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S8302-MAT Statistical Properties of Number Systems

Project 1 within the research area
"Numbertheoretical Algorithms"

Application for the third to fifth year

Project Leader

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1119	60	1118	30	1104	10

Jahr	PeK	GeK	MaK	ReK	SoK	WV	Summe
bisher							
−2	756.000[1]			32.871[2]	24.530[3]		813.401
−1	756.000[1]			43.225[2]	13.300[3]		812.525
Summe	1.512.000[1]			76.096[2]	37.830		1.625.926
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1	1.522.921[4]						1.522.921
2	1.522.921[4]						1.522.921
3	1.522.921[4]						1.522.921

[1] PeK 756.000 for the years 1 and 2 have been granted and approximately all is used. It is almost impossible to get the exact number.

[2] ReK 32.871 + 43.225 is the total number of travelling expenses for 2000 and 2001 till August. 53.416 have been paid from the coordination project and 22.680 from the project ("Tagungsgebühr").

[3] SoK 24.530 and 13.300 are the expenses for guests in the years 2000 and 2001 till August.

[4] 1 522 921 are PeK for 2.5 post-doc positions

Finally ReK and SoK for years 3–5 will be applied within the coordination project.

1 Short Description

Number systems naturally appear in number theoretical algorithms since every integer number is usually represented in the binary number system. However, the binary number system is not only used for representing numbers.

It is used in various algorithms. For example, the RSA enciphering algorithms (see [43]) needs the computation of powers $a^e \bmod m$, where a, e, m are large integers. By using the binary representation of e it is possible to compute $a^e \bmod m$ extremely fast. A completely different application of binary or q-ary representations of integers is the construction of (at the moment best known) low discrepancy sequences (see [18, 50]). Moreover, number systems appear in the theoretical analysis of various algorithms, for example, the sum-of-digits function plays an essential role in [23, 36, 37, 46, 42, 48, 51]. A completely different aspect comes from continued fraction expansions (which can be also viewed as a number system). Their generalization to formal Laurent series plays an essential rôle in the linear complexity profile of sequences over finite fields, see [49].

In all these applications the statistical behaviour of the digital expansions involved play an essential rôle in the (average case) analysis, e.g. the average value of the sum-of-digits function.

Therefore, the purpose of this project is to discuss statistical properties of number systems from various points of view. Within the second period (years 3–5) we want to discuss the following topics

1. **Digital expansions of polynomial sequences**
2. **Digital expansions and dynamical systems**
3. **Digital expansions and random sequences**
4. **Continued fraction expansions**
5. **Automatic asymptotic distributions**

ad 1. In [11, 12, 17, 68] the distribution of $f(P(n))$ and the joint distribution of $f_1(P_1(n))$ and $f_2(P_2(n))$ has been discussed for various cases, where f resp. f_1, f_2 are q- or G-additive functions (where the sequence $G = (G_j)_{j\ge 0}$ satisfies a linear recurrence) and $P(x)$ resp. $P_1(x), P_2(x)$ are positive integer polynomials. However, there are remaining cases which could not been covered so far. For example, if q_1 and q_2 are not coprime, the joint distribution of $f_1(P_1(n))$ and $f_2(P_2(n))$ (if f_1 is a q_1-additive and f_2 a q_2-additive function) is unknown.

There is also another challanging problem: Prove asymptotic expansions for the numbers $a_{N,k} = \#\{n < N : f(P(n)) = k\}$. For example, it is unknown whether the set $\{n : f(P(n)) \equiv 0 \bmod 2\}$ is finite or infinite.

ad 2. There is an intimate connection between G-ary digital expansions (where the sequence $G = (G_j)_{j\ge 0}$ satisfies a linear recurrence) and Markov partitions of toral automorphisms, see [17, 68] if the constant term of the characteristic polynomial equals 1. The case of larger constant term seems to be much more difficult, however, it seems that proper p-adic components could be helpful, see [66].

Another relation to dynamical systems are odometers (see [4, 30, 32]) where the space of digital expansions of non-negative integers is compactified in a natural way and the addition $+1$ is continuously continued. These dynamical systems have cocycles which are related to (the continuation) of the differences $s(n+1) - s(n)$. There are many interesting problems on these cocycles.

ad 3. In [15] the discrepancy of $(s(n)\alpha) \bmod 1$ has been estimated in terms of the continued fraction expansion from below and above. However, there is a gap between these two bounds and it is an open problem to determine the exact order of magnitude of the discrepancy.

Another problem which we want to treat is the construction of low discrepancy sequences with help of digital expansions different from the q-adic ones (e.g. there exist constructions of net sequences with help of q-adic expansions, see [50] (the simplest case is the Van-der-Corput sequence.)

ad 4. It is a classial fact that Diophantine approximation properties of a real number α are (more or less) determined by the continued fraction expansion of α. Thus, *statistical* properties of the contiuned fraction expansion directly translate to *metric* properties of Diophantine approximation. In [26] this has been worked out for formal Laurent series (by extending results of Niederreiter [49], where the linear complexity profile of sequences over finite fields is considered). Furthermore, in [27, 28] extensions of classical results of LeVeque and W. Philipp (concerning central limit laws and laws of the iterated logarithm) are given. Some problems concerning laws of the interated logarithm are still open. There are also many questions in the higher dimensional case (where there are no continued fraction) and of course for formal Laurent series.

ad 5. For many statistical questions concerning number systems one expects a central limit law for the parameter of interest (e.g. for the distribution of the sum-of-digits function). Thus, one needs asymptotic relations for expected valued and variance in order to normalize and finally a theorem showing that the normalized random variable converges weakly to $N(0,1)$. The purpose of this part of the project is to provide tools which do this automatically. The key is to use bivariate generating functions (e.g. in [13, 14, 33] bivariate generating function were the key to the proof of the central limit theorem). The MAPLE-program "equivalent" generates the asymptotic expansion of the Taylor coefficients of a given function in one variable. Hence, it is possible to get asymtptotic for mean value and variance automatically. However, at the moment there is no tool to extract a limiting distribution out from a bivariate generating function.

