SCALING LIMIT OF RANDOM k-TREES

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ABSTRACT. We consider a random k-tree $\mathsf{G}_{n,k}$ that is uniformly selected from the class of labelled k-trees with $n+k$ vertices. Since 1-trees are just trees, it is well-known that $\mathsf{G}_{n,1}$ (after scaling the distances by $1/(2\sqrt{n})$) converges to the Continuum Random Tree $\mathcal{T}_{\mathsf{e}}$. Our main result is that for $k \neq 1$, the random k-tree $\mathsf{G}_{n,k}$, scaled by $(k+1)/(2\sqrt{n})$, converges to the Continuum Random Tree $\mathcal{T}_{\mathsf{e}}$, too. In particular this shows that the diameter as well as the typical distance of two vertices in a random k-tree $\mathsf{G}_{n,k}$ are of order $\sqrt{n}$.

1. Introduction and Main result

A k-tree is a generalization of a tree and can be defined recursively: a *k-tree* is either a complete graph on k vertices (= a k-clique) or a graph obtained from a smaller k-tree by adjoining a new vertex together with k edges connecting it to a k-clique of the smaller k-tree (and thus forming a $(k+1)$-clique), see Figure 2.1. In particular, a 1-tree is a usual tree. (Note that the parameter k is always fixed.)

A k-tree is an interesting graph from an algorithmic point of view since many NP-hard problems on graphs have polynomial, in fact usually linear, dynamic programming algorithms when restricted to k-trees and their subgraphs for fixed values of k [6, 40, 27]; subgraphs of k-trees are called partial k-trees. Such NP-hard problems include maximum independent set size, minimal dominating set size, chromatic number, Hamiltonian circuit, network reliability and minimum vertex removal forbidden subgraph [5, 9]. Several graphs which are important in practice [32], have been shown to be partial k-trees, among them are

(1) Trees/ Forests (partial 1-trees)
(2) Series parallel networks (partial 2-trees)
(3) Outplanar graphs (partial 2-trees)
(4) Halin graphs (partial 3-trees).

However, other interesting graph classes like planar graphs or bipartite graphs are not partial k-trees. k-trees are also very interesting from a combinatorial point of view. For example, the enumeration problem for k-trees has been studied in various ways, see [7, 34, 23, 13, 29, 30, 24, 25, 26]. The number of labelled k-trees has been determined by Beineke and Pippert [7], Moon [34], Foata [23], Darrasse and Soria [13]; as usual a k-tree on n vertices is called *labelled* if the integers from $\{1, 2, \ldots, n\}$ have been assigned to its vertices (one-to-one) and two labelled k-trees are considered to be different if the corresponding edge sets are different. In order to analyze k-trees, it turns out that it is convenient to consider the number of *hedra* instead of the number of vertices as the size of a k-tree; we adopt the notions from [26]. A *hedron* is a $(k+1)$-clique in a k-tree, and by definition a k-tree with n hedra has $n+k$ vertices. A *front* of a k-tree is a k-clique. In what follows, we assume the k-trees are all labelled and a random k-tree with n hedra is uniformly selected from the class of labelled k-trees with n hedra.

Darrasse and Soria [13] showed a Rayleigh limiting distribution for the expected distance between pairs of vertices in a random k-tree, as it is known for usual trees and, thus, for 1-trees.

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Inspired by this results, we expect that a random k-tree with n hedra, after scaling the distances to the root by $1/\sqrt{n}$, converges to the *Continuum Random Tree* multiplied by a scaling factor. For $k = 1$ this is true by a result of Aldous. Actually Aldous has proved in a series of seminal papers [2, 3, 4] that a *critical Galton-Watson tree* conditioned on its size has the Continuum Random Tree (CRT) as its limiting object – and random 1-trees are a special case (with a Poisson offspring distribution). The concept *Continuum Random Tree* was also introduced by Aldous [2, 3, 4] and further developed by Duquesne and Le Gall [18, 19, 20].

Since Aldous's pioneering work on the Galton-Watson trees, the CRT has been established as the limiting object of a large variety of combinatorial structures [28, 38, 35, 36, 11, 31, 8, 12]. A key idea in the study of these combinatorial objects is to relate them to trees endowed with additional structures by using an appropriate bijection. In the present case of k-trees, we encode them as so-called k-front coding trees via a bijection due to Darrasse and Soria in [13], which was originally used to enumerate k-trees and to recursively count the distance between any two vertices in a random k-tree. Furthermore, in order to build a connection between the distance of two vertices in a random k-tree and the distance of two vertices in a critical Galton-Watson tree, we need to introduce the concept of a *size-biased enriched tree.* This is adapted from the *size-biased Galton-Watson tree* which was introduced by Addario-Berry, Devroye and Janson in [1] and was further generalized to the *size-biased $\mathcal{R}$-enriched trees* by Panagiotou, Stufler and Weller in [35, 36]. Our enriched tree is slightly different to the size-biased $\mathcal{R}$-enriched tree and we use their ideas in [38, 35] where an important step is to relate the distance between two vertices in a random graph to the distance between two blocks in a random size-biased $\mathcal{R}$-enriched tree.

Our main result establishes the convergence of a random k-tree to the CRT with respect to the Gromov-Hausdorff metric.

Theorem 1. *Let $\mathcal{G}_{n,k}$ be the class of vertex labelled k-trees with n hedra and denote by $\mathsf{G}_{n,k}$ a random k-tree that is uniformly selected from the class $\mathcal{G}_{n,k}$ and by $\mathsf{G}^{\circ}_{n,k}$ a random k-tree that is rooted at a front. Then*

$$\frac{\mathfrak{m}_k}{2\sqrt{n}}\mathsf{G}^{\circ}_{n,k} \xrightarrow{(d)} \mathcal{T}_e \qquad \textit{and} \qquad \frac{\mathfrak{m}_k}{2\sqrt{n}}\mathsf{G}_{n,k} \xrightarrow{(d)} \mathcal{T}_e$$

with respect to the Gromov-Hausdorff metric, where $\mathfrak{m}_k = k+1$ for $k \neq 1$ and $\mathfrak{m}_1 = 1$.

In particular this shows that the diameter as well as the typical distance of two vertices are of order $\sqrt{n}$ and they have the same limiting distribution as random trees.

The plan of the paper is as follows. In Section 2 we recall the combinatorial background for k-trees, introduce the Boltzmann sampler – a method of generating efficiently a uniform random combinatorial object, describe Darrasse and Soria's algorithm on computing the distances between two vertices in a k-tree, and present Aldous's result on the convergence of critical Galton-Watson trees to the CRT $\mathcal{T}_{\mathsf{e}}$. In Section 3 we prove our main result – Theorem 1.

