Generalized Thue-Morse Sequences of Squares and Uniform Distribution in Compact Groups

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joint work with Johannes Morgenbesser

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Summary

★ Thue-Morse sequence

★ Generalized Thue-Morse sequence and main result

★ Sketch of the proof

★ Applications

  ★ The sum-of-digits function

  ★ Automatic sequences

★ Thue-Morse sequence

Thue-Morse sequence $(t_n)_{n\geqslant 0}$:

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0

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01

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0110

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01101001

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$$\boxed{t_n = s_2(n) \bmod 2}$$

$$n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i \qquad \varepsilon_i(n) \in \{0, 1, \ldots, q-1\}, \qquad s_q(n) = \sum_{i=0}^{\ell-1} \varepsilon_i(n)$$

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Mauduit and Rivat (2009):

$$\#\{0 \leqslant n < N : t_{n^2} = 0\} \sim \frac{N}{2}$$

★ Generalized Thue-Morse sequences and main results

- H ... compact (Hausdorff) group
- $q \geqslant 2$ and $g_0, g_1, \ldots, g_{q-1} \in H$ with $g_0 = e$ (identity element)
- $G \leqslant H$ ... closure of the subgroup generated by $g_0, g_1, \ldots, g_{q-1}$

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q-multiplicative function:

$$\boxed{T(j + qn) = g_j T(n) = T(j)T(n)} \qquad 0 \leqslant j < q$$

$H = \langle \mathbb{Z}/2\mathbb{Z}, + \rangle$, $q = 2$, $g_0 = 0$, $g_1 = 1$: $\quad T(n) = s_2(n) \bmod 2 = t_n$

Theorem

Let μ denote the Haar measure of G. Then $(T(n))_{n\geqslant 0}$ is μ-uniformly distributed in G, that is,

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(for all continuous functions $f : G \to \mathbb{R}$.)

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Remark. Equivalently, a sequence (x_n) in G is μ-uniformly distributed if

$$\frac{1}{N}\left|\{n < N : x_n \in B\}\right| \to \mu(B)$$

holds for all μ-measurable sets $B \subseteq G$ with $\mu(\partial B) = 0$.

Theorem (D. and Morgenbesser, 2010)

There exists a positive integer $m = m(q, g_0, \ldots, g_{q-1})$ such that the following holds: Set

$$\mathrm{d}\nu = \sum_{v=0}^{m} \mathbf{1}_{g_v U} \cdot Q(v,m)\,\mathrm{d}\mu,$$

where

- $\mu \ldots$ *Haar measure on G,*
- $U = \mathrm{cl}(\{T(mn) : n \geqslant 0\}) \ldots$ *normal subgroup of G of index m,*
- $Q(v,m) = \#\{0 \leqslant n < m : n^2 \equiv v \bmod m\}$.

Then $(T(n^2))_{n \geqslant 0}$ is ν-uniformly distributed in G, that is,

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{T(n^2)} \to \nu.$$

Theorem (D. and Morgenbesser, 2010)

Let $a \geqslant 1$ and $b \geqslant 0$ be integers, $m = m(q, g_0, \ldots, g_{q-1})$ and $m' = \mathrm{gcd}(a, m)$. Set

$$\mathrm{d}\nu' = m' \cdot \mathbf{1}_{T(b)U'} \,\mathrm{d}\mu,$$

where

- $\mu \ldots$ *Haar measure on G,*
- $U' = \mathrm{cl}(\{T(m'n) : n \geqslant 0\}) \ldots$ *normal subgroup of G of index m'.*

Then $(T(an+b))_{n\geqslant 0}$ is ν'-uniformly distributed in G, that is,

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{T(an+b)} \to \nu'.$$

A **unitary group representation** is a continuous homomorphism $D : G \to U_n$ for some $n \geqslant 1$.

$U_n \ldots$ group of unitary $n \times n$ matrices over $\mathbb{C}$

D is irreducible if there is no proper subspace W of $\mathbb{C}^n$ with $D(x)W \subseteq W$ for all $x \in G$

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Lemma

Let G be a compact group and ν a regular normed Borel measure on G. Then a sequence $(x_n)_{n\geqslant 0}$ is ν-uniformly distributed in G iff

$$\frac{1}{N}\sum_{n=0}^{N-1} D(x_n) \to \int_G D \, \mathrm{d}\nu$$

holds for all irreducible unitary representations D of G.

