

# Limit Laws of Planar Maps with Prescribed Vertex Degrees

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We prove a general multi-dimensional central limit theorem for the expected number of vertices of a given degree in the family of planar maps whose vertex degrees are restricted to an arbitrary (finite or infinite) set of positive integers  $D$ . Our results rely on a classical bijection with mobiles (objects exhibiting a tree structure), combined with refined analytic tools to deal with the systems of equations on infinite variables that arise. We also discuss possible extensions to maps of higher genus and to weighted maps.

## 1. Introduction and Results

In this paper we study statistical properties of planar maps, which are connected planar graphs, possibly with loops and multiple edges, together with an embedding into the plane. Such objects are frequently used to describe topological features of geometric arrangements in two or three spatial dimensions. Thus, the knowledge of the structure and of properties of “typical” objects may turn out to be very useful in the analysis of particular algorithms that operate on planar maps. We say that a map is *rooted* if an edge  $e$  is distinguished and oriented. It is called the root edge. The first vertex  $v$  of this oriented edge is called the root vertex. The face to the right of  $e$  is called the root face and is usually taken as the outer (or infinite) face. Similarly, we call a planar map *pointed* if just a vertex  $v$  is distinguished. However, we have to be really careful with the model. In rooted maps the root edge *destroys* potential symmetries, which is not the case if we consider pointed maps.

The enumeration of rooted maps is a classical subject, initiated by Tutte in the 1960’s, see [13]. Among many other results, Tutte computed the number  $M_n$  of rooted maps with  $n$  edges, proving the formula

$$M_n = \frac{2(2n)!}{(n+2)!n!} 3^n$$

<sup>†</sup> Partially supported by the Austrian Science Fund FWF, Project SFB F50-02.

which directly provides the asymptotic formula

$$M_n \sim \frac{2}{\sqrt{\pi}} n^{-5/2} 12^n.$$

We are mainly interested in planar maps with degree restrictions. Actually, it turns out that this kind of asymptotic expansion is quite universal. Furthermore, there is always a (very general) central limit theorem for the number of vertices of given degree.

**Theorem 1.1.** *Suppose that  $D$  is an arbitrary set of positive integers but not a subset of  $\{1, 2\}$ . Let  $\mathcal{M}_D$  be the class of planar rooted maps with the property that all vertex degrees are in  $D$  and let  $M_{D,n}$  denote the number of maps in  $\mathcal{M}_D$  with  $n$  edges. Furthermore, if  $D$  contains only even numbers, then set  $\bar{d} = \gcd\{i : 2i \in D\}$ ; set  $\bar{d} = 1$  otherwise.*

*Then there exist positive constants  $c_D$  and  $\rho_D$  with*

$$M_{D,n} \sim c_D n^{-5/2} \rho_D^{-n}, \quad n \equiv 0 \pmod{\bar{d}}. \quad (1.1)$$

*Furthermore, let  $X_n^{(d)}$  denote the random variable counting vertices of degree  $d$  ( $\in D$ ) in maps in  $\mathcal{M}_D$ . Then  $\mathbb{E}(X_n^{(d)}) \sim \mu_d n$  for some constant  $\mu_d > 0$  and for  $n \equiv 0 \pmod{\bar{d}}$ , and the (possibly infinite) random vector  $\mathbf{X}_n = (X_n^{(d)})_{d \in D}$  ( $n \equiv 0 \pmod{\bar{d}}$ ) satisfies a central limit theorem, that is,*

$$\frac{1}{\sqrt{n}} (\mathbf{X}_n - \mathbb{E}(\mathbf{X}_n)), \quad n \equiv 0 \pmod{\bar{d}}, \quad (1.2)$$

*converges weakly to a centered Gaussian random variable  $\mathbf{Z}$  (in  $\ell^2$ ).*

Note that maps where all vertex degrees are 1 or 2 are very easy to characterise and are not really of interest, and that actually, their asymptotic properties are different from the general case. It is therefore natural to assume that  $D$  is not a subset of  $\{1, 2\}$ .

Since we can equivalently consider dual maps, this kind of problem is the same as the one considering planar maps with restrictions on the face valencies. This means that the same results hold if we replace *vertex degree* by *face valency*. For example, if we assume that all face valencies equal 4, then we just consider planar quadrangulations (which have also been studied by Tutte [13]). In fact, our proofs will refer just to face valencies.

Theorem 1.1 goes far beyond known results. There are some general results for the Eulerian case where all vertex degrees are even. First, the asymptotic expansion (1.1) is known for Eulerian maps by Bender and Canfield [2]. Furthermore, a central limit theorem of the form (1.2) is known for all Eulerian maps (without degree restrictions) [9]. However, in the non-Eulerian case there are almost no results of this kind; there is only a one-dimensional central limit theorem for  $X_n^{(d)}$  for all planar maps [10].

The uniform distribution of planar maps according to the number of edges is not the only distribution that has been studied. Many probabilistic results on planar maps have also been extended to other probability distributions, based on  $q$ -Boltzmann maps. Let  $\mathbf{q} = (q_1, q_2, \dots)$  be a sequence of non-negative weights. A  $\mathbf{q}$ -Boltzmann map is a random planar map with arbitrary vertex degrees, where the probability of choosing a given map

$M$  is proportional to

$$\prod_{i>0} q_i^{\#\text{vertices of degree } i \text{ in } M}.$$

When such a procedure describes a well-defined probability distribution,  $\mathbf{q}$  is called *admissible*. (By duality we could equivalently use weights of the form  $\prod q_i^{\#\text{faces of degree } i \text{ in } M}$ .)

In [11, 12], the authors showed that, under some integrability conditions, random  $\mathbf{q}$ -Boltzmann maps have the same profile as random uniform planar maps. In this spirit we will show that (under certain conditions) Theorem 1.1 also applies to  $\mathbf{q}$ -Boltzmann maps.

**Theorem 1.2.** *Let  $\mathbf{q} = (q_1, q_2, \dots)$  be a weight sequence with  $q_i = \Theta(i^\alpha)$  for some  $\alpha \geq -\frac{3}{2}$  and consider corresponding  $\mathbf{q}$ -Boltzmann maps. Furthermore, let  $Y_n^{(d)}$  denote the random variable counting vertices of degree  $d$ . Then  $\mathbb{E}(Y_n^{(d)}) \sim \mu_d n$  for some constant  $\mu_d > 0$  and the infinite random vector  $\mathbf{Y}_n = (Y_n^{(d)})_{d \geq 1}$  satisfies a central limit theorem.*

Section 2 introduces planar mobiles which, being in bijection with pointed planar maps, will reduce our analysis to simpler objects with a tree structure. Their asymptotic behaviour is derived in Section 3, first for the simpler case of bipartite maps (i. e. when  $D$  contains only even integers), then for families of maps without constraints on  $D$ . Sections 4 and 5 are devoted to the proof of the central limit theorem using analytic tools from [8, 9]. Finally, in Section 6 we discuss combinatorics and asymptotics of maps on orientable surfaces of higher genus. The expressions we obtain are much more involved than in the planar case, but we obtain similar analytic results.

## 2. Mobiles

Instead of investigating planar maps themselves, we will follow the principle presented in [5], whereby pointed planar maps are bijectively related to a certain class of trees called mobiles. (Their version of mobiles differs from the definition originally given in [3]; the equivalence of the two definitions is not shown explicitly in [5], but [7] gives a straightforward proof.)

**Definition.** A *mobile* is a planar tree – that is, a map with a single face – such that there are two kinds of vertices (black and white), edges only occur as black–black edges or black–white edges, and black vertices additionally have so-called “legs” attached to them (which are not considered edges), whose number equals the number of white neighbour vertices. A *bipartite mobile* is a mobile without black–black edges. The *degree* of a black vertex is the number of half-edges plus the number of legs that are attached to it. A mobile is called *rooted* if an edge is distinguished and oriented.

The essential observation is that mobiles are in bijection to pointed planar maps.

**Theorem 2.1.** *There is a bijection between mobiles that contain at least one black vertex and pointed planar maps, where white vertices in the mobile correspond to non-pointed vertices in the equivalent planar map, black vertices correspond to faces of the map, and the degrees of the black vertices correspond to the face valencies. This bijection induces a bijection on the edge sets such that the number of edges is the same. (Only the pointed vertex of the map has no counterpart.)*

*Similarly, rooted mobiles that contain at least one black vertex are in bijection to rooted and vertex-pointed planar maps.*

*Finally, bipartite mobiles with at least two vertices correspond to bipartite maps with at least two vertices, in the unrooted as well as in the rooted case.*

**Proof.** For the proof of the bijection between mobiles and pointed maps we refer to [7], where the bipartite case is also discussed. It just remains to note that the induced bijection on the edges can be directly used to transfer the root edge together with its direction.  $\square$

## 2.1. Bipartite Mobile Counting

We start with bipartite mobiles since they are more easy to count, in particular if we consider rooted bipartite mobiles, see [7].