2. Combinatorics, Boltzmann Sampler and Continuum Random Tree

It was shown in [7, 34, 23, 13] that the number $\mathsf{L}_k(n)$ of k-trees having n hedra is given by

$$\mathsf{L}_k(n) = \binom{n+k}{k}(kn+1)^{n-2} \tag{2.1}$$

and, thus, asymptotically by $\mathsf{L}_k(n) \sim n^k(kn)^{n-2}/k!$ as $n \to \infty$. Here we only review the generating function approach in [13] to count $\mathsf{L}_k(n)$. The key ingredient to count the number $\mathsf{L}_k(n)$ in [13] is a bijection between rooted k-trees and *k-front coding trees* rooted at a white node.

2.1. A bijection.

A rooted k-tree is a k-tree rooted at a front (or equivalently a k-clique). A k-front coding tree is a bipartite tree of black and white nodes which is rooted at a white node and where every black node has precisely k successors. The bijection will be built in a way that black nodes in k-front coding trees correspond to hedra in k-trees. Every black node also gets a label which is equal to the label of one of the vertices of the corresponding hedron. A white node in a k-front coding tree corresponds to a front of the k-trees and is labelled by the set $\{a_1, a_2, \ldots, a_k\}$ of labels of the corresponding front. A black node connects with a white node if the corresponding hedron contains the corresponding front and the label of the black node is just the label of the vertex that is not contained in the front. Thus, if we start with the root front of the k-tree we can recursively build up a corresponding k-front coding tree, see Figure 2.1.

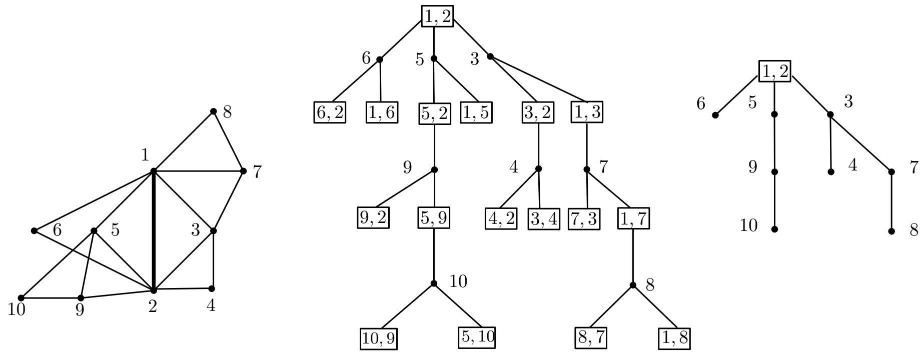

FIGURE 2.1. A 2-tree rooted at a front whose vertices are labelled by 1, 2 (left) and the corresponding 2-front coding tree $\mathsf{C}_{n,2}$ rooted at a white node labelled by $\{1, 2\}$ (middle); finally the derived tree $\mathsf{T}_{n,2}$ consists just of the white root and of all black nodes (right).

With the help of this correspondence, the problem of counting the k-trees with n hedra is reduced to count the corresponding k-front coding trees with n black nodes. We use the notation *$\circ$-rooted k-front coding trees* if the white root node has a fixed label $\{a_1, a_2 \ldots, a_k\}$. We call it *reduced k-front coding trees* (and denote it by $\circ - \bullet$ *k-front coding tree*) if the root $\circ$ has precisely one (black) child. Let $\mathcal{G}_k$ (resp. $\mathcal{G}_k^\circ$) be the class of k-trees (resp. $\circ$-rooted), let furthermore $\mathcal{C}_k$ be the class of $\circ$-rooted k-front coding trees and $\mathcal{B}_k$ be the class of $\circ - \bullet$ k-front coding trees. The bijection we mentioned above establishes a relation $\mathcal{G}_k^\circ \simeq \mathcal{C}_k$. Furthermore, every $\circ$-rooted k-front coding tree can be identified as a set of $\circ - \bullet$ k-front coding trees and every $\circ - \bullet$ k-front coding tree can be decomposed into a k-tuple of $\circ$-rooted k-front coding trees. Consequently, k-front coding trees satisfy the following specification:

$$\mathcal{C}_k = \textsc{Set}(\mathcal{B}_k) \quad \text{and} \quad \mathcal{B}_k = \{\bullet\} * \textsc{Seq}_k(\mathcal{C}_k). \tag{2.2}$$

In terms of exponential generating functions (where the *size* is always the number of black nodes), we thus get

$$C_k(x) = \exp(B_k(x)) \quad \text{and} \quad B_k(x) = x \cdot C_k(x)^k. \tag{2.3}$$

In particular $B_k(x)$ satisfies

$$B_k(x) = x \exp(kB_k(x)). \tag{2.4}$$

By applying the Lagrange inversion formula on (2.4), we obtain that the number of $\circ - \bullet$ k-front coding trees with n black nodes where the root $\circ$ has a fixed label $\{a_1, a_2, \ldots, a_k\}$, is

$$b_k(n) = n!\,[x^n]B_k(x) = (n-1)![x^{n-1}]\exp(knx) = (kn)^{n-1} \tag{2.5}$$

and the number of $\circ$-rooted k-front coding trees with n black nodes where the root $\circ$ has a fixed label $\{a_1, a_2, \ldots, a_k\}$ is

$$c_k(n) = n!\,[x^n]C_k(x) = (n-1)![x^{n-1}]\exp((kn+1)x) = (kn+1)^{n-1}. \tag{2.6}$$

Since there are $\binom{n+k}{k}$ ways to choose the root $\{a_1, a_2, \ldots, a_k\}$, the number of labelled k-trees having n hedra that are rooted at a front is

$$(kn+1)\mathsf{L}_k(n) = \binom{n+k}{k} c_k(n). \tag{2.7}$$

In view of (2.6), the closed formula for $\mathsf{L}_k(n)$ (2.1) is proved. It follows from (2.4) that the dominant singularity of $B_k(x)$ is $\rho_k = (ek)^{-1}$ and $B_k(\rho_k) = k^{-1}$, see [13, 16] for details.

Recall that the size of a k-front coding tree T, denoted by $|T|$, is the number of black nodes. Let $\mathcal{C}_{n,k}$ be the class of k-front coding trees of size n such that the white root has label $\{1, 2, \ldots, k\}$, then from (2.7) we find, the probability to uniformly choose a random rooted k-tree $\mathsf{G}^{\circ}_{n,k}$ is equal to the probability to uniformly choose a k-front coding tree $\mathsf{C}_{n,k}$ from the class $\mathcal{C}_{n,k}$.

Since $\mathcal{C}_k$ has a proper recursive specification (2.3), these random objects can be constructed (or sampled) by a so-called Boltzmann sampler $\Gamma C_k(x)$.

2.2. **Boltzmann Sampler.** Boltzmann samplers provide a way to efficiently generate a combinatorial object at random. They were introduced by Duchon, Flajolet, Louchard and Schaeffer [17] and were further developed by Flajolet, Fusy and Pivoteau [22]. Here we refer the readers to their papers [17, 22] for a detailed description of the Boltzmann samplers. An important property of Boltzmann samplers is that they generate objects of a given size n uniformly.

More precisely we will describe a Boltzmann sampler $\Gamma C_k(x)$ with parameter $x = \rho_k = (ek)^{-1}$ (which is possible since $\lambda_k = B_k(\rho_k) = k^{-1} < \infty$).