Remarks:

- The *characteristic integer* m is the largest integer such that $m \mid q-1$ and such that there exists a representation D of G with

$$D(g_u) = e^{-2\pi i \frac{u}{m}} \qquad \text{for all } u \in \{0,1,\ldots,q-1\}.$$

- $(T(n^2))_{n\geqslant 0}$ is uniformly distributed in G (i.e., $\nu = \mu$) iff $m \leqslant 2$.
- $(T(an+b))_{n\geqslant 0}$ is uniformly distributed in G (i.e., $\nu' = \mu$) iff $m' = \gcd(a,m) = 1$.

- If G is connected, then $T(n^2)$ and $T(an+b)$ are uniformly distributed in G.

★ Sketch of the proof

$$\boxed{\frac{1}{N}\sum_{0\leqslant n<N} D(T(n^2))}$$

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- The representation $D_0,\ldots,D_{m-1}$ are special but easy:

$$D_k(g_u) = e^{-2\pi i \frac{k}{m}u} \qquad \text{for all } 0\leqslant u<q \text{ and } 0\leqslant k<m$$

$$D_k(T(n^2)) = e^{-2\pi i \frac{k}{m}n^2} \qquad \textit{Gausssums}$$

- For all other irreducible unitary representations ...

Van der Corput type inequality:

$$\left\| \sum_{0 \leqslant n < N} Z(n) \right\|_{\mathbb{F}} \leqslant \left(\frac{dN}{R} \sum_{|r| < R} \left(1 - \frac{|r|}{R}\right) \left\| \sum_{\substack{0 \leqslant n \leqslant B \\ 0 \leqslant n + r \leqslant B}} Z(n+r) Z(n)^H \right\|_{\mathbb{F}} \right)^{1/2} + \frac{f}{2} R$$

Van der Corput type inequality:

$$\left\| \sum_{0 \leqslant n < N} D(T(n^2)) \right\|_{\mathbb{F}} \leqslant \left(\frac{dN}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \left\| \sum_{\substack{0 \leqslant n \leqslant B \\ 0 \leqslant n+r \leqslant B}} D(T(n+r)^2) D(T(n^2))^H \right\|_{\mathbb{F}} \right)^{1/2} + \frac{f}{2} R$$

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$$T\,(n) = g_{\varepsilon_0(n)} g_{\varepsilon_1(n)} \cdots g_{\varepsilon_{\lambda-1}(n)} g_{\varepsilon_\lambda(n)} \cdots g_{\varepsilon_{\ell-1}(n)}$$

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$$D(T((n+r)^2)) D(T(n^2))^H$$

$$= D(T_\lambda((n+r)^2)) D(g_{\varepsilon_\lambda}) \cdots D(g_{\varepsilon_{\ell-1}}) D(g_{\varepsilon_{\ell-1}})^H \cdots D(g_{\varepsilon_\lambda})^H D(T_\lambda(n^2))^H$$

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$$\begin{aligned} & D(T((n+r)^2))D(T(n^2))^H \\ & = D(T_{\lambda}((n+r)^2))D(g_{\varepsilon_\lambda}) \cdots \overbrace{D(g_{\varepsilon_{\ell-1}})D(g_{\varepsilon_{\ell-1}})^H}^{I_n} \cdots D(g_{\varepsilon_\lambda})^H D(T_{\lambda}(n^2))^H \\ & = D(T_{\lambda}((n+r)^2))D(T_{\lambda}(n^2))^H \end{aligned}$$

$$T_{\lambda}(n) = g_{\varepsilon_0(n)} g_{\varepsilon_1(n)} \cdots g_{\varepsilon_{\mu-1}(n)} g_{\varepsilon_\mu(n)} \cdots g_{\varepsilon_{\lambda-1}(n)}$$

Fourier terms:

$$F_{\lambda}(h) = \frac{1}{q^\lambda} \sum_{0 \leqslant u < q^\lambda} e^{-2\pi i \frac{hu}{q^\lambda}} D(T_{\lambda}(u))$$

$$T_{\mu,\lambda}(n) = \qquad g_{\varepsilon_\mu(n)} \cdots g_{\varepsilon_{\lambda-1}(n)}$$