**Proposition 2.2.** *Let  $R = R(t, z, x_1, x_2, \dots)$  be the solution of the equation*

$$R = tz + z \sum_{i \geq 1} x_{2i} \binom{2i-1}{i} R^i. \quad (2.1)$$

*Then the generating function  $M = M(t, z, x_1, x_2, \dots)$  of bipartite rooted maps satisfies*

$$\frac{\partial M}{\partial t} = 2(R/z - t), \quad (2.2)$$

*where the variable  $t$  corresponds to the number of vertices,  $z$  to the number of edges, and  $x_{2i}$ ,  $i \geq 1$ , to the number of faces of valency  $2i$ .*

**Proof.** Since rooted mobiles can be considered as ordered rooted trees (which means that the neighbouring vertices of the root vertex are linearly ordered and the subtrees rooted at these neighbouring vertices are again ordered trees), we can describe them recursively. This directly leads to a functional equation for  $R$  of the form

$$R = \frac{tz}{1 - z \sum_{i \geq 1} x_{2i} \binom{2i-1}{i} R^{i-1}}$$

which is apparently the same as (2.1). Note that the factor  $\binom{2i-1}{i}$  is precisely the number of ways of grouping  $i$  legs and  $i-1$  edges around a black vertex (of degree  $2i$ ; one edge is already there).

Hence, the generating function of rooted mobiles that are rooted by a white vertex is given by  $R/z$ . Since we have to discount the mobile that consists just of one (white) vertex, the generating function of rooted mobiles that are rooted at a white vertex and

contain at least two vertices is given by

$$R/z - t = \sum_{i \geq 1} x_{2i} \binom{2i-1}{i} R^i. \quad (2.3)$$

We now observe that the right-hand side of (2.3) is precisely the generating function of rooted mobiles that are rooted at a black vertex (and contain at least two vertices). Summing up, the generating function of bipartite rooted mobiles (with at least two vertices) is given by  $2(R/z - t)$ . Finally, if  $M$  denotes the generating function of bipartite rooted maps (with at least two vertices) then  $\frac{\partial M}{\partial t}$  corresponds to rooted maps where a non-root vertex is pointed (and discounted). Thus, by Theorem 2.1 we obtain (2.2).  $\square$

It is clear that Formula (2.2) can be specialised to count  $M_D$  for any subset  $D$  of even positive integers: It suffices to set  $x_{2i} = 1$  for  $2i \in D$  and  $x_{2i} = 0$  otherwise.

## 2.2. General Mobile Counting

We now proceed to develop a mechanism for general mobile counting that is adapted from [5]. For this, we will require Motzkin paths. A *Motzkin path* is a path starting at 0 and going rightwards for a number of steps; the steps are either diagonally upwards (+1), straight (0) or diagonally downwards (-1). A *Motzkin bridge* is a Motzkin path from 0 to 0. A *Motzkin excursion* is a Motzkin bridge which stays non-negative.

We define generating functions in the variables  $t$  and  $u$ , which count the number of steps of type 0 and -1, respectively. (Explicitly counting steps of type 1 is then unnecessary, of course.) The ordinary generating functions of Motzkin bridges, Motzkin excursions, and Motzkin paths from 0 to +1 shall be denoted by  $B(t, u)$ ,  $E(t, u)$  and  $B^{(+1)}(t, u)$ , respectively. By decomposing these three types of paths by their last passage through 0, we arrive at the equations (compare with [5]):

$$E = 1 + tE + uE^2, \quad B = 1 + (t + 2uE)B, \quad B^{(+1)} = EB.$$

In what follows we will also make use of bridges where the first step is either of type 0 or -1. Clearly, their generating function  $\bar{B}$  is given by  $\bar{B} = tB + uB^{(+1)} = B(t + uE)$ .

When Motzkin bridges are not constrained to stay non-negative, they can be seen as an arbitrary arrangement of a given number of steps +1, 0, -1. It is then possible to obtain explicit expressions for

$$B_{\ell, m} = [t^\ell u^m] B(t, u) = \binom{\ell + 2m}{\ell, m, m}, \quad (2.4)$$

$$B_{\ell, m}^{(+1)} = [t^\ell u^m] B^{(+1)}(t, u) = \binom{\ell + 2m + 1}{\ell, m, m + 1}, \quad (2.5)$$

$$\bar{B}_{\ell, m} = [t^\ell u^m] \bar{B}(t, u) = B_{\ell-1, m} + B_{\ell, m-1}^{(+1)} = \frac{\ell + m}{\ell + 2m} \binom{\ell + 2m}{\ell, m, m}. \quad (2.6)$$

Using the above, we can now finally compute relations for generating functions of proper classes of mobiles. We define the following series, where  $t$  corresponds to the number of white vertices,  $z$  to the number of edges, and  $x_i$ ,  $i \geq 1$ , to the number of black vertices of degree  $i$ :

- $L(t, z, x_1, x_2, \dots)$  is the series counting rooted mobiles that are rooted at a black vertex and where an additional edge is attached to the black vertex.
- $R(t, z, x_1, x_2, \dots)$  is the series counting rooted mobiles that are rooted at a white vertex and where an additional edge is attached to the root vertex.

Similarly to the above we obtain the following equations for the generating functions of mobiles and rooted maps.

**Proposition 2.3.** *Let  $L = L(t, z, x_1, x_2, \dots)$  and  $R = R(t, z, x_1, x_2, \dots)$  be the solutions of the equation*

$$\begin{aligned} L &= z \sum_{\ell, m} x_{2m+\ell+1} B_{\ell, m} L^\ell R^m, \\ R &= tz + z \sum_{\ell, m} x_{\ell+2m+2} B_{\ell, m}^{(+1)} L^\ell R^{m+1}, \end{aligned} \quad (2.7)$$

and let  $T = T(t, z, x_1, x_2, \dots)$  be given by

$$T = 1 + \sum_{\ell, m} x_{2m+\ell} \bar{B}_{\ell, m} L^\ell R^m, \quad (2.8)$$

where the numbers  $B_{\ell, m}$ ,  $B_{\ell, m}^{(+1)}$ , and  $\bar{B}_{\ell, m}$  are given by (2.4)–(2.6). Then the generating function  $M = M(t, z, x_1, x_2, \dots)$  of rooted maps satisfies

$$\frac{\partial M}{\partial t} = R/z - t + T, \quad (2.9)$$

where the variable  $t$  corresponds to the number of vertices,  $z$  to the number of edges, and  $x_i$ ,  $i \geq 1$ , to the number of faces of valency  $i$ .

**Proof.** The system (2.7) is just a rephrasing of the recursive structure of rooted mobiles. Note that the numbers  $B_{\ell, m}$  and  $B_{\ell, m}^{(+1)}$  are used to count the number of ways to circumscribe a specific black vertex and considering white vertices, black vertices and “legs” as steps  $-1$ ,  $0$  and  $+1$ . The generating function  $T$  given in (2.8) is then the generating function of rooted mobiles where the root vertex is black.

Finally, the equation (2.9) follows from Theorem 2.1 since  $R/z - t$  corresponds to rooted mobiles with at least one black vertex where the root vertex is white and  $T$  corresponds to rooted mobiles where the root vertex is black.  $\square$

### 3. Asymptotic Enumeration

In this section we prove the asymptotic expansion (1.1). It turns out that it is much easier to start with bipartite maps. Actually, the bipartite case has already been treated by Bender and Canfield [2]. However, we apply a slightly different approach, which will then be extended to cover the general case as well the central limit theorem.

#### 3.1. Bipartite Maps

Let  $D$  be a non-empty subset of even positive integers different from  $\{2\}$ . Then by Proposition 2.2 the counting problem reduces to the discussion of the solutions  $R_D =$

$R_D(t, z)$  of the functional equation

$$R_D = tz + z \sum_{2i \in D} \binom{2i-1}{i} R_D^i \quad (3.1)$$

and the generating function  $M_D(t, z)$  that satisfies the relation

$$\frac{\partial M_D}{\partial t} = 2(R_D/z - t). \quad (3.2)$$

Let  $\bar{d} = \gcd\{i : 2i \in D\}$ . Then for combinatorial reasons it follows that there only exist maps with  $n$  edges for  $n$  that are divisible by  $\bar{d}$ . This is reflected by the fact that the equation (3.1) can be rewritten in the form

$$\tilde{R} = t + \sum_{2i \in D} \binom{2i-1}{i} z^{i/\bar{d}} \tilde{R}^i, \quad (3.3)$$

where we have substituted  $R_D(t, z) = z \tilde{R}(t, z^{\bar{d}})$ . (Recall that we finally work with  $R_D/z$ .)