Lemma 2. *The following recursive procedure $\Gamma C_k(\rho_k)$ terminates almost surely and draws a random $\circ$-rooted k-front coding tree according to the Boltzmann distribution with parameter ρ_k, i.e., any $\circ$-rooted k-front coding tree of size n in the class $\mathcal{C}_{n,k}$ is drawn with probability $\rho_k^n/(n!\, C_k(\rho_k))$.*

```
ΓC_k(ρ_k): x_1 ← a white node ◦
           m ← Pois(λ_k)
           for i := 1 → m
               x_2 ← a single black node •
               merge x_2 into x_1 by adding an edge • − ◦
               F ← an m-tuple (ΓC_k(ρ_k), ..., ΓC_k(ρ_k))
               merge F into x_1 by connecting x_2 to the roots of F
           x_1 ← label the black nodes of x_1 uniformly at random
           return x_1
```

Note that k-front coding trees satisfy the specification (2.3), but they do not represent the distance relation in the k-trees. See Figure 2.1. Since we have fixed the label on the white root $\circ$, which is $\{1, 2, \ldots, k\}$, the labels on the black nodes of $\Gamma C_k(\rho_k)$ determine the corresponding labels on the white nodes.

2.3. ***k*-tree distance algorithm.** For a random k-front coding tree $\mathsf{C}_{n,k}$ of size n, let $\mathsf{M}_{n,k}$ be the corresponding k-tree under the bijection $\mathsf{C}_{n,k} \mapsto \mathsf{M}_{n,k}$ in subsection 2.1, then $\mathsf{M}_{n,k}$ is rooted at a front $\{1, 2, \ldots, k\}$. We use the notation (i^m, j^{k-m}) to represent the sequence $(i, \ldots, i, j, \ldots, j)$ of length k that has m occurrences of i and $(k-m)$ occurrences of j. Here we shall consider the distances to the vertex 1 in a k-tree $\mathsf{M}_{n,k}$. Darrasse and Soria [13] provided an algorithm to calculate the distances to the vertex 1 in a k-tree $\mathsf{M}_{n,k}$ by marking the distances on the corresponding k-front coding tree $\mathsf{C}_{n,k}$, which is similar to the algorithm given by Proskurowski in [37]. Note that every black node of the k-front coding tree is related to a vertex of the corresponding k-tree via its label, and the vertices that label a white node of the k-front tree represent k vertices that constitute a front of the corresponding k-tree. We will recall Darrasse and Soria's algorithm.