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$$\frac{2}{\pi}\log\left(\frac{4e^{\pi/2}q^{\lambda}}{\pi}\right) q^{\lambda/2} \max_{0\leqslant \ell < q^{\lambda}} \sum_{d \mid q^{\lambda}} d^{1/2}$$

$$\cdot \sum_{\substack{0\leqslant h_1,h_2,h_3,h_4<q^{\lambda} \\ (h_1+h_2+h_3+h_4,q^{\lambda})=d \\ d \mid 2r(h_1+h_2)+2sq^{\mu}(h_2+h_3)+\ell}} \|F_{\mu,\lambda}(h_1)\|_{\mathbb{F}} \|F_{\mu,\lambda}(h_2)\|_{\mathbb{F}} \|F_{\mu,\lambda}(h_3)\|_{\mathbb{F}} \|F_{\mu,\lambda}(h_4)\|_{\mathbb{F}}$$

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The analogue of this expression appears in Mauduit and Rivat's work and can be estimated as in their case.

★ Properties of the Fourier term

Lemma

Set

$$\Psi_D(t) = \sum_{0 \leqslant u < q} e(tu) D(g_u),$$

then

$$F_\lambda(h) = \frac{1}{q^\lambda} \Psi_D\left(-\frac{h}{q^\lambda}\right) \Psi_D\left(-\frac{h}{q^{\lambda-1}}\right) \cdots \Psi_D\left(-\frac{h}{q}\right).$$

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Lemma

Suppose that $D \notin \{D_0, \ldots, D_{m-1}\}$ *is an irreducible and unitary representation of G. Then there exists a constant* $c > 0$ *such that*

$$\boxed{\max_{h \in \mathbb{Z}} \|F_\lambda(h)\|_2 \ll q^{-c\lambda}}.$$

★ Exercise on linear subsequences

$$\sum_{n<N} D(T(an+b))$$

$$= \sum_{0\leqslant u<q^\nu} \sum_{n<N} D(T(u)) \cdot \frac{1}{q^\lambda} \sum_{0\leqslant h<q^\lambda} \mathrm{e}\left(\frac{h(an+b-u)}{q^\lambda}\right)$$

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$$\left\| \sum_{n<N} D(T(an+b)) \right\|_2 \ll \sum_{0\leqslant h<q^\lambda} \|F_\lambda(h)\|_2 \cdot \min\left(N, \frac{1}{\left|\sin \frac{\pi h a}{q^\lambda}\right|}\right).$$

★ Applications

★ The sum-of-digits function:

Theorem (Mauduit and Rivat, 2009)

Let $q, r \geqslant 2$ *and set* $m = \gcd(q-1, r)$*. Then we have*

$$\lim_{x\to\infty} \frac{1}{x} \# \{n \leqslant x : s_q(n^2) \equiv a \bmod r\} = \frac{1}{r} Q(a, m).$$

Furthermore, $(\alpha s_q(n^2))_{n\in\mathbb{N}}$ *is uniformly distributed modulo* 1 *iff* α *is irrational.*

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Proof:

- $H = \mathbb{Z}/r\mathbb{Z}$ and $g_j = j$, $0 \leqslant j < q$. Then $T(n) = s_q(n) \bmod r$.

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Proof:

- $H = \mathbb{Z}/r\mathbb{Z}$ and $g_j = j$, $0 \leqslant j < q$. Then $T(n) = s_q(n) \bmod r$.
- $H = \mathbb{R}/\mathbb{Z}$ and $g_j = \alpha j$, $0 \leqslant j < q$. Then $T(n) = \alpha s_q(n) \bmod 1$.

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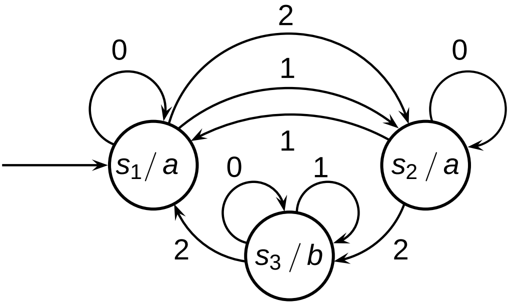
★ Automatic sequences:

Definition

A sequence $(u_n)_{n\geqslant 0}$ is called a *q-automatic sequence*, if u_n is the output of an automaton when the input is the q-ary expansion of n.



