THE INSTABILITY TIME DISTRIBUTION BEHAVIOUR OF SLOTTED ALOHA

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Abstract. This paper deals with a quantification of the instability behaviour of the slotted multiuser communication network system ALOHA. In [DS1] U. Schmid and the author could quantify the expected value $\mathbf{E}(Y)$ of the *instability time* $Y = \max\{n : X_n = 0\}$, where X_n denotes the backlog after n slots. Here it is shown that Y is approximately exponentially distributed. Moreover uniform asymptotic expansions for $y_m = \mathbf{P}[Y = m]$ and for the moments $M_m = \mathbf{E}(Y^m)$ are given.

1. Introduction

More than 20 years ago N. Abramson developed at the university of Hawaii the famous multiuser communication network system, the socalled ALOHA protocol. Since then many generalizations and improvements of such collosion resolution algorithms such as ETHERNET have been created and widely discussed. (See [Ga] for an overview.)

A well-known variety of original ALOHA is the slotted ALOHA algorithm which works as follows. An infinite number of stations share a single communication chanel. All stations are synchronized and the time axis is slotted so that one packet can be successfully transmitted in the slot $(n, n+1)$, $n \ge 1$. Packets for transmission arrive at a Poisson stream of rate $\lambda > 0$. If they arrive during the interval $(n-1, n)$ they are first transmitted in the slot $(n, n+1)$. When a station transmits a packet it learns at the end of the slot whether the packet has been successfully transmitted or whether a collsion has occurred. We will call the collection of packets that have collided and await transmission the backlog. Backlogged packets are independently retransmitted with probability $p \in (0,1)$. This means that the retransmission delay following an unsuccessful attempt is geometrically distributed with parameter $1-p$. Therefore the (slotted) ALOHA can be described as a Markov process $(X_n)_{n\ge 0}$ with $X_0 = 0$, where $X_n \in \{0, 1, 2, \dots\}$ represents the backlog after n slots. The transition stationary probabilities $p_{n,m} = \mathbf{P}[X_{l+1} = m | X_l = n]$ $(n, m \ge 0)$ are given by

$$p_{0,0} = e^{-\lambda}(1+\lambda), \quad p_{0,1} = 0, \quad p_{0,j} = e^{-\lambda}\frac{\lambda^j}{j!} \quad \text{for } j \ge 2 \tag{1.1}$$

and by

$$
\begin{aligned}
p_{n,n-j} &= 0 \qquad \text{for } j \ge 2 \\
p_{n,n-1} &= e^{-\lambda} np(1-p)^{n-1} \\
p_{n,n} &= e^{-\lambda}(1 - np(1-p)^{n-1} + \lambda(1-p)^n) \\
p_{n,n+1} &= e^{-\lambda}\lambda(1-(1-p)^n) \\
p_{n,n+j} &= e^{-\lambda}\frac{\lambda^j}{j!} \quad \text{for } j \ge 2.
\end{aligned} \tag{1.2}
$$

for $n > 0$.

The interesting (and at the first moment surprising) point is that this transmission algorithms is unstable. Moreover, with probability one only finitely many packets are successfully transmitted (see [KM]). But only few is known about the expected number of successfully transmitted packets or about the expected time of the last slot transmitting successfully a packet.

One method is to use large deviation techniques (cf. [GW] or [CFM]). There it is shown that the system operates near a stable point n_0 for a long period L of time, after that the system begins to follow an almost deterministic path out of the safe region $n \le n_1$, where $n_1 > n_0$ denotes the critical backlog. (This means that the drift $D_n = \mathbf{E}(X_{l+1} - X_l \,|\, X_l = n) > 0$ for $n \ge n_1$.) In [GW] it is shown that L is approximately exponentially distributed with a mean of

$$
\overline{N}(p,\lambda) = \mathbf{E}(L) = \exp\left(\frac{c(\lambda)}{p} + o\left(\frac{1}{p}\right)\right) \tag{1.3}
$$

for $p \to 0$, where $c(\lambda)$ is expressible via a certain integral and can be asymptotically expanded to

$$
c(\lambda) \sim \frac{1}{2}\log^2\frac{1}{\lambda} + \log\frac{1}{\lambda}\log\log\frac{1}{\lambda} \tag{1.4}
$$

for $\lambda \to 0$ (see [DS2]). However, the major deficit of large deviation techniques lies in the fact that some of the results have not been completely proven yet.

In [DS2] U. Schmid and the author use a completely different method (without any large deviation thechnique) to expand the expected value $N(p,\lambda) = \mathbf{E}(Y)$ of the last time $Y = \max\{n : X_n = 0\}$ with backlog $X_n = 0$. It is shown that

$$
N(p,\lambda) = \exp\left(\frac{1}{\log\frac{1}{1-p}}\left(\frac{1}{2}\log^2\frac{1}{\lambda} + \log\frac{1}{\lambda}\log\log\frac{1}{\lambda} + \mathcal{O}\left(\log\frac{1}{\lambda}\right)\right)\right) \tag{1.5}
$$

uniformly for $p \to 0$, $\lambda \to 0$. Note that (1.3) (combined with (1.4)) and (1.5) are quite similar although the underlying random variables L and Y are apparently different. Nevertheless both quantify the instability behaviour of slotted ALOHA.

The purpose of this paper is get more information about the distribution behaviour of $Y = \max\{n : X_n = 0\}$. In the following the random variable Y will be called *instablity time.* For simplicity we will write F for a factor that is bounded by $\exp\left(\frac{C}{p}\log\frac{1}{\lambda}\right)$ for some constant $C > 0$.

Theorem. *The instability time $Y = \max\{n : X_n = 0\}$ of the slotted ALOHA backlog process $(X_n)_{n\ge 0}$ is approximately exponentially distributed with expected* *value $N = N(p,\lambda)$. Moreover we can estimate the probablities $y_m = \mathbf{P}[Y = m]$ by*

$$\begin{aligned} y_m &= \frac{1-Q_0(1)}{\phi Q_0'(\phi)}\phi^{-m} + \mathcal{O}\left(\frac{F}{N}\left(1+\exp\left(-\frac{C}{p}\log\frac{1}{\lambda}\right)\right)^{-m}\right) \\ &= \frac{1}{N}\left(1+\mathcal{O}\left(\frac{F}{N}\right)\right)\exp\left(-\frac{m}{N}\left(1+\mathcal{O}\left(\frac{F}{N}\right)\right)\right) \\ &+ \mathcal{O}\left(\frac{F}{N}\left(1+\exp\left(-\frac{C}{p}\log\frac{1}{\lambda}\right)\right)^{-m}\right), \end{aligned} \tag{1.6}$$

where $C > 0$ is a proper chosen constant, and the moments $M_m = \mathbf{E}(Y^m)$ by

$$\begin{aligned} M_m &= m!\frac{1-Q_0(1)}{\phi Q_0'(\phi)}(\log\phi)^{-m-1} + \mathcal{O}\left(m!\frac{F^m}{N}\right) \\ &= m!\left(N\left(1+\mathcal{O}\left(\frac{F}{N}\right)\right)\right)^m + \mathcal{O}\left(m!\frac{F^m}{N}\right) \end{aligned} \tag{1.7}$$

uniformly for $p \to 0,\ \lambda \to 0$.

The function $Q_0(z)$ will be defined in (2.2) and satisfies $Q_0'(1)/(1-Q_0(1)) = N(p,\lambda)$ (see (2.6)). $\phi > 1$ is the unique positive solution of $Q_0(\phi) = 1$ and has the asymptotic expansion (2.15)

$$\phi = 1 + \frac{1}{N} - \frac{Q_0''(1)}{2Q_0'(1)}\frac{1}{N^2} + \mathcal{O}\left(\frac{F}{N^3}\right). \tag{1.8}$$

Remark. It is worth mentioning that Y can be described in terms of renewal theory. Applying the renewal theorem one gets

$$\mathbf{P}[Y \ge m] = \sum_{k\ge m} y_k \sim \frac{1-Q_0(1)}{(\phi-1)Q_0'(1)}\,\phi^{-m} \tag{1.9}$$

for $m \to \infty$, which is of course a direct consequence of (1.6). (Compare with [Fe, p 374 ff], where a continuous analogue is discussed.)

2. Proof of the Theorem

We will proceed similarly to [DS1] but note that this theorem is not a direct consequence of [DS1].

Let $q_{n,k}$ $(n \ge 0, k \ge 1)$be the conditional probabilitiy

$$q_{n,k} = \mathbf{P}[X_{l+1} \ne 0, X_{l+2} \ne 0, \dots, X_{l+k-1} \ne 0, X_{l+k} = 0 | X_l = n] \tag{2.1}$$

and $Q_n(z)$ $(n \ge 0)$ the generating function

$$Q_n(z) = \sum_{k\ge 1} q_{n,k} z^k. \tag{2.2}$$

Then $Q_n = Q_n(1) = \mathbf{P}[\min\{k > 0:\ X_{l+k} = 0\} < \infty | X_l = n]$. Especially

$$\begin{aligned} Q_0 = Q_0(1) &= \mathbf{P}[\min\{k > 0:\ X_{l+k} = 0\} < \infty | X_l = 0] \\ &= \mathbf{P}[\min\{k > 0:\ X_k = 0\} < \infty] \end{aligned} \tag{2.3}$$

is the probability that the process returns to backlog $X_k = 0$ sometimes. $Q_0 = 1$ would imply $\mathbf{P}[X_k = 0$ infinitely many often$] = 1$ which contradits the instability property mentioned above. Therefore we have $Q_0 < 1$. By Markov property it follows that

$$\begin{aligned}
Q_0(z) &= zp_{0,0} + z\sum_{k\ge 2} p_{0,k}Q_k(z) \\
&= ze^{-\lambda}\left(1+\lambda+\sum_{j\ge 2}\frac{\lambda^j}{j!}Q_j(z)\right) \\
Q_1(z) &= zp_{1,0} + z\sum_{k\ge 1} p_{1,k}Q_k(z) \\
&= ze^{-\lambda}\left(p+(1-p)(1+\lambda)Q_1(z)+\lambda p Q_2(z)+\sum_{j\ge 2}\frac{\lambda^j}{j!}Q_{j+1}(z)\right) \\
Q_n(z) &= z\sum_{k\ge n-1} p_{n,k}Q_k(z) \\
&= ze^{-\lambda}\Bigg(np(1-p)^{n-1}Q_{n-1}(z)+(1-np(1-p)^{n-1} \\
&\qquad +\lambda(1-p)^n)Q_n(z)+\lambda(1-(1-p)^n)Q_{n+1}(z) \\
&\qquad +\sum_{j\ge 2}\frac{\lambda^j}{j!}Q_{n+j}(z)\Bigg) \quad \text{for } n\ge 2.
\end{aligned} \tag{2.4}$$

It is an easy observation that the probability generating function $Y(z)$ of $Y = \max\{n:\ X_n = 0\}$ can be evaluated in terms of $Q_0(z)$ by

$$\begin{aligned}
Y(z) &= \sum_{m\ge 0} y_m z^m \\
&= (1-Q_0) + Q_0(z)(1-Q_0) + Q_0(z)^2(1-Q_0) + \cdots \\
&= \frac{1-Q_0}{1-Q_0(z)}
\end{aligned} \tag{2.5}$$

so that the expected value $N(p,\lambda) = \mathbf{E}(Y)$ is given by

$$N(p,\lambda) = Y'(1) = \frac{Q_0'(1)}{1-Q_0(1)}. \tag{2.6}$$

Hence the asymptotic expansion (1.5) follows from

Lemma 1 ([DS2])**.** *We have*

$$1-Q_0 = \exp\left(\frac{1}{\log(1-p)}\left(\frac{1}{2}\log^2\frac{1}{\lambda}+\log\frac{1}{\lambda}\log\log\frac{1}{\lambda}\right)+\mathcal{O}(F)\right) \tag{2.7}$$

uniformly for $p\to 0$, $\lambda\to 0$.

and from the estimate $1 \le Q_0' = \mathcal{O}(F)$. The essential part of this paper is a generalization of this bound for Q_0' to $Q_0^{(m)} = Q_0^{(m)}(1)$.

Lemma 2. *There exists a constant $C > 0$ such that $Q_0^{(m)}$ can be estimated by*

$$Q_0^{(m)} \le m! \exp\left(m \frac{C}{p} \log \frac{1}{\lambda}\right) \tag{2.8}$$

uniformly for $p \to 0$, $\lambda \to 0$.

We postpone the proof of Lemma 2 at the end of this section.

As a first consequence we get that the radius of convergence R of $Q_0(z)$ is at least

$$\overline{R} = 1 + \exp\left(-\frac{C}{p} \log \frac{1}{\lambda}\right) \tag{2.9}$$

and that $Q_0(z)$ can be estimated by

$$\begin{aligned} |Q_0(z)| &\le \sum_{m \ge 0} \left(\exp\left(\frac{C}{p} \log \frac{1}{\lambda}\right)(|z| - 1)\right)^m \\ &= \frac{1}{1 - \exp\left(\frac{C}{p} \log \frac{1}{\lambda}\right)(|z| - 1)} \end{aligned} \tag{2.10}$$

for $1 \le |z| < \overline{R}$. Similar bounds can be found for $Q_0^{(m)}(z)$ in the same region $1 \le |z| < \overline{R}$.

In order to show that Y is approximately exponentially distributed with mean $N(p, \lambda)$ we will prove that the distribution of $\hat{Y} = Y/N(p, \lambda)$ converges weakly to the exponential distribution with parameter 1 if p and λ converge to 0. For this purpose we consider the characteristic function $\hat{Y}^*(t)$ of $\hat{Y}$ which evaluates to

$$\hat{Y}^*(t) = Y\left(e^{it/N(p,\lambda)}\right) = \frac{1 - Q_0}{1 - Q_0\left(e^{it/N(p,\lambda)}\right)}. \tag{2.11}$$

Using the fact that

$$\max_{|z| \le 1} Q_0''(z) = Q_0'' \tag{2.12}$$

we can apply Taylor's formula

$$Q_0(z) = Q_0 + Q_0'(z - 1) + \mathcal{O}(Q_0''(z - 1)^2) \tag{2.13}$$

for $|z| \le 1$ to get

$$\begin{aligned} \hat{Y}^*(t) &= \frac{1 - Q_0}{1 - Q_0 - Q_0'\left(e^{it/N(p,\lambda)} - 1\right) + \mathcal{O}\left(Q_0''\left(e^{it/N(p,\lambda)} - 1\right)^2\right)} \\ &= \frac{1}{1 - it + \mathcal{O}\left(\left(1 + \frac{Q_0''}{Q_0'}\right)\frac{t^2}{N(p,\lambda)}\right)} \end{aligned} \tag{2.14}$$

for $t = \mathcal{O}(N(p, \lambda))$. Thus $\lim\limits_{p \to 0, \lambda \to 0} \hat{Y}^*(t) = 1/(1 - it)$ for each fixed $t \in \mathbf{R}$. But $1/(1 - it)$ is the characteristic function of the exponential distribution and so the distribution of $\hat{Y}$ converges weakly to the exponential distribution.

Next we will use singularity analysis to get more information about $y_m = [z^m]Y(z)$. Remember that we write F for a factor that is bounded by $\exp\left(\frac{C}{p} \log \frac{1}{\lambda}\right)$ for some constant $C > 0$.

Lemma 3. *The probability generating function $Y(z) = (1-Q_0)/(1-Q_0(z))$ has radius of convergence*

$$\phi = 1 + \frac{1}{N(p,\lambda)} - \frac{Q_0''}{2Q_0'}\frac{1}{N(p,\lambda)^2} + \mathcal{O}\left(\frac{F}{N(p,\lambda)^3}\right) \tag{2.15}$$

which is the solution of $Q_0(\phi) = 1$. In addition $Y(z)$ has only a single pole $z = \phi$ on its radius of convergence. The residue is

$$\begin{aligned} res(Y(z);\phi) &= -\frac{1-Q_0}{Q_0'(\phi)} \\ &= -\frac{1}{N(p,\lambda)}\left(1 - \frac{Q_0''}{Q_0'}\frac{1}{N(p,\lambda)} + \mathcal{O}\left(\frac{F}{N(p,\lambda)^2}\right)\right). \end{aligned} \tag{2.16}$$

Proof. Since $q_{0,n} \ge 0$ a simple geometric arguement shows that the unique real positive solution ϕ of $Q_0(\phi) = 1$ lies between 1 and $1 + 1/N(p,\lambda) < \overline{R}$. Thus $Y(z)$ has a simple pole at $z = \phi$ with residue

$$\mathrm{res}(Y(z);\phi) = -\frac{1-Q_0}{Q_0'(\phi)}. \tag{2.17}$$

Now $q_{0,n} > 0$ for $n > 2$ implies $|Q_0(z)| < Q_0(|z|)$ for $z \ne |z|$. So $z = \phi$ is the only singularity on the circle of convergence. The asymptotic expansion (2.15) for ϕ can easily be obtained by Taylor's formula and by boots-trapping starting with $\phi = 1 + \psi$ where $\psi = \mathcal{O}(1/N(p,\lambda))$. $\square$

Lemma 4. *There exist constants $C_1, C_2 > 0$ such that $Q_0(z) \ne 1$ for $z \le 1 + \exp\left(-\frac{C_1}{p}\log\frac{1}{\lambda}\right)$ with the only exception $z = \phi$. Moreover we have*

$$\min_{|z|=1+\exp\left(-\frac{C_1}{p}\log\frac{1}{\lambda}\right)} |1 - Q_0(z)| \ge \exp\left(-\frac{C_2}{p}\log\frac{1}{\lambda}\right) \tag{2.18}$$

uniformly for $p \to 0$, $\lambda \to 0$.

Proof. The bound for $Q_0''(z)$ and the lower bound $Q_0'(\phi) \ge 1$ imply that there exists some constant $C_3 > 0$ such that

$$|1 - Q_0(z)| \ge \frac{|z-\phi|}{2} \tag{2.19}$$

for $|z - \phi| \le \exp\left(-\frac{C_3}{p}\log\frac{1}{\lambda}\right)$. Using the explicit expressions

$$\begin{aligned} q_{0,1} &= e^{-\lambda}(1+\lambda) \\ q_{0,2} &= 0 \\ q_{0,3} &= e^{-3\lambda}\lambda^2 p^2(1-p) \\ q_{0,4} &= e^{-4\lambda}\Big(\lambda^3 p^3(1-p) + \lambda^2\left(1 - 2p(1-p) + \lambda(1-p)^2\right)p^2(1-p) \\ &\qquad + \lambda^2(1+\lambda)(1-p)p^2(1-p)\Big) \end{aligned} \tag{2.20}$$

and the trivial estimate

$$|Q_0(z)| \le |q_{0,1}z + q_{0,3}z^3 + q_{0,4}z^4| + \sum_{n>4} q_{0,n}|z|^n \tag{2.21}$$

it is easy to check that there is some constant $C_4 > 0$ with

$$\max_{\substack{|z|\le\phi \\ |z-\phi|\ge\frac{1}{2}\exp\left(-\frac{C_3}{p}\log\frac{1}{\lambda}\right)}} |Q_0(z)| \le 1 - \exp\left(-\frac{C_4}{p}\log\frac{1}{\lambda}\right). \tag{2.22}$$

This implies that there exist constants $C_5, C_6 > 0$ with

$$\max_{\substack{|z|\le\phi+\exp\left(-\frac{C_5}{p}\log\frac{1}{\lambda}\right) \\ |z-\phi|\ge\frac{1}{2}\exp\left(-\frac{C_3}{p}\log\frac{1}{\lambda}\right)}} |Q_0(z)| \le 1 - \exp\left(-\frac{C_6}{p}\log\frac{1}{\lambda}\right). \tag{2.23}$$

So (2.18) follows from (2.19) and (2.23). $\square$

Now a direct application of Cauchy's formula gives

$$\begin{aligned} y_m &= -\mathrm{res}(Y(z);\phi)\phi^{-m-1} + \frac{1}{2\pi i}\int_{|z|=1+\exp\left(-\frac{C_1}{p}\log\frac{1}{\lambda}\right)} \frac{Y(z)}{z^{m+1}}\,dz \\ &= \frac{1-Q_0}{\phi Q_0'(\phi)}\phi^{-m} + \mathcal{O}\left(\frac{F}{N(p,\lambda)}\left(1+\exp\left(-\frac{C_1}{p}\log\frac{1}{\lambda}\right)\right)^{-m}\right) \end{aligned} \tag{2.24}$$

which proves (1.6).

The asymptotic expansion for the moments M_m is an easy consequence of (2.24) and of the bivariate asymptotic expansion

$$\sum_{n\ge1} n^m e^{-\alpha n} = \frac{m!}{\alpha^{m+1}} - \frac{B_{m+1}}{m+1} + \frac{B_{m+2}}{m+2}\alpha + \mathcal{O}\left(\frac{m!}{(2\pi)^m}(m\alpha)^{3/2}\right) \tag{2.25}$$

which is uniform for $\alpha \to 0$ and $m \to \infty$ (see [DS1]).

For the proof of Lemma 2 we will use the following notation. Let $\mathbf{l}^\infty$ be the Banach space of all bounded sequences $\mathbf{x} = (x_n)_{n\ge1}$ with norm $||\mathbf{x}||_\infty = \sup_{n\ge1}|x_n|$ and $\mathbf{l}^\infty(0)$ the subset of all sequences $\mathbf{x} = (x_n)_{n\ge1} \in \mathbf{l}^\infty$ with $\lim_{n\to\infty} x_n = 0$. $\mathbf{x} = (x_n)_{n\ge1} \le \mathbf{y} = (y_n)_{n\ge1}$ will denote that $x_n \le y_n$ for all $n \ge 1$. Furthermore let $\mathrm{L}\mathbf{x} \in \mathbf{l}^\infty$ be defined by

$$\begin{aligned} (\mathrm{L}\mathbf{x})_1 &= \sum_{k\ge1} p_{1,k}x_k \\ (\mathrm{L}\mathbf{x})_n &= \sum_{k\ge n-1} p_{n,k}x_k \quad \text{for } n>1. \end{aligned} \tag{2.26}$$

Then $\mathbf{Q}(z) = (Q_n(z))_{n\ge1}$ satisfies

$$\mathbf{Q}(z) = z\mathbf{d} + z\mathrm{L}\mathbf{Q}(z), \tag{2.27}$$

where $\mathbf{d} = (d_n)_{n\ge1}$ is defined by $d_1 = p_{1,0} = e^{-\lambda}p$ and by $d_n = 0$ for $n > 1$. Thus $\mathbf{Q}^{(m)}(z) = (Q_n^{(m)}(z))_{n\ge1}$ fulfills

$$\begin{aligned}\mathbf{Q}'(z) &= \mathbf{d} + \mathrm{L}\mathbf{Q}(z) + z\mathrm{L}\mathbf{Q}'(z) = \frac{1}{z}\mathbf{Q}(z) + z\mathrm{L}\mathbf{Q}'(z)\\ \mathbf{Q}^{(m)}(z) &= m\mathrm{L}\mathbf{Q}^{(m-1)}(z) + z\mathrm{L}\mathbf{Q}^{(m)}(z) \quad \text{for } m>1.\end{aligned} \tag{2.28}$$

For $z = 1$ this reads as

$$\begin{aligned}\mathbf{Q} &= \mathbf{d} + \mathrm{L}\mathbf{Q}\\ \mathbf{Q}' &= \mathbf{d} + \mathrm{L}\mathbf{Q} + \mathrm{L}\mathbf{Q}' = \mathbf{Q} + \mathrm{L}\mathbf{Q}'\\ \mathbf{Q}^{(m)} &= m\mathrm{L}\mathbf{Q}^{(m-1)} + \mathrm{L}\mathbf{Q}^{(m)} \quad \text{for } m>1.\end{aligned} \tag{2.29}$$

So $\mathbf{Q}^{(m)} = (Q_n^{(m)})_{n\ge1}$ can be obtained by solving iteratively fixed point problems. Afterwards $Q_0^{(m)} = Q_0^{(m)}(1)$ evaluates to

$$\begin{aligned}Q_0' &= e^{-\lambda}(1+\lambda) + e^{-\lambda}\sum_{j\ge2}\frac{\lambda^j}{j!}(Q_j + Q_j') = Q_0 + e^{-\lambda}\sum_{j\ge2}\frac{\lambda^j}{j!}Q_j'\\ Q_0^{(m)} &= e^{-\lambda}\sum_{j\ge2}\frac{\lambda^j}{j!}\left(mQ_j^{(m-1)} + Q_j^{(m)}\right) \quad \text{for } m>1.\end{aligned} \tag{2.30}$$

Our aim is to show that $\mathbf{Q}^{(m)} \in \mathbf{l}^\infty(0)$ and that it is possible to find upper bounds $\mathbf{c}^{(m)} \in \mathbf{l}^\infty(0)$ for $\mathbf{Q}^{(m)}$ satisfying

$$||\mathbf{c}^{(m)}||_\infty \le m!\exp\left(m\frac{C}{p}\log\frac{1}{\lambda}\right) \tag{2.31}$$

for some constant $C > 0$. So (2.30) and (2.31) prove Lemma 2.

Lemma 5. *If there exist sequences* $\mathbf{0} \le \mathbf{c}^{(m)} \in \mathbf{l}^\infty(0)$ *satisfying* $\mathbf{c}^{(0)} \ge \mathbf{Q}$ *and*

$$\mathbf{c}^{(m)} \ge m\mathbf{c}^{(m-1)} + L\mathbf{c}^{(m)}, \tag{2.32}$$

then $\mathbf{Q}^{(m)} \in \mathbf{l}^\infty(0)$ *and* $\mathbf{Q}^{(m)} \le \mathbf{c}^{(m)}$.

Proof. First notice that (2.28) uniquely determine $\mathbf{Q}^{(m)}(z) \in \mathbf{l}^\infty$ for $|z| < 1$ by a direct application of Banach's fixed point theorem. You only have to observe that $||\mathrm{L}|| = \sup_{||\mathbf{x}||_\infty\le1} ||\mathrm{L}\mathbf{x}||_\infty = 1$. Now we can prove inductively that $\mathbf{0} \le \mathbf{Q}^{(m)} \le \mathbf{c}^{(m)}$ for $0 < z < 1$.

We start with $m = 1$. Set $\mathbf{a} = \mathbf{0} \in \mathbf{l}^\infty(0)$ and $\mathbf{b} = \mathbf{c}^{(1)} \in \mathbf{l}^\infty$. Then $\mathbf{0} \le \mathbf{a} \le \mathbf{b}$, $\mathbf{a} \le \frac{1}{z}\mathbf{Q}(z) + z\mathrm{L}\mathbf{a}$, and $\mathbf{b} \ge \mathbf{c}^{(0)} + \mathrm{L}\mathbf{b} \ge \frac{1}{z}\mathbf{Q}(z) + z\mathrm{L}\mathbf{b}$ force that there is a solution $\mathbf{x} \in \mathbf{l}^\infty(0)$ of $\mathbf{x} = \frac{1}{z}\mathbf{Q}(z) + z\mathrm{L}\mathbf{x}$ with $\mathbf{0} = \mathbf{a} \le \mathbf{x} \le \mathbf{b} = \mathbf{c}^{(1)}$. (See Lemma 3.4 of [DS2].) But this solution must be $\mathbf{Q}'(z)$. Thus the case $m = 1$ is settled.

Now suppose that $\mathbf{0} \le \mathbf{Q}^{(m-1)}(z) \le \mathbf{c}^{(m-1)}$ and that there exists a sequence $\mathbf{c}^{(m)}$ with (2.32). Set $\mathbf{a} = \mathbf{0} \in \mathbf{l}^\infty(0)$ and $\mathbf{b} = \mathbf{c}^{(m)} \in \mathbf{l}^\infty$. As above we have $\mathbf{0} \le \mathbf{a} \le \mathbf{b}$, $\mathbf{a} \le m\mathrm{L}\mathbf{Q}^{(m-1)}(z) + z\mathrm{L}\mathbf{a}$, and $\mathbf{b} \ge m\mathbf{c}^{(m-1)} + \mathrm{L}\mathbf{b} \ge m\mathrm{L}\mathbf{Q}^{(m-1)}(z) + z\mathrm{L}\mathbf{b}$. Thus there is a solution $\mathbf{x} \in \mathbf{l}^\infty(0)$ of $\mathbf{x} = m\mathrm{L}\mathbf{Q}^{(m-1)}(z) + z\mathrm{L}\mathbf{x}$ with $\mathbf{0} \le \mathbf{x} \le \mathbf{c}^{(m)}$ that must be equal to $\mathbf{Q}^{(m)}(z)$. This completes the general case.

Now it directly follows that $\mathbf{Q}^{(m)} = \mathbf{Q}^{(m)}(1)$ exists and that it is bounded by $\mathbf{0} \le \mathbf{Q}^{(m)} \le \mathbf{c}^{(m)}$. $\square$

Lemma 6. *There exists a sequence* $\mathbf{c} = (c_n)_{n\geq 1} \in \mathbf{l}^\infty(0)$ *such that* $||\mathbf{c}||_\infty \leq \exp\left(\frac{C}{p}\log\frac{1}{\lambda}\right)$ *for some constant* $C > 0$, *that* $\mathbf{c} \geq \mathbf{Q}$ *and that*

$$\left(1 - \exp\left(-\frac{C}{p}\log\frac{1}{\lambda}\right)\right)\mathbf{c} \geq L\mathbf{c}. \tag{2.33}$$

Proof. Let n_0 be the nearest integer to $\log\frac{1}{\lambda}/log(1/(1-p))$ and n_1 the nearest positive integer to $C_2\lambda^2/p$, where C_2 is some constant that appears in Lemma 6.4 of [DS2] which says that there are constants $C_1, C_2 > 0$ such that

$$c_n = \frac{C_1}{\lambda^2}2^n \tag{2.34}$$

satisfies $c_n - (\mathrm{L}\mathbf{c})_n \geq 1$ for $n_1 \leq n \leq n_0$. In the case $n_1 > 1$ Lemma 6.4 and Lemma 6.5 of [DS2] imply that there exist constants $C_3, C_4 > 0$ such that

$$c_n = \sum_{k=n_1-n+1}^{n_1} a_k \tag{2.35}$$

for $n \leq n_1$, where

$$a_k = \exp\left(C_3\log\frac{1}{\lambda}\right)\frac{(n_1-k)!}{n_1!}\left(\frac{C_4\lambda^2}{p}\right)^k \tag{2.36}$$

and

$$c_n = a_0 2^{n-n_1} + c_{n_1-1} \tag{2.37}$$

for $n > n_1$ satisfies $c_n - (\mathrm{L}\mathbf{c})_n \geq 1$ for $n \leq n_0$. Thus it is always possible to find a sequence $\mathbf{c} = (c_n)_{n\geq 1}$ with

$$\max_{n\leq n_0} c_n = \mathcal{O}(F) \tag{2.38}$$

satisfying $c_n - (\mathrm{L}\mathbf{c})_n \geq 1$ for $n \leq n_0$.

Now set

$$\begin{aligned} q_n &= \frac{np(1-p)^{n-1}}{e^\lambda - 1 - \lambda(1-p)^n}, \\ P_n &= q_1 q_2 \cdots q_n, \\ b_1 &= 1 - \frac{1}{1 + P_1 + P_2 + \cdots}, \\ b_n &= 1 - (1-b_1)(1 + P_1 + \cdots + P_{n-1}) \\ &= (1-b_1)P_n(1 + q_{n+1} + q_{n+1}q_{n+2} + \cdots). \end{aligned} \tag{2.39}$$

Then $\mathbf{b} = (b_n)_{n\geq 1} \in \mathbf{l}^\infty(0)$ satisfies

$$\mathbf{b} \geq \mathbf{d} + \mathrm{L}\mathbf{b}. \tag{2.40}$$

Since $\mathbf{x} = \mathbf{Q} = (Q_n)_{n\geq 1}$ is the only solution of $\mathbf{x} = \mathbf{d} + \mathrm{L}\mathbf{x}$ under the restriction $\mathbf{x} \in \mathbf{l}^\infty(0)$, $\mathbf{b}$ is an upper bound for $\mathbf{Q}$. (For details see [DS2].) More precisely $\mathbf{b} = (b_n)_{n\geq 1}$ satisfies the system ($b_0 = 1$)

$$\begin{aligned} b_n = e^{-\lambda}\big(&np(1-p)^{n-1}b_{n-1} + (1 - np(1-p)^{n-1} + \lambda(1-p)^n)b_n \\ &+ (e^\lambda - 1 - \lambda(1-p)^n)b_{n+1}\big) \end{aligned} \tag{2.41}$$

which is equivalent to

$$\frac{b_n - b_{n+1}}{b_{n-1} - b_n} = q_n. \tag{2.42}$$

Therefore we have

$$\begin{aligned} b_n - (\mathbf{Lb})_n &\ge e^{-\lambda}\frac{\lambda^2}{2}(b_{n+1} - b_{n+2}) \\ &= e^{-\lambda}\frac{\lambda^2}{2}(1-b_1)P_{n+1}. \end{aligned} \tag{2.43}$$

Let $n_2 \ge n_0$ be the smallest positive intger with $q_{n_2} \le 1/2$. By elementary calculations (see [DS2]) we get

$$n_2 = \frac{1}{\log\frac{1}{1-p}}\left(\log\frac{1}{\lambda} + \log\log\frac{1}{\lambda} + \mathcal{O}(1)\right), \tag{2.44}$$

$$b_{n_2} \ge \exp\left(-\frac{C_5}{p}\log\frac{1}{\lambda}\right) \tag{2.45}$$

for some constant $C_5 > 0$, and

$$1 + q_{n+1} + q_{n+1}q_{n+2} + \cdots = \mathcal{O}(F) \tag{2.45}$$

for $n \ge n_0$. Thus there is a constant $C_6 > 0$ such that

$$c_n = \exp\left(\frac{C_6}{p}\log\frac{1}{\lambda}\right) b_n \tag{2.46}$$

fulfills

$$\begin{aligned} c_n - (\mathbf{Lc})_n &\ge e^{-\lambda}\frac{\lambda^2}{2}(1-b_1)P_n q_{n+1}\exp\left(\frac{C_6}{p}\log\frac{1}{\lambda}\right) \\ &\ge (1-b_1)P_n(1 + q_{n+1} + q_{n+1}q_{n+2} + \cdots) \\ &= b_n \end{aligned} \tag{2.47}$$

for $n_0 \le n \le n_2$.

Now set

$$c_n = \exp\left(\frac{C_7}{p}\log\frac{1}{\lambda}\right) 2^{-n+n_2} \tag{2.48}$$

for $n \ge n_2$, where C_7 is a constant to be chosen in the sequel. Since $q_n \le 1/2$ for $n \ge n_2$ we surely have

$$\frac{c_n - c_{n+1}}{c_{n-1} - c_n} \ge q_n \tag{2.49}$$

or equivalently

$$\begin{aligned} c_n \ge e^{-\lambda}\big(&np(1-p)^{n-1}c_{n-1} + (1 - np(1-p)^{n-1} + \lambda(1-p)^n)c_n \\ &+ (e^\lambda - 1 - \lambda(1-p)^n)c_{n+1}\big) \end{aligned} \tag{2.50}$$

for $n \ge n_2$. From (2.45) and

$$c_n - (\mathbf{Lc})_n \ge e^{-\lambda}\frac{\lambda^2}{2}(c_{n+1} - c_{n+2}) \ge \frac{\lambda^2}{24}c_n \tag{2.51}$$

it easily follows that C_7 can be chosen in a way that $c_n \geq b_n$ for $n \geq n_2$. Putting all these things together it is easy to construct a sequence $\mathbf{c}$ that satisfies (2.33) $\square$

Now we are able to construct sequences $\mathbf{c}^{(m)} \in \mathbf{l}^\infty(0)$ that satisfy (2.31) and (2.32). (2.33) shows that $\mathbf{c}^{(1)} = \exp\left(\frac{C}{p}\log\frac{1}{\lambda}\right)\mathbf{c}$ is a proper choice for $m = 1$. In general we can use

$$\mathbf{c}^{(m)} = m!\exp\left(m\frac{C}{p}\log\frac{1}{\lambda}\right)\mathbf{c}. \tag{2.52}$$

It easily follows by induction that these sequences fulfill (2.32)

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