Algorithm 1: Distances in a k-tree

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Input: a k-front coding tree C and a sequence (a_i)_{i=1}^k = (0, 1^{k-1})
Output: an association table (vertex, distance)
p := min{a_i}_{i=1}^k + 1 and A = ∅
for all sons v of the root C do
    A := A ∪ {(v, p)}
    for i := 1 → k do
    A ← A ∪ the recursive call on the i-th son of v and (a_1, ..., a_{i−1}, p, a_{i+1}, ..., a_k)
return A
```

If we implement this algorithm on the 2-front coding tree (middle) in Figure 2.1, we get a distance table marked on every black node in Figure 2.2. The distance sequences on the white nodes help us to recursively mark the distances on the black nodes.

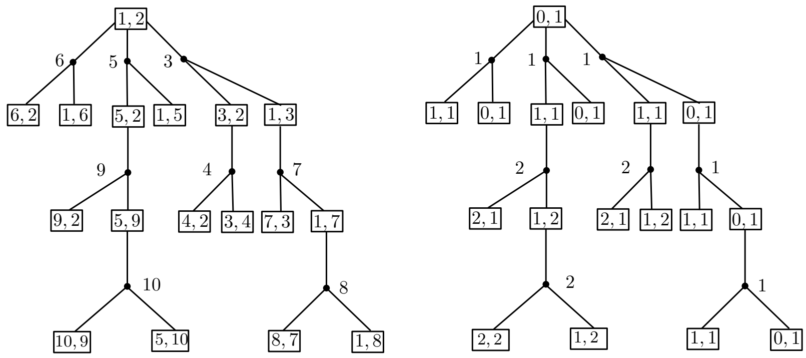

FIGURE 2.2. A 2-front coding tree (left) and the corresponding distance table on every black node (right).

2.4. **Gromov-Hausdorff convergence and the CRT.** Let $\mathsf{e} = (\mathsf{e}_t)_{0\le t\le 1}$ denote the *Brownian excursion of duration one*. Then this (random) continuous function e induces a pseudo-metric on the interval $[0,1]$ by

$$d_{\mathsf{e}}(u,v) = \mathsf{e}(u) + \mathsf{e}(v) - 2 \inf_{u\le s\le v} \mathsf{e}(s)$$

for $u \le v$. This defines a metric on the quotient $\mathcal{T}_{\mathsf{e}} = [0,1]/\sim$ where $u \sim v$ if and only if $d_{\mathsf{e}}(u,v) = 0$. The corresponding random pointed metric space $(\mathcal{T}_{\mathsf{e}}, d_{\mathsf{e}}, r_0(\mathcal{T}_{\mathsf{e}}))$, where $r_0(\mathcal{T}_{\mathsf{e}})$ is the equivalence class of the origin, is the *Continuum Random Tree* (CRT). We will simply use $\mathcal{T}_{\mathsf{e}}$ to denote the CRT. (Recall that the isometry classes of (pointed) compact metric spaces $\mathbb{K}^\bullet$, where a pointed space is a triple (X,d,r), where (X,d) is a metric space and $r \in X$ is a distinguished element, constitute a Polish space with respect to the (pointed) *Gromov-Hausdorff metric* d_{GH}. We refer the readers to [10, 21] for a full description of this metric.)

Let T_n be a Galton-Watson tree conditioned on having n vertices, T_n is critical if the offspring distribution ξ of T_n satisfies $\mathbb{E}(\xi) = 1$. T_n is aperiodic if $\gcd\{j : \mathbb{P}(\xi = j) > 0\} = 1$, see [15]. The convergence of T_n (properly scaled) to $\mathcal{T}_{\mathsf{e}}$ is due to Aldous [4].

Theorem 3. *Let T_n be a Galton-Watson tree conditioned on having n vertices, where the offspring distribution ξ of T_n is aperiodic, critical and has finite variance $\mathbb{V}ar\,\xi = \sigma^2$. As n tends to infinity, T_n with edges rescaled to length $\sigma/(2\sqrt{n})$ converges in distribution to the CRT, i.e.,*

$$\frac{\sigma}{2\sqrt{n}} T_n \xrightarrow{(d)} \mathcal{T}_{\mathsf{e}} \quad \text{in the metric space } (\mathbb{K}^\bullet, d_{GH}). \tag{2.8}$$

The Galton-Watson tree conditioned on having n vertices is also called the *conditioned Galton-Watson tree*. The conditioned Galton-Watson trees are essentially the same as the random *simply generated trees*, see [14, 15].

3. PROOF OF THE MAIN RESULT

Let $\mathsf{C}_{n,k}$ denote the the random k-front coding tree that is generated by the Boltzmann sampler $\Gamma C_k(\rho_k)$ having exactly n black nodes. Furthermore let $\mathsf{M}_{n,k}$ be the corresponding random k-tree under the bijection $\mathsf{C}_{n,k} \mapsto \mathsf{M}_{n,k}$. Finally let $\mathsf{T}^\circ_{n,k}$ be the *reduced* tree obtained from $\mathsf{C}_{n,k}$ by replacing every edge $\bullet - \circ - \bullet$ that passes a non-root white node, by an edge $\bullet - \bullet$ and removing all the white-node leaves, see Figure 2.1 for an example.

From the construction of the Boltzmann sampler $\Gamma C_k(\rho_k)$, it is clear that $\mathsf{T}^\circ_{n,k}$ is a Galton-Watson tree conditioned on having exactly n black nodes where the offspring ξ_1 of the white root node is Poisson distributed with parameter k^{-1} whereas the offspring ξ of every black node is Poisson distributed with parameter $k\lambda_k = kB_k(\rho_k) = 1$, that is, the probability that a black node has m offsprings in $\mathsf{T}^{\circ}_{n,k}$ is

$$\mathbb{P}(\xi = m) = \exp(-k\lambda_k) \cdot \frac{(k\lambda_k)^m}{m!} = \frac{\exp(-1)}{m!} \quad \text{and} \quad \mathbb{E}\,\xi = 1. \tag{3.1}$$

We can modify the tree $\mathsf{T}^{\circ}_{n,k}$ into a conditioned critical Galton-Watson tree $\mathsf{T}_{n,k}$ by replacing the offspring of the white root also by a Poisson distribution with parameter 1.

For any two black nodes x, y in $\mathsf{C}_{n,k}$, we set $d_{\mathsf{C}_{n,k}}(x, y) = \mathrm{dist}_{\mathsf{T}^{\circ}_{n,k}}(x, y)$, where dist denotes the usual graph theoretical distance. For the case $k \neq 1$, the distance $d_{\mathsf{C}_{n,k}}(x, y)$ of two black nodes x, y in $\mathsf{C}_{n,k}$ is different to the distance $\mathrm{dist}_{\mathsf{M}_{n,k}}(x, y)$ of x, y in the original k-tree $\mathsf{M}_{n,k}$. In order to represent the distances $\mathrm{dist}_{\mathsf{M}_{n,k}}(x, y)$ for any two nodes x, y in the tree $\mathsf{C}_{n,k}$, we need to decompose $\mathsf{C}_{n,k}$ into *blocks* according to the distance table from Algorithm 1. We implement the Algorithm 1 on the random tree $\mathsf{C}_{n,k}$ to have every black node marked with a distance and every white node marked with a distance sequence. For this random tree $\mathsf{C}_{n,k}$, denote by $\mathsf{C}_{i,n,k}$ a subtree of $\mathsf{C}_{n,k}$ that we call *block of type i*:

(1) $\mathsf{C}_{1,n,k}$ is rooted at the root and is induced by the root and all the black nodes that are in distance one to the vertex 1.
(2) $\mathsf{C}_{i,n,k}$, $i \geq 2$, is rooted at a white node with distance sequence $((i-1)^k)$ and is induced by this node and all its black descendents that have distance i to the vertex 1.

By construction, there is only one subtree $\mathsf{C}_{1,n,k}$ in $\mathsf{C}_{n,k}$, but there could be many subtrees $\mathsf{C}_{i,n,k}$ of $\mathsf{C}_{n,k}$ for $i \neq 1$, see Figure 3.1. For any two black nodes x, y in $\mathsf{C}_{n,k}$, let $\delta_{\mathsf{C}_{n,k}}(x, y) = a - 1$ where a is the minimal number of blocks necessary to cover the path connecting x and y. In particular if x, y are in the same block of $\mathsf{C}_{n,k}$, then $\delta_{\mathsf{C}_{n,k}}(x, y) = 0$. The following lemma will show, for any two black nodes x, y, the distance $\mathrm{dist}_{\mathsf{M}_{n,k}}(x, y)$ is almost the same as the block-distance $\delta_{\mathsf{C}_{n,k}}(x, y)$.

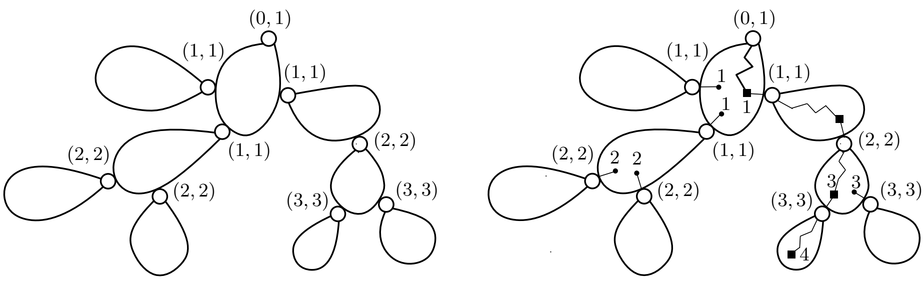

FIGURE 3.1. A decomposition of a random 2-front coding tree $\mathsf{C}_{n,2}$ into blocks $\mathsf{C}_{i,n,2}$ (left) where the pair (a, b) of integers represents the distance sequence on the root of a block. A spine (right) consists of selected good nodes in $\mathsf{C}_{n,2}$.

Lemma 4. *Let $\mathsf{C}_{n,k}$ denote the tree corresponding to the Boltzmann sampler $\Gamma C_k(\rho_k)$ conditioned on having n black nodes, let $\mathsf{M}_{n,k}$ be the corresponding k-tree under the bijection $\mathsf{C}_{n,k} \mapsto \mathsf{M}_{n,k}$. Then for any two black nodes x, y in $\mathsf{M}_{n,k}$,*

$$\mathrm{dist}_{\mathsf{M}_{n,k}}(x, y) = \delta_{\mathsf{C}_{n,k}}(x, y) + i \quad \textit{where } i \in \{0, 1, 2, 3\}. \tag{3.2}$$

Proof. If x, y are in the same block, i.e., $\delta_{\mathsf{C}_{n,k}}(x, y) = 0$. If both of them are in a block $\mathsf{C}_{1,n,k}$, then $\mathrm{dist}_{\mathsf{M}_{n,k}}(x, y) \leq \mathrm{dist}_{\mathsf{M}_{n,k}}(x, 1) + \mathrm{dist}_{\mathsf{M}_{n,k}}(y, 1) = 2 = \delta_{\mathsf{C}_{n,k}}(x, y) + 2$. If both of them are in a block $\mathsf{C}_{i+1,n,k}$ for some $i \geq 1$, recall that the root of $\mathsf{C}_{i+1,n,k}$ is a white node with distance sequence (i^k). Suppose the root of $\mathsf{C}_{i+1,n,k}$ has label $\{a_1, a_2, \ldots, a_k\}$, then for $x \in \mathsf{C}_{i+1,n,k}$, there exists an integer p such that $\mathrm{dist}_{\mathsf{M}_{n,k}}(a_p, x) = 1$. Otherwise if for all $m \leq k$, $\mathrm{dist}_{\mathsf{M}_{n,k}}(a_m, x) > 1$. It follows that $\mathrm{dist}_{M_{n,k}}(x, 1) > i + 1$, which contradicts with the fact $x \in \mathsf{C}_{i+1,n,k}$. Similarly, there is an integer q such that $\mathrm{dist}_{\mathsf{M}_{n,k}}(a_q, y) = 1$. Consequently $\mathrm{dist}_{\mathsf{M}_{n,k}}(x, y) \leq \mathrm{dist}_{\mathsf{M}_{n,k}}(a_p, x) + \mathrm{dist}_{\mathsf{M}_{n,k}}(a_q, y) + \mathrm{dist}_{\mathsf{M}_{n,k}}(a_q, a_p) = 3$, which implies (3.2).

If x, y are not in the same block, let b be the lowest common parent of x and y in $\mathsf{C}_{n,k}$, then b is either a black node or b is the white root. If b is a black node, let a_1 (resp. b_1) be the second black node on the path $b - \circ - a_1 - \cdots - \circ - x$ (resp. $b - \circ - b_1 - \cdots - \circ - y$) in $\mathsf{C}_{n,k}$. If b is the root, let

a_1 (resp. b_1) be its black child on the path $b - a_1 - \cdots - \circ - x$ (resp. $b - b_1 - \cdots - \circ - y$) in $\mathsf{C}_{n,k}$. Then one of the minimal paths connecting x and y in $\mathsf{M}_{n,k}$ must pass node b if b is a black node, and must pass one of the vertices from $\{1, 2, \ldots, k\}$ if b is the root. This is true because the k-tree corresponding to the subtree of $\mathsf{C}_{n,k}$ rooted at a_1 and the k-tree corresponding to the subtree of $\mathsf{C}_{n,k}$ rooted at b_1 are completely disjoint in $\mathsf{M}_{n,k}$. Without loss of generality, we assume b is a black node. Then our previous discussion implies $\mathrm{dist}_{\mathsf{M}_{n,k}}(x, y) = \mathrm{dist}_{\mathsf{M}_{n,k}}(x, b) + \mathrm{dist}_{\mathsf{M}_{n,k}}(y, b)$. Suppose $x \in \mathsf{C}_{i+1,n,k}$, there must exist a black node v_1 on the path $b - \circ - a_1 - \cdots - \circ - x$, such that $\mathrm{dist}_{\mathsf{M}_{n,k}}(x, v_1) = 1$ and $v_1 \in \mathsf{C}_{i,n,k}$. For the node v_1, there exists a black node v_2 on the path such that $v_2 \in \mathsf{C}_{i-1,n,k}$ and $\mathrm{dist}_{\mathsf{M}_{n,k}}(x, v_2) = 2$. We continue this process until we reach a black node v_t such that v_t and b are in the same block. Similarly, we can find a sequence of black nodes $w_1, \ldots, w_s$ from different blocks such that w_s and b are in the same block and $\mathrm{dist}_{\mathsf{M}_{n,k}}(y, w_s) = s$. It follows that

$$\begin{aligned}\mathrm{dist}_{\mathsf{M}_{n,k}}(x, b) + \mathrm{dist}_{\mathsf{M}_{n,k}}(y, b) &= \delta_{\mathsf{C}_{n,k}}(x, b) + \delta_{\mathsf{C}_{n,k}}(y, b) + \mathrm{dist}_{\mathsf{M}_{n,k}}(v_t, w_s) \\ &= \delta_{\mathsf{C}_{n,k}}(x, y) + \mathrm{dist}_{\mathsf{M}_{n,k}}(v_t, w_s).\end{aligned}$$

Since v_t and w_s are in the same block, we have $\mathrm{dist}_{\mathsf{M}_{n,k}}(v_t, w_s) \leq 3$ and the proof is complete. □