2 Introduction — Status of Research

Let $G = (G_j)_{j\ge 0}$ be a strictly increasing sequence of integers with $G_0 = 1$. Then every non-negative integer n has a (unique) proper G-ary digital expansion $n = \sum_{j\ge 0} \epsilon_j(n) G_j$ with integer digits $\epsilon_j(n) \ge 0$ provided that $\sum_{j<k} \epsilon_j(n) G_j < G_k$ for all $k \ge 0$. For example, for $G_j = q^j$ (with q a positive integer) we obtain the usual the q-ary expansion. Further examples are Cantor representations, where G_j is of the form $G_j = b_0 b_1 \cdots b_j$ (with integers $b_j > 1$), and the Zeckendorf representations, where $G_j = F_{j+2}$ are the Fibonacci numbers. More generally, one also considers G_j which are solutions of a linear recurrence with constant coefficients.

A very significant functional of number systems is the sum-of-digits function $s_G(n)$ which is given by $s_G(n) = \sum_{j\ge 0} \epsilon_j(n)$. It is convenient to consider a related sequence of (discrete) random variables X_N, $N \ge 0$, defined by

$$\mathbf{Pr}[X_N = k] = \frac{|\{n < N : s_G(n) = k\}|}{N}$$

in order to state some results of the literature concerning **statistical properties of number systems**. For example, the expected value is exactly the average value of the sum-of-digits function: $\mathbf{E}X_N = \frac{1}{N}\sum_{n<N} s_G(n)$ There is a vast literature concerning asymptotic properties of $\mathbf{E}X_N$ and $\mathbf{V}X_N$ and on the distribution of X_N.

Asymptotic and exact formulas for $\mathbf{E}X_N$ are due to Bush [7], Bellman and Shapiro [6], Delange [9], and Trollope [71] for q-ary digital expansions and due to Pethö and Tichy [53] and Grabner and Tichy [33, 34] for G-ary digital expansions with respect to linear recurrences. Corresponding formulas for higher moments $\mathbf{E}X_N^d$ and for the variance $\mathbf{V}X_N$ can be found in Coquet [8], Kirschenhofer [41], Kennedy and Cooper [38], Grabner, Kirschenhofer, Prodinger, and Tichy [31], and in Dumont and Thomas [19]. All known results are of the form

$$\mathbf{E}X_N \sim C_1 \log N \quad \text{and} \quad \mathbf{V}X_N \sim C_2 \log N$$

with constants $C_1, C_2 > 0$ depending on the number system $G = (G_j)$.

The asymptotic distribution of X_N and related problems are discussed in Schmidt [64], Schmid [61], Bassily and Katai [5], in Dumont and Thomas [20] and in Drmota and Gajdosik [14], where a new appoach using generating functions is presented. Again we always have the same result, a central limit theorem of the form

$$\frac{X_N - \mathbf{E}X_N}{\sqrt{\mathbf{V}X_N}} \to N(0,1).$$

It is also very interesting to study not only the sum-of-digits function of all non-negative integers but to restrict to a subsequence, e.g. to an arithmetic sequence $an + b$ (see Schmid [61]) or to values of a polynomial $P(n)$ (or the values of a polynomial $P(p_n)$ with prime argument). Bassily and Kátai [5] have proved that there is a central limit law of the same form.

Within the first two years the method of Bassily and Kátai [5] has been generalized and extended in various ways. We have proved that the q-adic number systems are asymptotically independent if the q-s are coprime [11]. This result has been extended to q-adic systems and the Zeckendorf expansion [17] and later to more general G-adic expansions [68]. Furthermore, functional versions of are given in [12]. In [25] the q-adic digits of the exponental sequence a^n ($n \le N$) are considered.

Furthermore within the first two years the block distribution for G-ary digital expansions is discussed in [67] (generalizing [10]) and the distribution modulo 1 of the sequence $s_q(n)\alpha$ is treated in [15].

We already extended our interest to the field of continued fraction expansions. Motivated by a paper by Niederreiter [49], where the linear complexity profile of sequences over finite fields is considered, M. Fuchs started to work on this subject. First he discussed metric results on the continued fraction expansion and diophantine approximation for formal Laurent series [26]. Furthermore, in [27, 28] extensions of classical results of LeVeque [44, 45] and of W. Philipp [54, 55] (concerning central limit laws and laws of the iterated logarithm) are given.

3 Aims, Methods, Working Program

As already mentioned we have divided our working program for the second period (years 3–5) into 5 topics which will be dicussed in detail. Afterward we present a working and time program.

3.1 Digital expansions of polynomial sequences

In a remarkable paper Bassily and Katai [5] have proved that the sequences $s_q(P(n))$, $n < N$, and $s_q(P(p))$, p a prime $< N$, where $P(x)$ is a positive integer polynomial, satisfy a central limit theorem. (More generally, sequences of the kind $f(P(n))$ and $f(P(p))$, where f are certain q-additive functions, have been considered. However, for the sake of brevity we will only discuss the sum-of-digits function $s_q(n)$

here.) The idea of [5] is – more or less – to approximate the sequence $s_q(P(n))$, $n < N$, by a sum of i.i.d. random variables. The *difference* it then quantified by estimates for exponential sums. Thus, it is natural that the method of [5] can be generalized in various ways. For example, in [11, 12] it is shown that the two-dimensional sequences $(s_{q_1}(P_1(n)), s_{q_2}(P_2(n))$ and $(s_{q_1}(P_1(p)), s_{q_2}(P_2(p))$ satisfy a central limit theorem, too, with asymtotically independent components if q_1, q_2 are coprime. (Interestingly one needs Baker's theorem on linar forms of logarithms to estimate the corresponding exponential sums.) The next step is to discuss the case of general q_1, q_2. The only known result here is due to Schmid [61] where linear polynomials are considered. The limiting distribution is again Gaussian but with non-zero correlation. It also follows from the methods of [11] that there is a central limit theorem with independent components if the degrees of $P_1(x)$ and $P_2(x)$ are different. Thus, it remains to cover the case where $P_1(x), P_2(x)$ are non-linear and of the same degree. It seems that again variations of the method of Bassily and Katai [5] are useful. Namely one is led to the conjecture that the sequence $(s_{q_1}(P_1(n)), s_{q_2}(P_2(n)), n < N$, can be approximated by a sum of weakly dependent random vectors (especially if $q_1 = q_2$).