**Lemma 3.1.** *There exists an analytic function  $\rho(t)$  with  $\rho(1) > 0$  and  $\rho'(1) \neq 0$  that is defined in a neighbourhood of  $t = 1$ , and there exist analytic functions  $g(t, z)$ ,  $h(t, z)$  with  $h(1, \rho(1)) > 0$  that are defined in a neighbourhood of  $t = 1$  and  $z = \rho(1)$  such that the unique solution  $R_D = R_D(t, z)$  of the equation (3.1) that is analytic at  $z = 0$  and  $t = 0$  can be represented as*

$$R_D = g(t, z) - h(t, z) \sqrt{1 - \frac{z}{\rho(t)}}. \quad (3.4)$$

Furthermore, the values  $z = \rho(t)e(2\pi i j/\bar{d})$ ,  $j \in \{0, 1, \dots, \bar{d}-1\}$ , are the only singularities of the function  $z \mapsto R_D(t, z)$  on the disc  $|z| \leq \rho(t)$ , and there exists an analytic continuation of  $R_D$  to the range  $|z| < |\rho(t)| + \eta$ ,  $\arg(z - \rho(t)e(2\pi i j/\bar{d})) \neq 0$ ,  $j \in \{0, 1, \dots, \bar{d}-1\}$ .

**Proof.** From general theory (see [8, Theorem 2.21]) we know that an equation of the form  $R = F(t, z, R)$ , where  $F$  is a power series with non-negative coefficients, has – usually – a square-root singularity of the form (3.4). We only have to assume that the function  $R \mapsto F(t, z, R)$  is neither constant nor a linear polynomial and that there exist solutions  $\rho > 0$ ,  $R_0 > 0$  of the system of equations

$$R_0 = F(1, \rho, R_0), \quad 1 = F_R(1, \rho, R_0)$$

which are inside the range of convergence of  $F$ . Furthermore, we have to assume that  $F_z(1, \rho, R_0) > 0$  and  $F_{RR}(1, \rho, R_0) > 0$  to ensure that (3.4) holds not only for  $t = 1$  but in a neighbourhood of  $t = 1$ , and the condition  $F_t(1, \rho, R_0) > 0$  ensures that  $\rho'(1) \neq 0$ .

This means that in our case we have to deal with the system of equations

$$R_0 = \rho + \rho \sum_{2i \in D} \binom{2i-1}{i} R_0^i, \quad 1 = \rho \sum_{2i \in D} i \binom{2i-1}{i} R_0^{i-1},$$

or just with a single equation (after eliminating  $\rho$ )

$$\sum_{2i \in D} (i-1) \binom{2i-1}{i} R_0^i = 1. \quad (3.5)$$

It is clear that (3.5) has a unique positive solution if  $D$  is finite. (We also recall that all  $i \geq 1$ , since  $2i$  has to be positive.) If  $D$  is infinite, we have to be more precise. Actually, we will show that (3.5) has a unique positive solution  $R_0 < 1/4$ . This follows from the fact that

$$(i-1) \binom{2i-1}{i} \sim \frac{4^i \sqrt{i}}{2\sqrt{\pi}}.$$

Thus, if  $D$  is infinite, it follows that the power series  $x \mapsto H(x) = \sum_{2i \in D} (i-1) \binom{2i-1}{i} x^i$  has radius of convergence  $1/4$  and we also have  $H(x) \rightarrow \infty$  as  $x \rightarrow 1/4^-$  since each non-zero term satisfies

$$\lim_{x \rightarrow 1/4^-} (i-1) \binom{2i-1}{i} x^i \sim \frac{\sqrt{i}}{2\sqrt{\pi}},$$

which is unbounded for  $i \rightarrow \infty$ . Finally, we set  $\rho = (\sum_{2i \in D} i \binom{2i-1}{i} R_0^{i-1})^{-1}$ .

It is clear that  $F_z(\rho, 1, R_0) > 0$ ,  $F_{RR}(\rho, 1, R_0) > 0$ , and  $F_t(\rho, 1, R_0) > 0$ . Hence we obtain the representation (3.4) in a neighbourhood of  $z = \rho = \rho(1)$  and  $t = 1$ .

Next, let us discuss the analytic continuation property. If  $\bar{d} = \gcd\{i : 2i \in D\} = 1$  then it follows from the equation (3.1) that the coefficients  $[z^n]R_D(1, z)$  are positive for  $n \geq n_0$  (for some  $n_0$ ). Consequently [8, Theorem 2.21] (see also [8, Theorem 2.16]) implies that there is an analytic continuation to the region  $|z| < |\rho(t)| + \eta$ ,  $\arg(z - \rho(t)) \neq 0$ . If  $\bar{d} > 1$ , then we can first reduce equation (3.1) to an equation (3.3) for the function  $\tilde{R}$  that is given by  $R_D(t, z) = z\tilde{R}(t, z^{\bar{d}})$ . We now apply the above method to this equation and obtain corresponding properties for  $\tilde{R}$ . Of course, these properties directly translate to  $R_D$ , and we are done.  $\square$

It is now relatively easy to obtain similar properties for  $M_D(t, z)$ .

**Lemma 3.2.** *The function  $M = M_D(t, z)$  that is given by (3.2) has the representation*

$$M_D = g_2(t, z) + h_2(t, z) \left(1 - \frac{z}{\rho(t)}\right)^{3/2} \quad (3.6)$$

*in a neighbourhood of  $t = 1$  and  $z = \rho(1)$ , where the functions  $g_2(t, z)$ ,  $h_2(t, z)$  are analytic in a neighbourhood of  $t = 1$  and  $z = \rho(1)$  and we have  $h_2(1, \rho(1)) > 0$ . Furthermore, the values  $z = \rho(t)e(2\pi i j / \bar{d})$ ,  $j \in \{0, 1, \dots, \bar{d} - 1\}$ , are the only singularities of the function  $z \mapsto M_D(t, z)$  on the disc  $|z| \leq \rho(t)$ , and there exists an analytic continuation of  $M_D$  to the range  $|z| < |\rho(t)| + \eta$ ,  $\arg(z - \rho(t)e(2\pi i j / \bar{d})) \neq 0$ ,  $j \in \{0, 1, \dots, \bar{d} - 1\}$ .*

**Proof.** This is a direct application of [8, Lemma 2.27].  $\square$

In particular it follows that  $M_D(1, z)$  has the singular representation of the form (3.6) with a dominant singularity  $(1 - z/\rho(1))^{3/2}$  near  $z = \rho(1)$ . The singular representations



are of the same kind near  $z = \rho(1)e(2\pi i j/\bar{d})$ ,  $j \in \{1, \dots, \bar{d} - 1\}$ , and we have the analytic continuation property. Hence it follows by usual singularity analysis (see for example [8, Corollary 2.15]) that there exists a constant  $c_D > 0$  such that

$$[z^n]M_D(1, z) \sim c_D n^{-5/2} \rho(1)^{-n}, \quad n \equiv 0 \pmod{\bar{d}},$$

which completes the proof of the asymptotic expansion in the bipartite case.

### 3.2. General Maps

We now suppose that  $D$  contains at least one odd number. It is easy to observe that in this case we have  $[z^n]M_D(1, z) > 0$  for  $n \geq n_0$  (for some  $n_0$ ), so we do not have to deal with several singularities. By Proposition 2.3 we have to consider the system of equations for  $L_D = L_D(t, z)$ ,  $R_D = R_D(t, z)$ :

$$L_D = z \sum_{i \in D} \sum_m B_{i-2m-1, m} L_D^{i-2m-1} R_D^m, \quad (3.7)$$

$$R_D = tz + z \sum_{i \in D} \sum_m B_{i-2m-2, m}^{(+1)} L_D^{i-2m-2} R_D^{m+1}, \quad (3.8)$$

and also the function

$$T_D = T_D(t, z) = 1 + \sum_{i \in D} \sum_m \bar{B}_{i-2m, m} L_D^{i-2m} R_D^m.$$

**Lemma 3.3.** *There exists an analytic function  $\rho(t)$  with  $\rho(1) > 0$  and  $\rho'(1) \neq 0$  that is defined in a neighbourhood of  $t = 1$ , and there exist analytic functions  $g(t, z)$ ,  $h(t, z)$  with  $h(1, \rho(1)) > 0$  that are defined in a neighbourhood of  $t = 1$  and  $z = \rho(1)$  such that*

$$R_D/z - t + T_D = g(t, z) - h(t, z) \sqrt{1 - \frac{z}{\rho(t)}}. \quad (3.9)$$

Furthermore, the value  $z = \rho(t)$  is the only singularity of the function  $z \mapsto R_D/z - t + T_D$  on the disc  $|z| \leq \rho(t)$ , and there exists an analytic continuation of  $R_D$  to the range  $|z| < |\rho(t)| + \eta$ ,  $\arg(z - \rho(t)) \neq 0$ .

**Proof.** The system of equations (3.7)–(3.8) – which we write in short-hand notation as  $L_D = F(t, z, L_D, R_D)$ ,  $R_D = G(t, z, L_D, R_D)$  – is a strongly connected system of two equations of the form where  $F$  and  $G$  are power series with non-negative coefficients. It is known that such a system of equations has in principle the same analytic properties (including the singular behaviour of its solutions) as a single equation, see [8, Theorem 2.33]; however, we have to be sure that the regions of convergence of  $F$  and  $G$  are large enough.