Lemma 4 allows us to transfer the distance $\mathrm{dist}_{\mathsf{M}_{n,k}}(x, y)$ of two vertices x, y in a random k-tree $M_{n,k}$ to the distance $\delta_{\mathsf{C}_{n,k}}(x, y)$ of two blocks in a random tree $\mathsf{C}_{n,k}$. In order to prove the convergence of $\mathsf{M}_{n,k}$ to the CRT $\mathcal{T}_{\mathsf{e}}$, it is sufficient to prove that the difference between $\mathfrak{m}_k \delta_{\mathsf{C}_{n,k}}(x, y)$ and $\mathrm{dist}_{\mathsf{T}_{n,k}}(x, y)$ is small with high probability where $\mathsf{T}_{n,k}$ is the above conditioned critical Galton-Watson tree and $\mathfrak{m}_k$ is a constant. For this purpose we consider the *spine* of a size-biased enriched tree, which was adapted from the size-biased Galton-Watson tree in [1]. This idea has been used in studying the scaling limit of random graphs from subcritical graph classes [35] and was further generalized to the random $\mathcal{R}$-enriched trees [39].

In fact, the block-distance $\delta_{\mathsf{C}_{n,k}}(v, 1)$ to the vertex 1 in the random tree $\mathsf{C}_{n,k}$ is not related to the depth of v in $\mathsf{C}_{n,k}$. It turns out that we have to choose a *good* black node ν_i from a block $\mathsf{C}_{i,n,k}$ of the random tree $\mathsf{C}_{n,k}$, such that they form a *spine* $\nu_1, \ldots, \nu_m$ so that $\delta_{\mathsf{C}_{n,k}}(\nu_i, 1)$ increases as the depth of ν_i on this spine increases, see Fig 3.1 and 3.2.

We call a black node v in a k-front coding tree *good* if one of its white children has distance sequence (i^k) for some integer $i \geq 1$. Let C_k denote the random k-front coding tree that is generated by the above Boltzmann sampler so that $\mathsf{C}_{n,k} = (\mathsf{C}_k : |\mathsf{C}_k| = n)$. In the same way, let $\mathsf{B}_{i,k}$ be a block of C_k which equals $\mathsf{C}_{i,n,k}$ if we condition C_k on size n. The next Lemma 5 will enable us to construct a size-biased enriched tree.

Lemma 5. *Suppose that $i \geq 1$ and let $\xi_{k,i}$ be the random variable counting the number of good black nodes v in a block $\mathsf{B}_{i,k}$ of type i in C_k. Then $\mathbb{E}\,\xi_{k,i} = 1$.*

Proof. The offspring η of the white root in C_k is Poisson distributed with parameter k^{-1} and the offspring of every black node in C_k is distributed as the sum of k independent and identically distributed random variables $\eta_1, \eta_2, \ldots, \eta_k$ where each η_i is a copy of η. The distance sequence on every white node of C_k determines if his children (black nodes) are good or not. We first compute $\mathbb{E}\,\xi_{k,1}$. The white root of $\mathsf{B}_{1,k}$ has distance sequence $(0, 1^{k-1})$ and all his children are good black nodes. So the first generation has $\mathbb{E}(\eta) = k^{-1}$ expected number of good black nodes. Assume μ_1 is a good black node in the first generation, μ_1 has k white-node children in C_k, among which $(k-1)$ have distance sequence $(0, 1^{k-1})$ and they have $1-k^{-1}$ expected number of good black-node children in C_k. By repeating this process on these good black-node children, we get

$$\mathbb{E}(\xi_{k,1}) = \frac{1}{k} + \frac{k-1}{k}\left(\frac{1}{k} + \frac{k-1}{k}\frac{1}{k} + \cdots\right) = \frac{1}{k}\sum_{i=0}^{\infty}\left(\frac{k-1}{k}\right)^i = 1.$$

For $i \neq 1$, we can compute $\mathbb{E}(\xi_{k,i})$ by repeating the same procedure as that for $\mathbb{E}(\xi_{k,1})$, namely the expected number $\mathbb{E}(\xi_{k,i})$ of good black nodes in a block $\mathsf{B}_{i,k}$ is equal to the expected number of good black nodes in its sub-block rooted at a black node ω_1, multiplied by k^{-1}, where ω_1 is the *first* good black node on its path to the root of block $\mathsf{B}_{i,k}$, from which it follows the expected number of good descendents of ω_1 is k. This implies $\mathbb{E}(\xi_{k,i}) = k^{-1} \cdot k = 1$. □

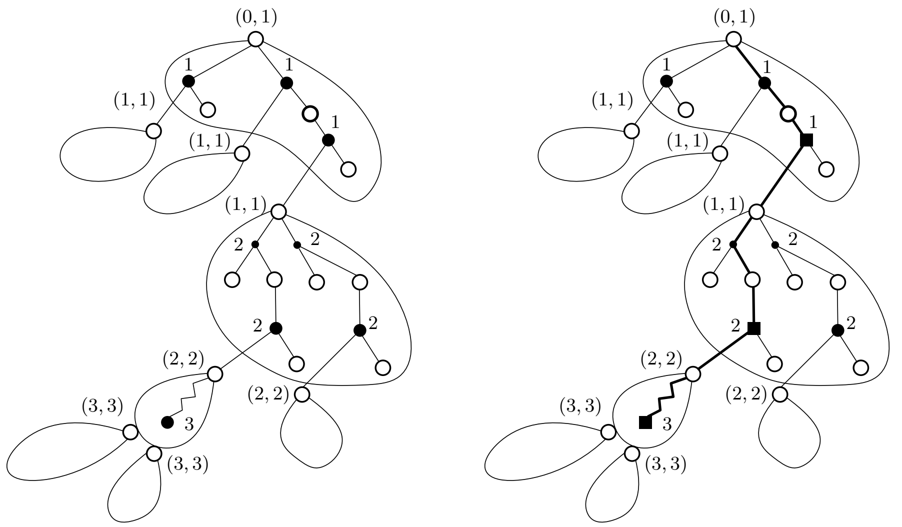

FIGURE 3.2. A 2-front coding tree C_2 with good nodes drawn in big $\bullet$ (left) and a size-biased enriched tree $\hat{\mathsf{C}}_2^{(3)}$ with a spine consisting of selected good nodes drawn in big $\blacksquare$ (right).

We will next define a *size-biased enriched tree* $\hat{\mathsf{C}}_k^{(m)}$ from a random k-front coding tree C_k. This construction comes from [1], which is a truncated version of the infinite size-biased Galton-Watson tree introduced by Lyons, Pemantle and Peres [33]. Let $\hat{\xi}_{k,i}$ be a random variable with the *size-biased* distribution

$$\mathbb{P}(\hat{\xi}_{k,i} = q) = q\,\mathbb{P}(\xi_{k,i} = q). \tag{3.3}$$

The expected value $\mathbb{E}(\xi_{k,i}) = 1$ in Lemma 5 guarantees that $\hat{\xi}_{k,i}$ is a probability distribution on $\mathbb{N}^+ = \{1, 2, \ldots\}$.

The size-biased enriched tree $\hat{\mathsf{C}}_k^{(m)}$ is now defined as follows. It starts with a *mutant block* $\mathsf{B}_{1,k}$ which is rooted at a usual root (that has distance sequence $(0, 1^{k-1})$) and contains good nodes. We now choose one of these good nodes (which number is distributed according to $\hat{\xi}_{k,1}$) and call it *heir* (and also *mutant*). The block $\mathsf{B}_{2,k}$ that is rooted at the child with distance sequence (1^k) of this heir will be the next mutant block, where we again assume that it has at least one good node. All other blocks that are adjacent to $\mathsf{B}_{1,k}$ are *normal.* We again choose one of the good nodes of the mutant block $\mathsf{B}_{2,k}$ (which number is distributed according to $\hat{\xi}_{k,2}$) and proceed inductively to choose mutant blocks and heirs till $\mathsf{B}_{m,k}$. All other blocks stay normal. We denote the heir in the m-th mutant block $\mathsf{B}_{m,k}$ by h. The path from the root to h is called *spine* of $\hat{\mathsf{C}}_k^{(m)}$, see Figure 3.2.