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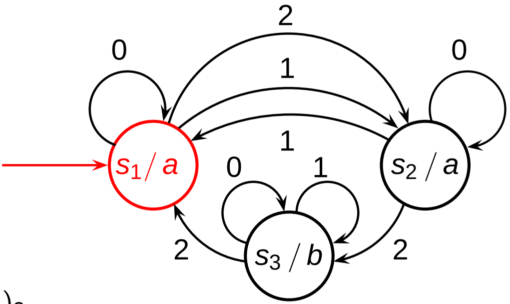

$32 = (1012)_3$

★ Applications

★ Automatic sequences:

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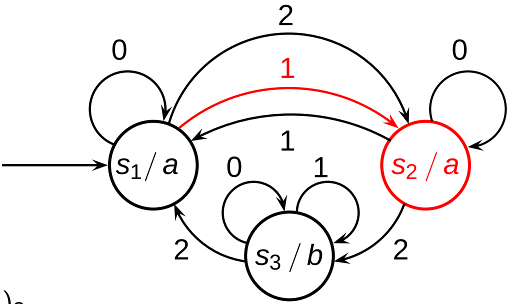

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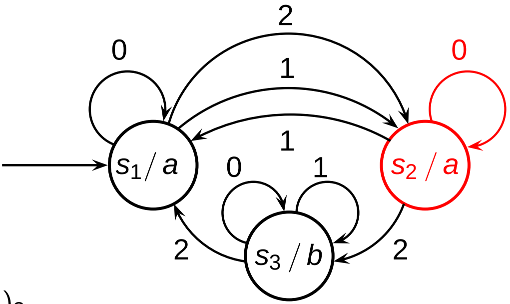

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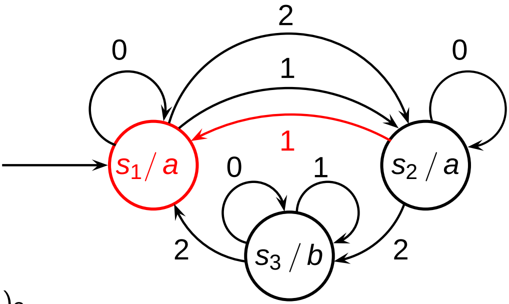

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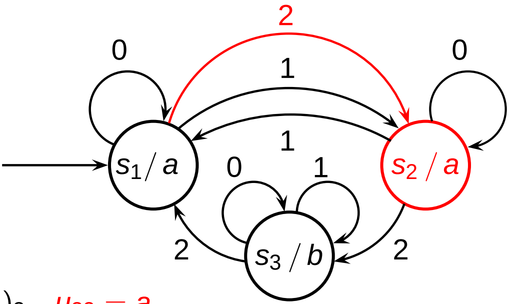

$32 = (1012)_3 \quad u_{32} = a,$

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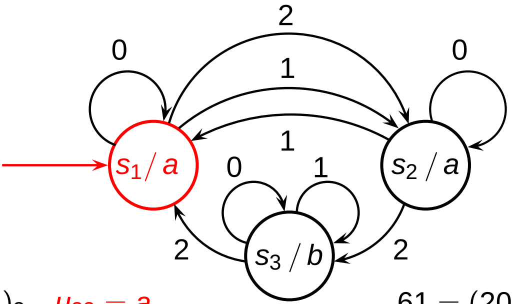

$32 = (1012)_3 \quad u_{32} = a,$ $\qquad 61 = (2021)_3$

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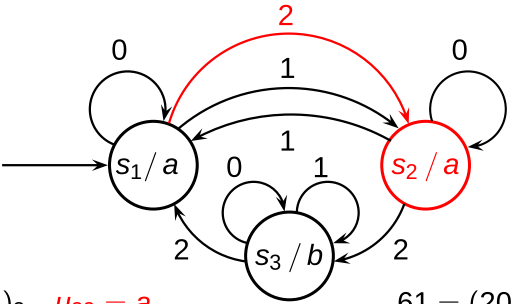