It is also a challanging problem to consider local versions of the above limit theorems, i.e. asymptotic relations for the numbers

$$|\{n < N : s_q(P(n)) = k\}|.$$

It is even an unsolved problem to decide whether the set

$$\{n : s_2(P(n)) \equiv 0 \bmod 2\}$$

is finite or infinite. The key problem is to derive non-trivial estimates for exponential sums of the kind

$$\sum_{n<N} e\left(x \cdot s_q(P(n))\right)$$

(where $e(x) = e^{2\pi i x}$). For linear polynomials these sums are very easy to estimate. However, for higher degrees nothing is known. It would be also interesting to consider primes (or polynomial values of primes). Here one would have to estimate exponential sums of the form

$$\sum_{p<N} e\left(x \cdot s_q(P(p))\right)$$

which is even more difficult.

3.2 Digital expansions and dynamical systems

Let $G = (G_j)_{j\ge 0}$ ge defined by the linear recurrence

$$G_j = a_1 G_{j-1} + a_2 G_{j-2} + \ldots + a_d G_{j-d} \text{ for } j \ge d$$

and

$$G_j = a_1 G_{j-1} + a_2 G_{j-2} + \ldots + a_j G_0 \text{ for } 1 \le j < d,\ G_0 = 1,$$

with non-negative integers a_i which satisfy the relations

$$(a_j, a_{j+1}, \ldots, a_d) \le (a_1, a_2, \ldots, a_{d-j+1}) \text{ for } 2 \le j \le d$$

(where "$<$" denotes the lexicographical order) and $a_d > 0$. Then in [17, 68] is has been proved that if $a_1 \ge a_2 \ge \cdots \ge a_d = 1$ then the sequences $s_G(P(n))$, $n < N$, and $s_G(P(p))$, p prime $< N$, satisfy a central limit theorem, where $s_G(n)$ denotes the G-ary sum-of-digits function. (Again a proper variation of Bassily and Katai's method [5] has been used.) One essential point in the proof is the obervation that there exists a tiling A_b ($b \in \{0, 1, \ldots, a_1\}$) of the d-dimensional unit cube such that $\epsilon_j(n) \in A_b$ (almost) if and only if

$$\left(\left\{\frac{n}{\alpha^j}\right\}, \left\{\frac{n}{\alpha^{j+1}}\right\}, \ldots, \left\{\frac{n}{\alpha^{j+d-1}}\right\}\right) \in A_b,$$

where $\alpha > 1$ is the dominant root of the chararcteristic polynomial of the above linear recurrence. The tiling A_b is a Markov partition of a toral automorphism directly related to the linear recurrence. For $d = 2$ the sets A_b are just rectangles modulo 1. However, for $d > 2$ these are fractal sets related the the Rauzy fractal (see [68]).

One task is to get rid of the restrictions on the coefficients $a_d = 1$ of the linear recurrence. The problem is that for $a_d > 1$ one is confronted with a toral endormophism which is not an automorphism any more. Probably this case is manageable by adjoining p-adic factors to the torus and considering tilings of this structure. Some properties of possible tilings are given by Siegel [66].

Tilings with fractal boundary appear also in the case of digital expansions of the Gaussian integers (cf. [29]), e.g. the well-known "twin dragon". The topological structure of these fractals has been studied extensively (by Akiyama and Thuswaldner [2]). The box counting dimension has been calculated by Scheicher and Thuswaldner [60]. In these works, the fundamental property of the fractals is their description by automata. Since Rauzy fractals are described by automata which seem to be rather similar, these methods can probably be applied to them, too.

A completely different problem related to digital expansions and dynamical systems are odometers (see [4, 30, 32]). Here the topological closure $\mathcal{K}$ of all digital expansions $(\epsilon_j(n))_{j\ge 0}$ is considered, and the addition $+1$ is continuously continued to a mapping $T : \mathcal{K} \to \mathcal{K}$. These dynamical systems have cocycles which are related to (the continuation) of the differences $\varphi(n) = s(n+1) - s(n)$. There are many interesting problems on these cocycles. For example, let $X_k := \varphi \circ T^{k-1}$ (considered as random variables) then the partial sums $S_k = X_1 + \cdots + X_k$ are related to the distribution of $s(n+k) - s(n)$, $n < N$. The applicant has already started to work on this subject together with P. Liardet (Marseille) and it is planed to continue this cooperation.

3.3 Digital expansions and random sequences

In [15] the distribution modulo 1 of the sequence $(s_q(n)\alpha)$ is considered in terms of the continued fraction expansion of $\alpha = [a_0; a_1, \ldots]$. It is proved that there is a positive constant $d(q)$ and for every $\gamma \ge 0$ there is a constant $c(q,\gamma) > 0$ such that the discrepancy D_N of $(\alpha s_q(n))$ mod 1 can be estimated by

$$\begin{aligned} D_N \quad \le \quad \frac{c(q,\gamma)}{\sqrt{\log N}} & \left(\sum_{i_0 - \gamma \log\log\log N \le l \le i_0} a_l + \frac{a_{i_0+1}}{(\log\log N)^{\gamma\frac{\log 2}{2} - \frac{1}{2}}} \right. \\ & \left. + \frac{(\log\log\log N)^{3/2}}{(\log\log N)^{\gamma\frac{\log 2}{2}}} \cdot \max_{l \le i_0 - \gamma \log\log\log N} a_l \right), \end{aligned}$$

where i_0 is defined by $q_{i_0} \le d(q)\sqrt{\log N \cdot \log\log N} < q_{i_0+1}$ (and $1 = q_0 \le q_1 < q_2 \ldots$ denote the best approximation denominators). Conversely there are positive constants $c'(q)$ and $d'(q)$ such that

$$D_N > \frac{c'(q)}{q_i}.$$

where the approximation denominator q_i satisfies $q_i \le d'(q)\sqrt{\log N} < q_{i+1}$. Especially if α has a bounded continued fraction expansion then

$$\frac{1}{\sqrt{\log N}} \ll D_N \ll \frac{\log\log\log N}{\sqrt{\log N}}.$$

It is therefore an interesting problem to characterize the exact order of magnitute of the discrepancy D_N in terms of the continued fraction expansion of α. Especially we are interested whether there exist α such that $D_N \ll (\log N)^{-\frac{1}{2}}$.

