The numbers A_k and B_k are defined as in (12) and (13). Recall that we suppose that $|q_{m_k}|_p = 1$ and $|A_k|_p \geq |B_k|_p$, hence $|A_k|_p \geq 1$. We note that we may also suppose that $|A_k - B_k|_p \geq |\omega - 1|_p$. Indeed, we may replace q_{m_k} either by q_{m_k+T} or q_{m_k+2T}, while preserving the condition $|q_{m_k}|_p = 1$, since (10) ensures that if q_{m_k} is not divisible by p, then either q_{m_k+T} or q_{m_k+2T} is not divisible by p. Now, if we replace q_{m_k} by q_{m_k+T} (resp. by q_{m_k+2T}), then A_k is replaced by $A_k\omega$ (resp. by $A_k\omega^2$), and B_k is replaced by $B_k\omega^{-1}$ (resp. by $B_k\omega^{-2}$). If $|A_k - B_k|_p < |\omega - 1|_p$, then we have $|A_k\omega - B_k\omega^{-1}|_p = |A_k(\omega - \omega^{-1}) + (A_k - B_k)\omega^{-1}|_p = |A_k(\omega - \omega^{-1})|_p \geq |\omega^2 - 1|_p = |\omega - 1|_p$ in virtue of the properties of the logarithm function, since $|\omega - 1|_p < p^{-1/(p-1)}$. In the same way, if $|A_k - B_k|_p < |\omega - 1|_p$, then we have $|A_k\omega^2 - B_k\omega^{-2}|_p \geq |\omega - 1|_p$. Accordingly, by these changes, we may suppose that $|q_{m_k}|_p = 1$ and

$$|A_k - B_k|_p \geq |\omega - 1|_p. \tag{30}$$

Let λ be the positive integer such that

$$|\omega - 1|_p^2 = p^{-\lambda+1}.$$

As $|q_{m_k}|_p = 1$, there exists an integer L_k with $0 < L_k < p^\lambda$ and $|L_k|_p = 1$, such that

$$L_k(A_k + B_k) \equiv 1 \bmod p^\lambda. \tag{31}$$

We shall prove that the p-adic closure of the set composed by the integers $L_k q_{m_k+nT}$, for $k \geq 1$ and $0 \leq n \leq h_k$, contains the ball $|y - 1|_p \leq p^{-\lambda}$. Note that, L_k being bounded, the integers $L_k q_{m_k+nT}$ satisfy (28), and thus the result will be proved.

Set $L_k A_k = A'_k$ and $L_k B_k = B'_k$. For $0 \leq n \leq h_k$, we have

$$L_k q_{m_k+nT} = A'_k\omega^n + B'_k\omega^{-n}.$$

Consider the map φ_k from the ball $\{x : |x - 1|_p < 1\}$ of $\mathbf{Q}_p(\omega)$ into $\mathbf{Q}_p(\omega)$ such that

$$\varphi_k(x) = A'_k x + B'_k x^{-1}.$$

Let y be a number of $\mathbf{Q}_p$ with

$$|y - 1|_p \leq p^{-\lambda}. \tag{32}$$

We shall find $x \in \mathbf{Q}_p(\omega)$ such that $\varphi_k(x) = y$. We must have $A'_k x^2 - yx + B'_k = 0$. Accordingly we take

$$x = \frac{y + \sqrt{y^2 - 4A'_k B'_k}}{2A'_k}. \tag{33}$$

However we must make precise the meaning of the symbol $\sqrt{\cdot}$. We define the function $\sqrt{\cdot}$ over the ball $\{z : |z - 1|_p \leq p^{-1}\}$ in $\mathbf{Q}_p$, with values in the same ball, by the fact that $\log\sqrt{z} = \frac{1}{2}\log z$ (that is to say $\sqrt{z} = \exp\frac{\log z}{2}$, where $\exp u = \sum_{n=0}^{+\infty}\frac{u^n}{n!}$ for $|u|_p \leq p^{-1}$). Therefore we have

$$|\sqrt{z} - 1|_p = |z - 1|_p. \tag{34}$$

Now, by (31) and (32), we have

$$|y^2 - 4A'_kB'_k - (A'_k - B'_k)^2|_p = |y^2 - (A'_k + B'_k)^2|_p \leq p^{-\lambda} \tag{35}$$

and, by (30) and $|A'_k - B'_k|_p^2 \geq p^{-\lambda+1}$, we get

$$\left|\frac{y^2 - 4A'_kB'_k}{(A'_k - B'_k)^2} - 1\right|_p \leq p^{-1}. \tag{36}$$