The probability that a given mutant block contains q good nodes and one of them is chosen as heir is, see (3.3), $q^{-1}\mathbb{P}(\hat{\xi}_{k,i} = q) = \mathbb{P}(\xi_{k,i} = q)$. For any given random k-front coding tree T, let T^α denote the tree T with a fixed spine α of block-depth m. Then the probability

$$\mathbb{P}(\hat{\mathsf{C}}_k^{(m)} = T^\alpha, \text{ with } \alpha \text{ as spine}) = \mathbb{P}(\mathsf{C}_k = T). \tag{3.4}$$

This shows, once the spine is fixed, that the probability that the size biased tree $\hat{\mathsf{C}}_k^{(m)}$ equals T^α is the same as the probability of generating T. In fact, (3.4) is true for any spine α, see Eq.(3.2) in [1]. We will need (3.4) to build a connection between $\mathfrak{m}_k \delta_{\mathsf{C}_{n,k}}(x,y)$ and $d_{\mathsf{C}_{n,k}}(x,y)$ with high probability in Lemma 6.

Lemma 6. *Let $\mathcal{C}_{n,k}$ be the class of rooted k-front coding trees of size n such that the white root has label $\{1, 2, \ldots, k\}$ and $\mathsf{C}_{n,k} \in \mathcal{C}_{n,k}$ is uniformly selected at random. Let $\mathfrak{m}_k = k+1$ for $k \geq 2$ and $\mathfrak{m}_1 = 1$. Then for all $s > 1$ and $0 < \epsilon < 1/2$ with $2\epsilon s > 1$, we have for all black nodes x, y in*

$\mathsf{C}_{n,k}$ *such that* x *is an ancestor of* y*, that one of these two properties*

$$\delta_{\mathsf{C}_{n,k}}(x,y) \geq \log^s(n) \;\; \text{and} \;\; |d_{\mathsf{C}_{n,k}}(x,y) - \mathfrak{m}_k \delta_{\mathsf{C}_{n,k}}(x,y)| \leq \delta_{\mathsf{C}_{n,k}}(x,y)^{1/2+\epsilon}, \tag{3.5}$$

$$\delta_{\mathsf{C}_{n,k}}(x,y) < \log^s(n) \;\; \text{and} \;\; d_{\mathsf{C}_{n,k}}(x,y) \leq \log^{s+2}(n) \tag{3.6}$$

holds with high probability.

Proof. Suppose the opposite of (3.5) is true, that is, there exist black nodes x, y in C_k such that x is an ancestor of y and they satisfy

$$\delta_{\mathsf{C}_k}(x,y) \geq \log^s(|\mathsf{C}_k|) \;\text{ and }\; |d_{\mathsf{C}_k}(x,y) - \mathfrak{m}_k \delta_{\mathsf{C}_k}(x,y)| > \delta_{\mathsf{C}_k}(x,y)^{1/2+\epsilon}. \tag{3.7}$$

We will denote by $\mathcal{F}_1$ the set of triples (C_k, x, y) (with x, y in C_k) that satisfy (3.7). Thus we just have to show that $\mathbb{P}((\mathsf{C}_k,x,y) \in \mathcal{F}_1 \big| |\mathsf{C}_k| = n) = o(1)$ as n tends to infinity.

Recall that $\mathsf{C}_{n,k}$ is the set of k-front coding trees generated by the Boltzmann sampler $\Gamma C_k(\rho_k)$ with n black nodes. Thus, from Lemma 2 it follows that

$$\mathbb{P}(\mathsf{C}_{n,k}) = \mathbb{P}(|\Gamma C_k(\rho_k)| = n) = \frac{c_k(n)\rho_k^n}{n!\, C_k(\rho_k)} = \frac{(kn+1)^{n-1}}{n! \exp(k^{-1})} \left(\frac{1}{ek}\right)^n \sim \frac{\exp(-k^{-1})}{k\sqrt{2\pi}} n^{-3/2} \;\text{ as }\; n \to \infty.$$

We apply (3.4) on the random k-front coding tree $\mathsf{C}_{n,k}$ with a spine that connects x to y. The block-depth of this spine is at least $\log^s n$ by assumption (3.7), which leads to

$$\begin{aligned} \mathbb{P}((\mathsf{C}_k,x,y) \in \mathcal{F}_1 \big| |\mathsf{C}_k| = n) &\leq \mathbb{P}(\mathsf{C}_{n,k})^{-1} \sum_{m=\log^s n}^{n-1} \mathbb{P}((\hat{\mathsf{C}}_k^{(m)}, x, y) \in \mathcal{F}_1 \;\text{ and }\; |\hat{\mathsf{C}}_k^{(m)}| = n) \\ &= \frac{k\sqrt{2\pi}}{\exp(-k^{-1})} n^{3/2} \sum_{m=\log^s n}^{n-1} \mathbb{P}((\hat{\mathsf{C}}_k^{(m)}, x, y) \in \mathcal{F}_1 \;\text{ and }\; |\hat{\mathsf{C}}_k^{(m)}| = n). \end{aligned} \tag{3.8}$$

Here the length of the spine in $\hat{\mathsf{C}}_k^{(m)}$ is distributed as the sum of m independent random variables $\zeta_{1,k}, \zeta_{2,k}, \ldots, \zeta_{m,k}$ where each $\zeta_{i,k}$ is distributed as the length of the path from the selected good node in some block $\mathsf{B}_{i,k}$ to the root of this block. We have for $k \geq 2$,

$$\mathbb{P}(\zeta_{i,k} = t) = \frac{1}{k} \cdot \left(1 - \frac{1}{k}\right)^{t-2} \;\text{ where }\; i \neq 1, t \geq 2$$

$$\mathbb{P}(\zeta_{1,k} = t) = \frac{1}{k} \cdot \left(1 - \frac{1}{k}\right)^{t-1} \;\text{ where }\; t \geq 1.$$

(We just have to extend the proof idea of Lemma 5.) For the case $k = 1$, every $\zeta_{i,1}$ is distributed with probability $\mathbb{P}(\zeta_{i,1} = 1) = 1$. As an immediate consequence, $\zeta_{i,k}$ has finite exponential moments for every i, k and $\mathbb{E}(\zeta_{i,k}) = k+1$, $\mathbb{E}(\zeta_{1,k}) = k$ for $k \geq 2, i \neq 1$. For the case $k = 1$ we have $\mathbb{E}(\zeta_{i,1}) = 1$ for every i. We set $\mathfrak{m}_k = k+1$ for $k \geq 2$ and $\mathfrak{m}_1 = 1$. Furthermore, the assumption in (3.7) implies

$$\mathbb{P}((\hat{\mathsf{C}}_k^{(m)}, x, y) \in \mathcal{F}_1 \;\text{ and }\; |\hat{\mathsf{C}}_k^{(m)}| = n) \leq \mathbb{P}(|\sum_{i=1}^m \zeta_{i,k} - m \cdot \mathfrak{m}_k| > m^{1/2+\epsilon}). \tag{3.9}$$

By applying the deviation inequality (see [1, 35, 36]) on the random variables $\zeta_{1,k}, \zeta_{2,k}, \ldots, \zeta_{m,k}$, we get for some positive constant c_1,

$$\mathbb{P}(|\sum_{i=1}^m \zeta_{i,k} - m \cdot \mathfrak{m}_k| > m^{1/2+\epsilon}) \leq 2\exp(-c_1(\log n)^{2s\epsilon}) = o(n^{-5/2}).$$

Together with (3.8) and (3.9), we can conclude that $\mathbb{P}((\mathsf{C}_k,x,y) \in \mathcal{F}_1 \big| |\mathsf{C}_k| = n) = o(1)$.

Now we turn to suppose the opposite of (3.6) is true, i.e., there exist black nodes x, y in C_k such that x is an ancestor of y. They satisfy

$$\delta_{\mathsf{C}_k}(x,y) < \log^s(|\mathsf{C}_k|) \;\text{ and }\; d_{\mathcal{C}_k}(x,y) > \log^{s+2}(|\mathsf{C}_k|). \tag{3.10}$$

We use the notation $\mathcal{F}_2$ to represent the set of triples (C_k, x, y) (with x, y in C_k) that satisfy (3.10). Again from (3.8) and from the deviation inequality, we obtain for some positive constant c_2,

$$\begin{aligned}\mathbb{P}((\mathsf{C}_k,x,y)\in\mathcal{F}_2\big||\mathsf{C}_k|=n) &\le \frac{k\sqrt{2\pi}}{\exp(-k^{-1})}n^{3/2}\sum_{m=1}^{\log^s n}\mathbb{P}((\hat{\mathsf{C}}_k^{(m)},x,y)\in\mathcal{F}_2 \text{ and } |\hat{\mathsf{C}}_k^{(m)}|=n)\\ &\le \frac{k\sqrt{2\pi}}{\exp(-k^{-1})}n^{3/2}\sum_{m=1}^{\log^s n}\mathbb{P}(\sum_{i=1}^{m}\zeta_{i,k}>\log^{s+2}n)\\ &= O(n^{3/2})(\log^s n)\exp(-c_2\log^{2s+4}(n)) = o(1)\end{aligned}$$

and the proof is complete. □

Now we are ready to prove our main result.

Proof of Theorem 1. It follows from Lemma 6 that with high probability

$$|d_{\mathsf{C}_{n,k}}(x,y)-\mathfrak{m}_k\delta_{\mathsf{C}_{n,k}}(x,y)|\le\delta_{\mathsf{C}_{n,k}}(x,y)^{1/2+\epsilon}+\log^{s+2}(n)$$

holds for all black nodes x, y where x is an ancestor of y in the random k-front coding tree $\mathsf{C}_{n,k}$. For any two black nodes μ,ν in $\mathsf{C}_{n,k}$, let α be the lowest common ancestor of μ and ν (α could be the white root of $\mathsf{C}_{n,k}$), then

$$\begin{aligned}|d_{\mathsf{C}_{n,k}}(\mu,\nu)-\mathfrak{m}_k\delta_{\mathsf{C}_{n,k}}(\mu,\nu)| &\le \delta_{\mathsf{C}_{n,k}}(\mu,\alpha)^{1/2+\epsilon}+\delta_{\mathsf{C}_{n,k}}(\nu,\alpha)^{1/2+\epsilon}+2\log^{s+2}(n)\\ &\le 2\mathrm{D}(\mathsf{C}_{n,k})^{1/2+\epsilon}+2\log^{s+2}(n)\end{aligned}$$

where $\mathrm{D}(\mathsf{C}_{n,k})$ is the diameter of random tree $\mathsf{C}_{n,k}$. We divide both sides of this inequality by $\sqrt{n}$ and obtain

$$\left|\frac{d_{\mathsf{C}_{n,k}}(\mu,\nu)}{\sqrt{n}}-\frac{\mathfrak{m}_k\delta_{\mathsf{C}_{n,k}}(\mu,\nu)}{\sqrt{n}}\right|\le\frac{2\mathrm{D}(\mathsf{C}_{n,k})^{1/2+\epsilon}}{\sqrt{n}}+\frac{2\log^{s+2}(n)}{\sqrt{n}}. \tag{3.11}$$

Recall that $\mathsf{T}_{n,k}$ contains the white root and all the black nodes of $\mathsf{C}_{n,k}$. $\mathsf{T}_{n,k}$ is a critical conditioned Galton-Watson tree. The only difference between $\mathsf{C}_{n,k}$ and $\mathsf{T}_{n,k}$ is that the offspring of the white root in $\mathsf{C}_{n,k}$ is Poisson distributed with parameter k^{-1}, while the offspring of the white root in $\mathsf{T}_{n,k}$ is Poisson distributed with parameter 1. If both black nodes μ,ν are contained in a random k-front coding tree $\mathsf{C}_{n,k}$, then they must be in the random tree $\mathsf{T}_{n,k}$, which indicates $d_{\mathsf{C}_{n,k}}(\mu,\nu)=\mathrm{dist}_{\mathsf{T}_{n,k}}(\mu,\nu)$ and $\mathrm{D}(\mathsf{C}_{n,k})\le\mathrm{D}(\mathsf{T}_{n,k})$. Consequently, (3.11) rewrites to

$$\left|\frac{\mathrm{dist}_{\mathsf{T}_{n,k}}(\mu,\nu)}{\sqrt{n}}-\frac{\mathfrak{m}_k\delta_{\mathsf{C}_{n,k}}(\mu,\nu)}{\sqrt{n}}\right|\le\frac{2\mathrm{D}(\mathsf{T}_{n,k})^{1/2+\epsilon}}{\sqrt{n}}+\frac{2\log^{s+2}(n)}{\sqrt{n}}.$$

The diameter $\mathrm{D}(\mathsf{T}_{n,k})$ of a random tree $\mathsf{T}_{n,k}$ is less than the height $\mathrm{H}(\mathsf{T}_{n,k})$ of $\mathsf{T}_{n,k}$ multiplied by 2. By applying the tails for the height of $\mathsf{T}_{n,k}$, see Theorem 1.2 in [1] and left-tail upper bounds for the height in [1], we obtain for the Gromov-Hausdorff distance

$$d_{\mathrm{GH}}\left(\frac{\mathsf{T}_{n,k}}{\sqrt{n}},\frac{\mathfrak{m}_k\mathsf{C}_{n,k}}{\sqrt{n}}\right)\le\max_{\mu,\nu}\left|\frac{\mathrm{dist}_{\mathsf{T}_{n,k}}(\mu,\nu)}{\sqrt{n}}-\frac{\mathfrak{m}_k\delta_{\mathsf{C}_{n,k}}(\mu,\nu)}{\sqrt{n}}\right|\xrightarrow{p}0.$$

Namely, for any fixed ε, the probability of the event $d_{\mathrm{GH}}(\frac{\mathsf{T}_{n,k}}{\sqrt{n}},\frac{\mathfrak{m}_k\mathsf{C}_{n,k}}{\sqrt{n}})\le\varepsilon$ converges to 1 as n tend to infinity.

Since the variance of the offspring distribution in the random tree $\mathsf{T}_{n,k}$ is 1, it follows from Theorem 3 that $\mathsf{T}_{n,k}/(2\sqrt{n})\xrightarrow{(d)}\mathcal{T}_e$. Hence we get $\mathfrak{m}_k\mathsf{C}_{n,k}/(2\sqrt{n})\xrightarrow{(d)}\mathcal{T}_e$ and consequently with the help of Lemma 4, we have

$$\frac{\mathfrak{m}_k\mathsf{M}_{n,k}}{2\sqrt{n}}\xrightarrow{(d)}\mathcal{T}_e.$$

Let $\mathcal{G}_{n,k}$ be the class of k-trees with n hedra, let $\mathsf{G}_{n,k}$ be the random k-tree that is uniformly selected from the class $\mathcal{G}_{n,k}$. Let $\mathsf{G}^{\circ}_{n,k}$ be the random k-tree $\mathsf{G}_{n,k}$ rooted at a front. Then the probability to uniformly choose $\mathsf{G}_{n,k}$ from the class $\mathcal{G}_{n,k}$ is equal to the probability to uniformly choose $\mathsf{C}_{n,k}$

from the class $\mathcal{C}_{n,k}$, namely the rooting and labeling process will not affect the probability space. This indicates,

$$\frac{\mathfrak{m}_k \mathsf{G}_{n,k}}{2\sqrt{n}} \xrightarrow{(d)} \mathcal{T}_e \quad \text{and} \quad \frac{\mathfrak{m}_k \mathsf{G}^\circ_{n,k}}{2\sqrt{n}} \xrightarrow{(d)} \mathcal{T}_e$$

where $\mathfrak{m}_k = k+1$ for $k \neq 1$ and $\mathfrak{m}_1 = 1$. □

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