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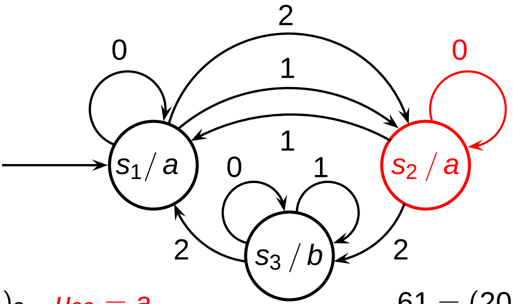

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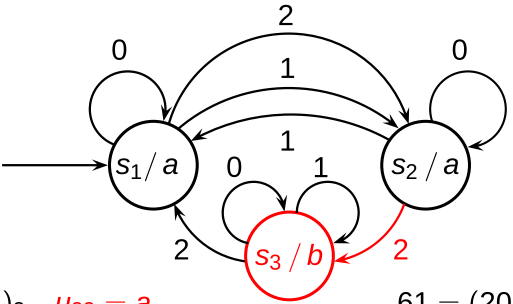

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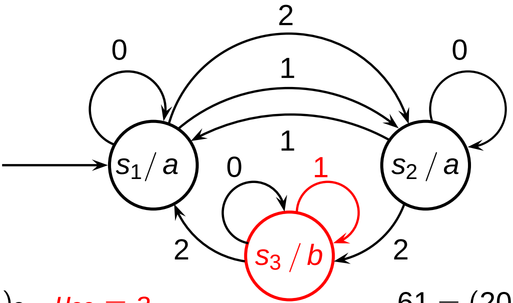

$32 = (1012)_3 \quad u_{32} = a,$ $\qquad 61 = (2021)_3 \quad u_{61} = b$

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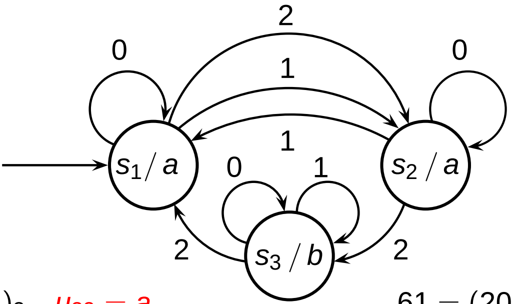
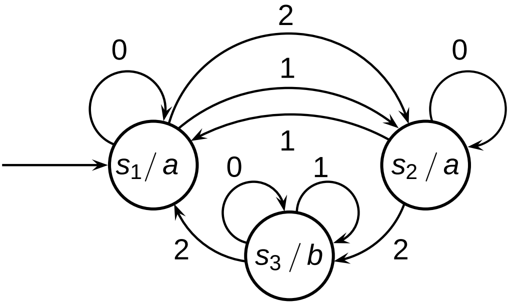
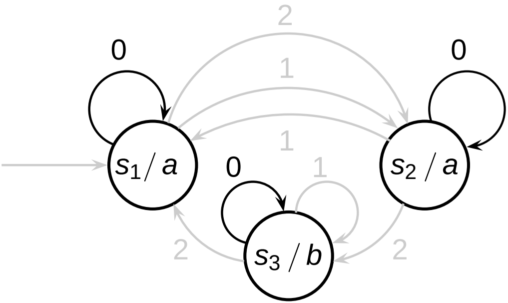


$32 = (1012)_3 \quad u_{32} = a,$ $\qquad 61 = (2021)_3 \quad u_{61} = b$

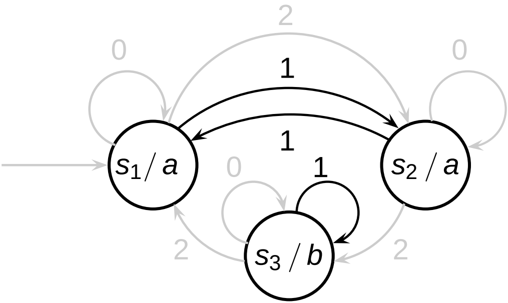
$(u_n)_{n\geqslant 0}$: *aaaaabaabaabaaabbaaabaaabbaaabaaabbaaaaaaba* . . .