In particular, if  $D$  is finite, then we have a positive algebraic system and we are done, see [1]. In the infinite case we have to argue in a different way. First of all, it is clear from the explicit solutions of  $E = E(t, u) = ((1 - t - \sqrt{(1 - t)^2 - 4u})/(2u))$  and  $B = B(t, u) = 1/\sqrt{(1 - t)^2 - 4u}$  that  $F$  and  $G$  (and all their derivatives with respect to  $L_D$  and  $R_D$ ) are certainly convergent if  $2|L_D| - |L_D|^2 + 2|R_D| < 1$ . On the other hand, it follows similarly to the bipartite case that the derivatives of  $F$  and  $G$  are divergent if

$L_D > 0$ ,  $R_D > 0$ , and  $2L_D - L_D^2 + 2R_D = 1$ . To see this we consider the function

$$B(t/s, us^2) = \frac{1}{1 - 2t/s - t^2/s^2 - 4uw^2} = \sum_{\ell, m} B_{\ell, m} s^{2m+\ell} t^\ell u^m = \sum_i s^i \sum_m B_{i-2m, m} t^{i-2m} u^m.$$

By singularity analysis it follows (for  $t, u > 0$ ) that

$$\sum_m B_{i-2m, m} t^{i-2m} u^m \sim c i^{-1/2} h(t, u)^{-i},$$

where  $c > 0$  and  $h = h(t, u) > 0$  satisfies the equation  $1 - 2t/h - t^2/h^2 - 4uh^2 = 0$ . Similarly, we can consider derivatives of  $F$  which correspond, for example, to sums of the form

$$\sum_m B_{i-2m, m} m t^{i-2m} u^m \sim c' i^{1/2} h(t, u)^{-i}.$$

In particular, if  $h(t, u) = 1$  (which is the case if  $2t - t^2 - 4u = 1$ ), then this term diverges for  $i \rightarrow \infty$ . Thus, the derivatives of  $F$  and  $G$  diverge if  $L_D > 0$ ,  $R_D > 0$ , and  $2L_D - L_D^2 + 2R_D = 1$ .

In order to determine the singularity of the system  $L_D = F(t, z, L_D, R_D)$ ,  $R_D = G(t, z, L_D, R_D)$  we have to find positive solutions of  $L_0, R_0, \rho$  of the system

$$L_0 = F(1, \rho, L_0, R_0), \quad R_0 = G(1, \rho, L_0, R_0), \quad 1 = \frac{G_{L_D} F_{R_D}}{1 - F_{L_D}} + G_{R_D}. \quad (3.10)$$

We do this in the following way. Starting with  $\rho = 0$ , we increase  $\rho$  and solve the first two equations to get  $L_0 = L_0(\rho)$ ,  $R_0 = R_0(\rho)$  till the third equation is satisfied. (Note that for  $\rho = 0$ , the right-hand side is 0 and, thus, smaller than 1.) As long as the right-hand side of the third equation is smaller than 1, it follows from the implicit function theorem that there is a local analytic continuation of the solutions  $L_0 = L_0(\rho)$ ,  $R_0 = R_0(\rho)$ . Furthermore, since  $L_0 > 0$  and  $R_0 > 0$ , we have to be in the region of convergence of the derivatives of  $F$  and  $G$ , that is,  $2L_0 - L_0^2 + 2R_0 < 1$ . From this it also follows that the solutions  $L_0 = L_0(\rho)$ ,  $R_0 = R_0(\rho)$  naturally extend to a point where the right-hand side of the third equation equals 1, so that the above system has a solution  $(1, \rho, L_0, R_0)$ . Of course, at this point the derivatives of  $F$  and  $G$  have to be finite, which implies that  $(1, \rho, L_0, R_0)$  lies inside the region of convergence of  $F$  and  $G$ .

This finally shows that all assumptions of [8, Theorem 2.33] are satisfied. Thus, singular representation of type (3.9) and the analytic continuation property follow for the functions  $L_D = L_D(t, z)$ ,  $R_D = R_D(t, z)$ . Hence, the same kind of properties follows for  $T_D = T_D(t, z)$  and consequently also for  $R_D/z - t + T_D$ .  $\square$

Lemma 3.3 shows that we are precisely in the same situation as in the bipartite case (actually, it is slightly easier since there is only one singularity on the circle  $|z| = \rho(t)$ ). Hence we immediately get the same property for  $M_D$  as stated in Lemma 3.2 and consequently the proposed asymptotic expansion (1.1).

#### 4. Central Limit Theorem for Bipartite Maps

Based on this previous result, we now extend our analysis to obtain a central limit theorem. Actually, this is immediate if the set  $D$  is finite, whereas the infinite case needs much more care.

Let  $D$  be a non-empty subset of even positive integers different from  $\{2\}$ . Then by Proposition 2.2 the generating functions  $R_D = R_D(t, z, (x_{2i})_{2i \in D})$  and  $M_D = M_D(t, z, (x_{2i})_{2i \in D})$  satisfy the equations

$$R_D = tz + z \sum_{2i \in D} x_{2i} \binom{2i-1}{i} R_D^i \quad \text{and} \quad \frac{\partial M_D}{\partial t} = 2(R_D/z - t). \quad (4.1)$$

If  $D$  is finite, then the number of variables is finite, too, and we can apply [8, Theorem 2.33] to obtain a representation of  $R_D$  of the form

$$R_D = g(t, z, (x_{2i})_{2i \in D}) - h(t, z, (x_{2i})_{2i \in D}) \sqrt{1 - \frac{z}{\rho(t, (x_{2i})_{2i \in D})}}. \quad (4.2)$$

A proper extension of the transfer lemma [8, Lemma 2.27] (where the variables  $x_{2i}$  are considered as additional parameters) leads to

$$M_D = g_2(t, z, (x_{2i})_{2i \in D}) + h_2(t, z, (x_{2i})_{2i \in D}) \left(1 - \frac{z}{\rho(t, (x_{2i})_{2i \in D})}\right)^{3/2}, \quad (4.3)$$

and finally [8, Theorem 2.25] implies a multivariate central limit theorem for the random vector  $\mathbf{X}_n = (X_n^{(2i)})_{2i \in D}$  of the proposed form.

Thus, we just have to concentrate on the infinite case. Actually, we proceed there in a similar way; however, we have to take care of infinitely many variables. There is no real problem to derive the same kind of representation (4.2) and (4.3) if  $D$  is infinite. Everything works in the same way as in the finite case, we just have to assume that the variables  $x_i$  are uniformly close to 1. And of course we have to use a proper notion of analyticity in infinitely many variables. We only have to apply the functional analytic extension of the above cited theorems that are given in [9]. Moreover, in order to obtain a proper central limit theorem we need a proper adaption of [9, Theorem 3]. In this theorem we have also a single equation  $y = F(z, (x_i)_{i \in I}, y)$  for a generating function  $y = y(z, (x_i)_{i \in I})$  that encodes the distribution of a random vector  $(X_n^{(i)})_{i \in I}$  in the form

$$y = \sum_n y_n \left( \mathbb{E} \prod_{i \in I} x_i^{X_n^{(i)}} \right) z^n,$$

where  $X_n^{(i)} = 0$  for  $i > cn$  (for some constant  $c > 0$ ) which also implies that all appearing potentially infinite products are in fact finite. (In our case this is satisfied since there is no vertex of degree larger than  $2n$  if we have  $n$  edges.) As we can see from the proof of [9, Theorem 3], the essential part is to provide tightness of the involved normalised random vector, and tightness can be checked with the help of moment conditions. It is clear that asymptotics of moments for  $X_n^{(i)}$  can be calculated with the help of derivatives of  $F$ , for example  $\mathbb{E} X_n^{(i)} = F_{x_i} / (\rho F_z) \cdot n + O(1)$ . This follows from the fact all information on the asymptotic behaviour of the moments is encoded in the derivatives of the singularity

$\rho(t, z, (x_i)_{i \in I})$ , and by implicit differentiation these derivatives relate to derivatives of  $F$ . More precisely, [9, Theorem 3] says that the following conditions are sufficient to deduce tightness of the normalised random vector:

$$\begin{aligned} \sum_{i \in I} F_{x_i} < \infty, & \quad \sum_{i \in I} F_{y x_i}^2 < \infty, & \quad \sum_{i \in I} F_{x_i x_i} < \infty, \\ F_{z x_i} = o(1), & \quad F_{z x_i x_i} = o(1), & \quad F_{y y x_i} = o(1), & \quad F_{y y x_i x_i} = o(1), \\ F_{z z x_i} = O(1), & \quad F_{z y x_i} = O(1), & \quad F_{z y y x_i} = O(1), & \quad F_{y y y x_i} = O(1), \end{aligned}$$

as  $i \rightarrow \infty$ , where all derivatives are evaluated at  $(1, \rho, (1)_{i \in I}, y(\rho))$ .