Moreover the number $(y^2 - 4A'_kB'_k)/(A'_k - B'_k)^2$ lies in $\mathbf{Q}_p$, since in the case where $\omega \notin \mathbf{Q}_p$, it is invariant under the above automorphism σ. Inequality (36) allows us to define the number $\sqrt{(y^2 - 4A'_kB'_k)/(A'_k - B'_k)^2}$ in $\mathbf{Q}_p$, and we put in (33)

$$\sqrt{y^2 - 4A'_kB'_k} = (A'_k - B'_k)\sqrt{\frac{y^2 - 4A'_kB'_k}{(A'_k - B'_k)^2}}.$$

Then the number x is well defined in $\mathbf{Q}_p(\omega)$. Further let us prove that

$$|x - 1|_p \leq |\omega - 1|_p. \tag{37}$$

Indeed, writing

$$\left|\sqrt{y^2 - 4A'_kB'_k} - (A'_k - B'_k)\right|_p = |A'_k - B'_k|_p \cdot \left|\sqrt{\frac{y^2 - 4A'_kB'_k}{(A'_k - B'_k)^2}} - 1\right|_p,$$

we deduce from (34) that

$$\left|\sqrt{y^2 - 4A'_kB'_k} - (A'_k - B'_k)\right|_p = \frac{|y^2 - 4A'_kB'_k - (A'_k - B'_k)^2|_p}{|A'_k - B'_k|_p},$$

and from (35), that

$$\left|\sqrt{y^2 - 4A'_kB'_k} - (A'_k - B'_k)\right|_p \leq \frac{p^{-\lambda}}{|A'_k - B'_k|_p}.$$

Since $|A'_k - B'_k|_p \geq p^{-\lambda/2}$, it follows that

$$\left|\sqrt{y^2 - 4A'_kB'_k} - (A'_k - B'_k)\right|_p \leq p^{-\lambda/2}.$$

As $|y - (A'_k + B'_k)|_p \leq p^{-\lambda}$, we thus get

$$\left|y + \sqrt{y^2 - 4A'_kB'_k} - 2A'_k\right|_p \leq p^{-\lambda/2}$$

which leads to (37) since $|A'_k|_p \geq 1$.

Lastly we prove that $\frac{\log x}{\log \omega}$ belongs to $\mathbf{Q}_p$. Indeed, in the case where $\omega \notin \mathbf{Q}_p$, we have

$$\sigma\left(\sqrt{\frac{y^2-4A'_kB'_k}{(A'_k-B'_k)^2}}\right)=\sqrt{\frac{y^2-4A'_kB'_k}{(A'_k-B'_k)^2}}$$

since this number lies in $\mathbf{Q}_p$. Hence, as $\sigma(A'_k-B'_k)=B'_k-A'_k$, we get $\sigma(\sqrt{y^2-4A'_kB'_k})=-\sqrt{y^2-4A'_kB'_k}$. We thus have

$$\sigma\left(\frac{y+\sqrt{y^2-4A'_kB'_k}}{2A'_k}\right)=\frac{y-\sqrt{y^2-4A'_kB'_k}}{2B'_k}.$$

It follows immediatly that

$$\sigma(x)=1/x \tag{38}$$

and as ω also satisfies (38), we conclude as above that $\frac{\log x}{\log \omega}$ belongs to $\mathbf{Q}_p$. Moreover, as $|\frac{\log x}{\log \omega}|_p=|x-1|_p/|\omega-1|_p$, we conclude from (37) that $\frac{\log x}{\log \omega}$ belongs to $\mathbf{Z}_p$. Accordingly, given a positive integer N, there exists an integer n, with $0\le n<p^N$, such that

$$\left|\frac{\log x}{\log \omega}-n\right|_p\le p^{-N},$$

i.e., $|\log x-\log \omega^n|_p\le p^{-N}|\omega-1|_p$, and thus $|x-\omega^n|_p\le p^{-N}|\omega-1|_p$. As $|B'_k|\le|A'_k|\le 1/|\omega-1|_p$, we thus get

$$|\varphi_k(x)-\varphi_k(\omega^n)|_p\le p^{-N}.$$

Now $\varphi_k(x)=y$, and if $0\le n\le h_k$, then $\varphi_k(\omega^n)=L_kq_{m_k+nT}$. Let us select k such that $h_k\ge p^N$, we thus have found an integer n, with $0\le n\le h_k$, such that

$$|L_kq_{m_k+nT}-y|\le p^{-N}. \tag{39}$$

Accordingly, the p-adic closure of the set of $L_kq_{m_k+nT}$, with $k\ge 1$ and $0\le n\le h_k$ contains the ball $|y-1|_p\le p^{-\lambda}$.

There are some minor changes when $p=2$. First note that, using the logarithm function over $\mathbf{Q}_2$ for $|z-1|\le 1/4$, we get $|z^2-1|_2=\frac{1}{2}|z-1|_2$. The function $\sqrt{\cdot}$ is then defined over the ball $|z-1|_2\le 1/8$ in $\mathbf{Q}_2$, and satisfies $|\sqrt{z}-1|_2=2|z-1|_2$. Also note that if $|A_k-B_k|_2<1$, then $|A_k|_2=|B_k|_2=2$, since $|A_k+B_k|_2=1$. Reasoning as above, we thus may suppose that

$$|A_k-B_k|_2\ge\frac{1}{2}\cdot|\omega-1|_2.$$

The number λ is then determined by

$$2^{-\lambda}=\frac{1}{16}|\omega-1|_2^2$$

and L_k is determined by

$$L_kq_{m_k}\equiv 1 \bmod 2^\lambda$$

and $0 < L_k < 2^\lambda$. Then, the 2-adic closure of the set of integers $L_k q_{m_k+nT}$ contains the ball $\{y : |y-1|_2 \le 2^{-\lambda}\}$, and (39) holds.

In order to obtain (7), it is enough to note that $\log |L_k q_{m_k+nT}| \ll m_k + n$, and thus, if condition (6) is satisfied, then

$$\log |L_k q_{m_k+nT}| \ll h_k.$$

We may choose the integer N above such that $h_k/p < p^N \le h_k$, i.e.,

$$p^N \asymp h_k,$$

and from (39), we get (7).

4. Proof of Theorem 3

– Proof of Theorem 3.

Let α be as in the statement of Theorem 3. Let p be a prime number. First, notice that if ω is any quadratic unit, then the index of the multiplicative group $G = \{x \in \mathbf{Q}_p(\omega) \ : \ |x-1|_p < p^{-1/(p-1)}\}$ in the unit ball $\{x \in \mathbf{Q}_p(\omega) \ : \ |x-1|_p = 1\}$ is a divisor of $p^2(p^2-1)$. Hence, we have $|\omega^{p^2(p^2-1)} - 1|_p < p^{-1/(p-1)}$. Accordingly, in the statement of Theorem 3, if we replace T by $p^2(p^2-1)T$, we can suppose, as in the proof of Theorem 1, that the eigenvalues ω_T, $1/\omega_T$ of the matrix $M = M_T \dots M_1$ are quadratic units with $|\omega_T^{p^2(p^2-1)} - 1|_p < p^{-1/(p-1)}$.

Then, in order to prove Theorem 3 for $y = 0$ it is enough to consider $Q_n = q_{nT-1}$ for $0 \le n \le h$. This sequence satisfies (10), hence we can write

$$Q_n = A(\omega_T^n - \omega_T^{-n})$$

since $Q_0 = 0$. We have

$$|A|_p \le 1/|\omega_T^2 - 1|_p.$$

Hence, for $0 < s \le \log h / \log p$, we have

$$|Q_{p^s}|_p \le \frac{|\omega_T^{2p^s} - 1|_p}{|\omega_T^2 - 1|_p} = p^{-s}.$$

Taking s such that $h/p < p^s \le h$, we get

$$|Q_{p^s}|_p \ll 1/h.$$

This proves the result, in virtue of (28).