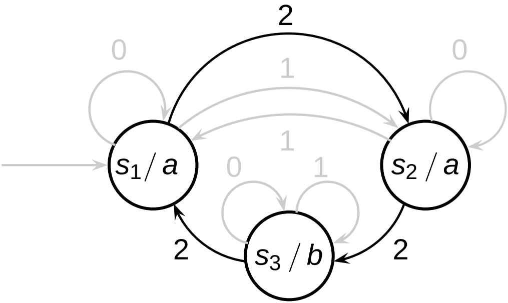
2
0
1
0
1
s_1 / a
0
1
s_2 / a
2
s_3 / b
2



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



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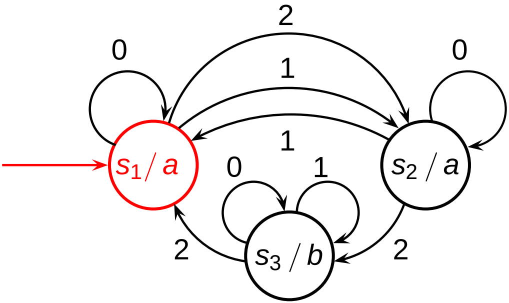

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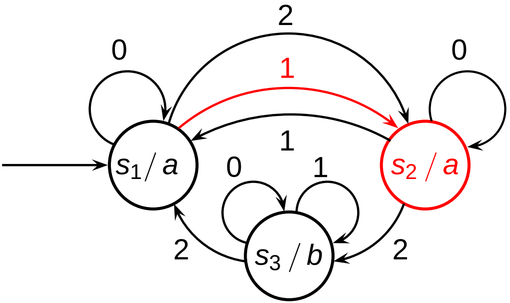
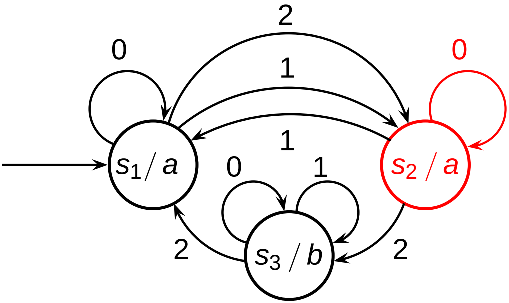


$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

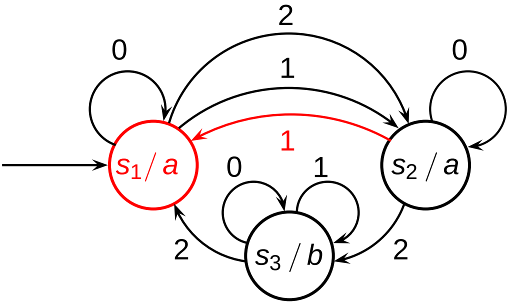
$32 = (1012)_3$:

$$M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



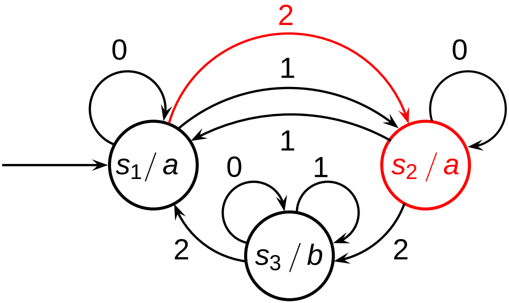
$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$32 = (1012)_3 : \qquad M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



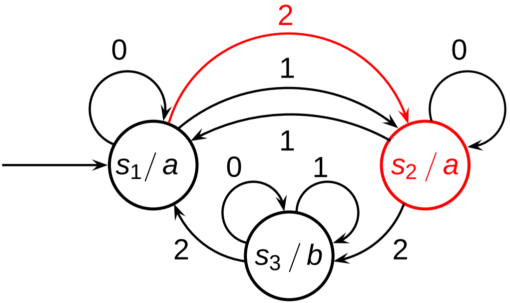
$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$32 = (1012)_3: \qquad M_1 \circ M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$32 = (1012)_3 : \qquad M_2 \circ M_1 \circ M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



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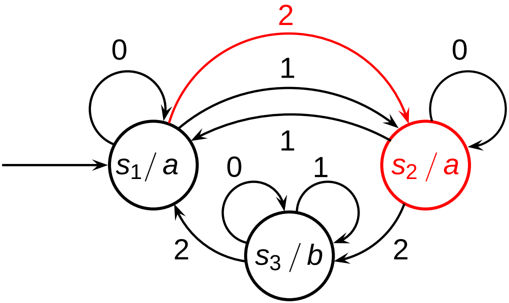
$$\boxed{S(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}}$$

$$\boxed{u_n = f(S(n)\mathbf{e}_1)} \qquad \mathbf{e}_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$$



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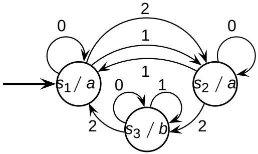
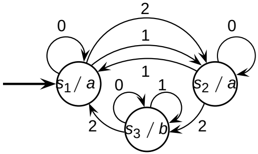
Definition

A q-automatic sequence is called *invertible* if there exists an automaton such that all transition matrices are invertible and M_0 is the identity matrix.