We also want to consider low discrepancy sequences. The best known low discrepancy sequences are net sequences which are sequences in the unit cube $[0,1)^s$ satisfying certain combinatorial properties (see [18, 50]). Due to their small discrepancy they are well suited for numerical integration (even if one uses only few points in high dimensional integrals). The usual construction of net sequences uses the digital expansions of n in the q-ary number system (the simplest case is the the Van der Corput sequence in the one dimensional case). We want to try to constuct low discrepancy sequences bases one digital expansions which are different from the q-adic ones.

3.4 Continued fraction expansions

Besides q-ary digital expansion there are lots of other possibilities to represent real numbers. One natural way, that takes the approximation proberty of a real number by rationals into account, is the continued fraction expansion.

Continued fraction theory can be used to get a better understanding of the diophantine approximation problem

$$\left| x - \frac{p}{q} \right| \le \frac{f(\log q)}{q^2} \tag{1}$$

where $x \in [0,1]$ and f is a positive, non-increasing, real-valued function defined on the non-negative real numbers. According to a famous result of Khintchine (see [39]) the inequality (1) has a.s. infinitely many solutions resp. a.s. finitely many solutions if $\sum_{k=1}^{\infty} f(k)$ diverges resp. converges (with respect to Lebesque measure which we are going to denote by λ). This result was obtained by Khintchine by using continued fraction theory.

Khintchine's Theorem was improved by several authors, see for instance [21, 44, 45, 54, 62, 69]. For example LeVeque [45] proved a central limit theorem for the following sequence of random variables

$$T_n^{(d)}(x) := \#\{(p,q) | 1 \le q \le n, (p,q) \le d, p/q \text{ is a solution of (1)}\},$$

where d is a fixed positive integer and f is a positive, non-increasing, real-valued function defined on the non-negative real numbers satisfying

$$f(x) = \mathrm{O}(x^{-1}), f'(x) = \mathrm{O}(x^{-2}), \text{ as } x \longrightarrow \infty; \quad \sum_{k=1}^{\infty} f(k) = \infty. \tag{2}$$

Fuchs [27] could generalize this result to

$$T_n(x) := \#\{(p,q) | 1 \le q \le n, p/q \text{ is a solution of (1)}\} \tag{3}$$

and more generally to

$$X_n(x) := \#\{(p,q) | 1 \le q \le n, q \equiv s \bmod r, p/q \text{ is a solution of (1)}\}. \tag{4}$$

(Philipp [54] has outlined a first proof of a central limit theorem for $T_n(x)$ but it turned out that his outline was not complete. Especially the order of magnitude of the variance was not correct.)

It is now an interesting quetion whether a law of the iterated logarithm holds for the sequence X_n, too. The results of [27] suggest that for almost all $x \in [0,1]$,

$$\limsup_{n\longrightarrow\infty} \frac{|X_n(x) - \sigma_1 F(n)|}{(2\sigma_2 F(n)\log F(n)\log\log F(n))^{1/2}} = 1,$$

where

$$F(n) := \sum_{k=1}^{n} f(k), \quad \sigma_1 = \frac{2}{r}, \quad \sigma_2 = \frac{12}{\pi^2 r C(r)} \sum_{0\le d<r,(d,r)|s} (d,r)\varphi((d,r)),$$

and

$$C(r) = r^2 \prod_{p|r}\left(1 - \frac{1}{p^2}\right).$$

For example, a result like this would imply, that Schmidt's theorem [62] is nearly best possible; especially, the log factor cannot be removed (notice that the factor $\Omega(h)^{1/2}$ in Schmidt's theorem can be removed as it was pointed out by Harman in [35]).

By using ideas of [55, 56, 57] it should be possible to prove at least

$$X_n(x) - \sigma_1 F(n) = \mathrm{O}((F(n)\log F(n))^{1/2}(\log\log F(n))^{3/2+\epsilon}) \text{ a.s.}$$

for all $\epsilon > 0$. This relation might help proving the above conjectural law of the iterated logarithm.

Harman proved in a sequence of papers (see references in [35]) analogues of Khintchine's Theorem for other restrictions different from arithmetic progressions for the denominators (e.g. primes, sum of two squares, square-free integers etc.). It would be interesting to obtain central limit theorems and iterated logarithm laws under these restrictions (of course, with suitable assumptions on f).

In a further paper [28] a common extension of LeVeque's and Philipp's result is given by proving almost sure and distribution type invariance principles. It is an interesting problem to obtain almost sure and distribution type invariance principles for the sequence (4). In order to prove such results, by using the method introduced in [27], we have to extend results in [57] to double arrays of random variables which is of own interest, too.

We further note that Khintchine's Theorem was extended in [40] to systems of diophantine inequalities

$$\left|x_i - \frac{p_i}{q}\right| \le \frac{f(\log q)}{q^2}, \ i = 1, \cdots n.$$

This result was in turn considerably improved by Schmidt in [62] who proved a strong law of large numbers with error term (compare also with [63]).

It is natural to ask whether it is possible or not to extend LeVeque's, Philipp's, and Fuchs' results to the higher dimensional case. Nothing is known in this direction. The problem in going up to higher dimensions is that there exists no suitable multidimensional continued fraction algorithm. The only method to handle the higher dimensional case works by looking more carefully on higher moments. Maybe, it is possible to prove convergence of all moments to the one of the normal distribution and applying the Frechet Shohat Theorem in order to obtain central limit theorems (compare with the remark on page 43 in [54])

Finally, in [26], we proved an analogue of LeVeque's result in the field of formal Laurent series that is the completion field of the rational function field $\mathbf{F}_q(T)$ (where $\mathbf{F}_q$ denotes a finite field) endowed with the following valuation

$$|\alpha| = q^n \text{ where } \alpha = \sum_{k \leq n} a_k T^k \in \mathbf{F}_q(T).$$

In the field of formal Laurent series, there is a continued fraction expansion (see [65]) and the metric theory is quite simple because of the fact that the valuation is ultrametric e.g. the sequence of partial quotients is an independent and identically distributed sequence of random variables (see [49]).

It is also interesting to find analogues of the results proved in [27] and [28] in the Laurent series case. In order to do this we have to obtain analogues of several classical results belonging to the metric theory of continued fraction expansion e.g. an analogue of a theorem due to Szüsz (see [70]) which extends the well known Theorem of Gauss-Kuzmin-Levy. Indeed, such an analogue would entail lots of consequences that are of own interest, too.

Besides continued fraction expansion also other digital expansions could be considered in the Laurent series case. Here polynomials play the rôle of integers, whereas normed polynomials are the positive ones. It seems that nearly nothing has been done concerning this question and there is a wide area of research possibilities. For instance, one could think to prove distribution results for the sum-of-digits function or more general G-additive functions (G is suitable number system) like distribution and joint distribution in residue class, asymptotic normality of the sum-of-digits function (or more general G-additive functions) considered on all polynomials, on a polynomial sequence of polynomials, on irreducible polynomials etc.

3.5 Automatic asymptotic distributions

In [13, 14] Drmota and Gajdosik have used bivariate generating functions (compare also with [33]) to prove that the G-ary sum-of-digits function $s_G(n)$ satisfies a central limit theorem. Set

$$G(z,u) = \sum_{j \geq 1} \left(\sum_{l=0}^{a_j - 1} z^l \right) z^{a_1 + \cdots + a_{j-1}} u^j$$

and let $1/\alpha(z)$ denote the (analytic) solution $u = 1/\alpha(z)$ of the equation

$$G(u,z) = 1$$

for z in a sufficiently small (complex) neighbourhood of $z_0 = 1$ such that $\alpha(1) = \alpha$. Then

$$\mathbf{E}_N := \frac{1}{N} \sum_{n<N} s_G(n) = \mu \frac{\log N}{\log \alpha} + \mathcal{O}(1),$$

$$\mathbf{V}_N := \frac{1}{N} \sum_{n<N} (s_G(n) - \mathbf{E} X_N)^2 = \sigma^2 \frac{\log N}{\log \alpha} + \mathcal{O}(1)$$

and

$$\frac{1}{N} |\{n < N : s_G(n) < \mathbf{E}_N + x \mathbf{V} X_N\}| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt + \mathcal{O}((\log N)^{-1/2+\varepsilon})$$

uniformly for all real x as $N \to \infty$, where

$$\mu = \frac{\alpha'(1)}{\alpha} \quad \text{and} \quad \sigma^2 = \frac{\alpha''(1)}{\alpha} + \mu - \mu^2.$$

The reason for this property is that the bivariate gererating function

$$B(z,u) = \sum_{j,k\ge 0} |\{n < G_j : s_G(n) = k\}|\, z^j u^j.$$

is given by

$$B(z,u) = \frac{P(z,u)}{1 - G(z,u)}$$

for some polynomial $P(z,u)$. Thus, $u = 1/\alpha(z)$ describes the surface of dominant singularities of $B(z,u)$ and after some work (see [14]) one obtains the above relations.

This kind of *transfer result* from properties of generating funcions to asymptotic properties of the coefficients is quite common in analytic combinatorics (see [22] for an overview). There even exists a MAPLE-program "equivalent" (see [59]) which generates in a totally formal and automatic (and rigorous) way the asymptotic expansion of the Taylor coefficients of a given function in one variable. However, at the moment there is no corresponding tool for distributional results (as above) where the input is a bivariate generating function.

The aim of this part of the project is to develop a MAPLE-program "gaussian" which decides whether the distribution behind the coefficients of a given bivariate generating function is Gaussian or not. The first step is to descrribe the surface of dominant singularities and the kind of singularity (polar, algebraic, essetial). The second step is then extract the kind of limiting distbution and the normalizing parameters. (The above example indicates that this can be done in a formal way, for a list of possible *transfer results* which can be implemented without major problems, see [22]).

It is also interesting to classify the kind of limiting distribution if it is not Gaussian and to obtain bivariate asymptotic expansions of the coefficients of the bivariate generating function (compare with [16]).

Thomas Klauser (who is employed by the present project) already has frequent contact to the group of P. Flajolet and B. Salvy (INRIA, Rocquencourt) who have developed the MAPLE-program "equivalent",. He has also an excellent knowledge in MAPLE and analytic combinatorics.

This program "gaussian" (and its extensiosn) will be applicable to several problems on digital expansions as well as to analytic combinatorics.

3.6 Working Program

Top 1. In "Year 3" Steiner should treat the joint distribution $(f_1(P_1(n)), f_2(P_2(n)))$, $n < N$, where f_1, f_2 are q- or G-additive functions (possibly with different q or G) and $P_1(x), P_2(x)$ are positive integer polynomials.

In "Year 4" Fuchs and Steiner should try to get estimates for the exponential sums $\sum e(x \cdot s_q(P(n)))$, where $P(x)$ is a non-linear positive integer polynomial.

Top 2. In "Year 3" Drmota should discuss (jointly with P. Liardet from Marseill) the distibution of $s_q(n+k) - s_q(n)$.

In "Year 4" Steiner should try to solve the problem concerning the relation of G-ary digital expansions and Markov partitions of corresponding automorphisms if the constant term $a_d > 1$.

Finally, in "Year 5" Steiner should discuss the box-counting and the Hausdorff dimensions of the boundaries of fractals which are related to automata.

Top 3. In "Year 4" Drmota should discuss (jointly with Larcher) the exact order of magnitude of the discrepancy of the sequence $(s_q(n)\alpha)$ mod 1.

In "Year 5" Drmota should discuss (jointly with Larcher) the possibility of constructing low discrepancy sequences with digital expansions different to the q-adic ones.

Top 4. In "Year 3" Fuchs should try to prove a law of the iterated logarithm for the sequence of random variables X_n (see (4)).

In "Year 4" Fuchs should discuss invariance principles for the sequence (4) and should try to provide multivariate analoga.

In "Year 5" Fuchs should consider distributional properties of digital expansions for formal Laurent series.

Top 5. In "Year 3" Klausner should complete a MAPLE-program "gaussian" which decides in formal way whether the coefficients of a bivariate generating functions obey a Gaussian limiting distribution.

In "Year 4" Drmota and Klausner should extend this program to specific non-Gaussian limiting distributions.

In "Year 5" Drmota and Klausner should provide a further extension. A MAPLE-program should produce bivariate asymptotic expansions for the coefficients of a bivariate generating functions.

	Top 1	Top 2	Top 3
Year 3	distribution of $(f_1(P_1(n)), f_2(P_2(n)))$ [Steiner]	cocyles in odometers [Drmota]	
Year 4	estimates for $\sum e(x \cdot s_q(P(n)))$ [Fuchs, Steiner]	Makov partitions for the case $a_d > 1$ [Steiner]	discrepancy of $(\alpha s_q(n))$ [Drmota]
Year 5		fracals, dimension, automata [Steiner]	low discrepancy sequences [Drmota]

	Top 4	Top 5
Year 3	iterated loarithm for X_n [Fuchs]	Gaussian limiting distributions [Klausner]
Year 4	invariance principles, higher dimensions [Fuchs]	non-Gaussian limiting distributions [Drmota, Klausner]
Year 5	digital expansions for formal Laurent series [Fuchs]	bivariate asymptotic expansions [Drmota, Klausner]

4 Position within the FSP, Internal Cooperations

The most intensive cooperations within the FSP are with *G. Larcher* (Analysis of Digital Point Sets) and to *R. F. Tichy* and *P. Kirschenhofer* (Algorithmic Diophantine Problems). It is also planed to cooperate with *P. Kirschenhofer's* new project.

The paper [17] was written together with Larcher and is devoted to the distribution of the sequence $(s_q(n)\alpha)$ mod 1 (compare with the above description). It is planed to continue this cooperation in the future. We also want to consider low discrepancy sequences.

There have been several contact to Tichy and Kirschenhofer and to their groups (in Graz and at conferences in Budapest and Marseille). There are several common interests, e.g. in Baker's method which has been used in papers [11, 25]. Kirschenhofer's new project ist also partly to statistical properties of digital expansions. Thus, these contacts will be surely more frequent in the second period of the FSP.

The transparent structure of the FSP makes cooperations between researchers in number theory much easier and much more efficient. Especially for young researchers (PhD students and post-docs) it is very important to have the possiblity to attend meetings and to have personal contact to other researchers (within and outside the FSP, at meetings, with guests, etc.). The FSP provides an excellent forum to realize all this – without an FSP this would be much more difficult.

5 External Cooperations

Within the first two years several conferences has been attended and several foreign guest have been invited. So it was easy to start cooperations with persons and institutions outside the FSP.

For example, E. Manstavicius (University of Vilnius, Lithuania) was invited and a joint paper [12] was written on functional limit theorems related to digital expansions.

Furthermore, Drmota was invited by P. Liardet (Université de Provence, Marseille, Febr. 2000) to stay in Marseille for one month. We started to work on the distribution of co-cycles of odometers, a problem which can be translated to the distribution of the sequence $s_q(n+k) - s_q(n)$.

Two further coopertations have been started with Austrian mathematicians who are not related to the FSP. It is planed to work together with *J. Schoissengeier* (University of Vienna) on digital expansions with respect to different bases and with *P. J. Grabner* (TU Graz) on the relation between the binary and the Zeckendorf representations of integers.

We will continue these cooperations, epecially with P. Liardet, J. Schoissengeier and P. J. Grabner.

6 Existing Infrastructure

The present project uses the infrastructure of the *Department of Geometry* of the *Vienna University of Technology.*
The *project leader* works at this department.
The project can use *2 rooms* and *4 computers.*

Name	Research Field	Dept.	%
M. Drmota	Number Theory	Geometry	50%
consulting	Since	Kind of Contract	%
P-leader	2000	Ass.Prof.	full position

7 Information on the Requested Funding

In total we apply for 2.5 post-doc positions.

For the first two years of this project 1.5 post-doc position have been granted. The amount of money for 1.5 post-doc positions is (almost) equivalent to 2.5 doctorands – at the moment the doctorands Michaes Fuchs (100%), Thomas Klausner (50%), Wolfgang Steiner (100%) are employed by the project. All three of them are very valuable for the present project (e.g. Fuchs has written 4 paper and Steiner

3 within the first two years). They will finish ther PhD-thesis in 2002 so that an *upgrade* of their position will be necessary.

The first (full) position is applied for *Michael Fuchs*. He should work on topic 4 (Continued fraction expansions) and partly on topic 1 (Digital expansions of polynomial sequences).

The second (full) position is applied for *Wolfgang Steiner*. He should work on topics 1 (Digital expansions of polynomial sequences) and 2 (Digital expansions and dynamical systems).

The third (half) position is applied for *Thomas Klausner*. He should work on topic 5 (Automatic asymptotic distributions).

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8 Curricula Vitae, Lists of Publications

Michael Drmota

Personal Data
Date of Birth: 17.07.1964
Place of Birth: Vienna, Austria
Nationality: Austria

Education
1982–1986: Study of *Technische Mathematik* at the Vienna University of Technology.
1982–1987: Study of *Elektrotechnik* at the Vienna University of Technology.
1986: Master's degree (Dipl.Ing.), Vienna University of Technology.
1986: PhD degree (Dr.techn.), Vienna University of Technology.

Career History
Since 04/1986: Univ.Assistent at the Dept. of Algebra (since 1999 Dept. of Geometry), Vienna University of Technology.
1990: Habilitation, Vienna University of Technology.
09/1992: Visiting Professor at Université Paris VI
09/1999: Visiting Professor at Université de Versailles
02/2001: Visiting Professor at Université de Provence, Marseille

Career-related Activities
Since 2000: Editor of *Internationale Mathematische Nachrichten* and member of the board of the Austrian Mathematical Society

Awards
1992: Hlawka-Prize of Austrian Academy of Sciences
1996: "Förderungspreis" of the Austrian Mathematical Society

Memberships
Österreichische Mathematische Gesellschaft (ÖMG)
Deutsche Mathematiker-Vereinigung (DMV)
American Mathematical Society (AMS)

Research Interests
Number Theory
Combinatorial Enumeration and Analysis of Algorithms
Stochastic Processes in Combinatorial Structures

Publications
70 publications in scientific journals, 10 invited lectures

Recent Publications

1. M. Drmota, Irregularities of Distributions with Respect to Polytopes, *Mathematika* **43** (1996), 108–119.

2. G. Baron, M. Drmota and L. Mutafchiev, Predecessors of a random mapping, *Combinatorics, Probability and Computing* **5** (1996), 317–335.

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5. M. Drmota and M. Soria, Marking in Combinatorial Constructions: Images and Preimages in Random Mappings, *SIAM J. Discrete Math.* **10** (1997), 246–269.

6. M. Drmota and B. Gittenberger, On the profile of random trees, *Random Structures and Algorithms* **10** (1997), 421–451.

7. M. Drmota, Systems of functional equations, *Random Structures and Algorithms* **10** (1997), 103–124.

8. M. Drmota, On nodes of given degree in random trees, in: *Probabilistic Methods in Discrete Mathematics* (V.F. Kolchin et al. eds.), VSP, Utrecht, 1997, 31–44.

9. M. Drmota, q-additive functions and well distribution modulo 1, *Demonstratio Math.* **30** (1998), 883–896.

10. M. Drmota and J. Gajdosik, The parity of the sum-of-digits-function of the Zeckendorff representation and related questions, *Fibonacci Q.* **36** (1998), 3–19.

11. M. Drmota and J. Gajdosik, The distribution of the sum of digits functions, *J. Theor. Nombres Bordx.* **10** (1998), 17–32.

12. T. Bruss, M. Drmota and G. Louchard, The Complete Solution of the Competitive Rank Selection Problem, *Algorithmica* **22** (1998), 413–447.

13. M. Drmota and B. Gittenberger, Strata of random mappings - A combinatorial approach, *Stoch. Proc. Appl.* **82** (1999), 157–171.

14. M. Drmota and B. Gittenberger, The distribution of nodes of given degree in random trees, *J. Graph Th.* **31** (1999), 227–253.

15. M. Drmota, Analysis of generating functions and probabilities on trees, in: *Probability Theory and Mathematical Statistics, Proceedings of the Seventh Vilnius Conference (1998)*, B. Grigelionis et al. (Eds), TEV/VSP, Vilnius/Utrecht, Tokyo, 1999, 213–228

16. M. Drmota and M. Skalba, Rarified Sums of the Thue-Morse Sequence, *Trans. Amer. Math. Soc.* **352** (2000), 609–642.

17. M. Drmota, The distribution of patterns in digital expansions, *in: Algebraic Number Theory and Diphantine Analysis* (F. Halter-Koch and R. F. Tichy eds.), de Gruyter, Berlin, 2000, 103–121.

18. M. Drmota and M. Skalba, The parity of the Zeckendorf sum-of-digits-function, *manuscripta math.* **101** (2000), 361–383.

19. M. Drmota, The saturation level in binary search trees, *in: Mathematics and Computer Science, Algorithms, Trees, Combinatorics and Probabilities (D. Gardy and A. Mokkadem eds)*, Birkhäuser, Basel, 2000, 41–51.

20. M. Drmota, An analytic approach to the height of binary search trees, *Algorithmica* **29** (2001), 89–119.

21. M. Drmota, D. Gardy, and B. Gittenberger, A unified presentation of some urn models, *Algorithmica.* **29** (2001)

22. M. Drmota and G. Larcher, The sum-of-digits-function and uniform distribution modulo 1, *J. Number Theory*, **89** (2001), 65–96.

23. M. Drmota, The asymptotic number of leftist trees, *Algorithmica*, to appear.

24. B. Chauvin, M. Drmota, and J. Jabbour-Hattab, The profile of binary search trees, *Ann. Appl. Prob.*, to appear.

25. M. Drmota, The variance of the height of binary search trees, *Theoret. Computer Science*, to appear.

26. M. Drmota, The joint distribution of q-additive functions, *Acta Arith.*, to appear.

Recent Invited Lectures

1. M. Drmota, *An Analytic Approach to Branching Processes and Brownian Excursion*, Probabilistic Methods in Discrete Mathematics, Petrozavodsk, Rußland, 3.–7. Juni 1996.

2. M. Drmota, *Die Parität der Ziffernsummenfunktion*, 14. Österreichischer Mathematikerkongreß, Salzburg, 22.–26. 9. 1997

3. M. Drmota, *Merkwürdige Dezimalzahlen*, Lehrervortbildungstag der ÖMG, Universität Wien, 17. 4. 1998

4. M. Drmota, *Analysis of generating functions and probabilities on trees,* 22^{nd} European Meeting of Statisticians and 7^{th} Vilnius Conference on Probability Theory and Mathematical Statistics, 12.–18. 8. 1998.

5. M. Drmota, *The parity of the sum-of-digits function and multiplies of primes*, Paul Erdős and his Mathematics, Budapest, Ungarn, 4.–11. 7. 1999.

6. M. Drmota, *Generating Functions and Probabilities on Trees*, University of Verailles-St. Quentin, Versailes, Frankreich, 17. 9. 1999.

7. M. Drmota, *The Joint Distribution of q-Additive Functions*, Workshop on Probabilistic Number Theory, Urbana-Champaign, 19.–20. 5. 2000.

8. M. Drmota, *The Profile and Height of Binary and m-Ary Search Trees*, The Sixth Seminar on the Mathematical Analysis of Algorithms, Krynica Morska, Polen. 3.–7. 7. 2000.

Recent Oral Contributions

1. M. Drmota, *On Zeckendorff representations of multiples of 3*, Seventh International Conference on Fibonacci Numbers and Their Applications, Graz, 15.–19. 7. 1996.

2. M. Drmota, *Irregularities with Respect to Polytopes*, Analytic and Elementary Number Theory, Wien, 18.–20. Juli 1996.

3. M. Drmota, *Polynomial Relations of Polynomial Roots*, International Conference on Number Theory, Eger, 29. Juli–2. August 1996.

4. M. Drmota, *On the profile of random trees*, $4^t h$ World Congress of the Bernoulli Society, 26.–31. 8. 1996, Wien.
5. M. Drmota, *Das Profil von Zufallsbäumen*, 24. Oktober 1996, Uiversität Wien.
6. M. Drmota, *An analytic approach to the height of binary trees*, "Average-Case"-Analyse von Algorithmen, Schloß Dagstuhl, BRD, 7.–11. 7. 1997.
7. M. Drmota, *The distribution of nodes of given degree in random trees*, "Random Structures and Algorithnms", Posnan, 4.–8. 8. 1997.
8. M. Drmota, *Die Höhe von binären Suchbäumen*, Universität Frankfurt, Frankfurt am Main, 18. 2. 1998.
9. M. Drmota, *Verzeigungprozesse und Brownsche Exkursion*, Universität Frankfurt, Frankfurt am Main, 19. 2. 1998.
10. M. Drmota, *Die Höhe von binären Suchbäumen*, Montanuniversität Leoben, 20. 3. 1998.
11. M. Drmota, *Neuere Entwicklungen der Zahlentheorie*, Seminarreihe: Wissenswertes aus der Mathematik, TU-Wien, 11. 5. 1998.
12. M. Drmota, *The Fourier Transform of Characteristic Functions and Estimates for the Discrepancy*, Complexity of Analytic Algorithms, Schloß Dagstuhl, BRD, 18.–22. 5. 1998.
13. M. Drmota, *An analytic approach to the height of binary search trees*, "Average-Case"-Analysis of Algorithms, DIMACS, University of Princeton, USA, 13.–17. 7. 1998.
14. M. Drmota, *The distribution of the sum-of-digits function*, Confererence on Algebraic Number Theory and Diophantine Analysis, TU-Graz, 31. 8.–4. 9. 1998.
15. M. Drmota, *Systems of Functional Equations and Patterns in Random Trees*, The Fifth Seminar on the Mathematical Analysis of Algorithms, CIRM, Barcelona, Spanien, 14.–18. 6. 1999.
16. M. Drmota, *Functional Equations and Probabilities on Trees* Labri, Bordeaux, Frankreich, 15. 9. 1999.
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22. M. Drmota, *The Joint Distribution of q-Additive Functions*, Universität Marseille (Luminy), Marseille, Frankreich, 26. 9. 2000.

23. M. Drmota, *The Height of Binary and Digital Search Trees*, Finite and Infinite Combinatorics, Budapest, Ungarn, 4.–10. 1. 2001.

24. M. Drmota, *Digital Expansions of Polynomial Sequences*, Workshop: Dynamic Odyssee, Luminy, Marseille, Frankreich, 11.–20. 2. 2001.

25. M. Drmota, *The Height of Binary and Digital Search Trees*, Purdue University, Indiana, USA, 28. 2. 2001.

26. M. Drmota, *The Height of Digital Search Trees*, Seventh Seminar on Analysis of Algorithms, Tatihou, France. 1.–7. 7. 2001

27. M. Drmota, *The Moment Method for the Sum-of-Digits Function*, Université de Provence, Marseille, 12. 7. 2001.

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Continued Fractions and Metric Diophantine Approximation
Limit Theorems for Weakly Dependent Random Variables

Recent Publications

1. M. Fuchs, *Digital Expansion of Exponential Sequences*, Journal de Theorie des Nombres de Bordeaux, to appear.
2. M. Fuchs, *On Metric Diophantine Approximation in the Field of Formal Laurent Series*, submitted.
3. M. Drmota, M. Fuchs, E. Manstavicius, *Functional Limit Theorems for Digital Expansions*, submitted.
4. M. Fuchs, *On a Problem of W.J.LeVeque*, manuscript.
5. M. Fuchs, *Invariance Principles in Metric Diophantine Approximation*, manuscript.

Posters and Oral Contributions

1. M. Fuchs: *Digital Expansions of Exponential Sequences*, Finite and Infinite Combinatorics, Budapest, Hungary, 08.01.2001.
2. M. Fuchs: *Digital Expansions of Exponential Sequences*, L'odyssée Dynamique, Marseille, France, 15.02.2001.

Diploma and Doctoral Theses

1. M. Fuchs, *Kettenbrüche im Körper der Formalen Laurentreihen und Anwendungen*, Master Thesis (2000), Vienna University of Technology.
2. M. Fuchs, *Limit Theorems in Metric Diophantine Approximation*, PhD-Thesis (1999-), ongoing, Vienna University of Technology.

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Diploma and Doctoral Theses

1. T. Klausner, *Erzeugende Funktionen von Graphen*, Master Thesis (2000), Vienna University of Technology.
2. T. Klausner, *Asymtptotic Expansions and Limiting Distributions with MAPLE*, PhD-Thesis (1999-), ongoing, Vienna University of Technology.

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Rauzy Fractals

Recent Publications

1. W. Steiner, *Generalized de Bruijn digraphs and the distribution of patterns in α-expansions*, Discrete Math., to appear.
2. M. Drmota, W. Steiner, *The Zeckendorf expansion of polynomial sequences* , submitted.
3. W. Steiner, *Parry expansions of polynomial sequences*, submitted.

Posters and Oral Contributions

1. W. Steiner: *Generalized de Bruijn graphs and digital expansions*, Finite and Infinite Combinatorics, Budapest, Hungary, 09.01.2001.
2. W. Steiner: *Markov partitions of toral automorphisms and digital expansions*, L'odyssée Dynamique, Marseille, France, 15.02.2001.
3. W. Steiner: *Digital expansions and Rauzy fractals*, Fractals in Graz, Graz, Austria, 07.06.2001.

Diploma and Doctoral Theses

1. W. Steiner, *Digital expansions and the distribution of related functions*, Master Thesis (2000), Vienna University of Technology.
2. W. Steiner, *Digital expansions of polynomial sequences*, PhD-Thesis (1999-), ongoing, Vienna University of Technology.