Let y be a non-zero integer. Consider $Q'_n = y q_{nT} = E\omega_T^n + F\omega_T^{-n}$, where

$$\max\{|E|_p, |F|_p\} \le 1/|\omega_T^2 - 1|_p.$$

Since $Q'_0 = y$, we have as above

$$|Q'_{2p^s} - y|_p \leq \frac{|\omega_T^{2p^s} - 1|_p}{|\omega_T^2 - 1|_p} = p^{-s},$$

which, again, proves the result by (28). □

– *Remarks on the proofs.*

Although Theorem 2 for $y = 0$ is a consequence of Theorem 1, and was already proved in [12], we would like to note that in this case, the proof of (7) is very simple, and the use of the p-adic logarithm function is actually not necessary. Indeed, the assumptions of Theorem 1 are satisfied for some positive integer m, and we have $a_{m+j+nT} = b_j$ for every non-negative integer n and for any $j = 1, \ldots, T$. For any $k \geq 1$, we set $m_k = m$, and we choose the integers $x_k = q_{m+T}$ and $y_k = -q_m$. Then, as in the proof of Theorem 1, we consider integers $Q_n = q_{m+T}q_{m+nT} - q_m q_{m+(n+1)T}$. These integers satisfy (28) since m is now a fixed number. In this case, as $Q_0 = 0$, we have $C_k = -D_k$, that is to say that we can write $Q_n = C(\omega^n - \omega^{-n})$, with $C \neq 0$ since $q_{m+T} - \omega q_m \neq 0$. Hence, we have $Q_n \neq 0$ for $n > 0$, and

$$|Q_n|_p \ll |\omega^{2n} - 1|_p.$$

Then, we only have to check that

$$|\omega^{2p^s} - 1|_p \leq p^{-s}|\omega^2 - 1|_p.$$

This follows from an elementary induction by writing $\omega^{2p^s} = 1 + u_s$, and using Newton's formula. As $\log Q_n \ll n$, we thus see that the integers Q_{p^s} satisfy (28), with $\log |Q_{p^s}| \ll p^s$ and $|Q_{p^s}|_p \ll p^{-s}$. This provides the estimation (7). The same holds when $y \neq 0$ is an integer. Indeed, by the Bezout Theorem, q_m and q_{m+1} being coprime, we can take integers x' and y' such that $x'q_m + y'q_{m+1} = y$ (where $m \geq 0$ is chosen in a such way that $a_{m+j+nT} = b_j$ for every non-negative integer n and for any $j = 1, \ldots T$). Since both the sequences $(q_{m+nT})_{n\geq 0}$ and $(q_{m+1+nT})_{n\geq 0}$ satisfy (10), then for every non-negative integer n we can write

$$x'q_{m+nT} + y'q_{m+1+nT} = E\omega^n + F\omega^{-n}$$

where E and F are numbers in $\mathbf{Q}_p(\omega)$, with $|E|_p \leq 1/|\omega^2 - 1|_p$ and $|F|_p \leq 1/|\omega^2 - 1|_p$. The integers $Q'_n = x'q_{m+nT} + y'q_{m+1+nT}$ satisfy (28). Since $|\omega^{2p^s} - 1|_p \leq p^{-s}|\omega^2 - 1|_p$, we see that for $n = 2p^s$, we have $|Q'_{2p^s} - Q'_0|_p \leq p^{-s}$, that is to say

$$|Q'_{2p^s} - y|_p \leq p^{-s}.$$

Further, it is easy to see that Theorem 2 and Theorem 3 remain valid when replacing in (8) the p-adic absolute value $|\cdot|_p$ by $|\cdot|_{\mathcal{D}}$, if we take $y \in \mathbf{Z}$.

In the other parts of the proof, the role played by the p-adic logarithm function seems deeper. For instance, it is easy to prove the analogue of Theorem 2 for formal power series over a finite field, when y is a polynomial. But we do not know whether Theorem 1 has

an analogue in this setting. The crucial point is the use of the p-adic logarithm function, that we lose in the formal power series case.

5. Proof of Theorem 4

Let p be a prime number and y be an integer. Let μ denote the *Kaufman measure*, whose existence has been proved in [7]. The measure μ is supported on the set $\boldsymbol{Bad}$. For any $n \geq 1$, set $q_n := p^n + y$. For any real number α we have

$$|q_n| \cdot \|q_n\alpha\| \cdot |q_n - y|_p = |q_n| \cdot \|q_n\alpha\| \cdot |p^n|_p \leq (1 + |y|) \, \|q_n\alpha\|.$$

Thus, (4) holds as soon as α satisfies

$$\inf_{n\geq 1} \|q_n\alpha\| = 0. \tag{40}$$

Using the exponential growth of the sequence $(q_n)_{n\geq 1}$ and a result of Davenport, Erdős, and LeVeque [4], as explained in Section 4 of [13], we get that (40) holds for μ-almost all α. Hence, μ-almost all α satisfy (4) for any prime p and any integer y. Arguing then as in page 294 from [13], we obtain that the Hausdorff dimension of the set of real numbers α in $\boldsymbol{Bad}$ for which (4) holds for any prime p and any integer y is equal to 1, as claimed.

6. Proof of Theorem 5

Our proof is very much inspired by [8]. It rests on the Folding Lemma, recalled below.

Lemma F. *If $p_n/q_n = [a_0; a_1, a_2, \ldots, a_n]$ with $a_n \geq 2$, then*

$$\frac{p_n}{q_n} + \frac{(-1)^n}{q_n^2} = [a_0; a_1, a_2, \ldots, a_{n-1}, a_n + 1, a_n - 1, a_{n-1}, \ldots, a_2, a_1].$$

Lemma F follows from Propositions 2 and 3 of [14].

Recall that $[0; a_1, \ldots, a_n, 1, 1] = [0; a_1, \ldots, a_n, 2]$ for any positive integers $a_1, \ldots, a_n$. We display an immediate consequence of Lemma F.

Lemma 3. *If $a/m = [0; 1, 1, a_3, \ldots, a_{h-1}, a_h]$ with $h \geq 4$ and $a_h \geq 2$, then*

$$\frac{ma + (-1)^h}{m^2} = [0; 1, 1, a_3, \ldots, a_{h-1}, a_h + 1, a_h - 1, a_{h-1}, \ldots, a_3, 2].$$

Lemma 3 is the main tool for the proof of Theorem 5. Let p be a prime number. There are positive integers a and b, with $1 \leq a \leq p^b$ and a coprime with p, such that the continued fraction of a/p^b reads

$$\frac{a}{p^b} = [0; 1, 1, a_3, \ldots, a_{h-1}, a_h],$$

with $h \geq 4$ and $a_h \geq 2$. Set

$$M = \max\{a_3, \ldots, a_{h-1}, a_h + 1\}.$$

By repeated applications of Lemma 3, we see that, for any $j \geq 2$, the continued fraction of the rational number

$$\alpha_j := \frac{a}{p^b} + \frac{(-1)^h}{p^{2b}} - \frac{1}{p^{2^2 b}} \cdots - \frac{1}{p^{2^j b}}$$

reads

$$[0; 1, 1, \ldots, 2]$$

and has all its partial quotients bounded by M.

Set

$$\alpha = \lim_{j \to +\infty} \alpha_j = \frac{a}{p^b} + \frac{(-1)^h}{p^{2b}} - \sum_{j \geq 2} \frac{1}{p^{2^j b}}.$$

By construction, all the partial quotients of α are less than or equal to M, hence α is in ***Bad***. Furthermore, it is easily checked that, for any $j \geq 2$, we have

$$p^{2^j b} \cdot ||p^{2^j b}\alpha|| \leq 1 + 2^{-2^{j-1}}.$$

This implies that

$$\liminf_{q \to +\infty} q^2 \cdot \|q\alpha\| \cdot |q|_p \leq 1,$$

as asserted.

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