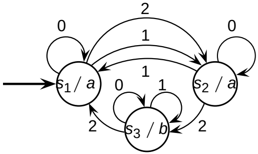
$(u_n)_{n\geqslant 0}$: *aaaaabaabaabaaabbaaabaaabbaaabaaabbaaaaaaaba* ...

Frequency of a in $(u_n)_{n\geqslant 0}$: $\lim_{N\to\infty} \frac{1}{N} \#\{0 \leqslant n < N : u_n = a\}$



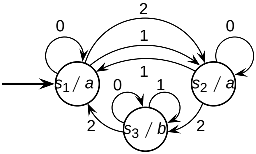
$(u_n)_{n\geqslant 0}$: $aaaaabaabaabaaabbaaabaaabbaaabaaabbaaaaaaba\ldots$

Frequency of a in $(u_{3n})_{n\geqslant 0}$: $\lim_{N\to\infty} \frac{1}{N}\#\{0 \leqslant n < N : u_{3n} = a\}$



$(u_n)_{n \geqslant 0}$: $aaaaabaababaabaaabbaaabaaabbaaabaaabbaaaaaaaba\ldots$

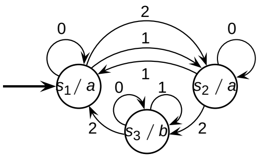
Frequency of a in $(u_{n^2})_{n \geqslant 0}$: $\lim_{N\to\infty} \frac{1}{N} \#\{0 \leqslant n < N : u_{n^2} = a\} = ?$



$(u_n)_{n \geqslant 0}$: *aaaaabaababaabaaabbaaabaaabbaaabaaabbaaaaaaba* . . .

Frequency of a in $(u_{n^2})_{n \geqslant 0}$: $\lim_{N \to \infty} \frac{1}{N} \#\{0 \leqslant n < N : u_{n^2} = a\} = ?$

$$H = SO(n, \mathbb{R}),\ g_i = M_i,\ 0 \leqslant i < q :$$



$(u_n)_{n\geqslant 0}$: aaaaabaabaabaaabbaaabaaabbaaabaaabbaaaaaaba ...

Frequency of a in $(u_{n^2})_{n\geqslant 0}$: $\lim_{N\to\infty} \frac{1}{N} \#\{0 \leqslant n < N : u_{n^2} = a\} = ?$

$$H = SO(n,\mathbb{R}),\ g_i = M_i,\ 0 \leqslant i < q :$$

Theorem (D. and Morgenbesser, 2010)

Let $q \geqslant 2$ and $(u_n)_{n\geqslant 0}$ be an invertible q-automatic sequence. Then the frequency of each letter of the subsequence $(u_{n^2})_{n\geqslant 0}$ exists.

Thank you!

1 LAUWERENS KUIPERS AND HARALD NIEDERREITER: *Uniform Distribution of Sequences*. Wiley-Interscience Publication, 1974

2 JEAN-PAUL ALLOUCHE AND JEFFREY SHALLIT: *Automatic sequences*. Cambridge University Press, 2003

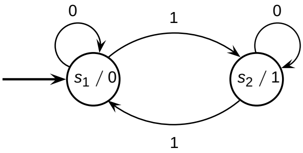
3 ALEKSANDR O. GELFOND: *Sur les nombres qui ont des propriétés additives et multiplicatives données*. Acta Arithmetica, 1968

4 CHRISTIAN MAUDUIT AND JOËL RIVAT: *La somme des chiffres des carrés*. Acta Mathematica, 2009

5 MICHAEL DRMOTA AND JOHANNES F. MORGENBESSER: *Generalized Thue-Morse Sequence of Squares*. submitted

Examples of automatic sequences

Thue-Morse sequence:



Rudin-Shapiro sequence: