Algorithmica manuscript No.
(will be inserted by the editor)

A Central Limit Theorem for the Number of Degree-k Vertices in Random Maps

Michael Drmota · Konstantinos Panagiotou

Received: date / Accepted: date

Abstract We prove that the number of vertices of given degree in (general or 2-connected) random planar maps satisfies a central limit theorem with mean and variance that are asymptotically linear in the number of edges. The proof relies on an analytic version of the quadratic method and singularity analysis of multivariate generating functions.

Keywords Planar Map · Degree Distribution · Central limit theorem

Mathematics Subject Classification (2000) 05A15 · 05A16 · 60F05

1 Introduction

In this paper we study statistical properties of planar maps, which are connected planar graphs, possibly with loops and multiple edges, together with an embedding in the plane. Such objects are frequently used to describe topological features of geometric arrangements in two or three spatial dimensions.

The first author is supported by the Austrian Science Foundation FWF, Project S9604. The second author is supported by the German Research Foundation, Grant PA 2080/2-1. An extended abstract of this work appeared in the Proceedings of the Meeting on Analytic Algorithmics and Combinatorics (ANALCO12), 2012.

M. Drmota
Institute of Discrete Mathematics and Geometry, TU Wien, Wiedner Hauptstr. 8–10, A-1040 Wien, Austria
Tel.: +43-1-58801-10450
Fax: +43-1-58801-10499
E-mail: michael.drmota@tuwien.ac.at

K. Panagiotou
Department of Mathematics, Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 Munich, Germany
Tel.: +49-89-2180-4419
Fax: +49-89-2180-4043
E-mail: kpanagio@math.lmu.de

Thus, the knowledge of the structure and of properties of "typical" objects may turn out to be very useful in the analysis of particular algorithms that operate on planar maps.

We say that map is *rooted* if a vertex v and an edge e incident with v are distinguished. We call v the *root-vertex* and e the *root-edge*. The face to the right of e is called the *root-face* and is usually taken as the outer (or infinite) face. In this paper we only consider rooted maps.

The enumeration of rooted maps is a classical subject, initiated by Tutte in the 60's, see [16]. Among many other results, Tutte computed the number M_n of rooted maps with n edges, proving the formula

$$M_n = \frac{2(2n)!}{(n+2)!n!}3^n.$$

Our main interest is the degree distribution of random planar maps. We denote by $d_{n,k}$ the probability that the root-vertex has degree k in a random map with n edges, i.e., a map that is drawn uniformly at random from the set of all maps with n edges. It is known that the limit $d_k = \lim_{n\to\infty} d_{n,k}$ exists. Actually, the values d_k are almost explicit. They are given by the generating function

$$\sum_{k\geq 2} d_k u^k = \frac{u\sqrt{3}}{\sqrt{(2+u)(6-5u)^3}}. \quad (1)$$

For more details see Liskovets [14] and the references therein.

The problem is slightly different when we look at the degree of of *random vertex* in a random map. Let $p_{n,k}$ denote the probability that a randomly chosen vertex in a random map with n edges has degree k. Then the limit $p_k = \lim_{n\to\infty} p_{n,k}$ exists, too, and we have

$$p_k = \mu d_k / k$$

for a certain constant $\mu > 0$, see also [14]. By integration it is possible to obtain an explicit (but involved) representation of the generating function for the sequence p_k. Note that p_k is closely related to the average behavior of the number $X_n^{(k)}$ of vertices of degree k in a random planar map with n edges. It is well known that the number of vertices V_n in maps of size n satisfies a central limit theorem with $\mathbb{E}[V_n] \sim n/2$. Consequently, as $n \to \infty$

$$\mathbb{E}[X_n^{(k)}] \sim \frac{1}{2} p_k n.$$

2 Our Result

The main goal of this paper is to study the random variable $X_n^{(k)}$ in more detail. In particular we will prove a central limit theorem and tail estimates in the general and in the 2-connected case.

Theorem 1 *Let $k \in \mathbb{N}$. The number $X_n^{(k)}$ of vertices of degree k in a random planar map with n edges satisfies a central limit law, i.e.,*

$$\frac{X_n^{(k)} - \mathbb{E}[X_n^{(k)}]}{\mathrm{Var}[X_n^{(k)}]^{1/2}} \to \mathcal{N}(0,1),$$

where $\mathbb{E}[X_n^{(k)}] = \mu_k n + O(1)$ and $\mathrm{Var}[X_n^{(k)}] = \sigma_k^2 n + O(1)$, and $\mu_k, \sigma_k > 0$ are computable constants. Moreover, $X_n^{(k)}$ has exponential tails, i.e., there is an $\varepsilon_0 > 0$ and a $c_k > 0$ such that for any $0 < \varepsilon < \varepsilon_0$

$$\Pr\left[|X_n^{(k)} - \mathbb{E}[X_n^{(k)}]| \geq \varepsilon \mathbb{E}[X_n^{(k)}]\right] \leq e^{-\varepsilon^2 c_k n}.$$

The same is true for random 2-connected planar maps.

Note that μ_k in the above statement equals (in the case of planar maps) the quantity $p_k/2$ from Section 1. Since the dual of a planar map is again a planar map and the degree of a vertex corresponds to the valency of a face in the dual, the same result holds for the number of faces of given valency. Actually, our combinatorial approach will make use of this correspondence.

Tail estimates for $X_n^{(k)}$ in the case of general maps have been also obtained by Gao and Wormald [11]. They are on the one hand weaker, but on the other hand more uniform (for $k \leq c \log n$, where $c > 0$ is sufficiently small). Moreover, Johannsen and the second author [13] studied, among others, the case of 2-connected planar maps. However, the central limit theorem was in both cases unknown. Note that our work does not answer the question for the 3-connected case; it seems that there is no (known) method for counting directly 3-connected planar maps that is similar to the presented methods for general and 2-connected planar maps. On the other hand, the expected number of vertices of a given degree for 3-connected planar maps was given in [1], see also [7].

Nevertheless, there are several classes of planar maps and *graphs*, where a central limit theorem holds. For example, Gao and Wormald [12] showed such a result for certain classes of triangulations. Moreover, for labeled outerplanar graphs and labeled series-parallel graphs this was shown by Drmota, Gimenéz and Noy [6]. This result was extended to so-called *subcritical graph classes* by Drmota, Fusy, Kang, Kraus and Rue [5], even in the unlabeled case.

For random planar graphs the current picture is unfortunately incomplete. It was shown by Drmota, Gimenez and Noy [7] and Panagiotou and Steger [15] that limiting degree distribution exists (and that there is at least a weak concentration result for $X_n^{(k)}$ [15]) but – at the moment – there is no central limit theorem although there is *no doubt* that a central limit theorem should hold. Indeed, there is a strong relation between planar maps and planar graphs. For example, by Whitney's theorem 3-connected planar maps and 3-connected planar graphs coincide. It is therefore very likely that several shape characteristics have (up to a scaling constant) the same limiting behavior. Hence, our results supports the conjecture that there is also a central limit theorem for planar graphs.

Plan of the proof The proof of Theorem 1 is divided into several steps, where we mostly concentrate on the case of general maps; the modifications that are needed for the 2-connected case are discussed in Section 6. The first step is to solve the counting problem in terms of generating functions. More precisely, we provide a quadratic catalytic equation for the ordinary generating function enumerating planar maps, where also the number of faces of a certain valency is taken into account, see Section 3. The basic idea is to generalize Tutte's classical approach [16] to counting planar maps. However, it turns out that the direct generalization leads to an equation, where the catalytic variable u has to be evaluated at $u = 1$ and $u = 0$. We overcome this problem by showing that the function (and their derivatives) when it is evaluated at $u = 0$ can be expressed analytically in terms of the variables and of the function evaluated at $u = 1$. This leads to a catalytic quadratic equation, see Lemma 2.

In principle this equation could be handled with the help of the so-called *quadratic method* by Brown [2]. However, this cannot be done explicitly due to the complexity of the equation. The next step is therefore to mimic the quadratic method analytically, which is done Section 4. The main problem is to characterize the kind of dominating singularities that are responsible for the asymptotic behavior of the parameters of interest. The main observation is that, under mild analytic conditions, there is a universal behavior for the dominant singularity, namely a critical exponent 3/2, as it has been observed in all known map counting problems; see Lemma 3.

Finally, in Section 5 it is shown that this method is applicable to determine the asymptotic behavior of the probability generating function of $X_n^{(k)}$. This leads directly to the central limit theorem with the help of Hwang's Quasi-Power-Theorem (Lemma 7, see [10] or [4]).

The case of 2-connected planar maps is discussed in Section 6. There, the combinatorial part is slightly more involved, but it finally leads to a polynomial equation that can be rewritten as a catalytic quadratic equation. Hence, the same analytic approach as developed for the connected case applies. Finally, in Section 7 we outline an extension of Theorem 1 for random planar maps with a given vertex-density.

3 Combinatorics

We use ordinary generating functions, where z marks edges and x non-root faces. The next statement is classical in the area of map enumeration and goes back to Tutte [16]. We include a proof for completeness, as we shall use similar considerations in Lemma 2 to determine a functional equation for the generating function that takes faces of degree k into account, too. Note that it is sufficient to study the valency distribution of faces, since it is the same as the degree distribution.

Lemma 1 *Let $M(z, x, u)$ be the ordinary generating function enumerating general maps with respect to edges and non-root faces, where additionally u*

marks the valency of the root face. Then

$$M(z,x,u) = 1 + zu^2 M(z,x,u)^2 + \frac{zxu}{1-u}(M(z,x,1) - uM(z,x,u)). \tag{2}$$

Proof We replicate the argument by Tutte [16], tailored to our specific purpose. A general map belongs to precisely one of the following three categories. First, it contains no edge, so that the corresponding generating function is the constant 1. Second, the root edge is a bridge, i.e., if removed, the map falls apart in two general maps. Clearly, the generating function enumerating such maps is given by $zu^2 M(z,x,u)^2$.

All maps that belong to none of the two categories above are obtained by taking a map and adding an edge that preserves its root node and "cuts across" the root face in some unambiguous fashion, i.e., so that the construction can be reverted. This operation results in $r+1$ new distinct maps with root-face degrees in $\{1,\dots,r+1\}$, and one edge and one non-root face more than the map we started with; see Figure 1 for an illustration. By putting everything together we infer that this construction translates the monomial u^r to

$$zx(u + u^2 + \cdots + u^{r+1}) = \frac{zxu(1-u^{r+1})}{1-u}.$$

Consequently, the maps in the last category are enumerated by $\frac{zxu}{1-u}(M(z,x,1) - uM(z,x,u))$. This completes the proof. □

In a second step we also take into account the number of (non-root) faces of valency k.

Lemma 2 *Let $k \geq 2$ be a fixed integer and let $M(z,x,w,u)$ be the ordinary generating function enumerating general maps with respect to edges and non-root faces, where additionally u marks the valency of the root face and w the non-root faces of valency k. Then*

$$\begin{aligned} M(z,x,w,u)&\left(1 - zx(w-1)u^{-k+2}\right) \\ &= 1 + zu^2 M(z,x,w,u)^2 \\ &+ \frac{zxu}{1-u}(M(z,x,w,1) - uM(z,x,w,u)) \\ &- zx(w-1)u^{-k+2} G\left(z,x,w,M(z,x,w,1),u\right) \end{aligned} \tag{3}$$

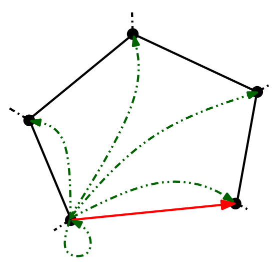

Fig. 1 All maps that can be obtained by "cutting across" the root face of size five.

where $G(z,x,w,y,u)$ is a polynomial of degree $k-2$ in u with coefficients that are analytic functions in (z,x,w,y) for $|z| \le 1/10$, $|x| \le 2$, $|w-1| \le 2^{1-k}$, and $|y| \le 2$.

Proof For the sake of brevity let us write $M(u)$ for $M(x,z,w,u)$ and $M_\ell = M_\ell(x,z,w)$ for the coefficient $[u^\ell]M(u)$. By using the same decomposition as in the proof of Lemma 1 we get

$$\begin{aligned} M(u) &= 1 + zu^2 M(u)^2 + \frac{zxu}{1-u}(M(1) - uM(u)) \\ &\quad + zx(w-1)u^{-k+2}\left(M(u) - \sum_{\ell=0}^{k-2} M_\ell u^\ell\right). \end{aligned} \tag{4}$$

The difference to the proof of Lemma 1 is that u^r is replaced now by

$$\begin{aligned} &zx(u + u^2 + \cdots + u^{r+1}) + zx(w-1)u^{r-k+2} \\ &= \frac{zxu(1-u^{r+1})}{1-u} + zx(w-1)u^{r-k+2} \end{aligned}$$

if $r \ge k-1$ (and by $zx(u + u^2 + \cdots + u^{r+1})$ if $r < k-1$).

What we show next is that $M_\ell = M_\ell(z,x,w)$ can be represented as an analytic function in z, x, w, and $M(1)$. Of course we have $M_0 = 1$. Moreover, by differentiating (4) with respect to u, noting that $M(u) - \sum_{\ell=0}^{k-2} M_\ell u^\ell$ can be written as $u^{k-1}N(u)$, where $N(u)$ is a power series in u, and by setting $u = 0$ we get $M_1 = zxM(1) + zx(w-1)M_{k-1}$. Furthermore for $\ell \ge 2$ we have

$$\frac{\partial^\ell}{\partial u^\ell} u^2 M(u)^2 \Big|_{u=0} = \ell! \sum_{j=0}^{\ell-2} M_j M_{\ell-2-j}$$

and

$$\frac{\partial^\ell}{\partial u^\ell} \frac{u}{1-u}(M(1) - uM(u)) \Big|_{u=0} = \ell! \left(M(1) - \sum_{j=0}^{\ell-2} M_j\right).$$

Finally, the ℓth derivative of $u^{-k+2}M_{\ge k-1}(u)$ evaluated at $u = 0$ equals $\ell! M_{k+\ell-2}$. Thus, we arrive at the relation

$$M_\ell = z\sum_{j=0}^{\ell-2} M_j M_{\ell-2-j} + zxM(1) - zx\sum_{j=0}^{\ell-2} M_j + zx(w-1)M_{k+\ell-2}. \tag{5}$$

We consider now $M(1)$ as an additional variable Y and the infinite system (5) as an equation for the sequence $(M_\ell)_{\ell \ge 1}$. By introducing a proper functional analytic frame it is easy to show (by a contraction argument) that the infinite system (5) has a unique and analytic solution.

We set $y_\ell = M_\ell 2^{-\ell}$ and consider the ℓ^1 norm $\|\mathbf{y}\|_1 = \sum_{\ell \ge 1} |y_\ell|$ of $\mathbf{y} = (y_\ell)_{\ell \ge 1}$. Furthermore, we define the mapping $\mathbf{T} : \ell^1(\mathbb{C}) \to \ell^1(\mathbb{C})$ by

$$(\mathbf{T}(\mathbf{y}))_\ell = \frac{z}{4}\sum_{j=0}^{\ell-2} y_j y_{\ell-2-j} + zxY - zx\sum_{j=0}^{\ell-2} y_j 2^{-\ell+j} + zx(w-1)2^{k-2} y_{k+\ell-2}.$$

Then a fixed point of $\mathbf{T}$ is directly related to a solution of the system (5), that is, if $\mathbf{y} = (y_\ell)_{\ell \ge 1}$ satisfies $\mathbf{y} = \mathbf{T}(\mathbf{y})$ then $M_\ell = 2^\ell y_\ell$ satisfies (5) with $Y = M(1)$.

By definition it follows directly that

$$\|\mathbf{T}(\mathbf{y})\|_1 \le \frac{|z|}{4}(1+\|\mathbf{y}\|_1)^2 + |zx|\left(\frac{1}{2} + |w-1|2^{k-2}\right)(1+\|\mathbf{y}\|_1) + |zx||Y|$$

and

$$\|\mathbf{T}(\mathbf{y}) - \mathbf{T}(\mathbf{z})\|_1 \le \left(\frac{|z|}{4}(2+\|\mathbf{y}\|_1+\|\mathbf{z}\|_1) + |zx|\left(\frac{1}{2} + |w-1|2^{k-2}\right)\right)\|\mathbf{y}-\mathbf{z}\|_1.$$

Let us assume that $|z| \le 1/10$, $|x| \le 2$, $|w-1| \le 2^{1-k}$, and $|Y| \le 2$. Furthermore, set $B = 15 - 4\sqrt{13} < 1$. Then we have that $\|\mathbf{y}\|_1 \le B$ implies $\|\mathbf{T}(\mathbf{y})\|_1 \le B$ and that $\mathbf{T}$ is a contraction on the set of sequences $\mathbf{y}$ with $\|\mathbf{y}\|_1 \le B$. Hence, by Banach's fixed point theorem the equation $\mathbf{y} = \mathbf{T}(\mathbf{y})$ has a unique solution. Moreover, if we start with $\mathbf{y}_0 = \mathbf{0}$ and $\mathbf{y}_{k+1} = \mathbf{T}(\mathbf{y}_k)$ (for $k \ge 0$) then $\mathbf{y}_k$ converges to $\mathbf{y}$ uniformly for $|z| \le 1/10$, $|x| \le 2$, $|w-1| \le 2^{1-k}$, and $|Y| \le 2$. By induction all components of $\mathbf{y}_k$ are polynomial in z, x, w, Y. Hence the components of the (uniform) limit $\mathbf{y}$ are analytic functions in x, z, w, Y, which also implies that M_ℓ, $\ell \ge 1$, can be written as an analytic function in z, x, w and $M(1)$. Finally setting

$$G(z,x,w,y,u) = \sum_{\ell=0}^{k-2} M_\ell u^\ell$$

completes the proof of the lemma. □

4 Analytic Quadratic Method

We first recall the principle of the *quadratic method*. Suppose that we fix $x = x_0 = 1$ and $w = w_0 = 1$ in Equation (3). Furthermore, we use the abbreviations $M(z,u) = M(z,x_0,w_0,u)$ and $y(z) = M(z,x_0,w_0,1)$. By completing the square (3) can be rewritten as

$$[G_1(z,u)M(z,u) + G_2(z,u)]^2 = H(z,y(z),u), \tag{6}$$

where the G_i and H are polynomials. More generally it is sufficient to assume that G_i and H are analytic function and it is also possible to assume that the G_i depend on $y(z)$, too:

$$[G_1(z,y(z),u)M(z,u) + G_2(z,y(z),u)]^2 = H(z,y(z),u). \tag{7}$$

For example, if $x = x_0 = 1$ and $w = w_0 = 1$ then

$$H = 4(u-1)u^3z^2y + u^4z^2 - 4u^4z + 6u^3z - 2u^2z + u^2 - 2u + 1. \tag{8}$$

The *quadratic method* consists in binding variables z and u, assuming that there exists a function $u(z)$ such that $H(z, y(z), u(z)) = 0$ identically. Because of the square in the left-hand side of (6), the derivative $H_u(z, y(z), u(z))$ with respect to u also vanishes. From the system of equations

$$H(z, y(z), u(z)) = 0, \quad H_u(z, y(z), u(z)) = 0. \tag{9}$$

one eliminates $y(z)$ to find $u(z)$, and then finds $y(z)$ from $H(z, y(z), u(z)) = 0$. Once we know $y(z) = M(z, 1)$, from Equation (3) we obtain $M(z, u)$.

If we carry out this program in this particular case of maps (recall that we set $x = x_0 = 1$ and $w = w_0 = 1$), we find that $u(z) = (5 - \sqrt{1-12z})/2(z+2)$ and

$$y(z) = \frac{18z - 1 + (1-12z)^{3/2}}{54z^2} = 1 + 2z + 9z^2 + 54z^3 + \cdots ,$$

from which we can deduce the explicit form for the numbers M_n. An explicit expression is obtained also for $M(z, u)$, which encodes completely the distribution of the degree of the root-face. Since planar maps are closed under duality, this is the same distribution as the degree of the root-vertex. It is an easy exercise to derive the limiting distribution encoded in (1).

The system (9) can be also used to detect the singularity $z_0 = 1/12$ of $y(z)$. We have to look at critical points (z_0, y_0, u_0) of the system (9). Its Jacobian

$$\begin{vmatrix} H_y & H_u \\ H_{uy} & H_{uu} \end{vmatrix} = \begin{vmatrix} H_y & 0 \\ H_{uy} & H_{uu} \end{vmatrix} = H_y H_{uu}$$

must vanish, that is, $H_y H_{uu} = 0$ at (z_0, y_0, u_0). It turns out that $H_{uu} = 0$ and $H_y \neq 0$ is the *correct choice* for map counting problems. The critical condition is then

$$H_{uu}(z_0, y_0, u_0) = 0.$$

This condition is easy to check, since we always work in the realm of algebraic functions and algebraic numbers. Actually the system $H = H_u = H_{uu} = 0$ has (usually) only finitely many solutions. For the running example we are using, we have $(z_0, y_0, u_0) = (1/12, 4/3, 6/5)$ and a simple calculation confirms that indeed $H_{uu}(1/12, 4/3, 6/5) = 0$.

The most interesting observation in this context is that $y(z)$ has a singularity of the kind $(1 - z/z_0)^{3/2}$. This behavior is typical in the context of planar map enumeration, and it turns out that there is also a universal analytic reason for this behavior. This was observed recently by Drmota and Noy [8], who proved the following lemma. For the reader's convenience we include a proof, since we will generalize it.[1]

Lemma 3 *Suppose that* z_0, y_0, u_0 *are complex numbers and that* $H(z, y, u)$ *is a function that is analytic at* (z_0, y_0, u_0) *and satisfies the properties*

$$H = 0, \quad H_u = 0, \quad H_{uu} = 0$$

[1] This proof is not included in the final version of the proceedings paper [8].

and

$$H_y \neq 0, \quad H_{uy} \neq 0, \quad H_{uuu} \neq 0, \quad H_z H_{uy} \neq H_y H_{uz}$$

for $(z, y, u) = (z_0, y_0, u_0)$. *Then the system of functional equations*

$$H(z, y(z), u(z)) = 0, \qquad H_u(z, y(z), u(z)) = 0 \tag{10}$$

has precisely two (local) solutions $u(z)$ *and* $y(z)$ *with* $u(z_0) = u_0$ *and* $y(z_0) = y_0$ *given by*

$$\begin{aligned} u(z) &= g_1(z) \pm g_2(z)\sqrt{1 - \frac{z}{z_0}}, \\ y(z) &= h_1(z) \pm h_2(z)\left(1 - \frac{z}{z_0}\right)^{3/2} \end{aligned} \tag{11}$$

in a neighborhood of z_0 *(except in the part, where* $1 - z/z_0 \in \mathbb{R}^-$*), where* $g_1(z)$, $g_2(z)$, $h_1(z)$, *and* $h_2(z)$ *are analytic functions at* z_0 *and satisfy*

$$\begin{aligned} g_1(z_0) &= u_0, \\ g_2(z_0) &= \sqrt{\frac{2z_0(H_y H_{uz} - H_z H_{uy})}{H_y H_{uuu}}} \neq 0, \\ h_1(z_0) &= y_0, \\ h_2(z_0) &= g_2(z_0)\frac{2z_0(H_y H_{uz} - H_z H_{uy}) \cdot B}{3H_{uuu} H_{uy}^2 H_y^2}, \end{aligned}$$

where

$$B = H_{uuu} H_{uy}^2 - H_y H_{uy} H_{uuuu} + 3H_y H_{uuy} H_{uuu}$$

and all derivatives of H *are evaluated at* $(z, y, u) = (z_0, y_0, u_0)$.

Proof We solve the system (10) by first considering the equation $H_u(z, y, u) = 0$, where z and u are considered as independent variables and $y = Y(z, u)$ is the unknown function. In a second step we solve the equation $H(z, Y(z, u), u) = 0$, where z is the independent variable. Then the solution $u = u(z)$ is the function that we are looking for and $y(z) = Y(z, u(z))$.

Since we assume that $H_{uy}(z_0, y_0, u_0) \neq 0$ it follows from the implicit function theorem that there exists a function $Y(z, u)$ with $Y(z_0, u_0) = y_0$ that is analytic at (z_0, u_0) and solves (locally) the equation $H_u(z, Y(z, u), u) = 0$. Observe that $(z, y, u) = (z_0, y_0, u_0)$ we have the relations

$$Y_u(z_0, u_0) = -\frac{H_{uu}}{H_{uy}} = 0, \qquad Y_{uu}(z_0, u_0) = -\frac{H_{uuu}}{H_{uy}} \neq 0,$$

and

$$Y_{uu}(z_0, u_0) = \frac{3H_{uuy} H_{uuu} - H_{uy} H_{uuuu}}{H_{uy}^2},$$

and

$$Y_z(z_0,u_0) = -\frac{H_{uz}}{H_{uy}}, \qquad Y_{uz}(z_0,u_0) = \frac{H_{uuy}H_{uz} - H_{uuz}H_{uy}}{H_{uy}^2}.$$

Next we set $F(z,u) = H(z,Y(z,u),u)$ and solve the equation $F(z,u) = 0$ for $u = u(z)$. By assumption we have

$$\begin{aligned} F(z_0,u_0) &= H(z_0,y_0,u_0) = 0,\\ F_u(z_0,u_0) &= H_y(z_0,y_0,u_0)Y_u(z_0,u_0) + H_u(z_0,y_0,u_0) = 0,\\ F_{uu}(z_0,u_0) &= H_y(z_0,y_0,u_0)Y_{uu}(z_0,u_0) \neq 0,\\ F_z(z_0,u_0) &= H_y(z_0,y_0,u_0)Y_z(z_0,u_0) + H_z(z_0,y_0,u_0)\\ &= \frac{(H_zH_{uy} - H_yH_{uz})(z_0,y_0,u_0)}{H_{uy}(z_0,y_0,u_0)} \neq 0. \end{aligned}$$

Hence the equation $F(z,u) = 0$ satisfies the assumptions of a classical lemma on the singular structure of the solution of a single equation (for example, it can be found in [4]). Thus, the only two solutions $u(z)$ have local expansions of the form

$$u(z) = g_1(z) \pm g_2(z)\sqrt{1 - \frac{z}{z_0}},$$

where $g_1(z)$ and $g_2(z)$ are analytic and satisfy $g_1(z_0) = u_0$ and

$$g_2(z_0) = \sqrt{\frac{2z_0F_z(u_0,z_0)}{F_{uu}(u_0,z_0)}} = \sqrt{\frac{2z_0(H_yH_{uz} - H_zH_{uy})}{H_yH_{uuu}}}.$$

From a simple calculation (by using Taylor's theorem and by comparing coefficients) we also obtain an expression for

$$g_1'(z_0) = \frac{F_zF_{uuu} - 3F_{uz}F_{uu}}{F_{uu}^2} = \frac{A}{3H_y^2H_{uuu}^2},$$

where

$$\begin{aligned} A = &-3H_y^2H_{uuu}H_{uuz} + 2H_{uuu}H_{uy}H_yH_{uz}\\ &- 2H_{uuu}H_{uy}^2H_z + 3H_yH_{uuy}H_{uuu}H_z\\ &+ H_y^2H_{uuuu}H_{uz} - H_yH_{uy}H_{uuuu}H_z. \end{aligned}$$

Finally we use the expansion of $u(z)$ to derive the local behavior of

$$\begin{aligned} y(z) = Y(z,u(z)) = y_0 &+ Y_z(z_0,u_0)(z-z_0) + \frac{Y_{uu}(z_0,u_0)}{2}(u(z)-u_0)^2\\ &+ Y_{uz}(z_0,u_0)(z-z_0)(u(z)-u_0)\\ &+ \frac{Y_{uuu}(z_0,u_0)}{6}(u(z)-u_0)^3 + O((z-z_0)^2). \end{aligned}$$

Note that the property $Y_u(z_0,u_0) = 0$ implies that $y(z)$ has no $\sqrt{1 - z/z_0}$ term in its expansion. Precise expressions for the coefficients (like $h_2(z_0)$) can be determined easily. □

For completeness, we check all the conditions in the statement for $H(z,y,u)$ as in (8), evaluated at the critical point $(z_0,y_0,u_0)=(1/12,4/3,6/5)$. In addition to H, we need $G_1=2(1-u)u^2z$. Then

$$G_1=-\frac{6}{125},\quad H_y=\frac{6}{625},\quad H_{uy}=\frac{9}{125},\quad H_{uuu}=-\frac{50}{9},$$
$$H_zH_{uy}-H_yH_{zu}=\frac{288}{15625},$$
$$H_{uuu}H_{uy}^2-H_yH_{uy}H_{uuuu}+3H_yH_{uuy}H_{uuu}=\frac{43}{625}.$$

Hence, Lemma 3 is applicable, and we obtain the singular behavior of $y(z)=M(z,1)$ around z_0, which is of the form $(1-z/z_0)^{3/2}$.

In general it is not immediately clear which critical point (z_0,y_0,u_0) is responsible for the dominating singularity on the radius of convergence of $y(z)=M(z,1)$ (when there are several ones). Nevertheless in the case of planar maps there is no doubt that we have chosen the *correct critical point*, since $y(z)=M(z,1)$ is known explicitly in this case. Moreover, the system $H=H_u=H_{uu}=0$ has only the two solutions $(1/12,4/3,6/5)$ and $(-1/4,-4,2)$, and the latter cannot be the sought critical point since the first two components are negative.

In addition to the above considerations we can vary the variables x and w (at least) in a (small) neighborhood of $x_0=1$ and $w_0=1$ without changing the kind of the dominating singularity. For this purpose we rewrite (3) as

$$\left(\overline{G}_1(z,x,w,u)M(z,x,w,u)+\overline{G}_2(z,x,w,u)\right)^2=\overline{H}(z,x,w,\overline{y}(z,x,w),u),$$

where $\overline{y}(z,x,w)$ abbreviates $M(z,x,w,1)$.[2] As above we will consider the system of equations

$$\overline{H}=\overline{H}_u=\overline{H}_{uu}=0, \tag{12}$$

and consider (if possible the solutions) $y_0(x,w)$, $z_0(x,w)$, $u_0(x,w)$.

Lemma 4 *There exist complex neighborhoods X and W of $x_0=1$ and $w_0=1$ such that the system (12) has a unique solution $y_0(x,w)$, $z_0(x,w)$, $u_0(x,w)$ with $y_0(x_0,w_0)=y_0$, $z_0(x_0,w_0)=z_0$, $u_0(x_0,w_0)=u_0$. Furthermore, the function $\overline{y}(z,x,w)=M(z,x,w,1)$ has a local representation of the form*

$$\overline{y}(z,x,w)=\overline{h}_1(z,x,w)+\overline{h}_2(z,x,w)\left(1-\frac{z}{z_0(x,w)}\right)^{3/2}, \tag{13}$$

where $\overline{h}_1(z,x,w)$ and $\overline{h}_2(z,x,w)$ are non-zero analytic functions. Moreover, there exists an analytic continuation to

$$\Delta=\{z:|z|<|z_0(x,w)|+\eta,|\arg(z/z_0(x,w)-1)|>\delta\},$$

for some real numbers $\eta>0$ and $0<\delta<\pi/2$.

[2] As in previous case the same approach applies if the functions $\overline{G}_i$ depend on $y(z,x,w)$, too.

Proof The first step is to show that the system of equation (12) has a proper solution in a neighborhood of $x_0 = 1$ and $w_0 = 1$. Actually we know that it has a solution for $x = 1$ and $w = 1$. Furthermore, note that the Jacobian

$$\begin{vmatrix} \overline{H}_y & \overline{H}_z & \overline{H}_u \\ \overline{H}_{uy} & \overline{H}_{uz} & \overline{H}_{uu} \\ \overline{H}_{uuy} & \overline{H}_{uuz} & \overline{H}_{uuu} \end{vmatrix} = \begin{vmatrix} \overline{H}_y & \overline{H}_z & 0 \\ \overline{H}_{uy} & \overline{H}_{uz} & 0 \\ \overline{H}_{uuy} & \overline{H}_{uuz} & \overline{H}_{uuu} \end{vmatrix}.$$

If we set $x = x_0 = 1$ and $w = w_0 = 1$, then this expression becomes

$$\overline{H}_{uuu}\left(\overline{H}_y\overline{H}_{uz} - \overline{H}_z\overline{H}_{uy}\right) \ne 0.$$

Hence, the implicit function theorem asserts that there exists a neighborhood X of $x_0 = 1$ and neighborhood W of $w_0 = 1$ such that the system (12) has an analytic solution $y_0(x,w)$, $z_0(x,w)$, $u_0(x,w)$ with $y_0(x_0,w_0) = y_0$, $z_0(x_0,w_0) = z_0$, $u_0(x_0,w_0) = u_0$. By continuity we can choose X and W also in a way that the non-zero conditions ($\overline{H}_y \ne 0$ etc.) of Lemma 3 are satisfied in X and W.

By checking the proof of Lemma 3 it follows that it generalizes to $x \in X$ and $w \in W$ so that all appearing functions are analytic in x and w. In particular, we use the fact that this is true for equations of the form $F(z,x,w,u) = 0$, where we observe that the solution $u = u(z,x,w)$ can be locally represented as

$$u(z,x,w) = g_1(z,x,w) \pm g_2(z,x,w)\sqrt{1 - \frac{z}{z(x,w)}}$$

with analytic functions $g_1(z,x,w)$ and $g_2(z,x,w)$ (for details see [4]). This leads to (13), where we have chosen the "+" sign since $\overline{y}$ has non-negative coefficients.

Finally we have to check that $\overline{y}(z,x,w)$ has a proper analytic continuation to the region Δ. For this purpose we have to study $y(z)$ not only around $z = z_0$. Since we know $y(z)$ and $u(z)$ explicitly it is easy to check that $H_{uu}(z,y(z),u(z)) \ne 0$ if $z \ne z_0$ and $H_y(z,y(z),u(z)) \ne 0$ if $z \ne 0$. Hence the Jacobian

$$\begin{vmatrix} H_y & H_u \\ H_{uy} & H_{uu} \end{vmatrix} = \begin{vmatrix} H_y & 0 \\ H_{uy} & H_{uu} \end{vmatrix} = H_y H_{uu}$$

is non-zero for $|z| = z_0$ but $z \ne z_0$. Now suppose that x and w vary in properly chosen neighborhoods of $x_0 = 1$ and $w_0 = 1$. If $|z - z_0(x,w)| < \varepsilon$ we use the local representation (13) and obtain an analytic continuation. If $|z|$ is close to $|z_0(x,w)|$ but $|z - z_0(x,w)| \ge \varepsilon$ we obtain by continuity that the Jacobian $\overline{H}_{uu}\overline{H}_y$ stays non-zero (if x and w are sufficiently close to $x_0 = 1$ and $w_0 = 1$). Hence, by the implicit function theorem $y(z,x,w)$ and $u(z,x,w)$ can be analytically continued. By compactness it is sufficient to continue $y(z,x,w)$ and $u(z,x,w)$ only at finitely many points. Hence, there exist $\delta > 0$ and $\eta > 0$ such that $y(z,x,w)$ continues analytically to Δ. □

5 The Central Limit Theorem

It is now easy to complete the proof of Theorem 1. For this purpose we use the following transfer principle by Flajolet and Odlyzko [9] (see also the book of Flajolet and Sedgewick [10] and many references therein).

Lemma 5 *Let W be a compact set and assume that there exist functions $C(w)$, $z_0(w)$, $\alpha(w)$, $\beta(w)$ such that $\beta(w) > \Re(\alpha(w))$, $\alpha(w) \in \mathbb{C} \setminus \mathbb{N}_0$ and $z_0(w) > 0$ for all $w \in W$ with the following property. Suppose that $f(z;w)$ is a power series in z and a parameter $w \in W$ such that there is an expansion of the form*

$$f(z;w) = C(w)\left(1 - \frac{z}{z_0(w)}\right)^{\alpha(w)} + O\left(\left(1 - \frac{z}{z_0(w)}\right)^{\beta(w)}\right)$$

that is uniform for $w \in W$ and $z \in \Delta$, where

$$\Delta = \{z : |z| < |z_0(w)| + \eta, |\arg(z/z_0(w) - 1)| > \delta\},$$

for some real numbers $\eta > 0$ and $0 < \delta < \pi/2$. Then, uniformly for $w \in W$

$$[z^n]f(z;w) = (1 + o(1)) \cdot C(w)\frac{n^{-\alpha(w)-1}}{\Gamma(-\alpha(w))} z_0(w)^{-n}.$$

With the help of this lemma we can prove the following property.

Lemma 6 *Let $k \geq 2$ and $X_n^{(k)}$ as in Theorem 1. Then there exists a neighbourhood W of $w_0 = 1$ such that*

$$\mathbb{E}\left(w^{X_n^{(k)}}\right) = (1 + o(1))C(w)\left(\frac{z_0}{z_0(x_0, w)}\right)^n$$

uniformly for $w \in W$, where $C(w)$ is a non-zero analytic function.

Proof We apply Lemma 5 for $f(z;w) = \overline{y}(z, x_0, w) - \overline{h}_1(z, x_0, w)$, which can be rewritten as

$$f(z;w) = \overline{h}_2(z_0(x_0, w), x_0, w)\left(1 - \frac{z}{z_0(x_0, w)}\right)^{3/2} + O\left(\left(1 - \frac{z}{z_0(x_0, w)}\right)^{5/2}\right).$$

The additive term $\overline{h}_1(z, x_0, w)$ is analytic in a larger region and, thus, provides a negligible contribution. Consequently we obtain

$$[z^n]\overline{y}(z, x_0, w) = (1 + o(1))\overline{h}_2(z_0(x_0, w), x_0, w)\frac{n^{-5/2}}{\Gamma(-3/2)} z_0(x_0, w)^{-n}.$$

and also

$$\mathbb{E}\left(w^{X_n^{(k)}}\right) = \frac{[z^n]\overline{y}(z, x_0, w)}{[z^n]\overline{y}(z, x_0, 1)} = (1 + o(1))\frac{\overline{h}_2(z_0(x_0, w), x_0, w)}{\overline{h}_2(z, x_0, 1)}\left(\frac{z_0}{z_0(x_0, w)}\right)^n.$$

□

The final step of the proof is to apply Hwang's Quasi-Power-Theorem (see [10] or [4]). The central limit theorem and the exponential tail estimates follow immediately. For the reader's convenience we state a proper version of it.

Lemma 7 (Quasi-Power-Theorem) *Let X_n be a sequence of random variables with the property that*

$$\mathbb{E}\, w^{X_n} = e^{\lambda_n \cdot A(w)+B(w)} \left(1+O\left(\frac{1}{\phi_n}\right)\right) \tag{14}$$

holds uniformly in a complex neighborhood of $w=1$, where λ_n and ϕ_n are sequences of positive real numbers with $\lambda_n \to \infty$ and $\phi_n \to \infty$, and $A(w)$ and $B(w)$ are analytic functions in this neighbourhood of $w=1$ with $A(1) = B(1) = 0$. Then X_n satisfies a central limit theorem of the form

$$\frac{1}{\sqrt{\lambda_n}}\left(X_n - \mathbb{E}\, X_n\right) \to N\left(0,\sigma^2\right) \tag{15}$$

and we have

$$\mathbb{E}\, X_n = \lambda_n \mu + O\left(1+\lambda_n/\phi_n\right)$$

and

$$\mathbb{V}\mathrm{ar}\, X_n = \lambda_n \sigma^2 + O\left(\left(1+\lambda_n/\phi_n\right)^2\right),$$

where $\mu = A'(1)$ and $\sigma^2 = A''(1)+A'(1)$. Finally, if we additionally assume that $\lambda_n = \phi_n$ there exist positive constants c_1, c_2, c_3 such that

$$\mathbb{P}\left\{\|X_n - \mathbb{E}\, X_n\| \ge \varepsilon\sqrt{\lambda_n}\right\} \le c_1 e^{-c_2\varepsilon^2} \tag{16}$$

uniformly for $\varepsilon \le c_3\sqrt{\lambda_n}$.

6 Two-Connected Planar Maps

The analysis of 2-connected planar maps is similar to that of connected ones. However, the combinatorial part is slightly more involved. We first recall a method by Brown and Tutte [3] to obtain a functional equation for the counting generating function, where the quadratic method can be applied. The second step is to extend this procedure in order to take faces of degree k into account. The analytic part is then (almost) a direct application of the methods developed above.

The following lemma goes back to Brown and Tutte [3] and provides a proper functional equation representation of the trivariate generating function for 2-connected maps. Since we will generalize this result in Lemma 9 we first give a proof.

Lemma 8 *Let $B(z,x,u)$ be the ordinary generating function enumerating 2-connected maps with respect to edges and non-root faces, where additionally u marks the valency of the root face. Then*

$$\begin{aligned} B(z,x,u)^2 + ((1-u)(1-zu) + zxu - uB(z,x,1))B(z,x,u) \\ -zu^2x(z(1-u)+B(z,x,1)) = 0. \end{aligned} \tag{17}$$

Proof Set $B_\ell(z,x) = [u^\ell]B(z,x,u)$ and consider the rooted edge e of the root face of the 2-connected map. Of course, e belongs to the root face and to another face f. Denote by γ_1 the remaining edges on the root face and by γ_2 the remaining edges on the face f. If one deletes e, then $M \setminus e$ might have $\ell \geq 0$ cut-vertices $a_1, \ldots, a_\ell$ that are exactly the common points of γ_1 and γ_2 (except from the two vertices incident to e, see Figure 2). A careful look at

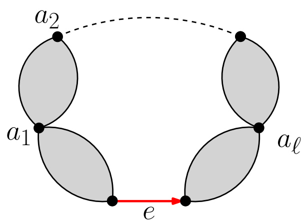

Fig. 2 The decomposition of 2-connected maps.

this recursive structure leads to the relation

$$B(z,x,u) = zxu \sum_{\ell \geq 0} \left(\sum_{m \geq 2} B_m(z,x)(u + u^2 + \cdots u^{m-1}) + zu \right)^{\ell+1}.$$

Summing up the geometric series yields

$$zxu \frac{\sum_{m\geq 2} B_m(z,x)(u+u^2+\cdots u^{m-1}) + zu}{1 - \sum_{m\geq 2} B_m(z,x)(u+u^2+\cdots u^{m-1}) - zu} = zxu \frac{\frac{uB(1)-B(u)}{1-u} + zu}{1 - \frac{uB(1)-B(u)}{1-u} - zu}$$

that rewrites to (17). □

It is clear that the equation (17) can be handled with the help of the (analytic) quadratic method. We just have to set $x = 1$ and to choose $(z_0, y_0, u_0) = (4/27, 1/27, 3/2)$, where the corresponding function H is given by

$$H = zu^2(z(1-u)+y) + \frac{1}{4}((1-u)(1-zu) + zu - uy)^2.$$

Next we adapt the previous lemma in order to take into account the number of faces of degree k.

Lemma 9 *Let $k \ge 2$ be a fixed integer and let $B(z,x,w,u)$ be the ordinary generating function enumerating general maps with respect to edges and non-root faces, where additionally u marks the valency of the root face and w the number of non-root faces of valency k. Then*

$$\begin{aligned} B(z,x,w,u)^2 &+ ((1-u)(1-zu) + zxu - uB(z,x,1))B(z,x,w,u) \\ &- zu^2x(z(1-u) + B(z,x,w,1)) \qquad (18) \\ &= zx(w-1)u^{-k+2} G\left(z,x,w,B(z,x,w,u),B(z,x,w,1),u\right), \end{aligned}$$

where $G(z,x,w,y_1,y_2,u)$ is a polynomial of degree $k-2$ in u and of degree k in y_1 with coefficients that are analytic functions in (z,x,w,y_2) for $|z| \le 1/6$, $|x| \le 2$, $|w-1| \le C8^{-k}$, and $|y_2| \le 2$ (where $C > 0$ is a sufficiently small constant).

Proof For the sake of brevity let us write $B(u)$ (or just B) for $B(x,z,w,u)$ and $B_\ell = B_\ell(x,z,w)$ for the coefficient $[u^\ell]B(u)$. By using the same decomposition as in the proof of Lemma 8 we obtain

$$\begin{aligned} B(z,x,w,u) &= zxu\frac{\sum_{m\ge 2} B_m(z,x,w)(u+u^2+\cdots u^{m-1}) + zu}{1-\sum_{m\ge 2} B_m(z,x,w)(u+u^2+\cdots u^{m-1}) - zu} \qquad (19) \\ &+ (w-1)[v^k]zxuv\frac{\sum_{m\ge 2} B_m(z,x,w)(uv^{m-1}+u^2v^{m-2}+\cdots u^{m-1}v) + zuv}{1-\sum_{m\ge 2} B_m(z,x,w)(uv^{m-1}+u^2v^{m-2}+\cdots u^{m-1}v) - zuv} \\ &= zxu\frac{\frac{uB(1)-B(u)}{1-u} + zu}{1-\frac{uB(1)-B(u)}{1-u} - zu} + (w-1)P(z,x,u,D_1(u),D_2(u),\ldots,D_{k-1}(u)), \end{aligned}$$

where P is an appropriate polynomial and $D_\ell(u)$ abbreviates

$$D_\ell(u) = \frac{1}{u^\ell}\left(B(u) - \sum_{j\le \ell} B_j u^\ell\right) = \sum_{j>0} B_{j+\ell}u^j. \qquad (20)$$

Note that the (total) degree of P in $D_1,\ldots,D_{k-1}$ equals $k-1$. For example, for $k=4$ we have

$$P(z,x,u,D_1,D_2,D_3) = xzu\left(D_3 + 2D_2(D_1+zu) + (D_1+zu)^3\right).$$

From this it is possible to obtain explicitly the coefficient of u^ℓ. For example, for $k=4$ we have

$$\begin{aligned} [u^\ell]P(z,x,u,D_1(u),D_2(u),\ldots,D_{k-1}(u)) &= zxB_{\ell+2} + 2zx\sum_{\ell_1+\ell_2=\ell-1} B_{\ell_1+2}B_{\ell_2+1} \\ &+ 2xz^2B_{\ell+1} + xz\sum_{\ell_1+\ell_2+\ell_3=\ell-1} B_{\ell_1+1}B_{\ell_2+1}B_{\ell_3+1} \\ &+ 3xz^2\sum_{\ell_1+\ell_2=\ell-2} B_{\ell_1+1}B_{\ell_2+1} + 3xz^3B_{\ell-2} + xz^4\delta_{\ell,4} \end{aligned}$$

where $\delta_{i,j}$ denotes the Kronecker delta. Next we rewrite the above equation to

$$B = -B^2 + (u(1-xz) + zu^2 - zu^3 + uB(1))B + xzu^2B(1) + xz^2u^2 - yz^2u^3 + (w-1)P(z,x,u,D_1(u),\ldots,D_{k-1}(u))(1-u(1+z+B(1)) + zu^2 + B).$$

As indicated above it is easy to compute the ℓ-th coefficient

$$Q_\ell = [u^\ell]P(z,x,u,D_1(u),\ldots,D_{k-1}(u))(1-u(1+z+B(1)) + zu^2 + B)$$

as a polynomial in $x, z, B(1)$ and $B_{\ell+j}^{(r)}$, where j is contained in $\{-(k-1), -(k-2), \ldots, k-2, k-1\}$ and r is contained in $\{1,2,\ldots,k-1\}$; here $B_\ell^{(r)}$ denotes the r-fold convolution of the sequence B_ℓ.

As in the proof of Lemma 1 the above relation gives rise to an infinite system of equations for B_j:

$$B_\ell = -B_\ell^{(2)} + (1-xz+B(1))B_{\ell-1} + zB_{\ell-2} - zB_{\ell-3} + (xzB(1) + xz^2)\delta_{\ell,2} - yz^2\delta_{\ell,3} + (w-1)Q_\ell, \tag{21}$$

Following the concept of the proof of Lemma 2 we consider $B(1)$ as an additional variable Y and the infinite system (21) as an equation for the sequence $(B_\ell)_{\ell\ge1}$ and show (again by a contraction argument) that the infinite system (21) has a unique and analytic solution. We set $y_\ell = B_\ell 8^{1-\ell}$ and consider the ℓ^1 norm. Then the system (21) rewrites to a fixed point equation that turns out to be a contraction on the set of sequences $\mathbf{y} = (y_\ell)_{\ell\ge1}$ for $\|\mathbf{y}\|_1 \le 2$, $|x| \le 2$, $|Y| \le 2$, $|z| \le \frac{1}{6}$ and $|w-1| \le C8^{-k}$ (for a sufficiently small $C > 0$):

$$y_\ell = -\frac{1}{8}y_\ell^{(2)} + \frac{1}{8}(1-xz+Y)y_{\ell-1} + \frac{z}{64}y_{\ell-2} - \frac{z}{256}y_{\ell-3} + \frac{1}{8}(xzY + xz^2)\delta_{\ell,2} - \frac{1}{64}yz^2\delta_{\ell,3} + (w-1)\tilde{Q}_\ell,$$

where $\tilde{Q}_\ell$ is a polynomial in x, z, Y and $y_{\ell+j}^{(r)}$, j is contained in $\{-(k-1), -(k-2), \ldots, k-2, k-1\}$ and r is contained in $\{1,2,\ldots,k-1\}$. Hence, we can rewrite (19) in the form (18), where $D_\ell(u)$ is as in (20). $\square$

Now we are almost in the same situation as in the case of connected planar maps. The main difference is that the function G on the right hand side depends also polynomially on $B = B(u)$. This means the final equation is not a quadratic polynomial in B any more, except if $w = 1$. Nevertheless it is just a *pertubation* of a quadratic equation. Actually we can rewrite it as

$$(B - G_2(z,x,w,u))^2 = H_1(z,x,w,B(1),u) + (w-1)H_2(z,x,w,B,B(1),u)$$

for proper (analytic) functions G_2, H_1, H_2. We consider this equation locally at $(z,x,w,u) = (4/27, 1, 1, 3/2)$, $B(1) = 1/27$, and $B = G_2(4/27, 1, 1, 3/2) = 1/9$.

Note that the right hand side vanishes at this point and that all derivatives with respect to B are zero, too. Furthermore, the left hand side as well as the the first derivative with respect to B are equal to zero whereas the second derivative is non-zero. Consequently, by the Weierstrass preparation theorem the above equation is locally equivalent to an equation of the form

$$(B - \overline{G}_2(z,x,w,B(1),u))^2 = \overline{H}(z,x,w,B(1),u),$$

where the right hand side $\overline{H}$ satisfies (again) the necessary conditions $\overline{H} = \overline{H}_u = \overline{H}_{uu} = 0$ (and the other regularity conditions at the critical point).

Now we are precisely in the same situation as in connected case. We can apply the analytic quadratic method to obtain asymptotics for $\mathbb{E}(w^{X_n^{(k)}})$. Finally the Quasi-Power-Theorem applies to obtain the central limit theorem (and the corresponding exponential tail estimates).

7 Fixed Vertex Density

In this final section we indicate that Theorem 1 can be generalized to the situation that the numbers m of vertices and n of edges has a fixed ratio μ.

Theorem 2 *Let $k \in \mathbb{N}$. Furthermore, let $\mu \in (0,1)$ be a fixed number and let (m_n) be a sequence of positive integers with $\lim_{n\to\infty} m_n/n = \mu$. Then, the number $X_{n,m_n}^{(k)}$ of vertices of degree k in a random planar map with n edges and m_n vertices satisfies a central limit law, i.e.,*

$$\frac{X_{n,m_n}^{(k)} - \mathbb{E}[X_{n,m_n}^{(k)}]}{(\mathrm{Var}[X_{n,m_n}^{(k)}])^{1/2}} \to N(0,1),$$

where $\mathbb{E}[X_{n,m_n}^{(k)}] = \mu_k n + O(1)$ and $\mathrm{Var}[X_{n,m_n}^{(k)}] = \sigma_k^2 n + O(1)$, and $\mu_k = \mu_k(\mu), \sigma_k = \sigma_k(\mu) > 0$ are computable constants that depend on μ. Moreover, $X_{n,m_n}^{(k)}$ has exponential tails, i.e., there is an $\varepsilon_0 > 0$ and a $c_k > 0$ (depending on μ) such that for any $0 < \varepsilon < \varepsilon_0$

$$\Pr\left[|X_{n,m_n}^{(k)} - \mathbb{E}[X_{n,m_n}^{(k)}]| \ge \varepsilon \mathbb{E}[X_{n,m_n}^{(k)}]\right] \le e^{-\varepsilon^2 c_k n}.$$

The proof is an extension of the proof of Theorem 1. For the sake of simplicity we only discuss the case of connected planar maps. The proof of for the 2-connected case follows exactly the same lines.

First, observe that the variable x takes into account the number of vertices. In the proof of Theorem 1 we have set $x = 1$ in oder to discount the number of vertices. However, we can use it as a parameter. In particular, if x is positive (or sufficiently close to the positive real axis), we observe that the generating function $M(z,x,w,1)$ can be represented as

$$M(z,x,w,1) = h_1(z,x,w) + h_2(z,x,w)\left(1 - \frac{z}{z(x,w)}\right)^{3/2}$$

with analytic functions h_1, h_2 and z.

Let us first discuss the counting problem of planar maps with n edges and m vertices, that is, we set $w = 1$. We will consider x as a parameter; the case $x = 1$ corresponds to the classical map enumeration problem discussed in Section 4. By completing the square in (2) we obtain the equation

$$[G_1(z,x,u)M(z,x,u) + G_2(z,x,u)]^2 = H(z,x,M(z,x,1),u),$$

where, by abbreviating $M(z,x,1) = M$,

$$H = 4(u-1)u^3z^2xM - 4\,zu^2 + 8zu^3 - 4zu^4 - 2u + 2xzu^2 + u^2 - 2xzu^3 + x^2z^2u^4 + 1,$$

and G_1 and G_2 are also given explicitly. We are now in a situation where we can apply Lemma 3. Consider the solutions $(z_0, y_0, u_0) = (z_0(x), y_0(x), u_0(x))$ to the system

$$H(z_0,x,y_0,u_0) = H_u(z_0,x,y_0,u_0) = H_{uu}(z_0,x,y_0,u_0) = 0.$$

Using resultants this system can be solved, and a rational parametrization of the solution can be computed. Set

$$x(t) = \frac{93312t^3}{(27t+8)(8-9t)^3} \qquad (22)$$

and

$$z(t) = \frac{(27t+8)(8-9t)^3}{4(243t^2+64)^2}, \quad y(t) = \frac{243t^2+64}{108t^2+32}, \quad u(t) = \frac{243t^2+64}{81t^2+64}. \qquad (23)$$

If t is chosen such that $x(t) = x$ then $z_0(x) = z(t)$, $y_0(x) = y(t)$ and $u_0(x) = u(t)$. Moreover, the higher order conditions in Lemma 3 can be easily verified with this explicit parametrization.

With those facts at hand and a similar reasoning as in the proof of Lemma 4 it follows that we can apply the transfer principle, c.f. Lemma 5, to obtain

$$[z^n]\, M(z,x,1,1) = (1+o(1))\, \frac{h_2(z_0(x),x,1)}{\Gamma(-3/2)} n^{-5/2} z_0(x)^{-n}.$$

This bound holds uniformly for x is a compact set that is sufficiently close to the positive real axis. Next, by Cauchy's formula

$$[z^n x^m]\, M(z,x,1,1) = \frac{1}{2\pi i} \int_{|x|=x_0} [z^n]\, M(z,x,1,1) \frac{dx}{x^{m+1}},$$

where we choose the radius x_0 of the circle of integration in a way that

$$\frac{m}{n} = -\frac{x_0 \frac{\partial}{\partial x} z_0(x_0)}{z_0(x_0)}.$$

Note that x_0 is just the saddle point of the function $x \mapsto z_0(x)^{-n} x^{-m}$. Hence, it follows that

$$[z^n x^m]\, M(z,x,1,1) \sim C(x_0) n^{-3} z_0(x_0)^{-n} x_0^{-m},$$

(where we have to use proper upper bounds for $[z^n]M(z,x,1,1)$ if x is away from the real axis, but they are standard). Furthermore, since

$$-\frac{x_0 \frac{\partial}{\partial x} z(x_0,1)}{z(x_0,1)} = -\frac{x_0(t)z_0'(t)}{z_0(t)x_0'(t)} = \frac{5184t^2}{(9t+8)(243t^2+64)}$$

it is easy to relate the ratio $m/n \in (0,1)$ bijectively with the parameter $t \in (0,8/9)$. For example, if $t = 8/27$ then $x_0(t) = 1$ and $z_0(t) = 1/12$.

Finally, we can use a similar procedure if w is close to 1 and we obtain

$$\begin{aligned} \mathbb{E}\left(w^{X_{n,m}^{(k)}}\right) &= \frac{[z^n x^m]M(z,x,1,w)}{[z^n x^m]M(z,x,1,1)} \\ &= (1+o(1))\frac{\overline{h}_2(z_0(x_0,w),x_0,w)}{\overline{h}_2(z_0(x_0),x_0,1)}\left(\frac{z_0(x_0,1)}{z_0(x_0,w)}\right)^n. \end{aligned}$$

Again, by Hwang's Quasi-Power-Theorem the central limit theorem (and also the exponential tail estimates) follow. We just add that all these estimates are uniform if the ratio m/n varies is a compact interval that is contained in $(0,1)$ (compare with Lemma 5). Thus we have also covered the situation where $m_n/n \to \mu$ for a fixed constant $\mu \in (0,1)$.

References

1. E. A. Bender and E. R. Canfield, Face sizes of 3-polytopes, *Journal of Combinatorial Theory, Series B*, **46**, 58–65, 1989.
2. W. G. Brown, On the existence of square roots in certain rings of power series, *Math. Annalen* **158**, 82–89 (1965).
3. W. G. Brown and W. T. Tutte, On the enumeration of rooted non-separable planar maps, *Canadian Journal of Mathematics*, **16**, 572–577, 1964.
4. M. Drmota, *Random Trees: an Interplay between Combinatorics and Probability*, Springer, Wien (2009).
5. M. Drmota, E. Fusy, M. Kang, V. Kraus, and J. Rue, Asymptotic study of subcritical graph classes, *SIAM Journal of Discrete Mathematics*, **25**, 1615–1651 (2011).
6. M. Drmota, O. Gimenez and M. Noy, Vertices of given degree in series-parallel graphs, *Random Structures & Algorithms*, **36**, 273–314 (2010),
7. M. Drmota, O. Gimenez and M. Noy, Degree distribution in random planar graphs, *Journal of Combinatorial Theory, Series A*, **118**, 2102–2130 (2011).
8. M. Drmota and M. Noy, Universal exponents and tail estimates in the enumeration of planar maps, *Proceedings of Eurocomb 2011*, Electronic Notes in Discrete Math, **38**, 309–317 (2012).
9. P. Flajolet and A. Odlyzko, Singularity analysis of generating functions, *SIAM Journal of Discrete Mathematics* **3**, 216–240 (1990).
10. F. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge (2009).
11. Z. Gao and N. Wormald, Sharp concentration of the number of submaps in random planar triangulations, *Combinatorica*, **23**(3), 467–486 (2003).
12. Z. Gao and N. Wormald, Asymptotic normality determined by high moments, and submap counts of random maps, *Probability Theory and Related Fields*, **130**(3), 368–376 (2004).
13. D. Johannsen and K. Panagiotou, Vertices of Degree k in Random Maps, *Proceedings of the 21st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '10)*, 1436–1447 (2010).

14. V. A. Liskovets, A pattern of asymptotic vertex valency distributions in planar maps, *Journal of Combinatorial Theory, Series B* **75**, 116–133 (1999).
15. K. Panagiotou and A. Steger, On the Degree Distribution of Random Planar Graphs, *Proceedings of the 22nd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '11)*, 1198–1210 (2011).
16. W. T. Tutte, A census of planar maps, *Canadian Journal of Mathematics* **15**, 249–271 (1963).