The situation is slightly different in our case since we have to work with  $M_D$  instead of  $R_D$ . However, the only real difference between  $R_D$  and  $M_D$  is that the critical exponents in the singular representations (4.2) and (4.3) are different, but the behaviour of the singularity  $\rho(t, (x_i)_{i \in I})$  is precisely the same. Note that after the integration step we can set  $t = 1$ . Now tightness for the normalised random vector that is encoded in the function  $M_D$  follows in the same way as for  $R_D$ , and since the singularity  $\rho(1, (x_i)_{i \in I})$  is the same, we get precisely the same conditions as in the case of [9, Theorem 3].

This means that we just have to check the above conditions hold for

$$F = F(1, z, (x_{2i})_{2i \in D}, y) = z + z \sum_{2i \in D} x_{2i} \binom{2i-1}{i} y^i,$$

where all derivatives are evaluated at  $z = \rho$ ,  $x_{2i} = 1$ , and  $y = R_D(\rho) < 1/4$ . However, they are trivially satisfied since  $\sum_{i \geq 1} \binom{2i-1}{i} i^K y^i < \infty$  for all  $K > 0$  and for positive real  $y < 1/4$ .

## 5. Central Limit Theorem for General Maps

We now assume that  $D$  contains at least one odd number. By Proposition 2.3 we have to consider the system of equations

$$\begin{aligned} L_D &= z \sum_{i \in D} x_i \sum_m B_{i-2m-1, m} L_D^{i-2m-1} R_D^m, \\ R_D &= tz + z \sum_{i \in D} x_i \sum_m B_{i-2m-2, m}^{(+1)} L_D^{i-2m-2} R_D^{m+1}, \end{aligned}$$

for the generating functions  $L_D = L_D(t, z, (x_i)_{i \in D})$  and  $R_D = R_D(t, z, (x_i)_{i \in D})$ , the generating function

$$T_D = T_D(t, z, (x_i)_{i \in D}) = 1 + \sum_{i \in D} x_i \sum_m \bar{B}_{i-2m, m} L_D^{i-2m} R_D^m$$

and finally the generating function  $M_D = M_D(t, z, (x_i)_{i \in D})$  that satisfies the relation

$$\frac{\partial M_D}{\partial t} = R_D/z - t + T_D.$$

Again, if  $D$  is finite, we can proceed as in the bipartite case by applying [8, Theorem 2.33, Lemma 2.27, and Theorem 2.25] which implies the proposed central limit theorem.

If  $D$  is infinite, we argue in a similar way as in the bipartite case. The only difference

is that we are not starting with one equation but with a system of two equations that have the (general) form

$$L = F(t, z, (x_i)_{i \in D}, L, R), \quad R = G(t, z, (x_i)_{i \in D}, L, R).$$

Nevertheless, it is possible to reduce two equations of this form to a single one. The proof of [8, Theorem 2.33] shows that there are no analytic problems since we have a positive and strongly connected system. We use the first equation to obtain an implicit function solution  $f = f(t, z, (x_i)_{i \in D}, r)$  that satisfies

$$f = F(t, z, (x_i)_{i \in D}, f, r).$$

Then we substitute  $f$  for  $L$  in the second equation and arrive at a single functional equation

$$R = G(t, z, (x_i)_{i \in D}, f(t, z, (x_i)_{i \in D}, R), R)$$

for  $R = R_D(t, z, (x_i)_{i \in D})$ . Note that the proof of [8, Theorem 2.33] assures that  $f$  is analytic although  $L$  and  $R$  get singular. Hence by setting

$$H(t, z, (x_i)_{i \in D}, r) = G(t, z, (x_i)_{i \in D}, f(t, z, (x_i)_{i \in D}, r), r)$$

we obtain a single equation  $R = H(t, z, (x_i)_{i \in D}, R)$  for  $R = R_D$  and we can apply the same method as in the bipartite case. Of course, the calculations get more involved. For example, we have

$$H_{x_i} = G_{x_i} + \frac{G_L F_{x_i}}{1 - F_L},$$

where

$$\begin{aligned} F_L &= \rho \sum_{i \in D} \sum_m (i - 2m - 1) B_{i-2m-1, m} L_0^{i-2m-2} R_0^m, \\ F_{x_i} &= \rho \sum_m B_{i-2m-1, m} L_0^{i-2m-1} R_0^m, \\ G_L &= \rho \sum_{i \in D} \sum_m (i - 2m - 2) B_{i-2m-2, m}^{(+1)} L_0^{i-2m-3} R_0^m, \\ G_{x_i} &= \rho \sum_m B_{i-2m-2, m}^{(+1)} L_0^{i-2m-2} R_0^m. \end{aligned}$$

From the proof of Lemma 3.3 we already know that  $2L_0 - L_0^2 + 4R_0 < 1$ , which implies that

$$\sum_{i \geq 1} \sum_m m^K (i - 2m - 1) B_{i-2m-1, m} L_0^{i-2m-2} R_0^m < \infty$$

for all  $K > 0$ . Furthermore, we have  $F_L < 1$  and  $G_R < 1$ . Hence it follows that  $\sum_{i \in D} H_{x_i} < \infty$ . In the same way, we can handle the other conditions which completes the proof of Theorem 1.1.

### 5.1. Weighted Maps

In order to cope with weighted maps we just have to substitute  $x_i = q_i$ . Then the coefficient  $M_{\mathbf{q}, n} := [z^n] M(1, z, \mathbf{q})$  is just the weighted sum of all maps with  $n$  edges.

Actually, under the condition that  $q_i = \Theta(i^\alpha)$  with  $\alpha \geq -\frac{3}{2}$  it follows that  $M_{\mathbf{q},n} \sim cn^{-5/2}\gamma^n$  for some positive constants  $c, \gamma$ . The reason is that we can show (almost in the same way as in Lemma 3.3) that there exist solutions  $L_0 > 0, R_0 > 0, \rho > 0$  with  $2L_0 - L_0^2 + 2R_0 < 1$  of the corresponding system (3.10). The simple reason is that the series  $\sum_{i \geq 1} \sqrt{i} \cdot i^\alpha$  diverges for  $\alpha \geq -\frac{3}{2}$ . This proves that we have a square-root singularity for the functions  $L$  and  $R$  etc.

The central limit theorem can be proved also in the same way as above; we just have to replace  $x_i$  by  $x_i q_i$ . We leave the details to the reader.

## 5.2. Mean and Covariance

Recall that we have used [8, Theorem 2.33, Lemma 2.27, and Theorem 2.25] to prove the central limit theorem. Actually this method provides us also expressions for the constants  $\mu_d$  ( $d \in D$ ) and a covariance matrix  $\Sigma = (\sigma_{d_1, d_2})_{d_1, d_2 \in D}$  for the limiting Gaussian random variable  $Z$ .

In the bipartite case we get

$$\mu_{2j} = \rho \binom{2j}{j} R_0^{j-1} \quad \text{and} \quad s_{2j, 2k} = \mu_{2j} \delta_{j,k} - \mu_{2j} \mu_{2k} (1 + (j-1)(k-1)c)$$

for some positive constant  $c$  and where  $\rho$  and  $R_0$  are defined in the proof of Lemma 3.3.<sup>1</sup>

In principle the same procedure also works for non-bipartite maps, however, the expressions are much more involved. Therefore we only state the results for the basic case  $D = \mathbb{N}$ . The corresponding constants  $\mu_d$  and  $\sigma_{d_1, d_2}$  are given by

$$\mu_d = A_d + 2\bar{A}_d$$

and

$$\begin{aligned} \sigma_{d_1, d_2} &= \mu_{d_1} \delta_{d_1, d_2} - \frac{3}{2} \mu_{d_1} \mu_{d_2} + \frac{729}{4} (d_1 - 1)(d_2 - 1) A_{d_1-1} A_{d_2-1} \\ &\quad + \frac{9}{2} ((\bar{A}_{d_1} + (d_1 - 1)\bar{A}_{d_1-1})\bar{A}_{d_2} + (\bar{A}_{d_2} + (d_2 - 1)\bar{A}_{d_2-1})\bar{A}_{d_1}) \\ &\quad - \frac{1}{18} (39\bar{A}_{d_1} + (d_1 - 1)\mu_{d_1-1}) (39\bar{A}_{d_2} + (d_2 - 1)\mu_{d_2-1}) \\ &\quad - \frac{1}{2} (\mu_{d_1} A_{d_2} + \mu_{d_2} A_{d_1}) + \frac{1}{12} ((d_2 - 1)\mu_{d_1} \mu_{d_2-1} + (d_1 - 1)\mu_{d_2} \mu_{d_1-1}), \end{aligned}$$

where

$$A_d = \frac{1}{6^d} \sum_{m \geq 0} \binom{d-1}{d-2m-1, m, m} 4^m \quad \text{and} \quad \bar{A}_d = \frac{1}{6^d} \sum_{m \geq 0} \binom{d-1}{d-2m-2, m, m+1} 4^m.$$

## 6. Maps of Higher Genus

The bijection used in Section 2 relies solely on the orientability of the surface on which the maps are embedded. Therefore it can easily be extended to maps of higher genus,

<sup>1</sup> Gregory Miermont has pointed out to the second author a very nice probabilistic interpretation of these representations in terms of monotone Galton–Watson trees and infinite sequences of Gaussian random variables.

i. e. embedded on an orientable surface of genus  $g \in \mathbb{Z}_{>0}$  (while planar maps correspond to maps of genus 0). The main difference lies in the fact that the corresponding mobiles are no longer trees but rather *one-faced* maps of higher genus, while the other properties still hold.

However, due to the appearance of cycles in the underlying structure of mobiles, another difficulty arises. Indeed, in the original bijection, vertices and edges in mobiles could carry labels (related to the geodesic distance in the original map), subject to local constraints. In our setting, the legs actually encode the *local* variations of these labels, which are thus implicit. Local constraints on labels are naturally translated into local constraints on the number of legs. But the labels have to remain consistent along each cycle of the mobiles, which gives rise to non-local constraints on the repartition of legs.

In order to deal with these additional constraints, and to be able to control the degrees of the vertices at the same time, we will now use a hybrid formulation of mobiles, carrying both labels and legs. As before, we will focus on the simpler case of mobiles coming from bipartite maps and establish the direct generalization of Theorem 1.1:

**Theorem 6.1.** *Suppose that  $D \neq \{2\}$  is an arbitrary set of positive even integers, let  $\mathcal{M}_D^{(g)}$  be the class of rooted bipartite maps of genus  $g$  with the property that all vertex degrees are in  $D$  and let  $M_{D,n}^{(g)}$  denote the number of maps in  $\mathcal{M}_D^{(g)}$  with  $n$  edges. Furthermore, set  $\bar{d} = \gcd\{i : 2i \in D\}$ .*

*Then there exist positive constants  $c_D^{(g)}$  and  $\rho_D^{(g)}$  with*

$$M_{D,n}^{(g)} \sim c_D^{(g)} n^{5(g-1)/2} (\rho_D^{(g)})^{-n}, \quad n \equiv 0 \pmod{\bar{d}}. \quad (6.1)$$

*Furthermore, let  $X_n^{(d)}$  denote the random variable counting vertices of degree  $d$  ( $\in D$ ) in maps in  $\mathcal{M}_D^{(g)}$ . Then  $\mathbb{E}(X_n^{(d)}) \sim \mu_d n$  for some constant  $\mu_d > 0$  and for  $n \equiv 0 \pmod{\bar{d}}$ , and the (possibly infinite) random vector  $\mathbf{X}_n = (X_n^{(d)})_{d \in D}$  ( $n \equiv 0 \pmod{\bar{d}}$ ) satisfies a central limit theorem.*

Theorem 1.1 can be easily recovered for planar bipartite maps, by setting  $g = 0$ . The main difference lies in the exponent  $5(g-1)/2$ , which appears to be also universal for rooted maps of genus  $g$ . Hence Theorem 6.1 is expected to hold for  $D$  without any restriction.

### 6.1. $g$ -Mobiles

Given  $g \in \mathbb{Z}_{\geq 0}$ , a  $g$ -mobile is a one-faced map of genus  $g$  – embedded on the  $g$ -torus – such that there are two kinds of vertices (black and white), edges only occur as black–black edges or black–white edges, and black vertices additionally have so-called “legs” attached to them (which are not considered edges), whose number equals the number of white neighbour vertices.

Furthermore, for each cycle  $c$  of the  $g$ -mobile, let  $n_{\circ}$ ,  $n_{\rightarrow}$  and  $n_{\leftarrow}$  respectively be the numbers of white vertices on  $c$ , of legs dangling to the left of  $c$  and of white neighbours to the left of  $c$ . One has the following constraint (see Figure 6.1):

$$n_{\rightarrow} = n_{\circ} + n_{\leftarrow} \quad (6.2)$$

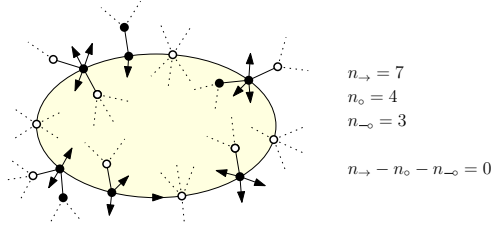


Figure 1. An oriented cycle in a  $g$ -mobile and the constraint on its left (coloured area). Notice that a similar constraint holds on its right, but is necessarily satisfied thanks to the properties of a  $g$ -mobile.

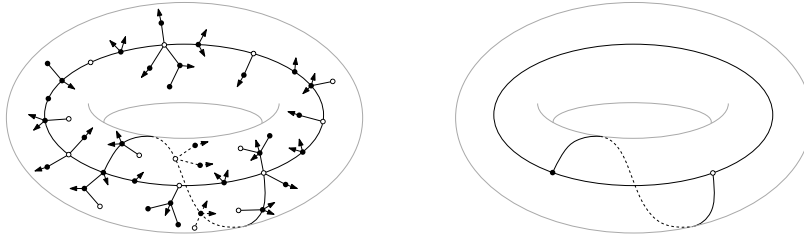


Figure 2. A 1-mobile on the torus and its scheme.

The *degree* of a black vertex is the number of half-edges plus the number of legs that are attached to it. A *bipartite*  $g$ -mobile is a  $g$ -mobile without black–black edges. A  $g$ -mobile is called *rooted* if an edge is distinguished and oriented.

Notice that a 0-mobile is simply a mobile as described in Definition 2.

Actually there is a direct analogue of Theorem 2.1:  $g$ -mobiles are in bijection to pointed maps of genus  $g$  (with precisely the same properties as stated in Theorem 2.1. This generalization of the bijection to higher genus was first given in [6] for quadrangulations and [4] for Eulerian maps (from which we will exploit many ideas in this section).

$g$ -mobiles are not as easily decomposed as planar mobiles, due to the existence of cycles. However, they still exhibit a rather simple structure, based on *scheme* extraction. The  $g$ -*scheme* (or simply the *scheme*) of a  $g$ -mobile is what remains when we apply the following operations (see Figure 2): first remove all legs, then remove iteratively all vertices of degree 1 and finally replace any maximal path of degree-2-vertices by a single edge.

Once these operations are performed, the remaining object is still a one-faced map of genus  $g$ , with black and white vertices (white–white edges can now occur), where the vertices have minimum degree 3. The important property is that there is a finite number of schemes of a given genus.

## 6.2. Bipartite $g$ -Mobile Counting

A  $g$ -mobile can now be uniquely decomposed as a scheme where each edge is substituted by a sequence of elementary cells. By definition of a  $g$ -mobile, one needs to track the *increment* of each cell to ensure that the overall cycle constraints are satisfied.

An *elementary cell* is a half-edge connected to a black vertex itself connected to a white vertex with a dangling half-edge. The white vertex has a sequence of black-rooted mobiles attached on each side. For an elementary cell of increment  $i$ , the black vertex



has  $k \geq 0$  white-rooted mobiles and  $k + i + 1 \geq 0$  legs on its left,  $l \geq 0$  white-rooted mobiles and  $l - i + 1$  legs on its right, and its degree is  $2(k + l + 2)$ .

The generating series  $P := P(t, z, R, (x_{2i}), s)$  of a cell, where  $s$  marks the increment, is:

$$P(t, z, R, (x_{2i}), s) = \frac{R^2}{t} \sum_{k, l \geq 0} x_{2(k+l+2)} R^{k+l} \sum_{i=-k-1}^{l+1} \binom{2k+i+1}{k} \binom{2l-i+1}{l} s^i.$$

Depending on the edge end colours, there might be an additional black or white vertex inserted at the end of the sequence of elementary cells. This is reflected by an extra factor in the generating series  $S_e := S_e(t, z, (x_{2i}), s)$ :

$$S_{(u,v)}(t, z, (x_{2i}), s) = \begin{cases} \frac{1}{1-P} & \text{if } (u, v) = (\circ, \bullet) \text{ or } (\bullet, \circ), \\ \frac{stz}{R^2} \left( \frac{1}{1-P} - 1 \right) & \text{if } (u, v) = (\circ, \circ), \\ \frac{R^2}{stz} \frac{1}{1-P} & \text{if } (u, v) = (\bullet, \bullet). \end{cases}$$

We can now express the generating series  $Q_S := Q_S(t, z, (x_{2i}))$  of rooted bipartite  $g$ -mobiles with scheme  $S$  (for the definition of a labelling see Appendix B):

$$Q_S(t, z, R, (x_{2i}); s) = 2 \frac{z \partial}{\partial z} \frac{1}{2|E|} z^{|E|} t^{|V_\circ|} \left( \frac{R}{tz} \right)^{|C_\circ|} \sum_{(l_c) \text{ labelling}} \left[ \prod_{e \in E} [s^{\text{incr}(e)}] S_{(e_-, e_+)} \right. \\ \left. \prod_{v \in V_\bullet} \sum_{i_1, \dots, i_{\deg(v)} \geq 0} \left( \prod_{k=1}^{\deg(v)} \binom{2i_k + l_{c_{k+1}} - l_{c_k} + 1}{i_k} R^{i_k} \right) x_{2(\deg(v) + \sum i_k)} \right]. \quad (6.3)$$

**Proposition 6.2.** *The generating series  $M_D^{(g)} := M_D^{(g)}(t, z, (x_{2i}))$  for the family of rooted bipartite maps of genus  $g$ , where the vertex degrees belong to  $D$ , satisfies the relation:*

$$\frac{\partial M_D^{(g)}}{\partial t} = \frac{2}{z} \sum_{\substack{S \text{ scheme} \\ \text{of genus } g}} Q_S(t, z, (x_{2i} \mathbb{1}_{\{2i \in D\}})). \quad (6.4)$$

**Proof.** This follows directly from the bijections between  $g$ -mobiles and maps of genus  $g$  and equation (6.3).  $\square$

### 6.3. Asymptotics of $g$ -Mobiles

We proceed similarly to [4]. However, for the sake of brevity we will not work out all technical details. For example, we will take only care of the (local) singular expansion and restrict ourselves to the case  $\bar{d} = 1$ .

First we need proper expansion of the coefficients of  $(1 - P)^{-1}$ .

**Lemma 6.3.** *We have, as  $|\Delta| \rightarrow \infty$ ,*

$$[s^\Delta] \frac{1}{1 - P(t, z, R(t, z), (x_{2i}); s)} = C_{\text{sgn } \Delta}(t, z) \alpha_{\text{sgn } \Delta}(t, z)^{|\Delta|} + (|\alpha_\pm(t, z)| - \delta)^{|\Delta|},$$

where  $\alpha_{\pm}(t, z) \sim 1 - c_1(1 - z/\rho)^{1/4}$  and  $C_{\pm}(t, z) \sim c_2(1 - z/\rho)^{-1/4}$  for some positive constants  $c_1, c_2$  and  $\rho = \rho(t, (x_{2i}))$

**Proof.** With the help of (3.5) it is easy to check that the following three relations hold when we evaluate at  $t$  close to 1,  $x_{2i}$ ,  $2i \in D$ , close to 1,  $R = R_0(t, (x_{2i}))$ ,  $z = \rho = \rho(t, (x_{2i}))$ , and  $s = 1$ :

$$P = 1, \quad P_s = 0, \quad P_{ss} \neq 0, \quad P_R \neq 0.$$

Thus we have locally two solutions  $s = \alpha_{1,2}(t, z)$  of the equation  $P(t, z, R(t, z), (x_{2i}), s) = 1$  that are of the form  $\alpha_{1,2}(t, z) = 1 \mp c_1(1 - z/\rho)^{1/4} + O\left((1 - z/\rho)^{1/4}\right)$ . For  $s$  with  $|\alpha_1(t, z)| < |s| < |\alpha_2(t, z)|$  we also have  $|P| < 1$  and consequently by Cauchy integration applied to the Laurent series  $s \mapsto P$

$$[s^{\Delta}] \frac{1}{1 - P(t, z, R(t, z), (x_{2i}); s)} = \frac{1}{2\pi i} \int_{|s|=s_0} \frac{1}{1 - P(t, z, R(t, z), (x_{2i}); s)} \frac{ds}{s^{\Delta+1}},$$

where  $|\alpha_1(t, z)| < s_0 < |\alpha_2(t, z)|$ . Clearly  $s = \alpha_{1,2}(t, z)$  are polar singularities of  $1/(1-P)$ . Thus, if we shift the integral to a circle  $|s| = |\alpha_2(t, z)| + \delta$  (for some  $\delta > 0$ ) and by collecting the residue at  $s = \alpha_2(t, z)$ , we get, as  $\Delta \rightarrow +\infty$ ,

$$[s^{\Delta}] \frac{1}{1 - P(t, z, R(t, z), (x_{2i}); s)} = C_1(t, z) \alpha_2(t, z)^{-\Delta} + (|\alpha_2(t, z)| + \delta)^{-\Delta},$$

where  $C_2(t, z) = 1/P_s(t, z, R(t, z), (x_{2i}))$ ,  $\alpha_2(t, z) = c_2(1 - z/\rho)^{-1/4} + O(1)$ . Similarly we obtain the corresponding expansion for  $\Delta \rightarrow -\infty$ . Thus, setting  $\alpha_+(t, z) = \alpha_2(t, z)^{-1}$ ,  $\alpha_-(t, z) = \alpha_1(t, z)$ ,  $C_+(t, z) = C_2(t, z)$ , and  $C_-(t, z) = C_1(t, z)$  completes the proof of the lemma.  $\square$

Next we observe that for any  $d \in \mathbb{Z}_{>0}$ , and any  $\Delta_1, \dots, \Delta_d \in \mathbb{Z}$  such that  $\sum \Delta_k = d$  we have

$$\begin{aligned} \sum_{i_1, \dots, i_d \geq 0} x_{2 \sum i_k + 2d} \prod_{k=1}^d \binom{2i_k + \Delta_k}{i_k} R^{i_k} &= \sum_{m \geq 0} x_{2m+2d} R^m \binom{2m+2d-1}{m+d} \binom{m+d-1}{d-1} \\ &= \sum_{m \geq d} x_{2m} R^{m-d} \binom{2m-1}{m} \binom{m-1}{d-1}. \end{aligned}$$

With the help of these preliminaries we can determine the singular structure of the generating functions  $Q_S(t, z, (x_{2i}))$  related to a scheme  $S$ . For the sake of brevity we will only discuss labelled schemes where all vertices are white. Thus all edges are white–white and labels are carried by the white vertices. Without loss of generality, one can assume that the minimal label is 0 (by shifting all labels, as only the differences matter).

In order to handle the sums over all labellings, define  $\lambda : V_{\circ} \rightarrow [0, M]$  (where  $M = \text{card}(\{\text{labels of } V_{\circ}\}) - 1$ ), the relative order of the labels. Labels can then be rewritten as:

$$\forall v \in V_{\circ}, l_v = \sum_{i=1}^{\lambda(v)} \delta_i, \quad \text{with } \delta_i \in \mathbb{Z}_{>0}.$$

For a given edge  $e = (u, v)$  of the labelled scheme, one substitutes a sequence of cells of overall increment  $\Delta(e) = l_v - l_u + 1 = \sum_j A_{e,j} \delta_j + 1$ , where  $A_{e,j} = \mathbb{1}_{\lambda(u) < j \leq \lambda(v)}$ .

Hence, we obtain, after rewriting, that:

$$\begin{aligned}
Q_S(t, z, (x_{2i})) &\sim \frac{z\partial}{\partial z} \frac{1}{|E|} z^{|E|} t^{|V_0|} \left(\frac{R}{tz}\right)^{|C_0|} \sum_{(l_c)} \prod_{\text{labelling } e \in E} [s^{\Delta(e)}] S_{(e_-, e_+)} \\
&\sim \frac{z\partial}{\partial z} \frac{1}{|E|} z^{|E|} t^{|V_0|} \left(\frac{R}{tz}\right)^{|C_0|} \sum_{\delta_1, \dots, \delta_M > 0} \prod_{e \in E} \frac{tz}{R^2} [s^{\sum_j A_{e,j} \delta_j}] \left(\frac{1}{1-P} - 1\right) \\
&\sim \frac{z\partial}{\partial z} \frac{1}{|E|} z^{|E|} t^{|V_0|} \left(\frac{R}{tz}\right)^{|C_0|} \left(\frac{tz}{R^2}\right)^{|E|} \sum_{\delta_1, \dots, \delta_M > 0} \prod_{e \in E} C_+(t, z) \alpha_+(t, z)^{\sum_j A_{e,j} \delta_j} \\
&\sim \frac{z\partial}{\partial z} \frac{1}{|E|} z^{|E|} t^{|V_0|} \left(\frac{R}{tz}\right)^{|C_0|} \left(\frac{tzC(t, z)}{R^2}\right)^{|E|} \sum_{\delta_1, \dots, \delta_M > 0} \prod_j \left(\prod_e \alpha_+(t, z)^{A_{e,j}}\right)^{\delta_j} \\
&\sim \frac{z\partial}{\partial z} \frac{1}{|E|} z^{|E|} t^{|V_0|} \left(\frac{R}{tz}\right)^{|C_0|} \left(\frac{tzC(t, z)}{R^2}\right)^{|E|} \prod_{j=1}^M \frac{\prod_e \alpha_+(t, z)^{A_{e,j}}}{1 - \prod_e \alpha_+(t, z)^{A_{e,j}}} \\
&\sim \frac{z\partial}{\partial z} \frac{1}{|E|} t^{|V_0| - |E|} C(t, z)^{|E|} \frac{1}{(1 - \alpha_+(t, z))^M} \\
&\sim c_3 \frac{z\partial}{\partial z} \frac{1}{|E|} \left(1 - \frac{z}{\rho}\right)^{(-|E| - M)/4}
\end{aligned}$$

The main contribution will then come from cubic schemes with maximal  $M$ , i. e. where all labels are distinct. Thus  $|E| = 6g - 3$ ,  $M = |V| - 1 = 4g - 3$ .

$$Q_S(t, z, (x_{2i})_{2i \in D}) \sim c_3 \frac{z\partial}{\partial z} \frac{1}{|E|} \left(1 - \frac{z}{\rho(t, (x_{2i})_{2i \in D})}\right)^{-5g/2 + 3/2}$$

Similar asymptotics can be derived — with more technical computations — for the mobiles where the scheme also has black vertices.

Summing up over all the maximal cubic schemes of genus  $g$ , and after an integration step, we recover the expected singular behaviour

$$M_D^{(g)}(t, z, (x_{2i})_{2i \in D}) \sim c_4 \frac{z\partial}{\partial z} \frac{1}{2|E|} \left(1 - \frac{z}{\rho(t, (x_{2i})_{2i \in D})}\right)^{-5g/2 + 5/2}$$

which corresponds to the asymptotics given in Theorem 6.1 (when we set  $t = 1$  and  $x_{2i} = 1$ ,  $2i \in D$ ). The central limit theorem follows as in the planar case by varying  $x_{2i}$  around 1.

**Acknowledgements.** The authors are very grateful to Mireille Bousquet-Melou, Guillaume Chapuy, and Gregory Miermont for their help and for several valuable remarks.

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### Appendix A. Schemes of $g$ -Mobiles

A  $g$ -mobile is not as easily decomposed as a planar one, due to the existence of cycles. However, it still exhibits a rather simple structure, based on *scheme* extraction.

The  $g$ -*scheme* (or simply the *scheme*) of a  $g$ -mobile is what remains when we apply the following operations (see Figure 2): first remove all legs, then remove iteratively all vertices of degree 1 and finally replace any maximal path of degree-2-vertices by a single edge.

Once these operations are performed, the remaining object is still a one-faced map of genus  $g$ , with black and white vertices (white–white edges can now occur), where the vertices have minimum degree 3.

To count  $g$ -mobiles, one key ingredient is the fact that there is only a finite number of schemes of a given genus. Indeed, let  $d_i$  be the number of vertices of degree  $i$  in a  $g$ -scheme:

$$\sum_{k \geq 3} (i - 2)d_i = \sum_{k \geq 3} id_i - 2 \sum_{k \geq 3} d_i = 2(\#\text{edges} - \#\text{vertices}) = 4g - 2.$$

The number of vertices (respectively edges) is then bounded by  $4g - 2$  (respectively  $6g - 3$ ), where this bound is reached for cubic schemes (see an example in Figure 2).

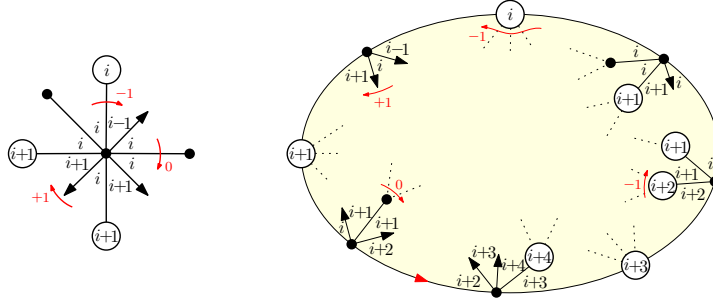


Figure 3. The variations of labels around a black vertex and along an oriented cycle.

To recover a proper  $g$ -mobile from a given  $g$ -scheme, one would have to insert a suitable planar mobile into each corner of the scheme and to substitute each edge with some kind of path of planar mobiles. Unfortunately, this cannot be done independently: Around each black vertex, the total number of legs in every corner must equal the number of white neighbours, and around each cycle, (6.2) must hold.

In order to make these constraints more transparent, we will equip schemes with labels on white vertices and black corners. Now, when trying to reconstruct a  $g$ -mobile from a scheme, one has to ensure that the local variations are consistent with the global labelling. To be precise, the label variations are encoded as follows (see Figure 3):

- Around a black vertex of degree  $d$ , let  $(l_1, \dots, l_d)$  be the labels of its corners read in clockwise order:

$$\text{for all } i, l_{i+1} - l_i = \begin{cases} +1 & \text{if there is a leg between the two corresponding corners,} \\ 0 & \text{if there is a black neighbour,} \\ -1 & \text{if there is a white neighbour.} \end{cases}$$

- Along the left side of an oriented cycle, the label decreases by 1 after a white vertex or when encountering a white neighbour and increases by 1 when encountering a leg.

The above statements hold for general – as well as bipartite – mobiles. In the following, we will only consider bipartite mobiles, as they are much easier to decompose.

## Appendix B. Reconstruction of Bipartite Maps of Genus $g$

In the following, it will be convenient to work with rooted schemes. One can then define a canonical labelling and orientation for each edge of a rooted scheme. An edge  $e$  now has an origin  $e_-$  and an endpoint  $e_+$ . The  $k$  corners around a vertex of degree  $k$  are clockwise ordered and denoted by  $c_1, \dots, c_k$ .

Given a scheme  $S$ , let  $V_\circ, V_\bullet, C_\circ, C_\bullet$  be respectively the sets of white and black vertices and of white and black corners. A *labelled scheme*  $(S, (l_c)_{c \in V_\circ \cup C_\bullet})$  is a pair consisting of a scheme  $S$  and a labelling on white vertices and black corners, with  $l_c \geq 0$  for all  $c$ . Labellings are considered up to translation, as they will not affect local variations. For an edge of  $S$   $e \in E_S$ , we associate a label to each extremity  $l_{e_-}, l_{e_+}$ . If an extremity is

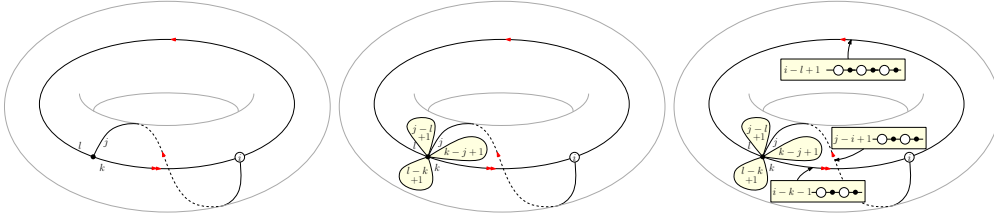


Figure 4. Steps (1)–(3) of the algorithm.

a white vertex of label  $l$ , its label is  $l$ . If the extremity is a black vertex, its label is the same as the next clockwise corner of the black vertex.

Let a *doubly-rooted planar mobile* be a rooted (on a black or white vertex) planar mobile with a secondary root (also black or white). These two roots are the extremities of a path  $(v_1, \dots, v_k)$ . The *increment* of the doubly-rooted mobile is then defined as  $n_{\rightarrow} - n_{\circ} - n_{\leftarrow}$ , which is not necessarily 0, as the path is not a cycle.

Similarly as in [4], we present a non-deterministic algorithm to reconstruct a  $g$ -mobile:

#### Algorithm.

- (1) Choose a labelled  $g$ -scheme  $(S, (l_c)_{c \in V_{\circ} \cup C_{\bullet}})$ .
- (2) For all  $v \in V_{\bullet}$ , choose a sequence of non-negative integers  $(i_k)_{1 \leq k \leq \deg(v)}$ , then attach  $i_k$  planar mobiles and  $i_k + l_{c_{k+1}} - l_{c_k} + 1$  legs to  $c_k$  (the  $k^{\text{th}}$  corner of  $v$ ).
- (3) For all  $e \in S$ , replace  $e$  by a doubly-rooted mobile of increment  $\text{incr}(e) = l_{e_+} - l_{e_-} + \begin{cases} +1 & \text{if } e_- \text{ is white,} \\ -1 & \text{if } e_- \text{ is black.} \end{cases}$
- (4) On each white corner of  $S$ , insert a planar mobile.
- (5) Distinguish and orient an edge as the root.

**Proposition B.1.** *Given  $g > 0$ , the algorithm generates each rooted bipartite  $g$ -mobile whose scheme has  $k$  edges in exactly  $2k$  ways.*

**Proof.** One can easily see that the obtained object is indeed bipartite. Attaching planar mobiles and legs added at step (2) in a corner  $c_k$  creates new corners, such that:

- The first carries the same label  $l_{c_k}$  as  $c_k$ , and
- the last carries the label  $l_{c_k} + (i_k + l_{c_{k+1}} - l_{c_k} + 1) - i_k = l_{c_{k+1}} + 1$ .

The next corner should then be labelled  $(l_{c_{k+1}} + 1) - 1 = l_{c_{k+1}}$ , due to the next white neighbour, which is precisely what we want.

In the same fashion, at step (3), a simple counting shows that each edge is replaced by a path such that the labels along it evolve according to the scheme labelling.

We thus obtain a well-formed rooted bipartite  $g$ -mobile, with a secondary root on its scheme. Since the first root destroys all symmetries, there are exactly  $2k$  choices for the secondary root which would give the same rooted  $g$ -mobile.  $\square$