PROFILES OF m-ARY SEARCH TREES

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Summery

- Quicksort
- Probabilistic Model
- Profile of Trees
- Function Spaces
- Contraction Method
- Cauchy Integral

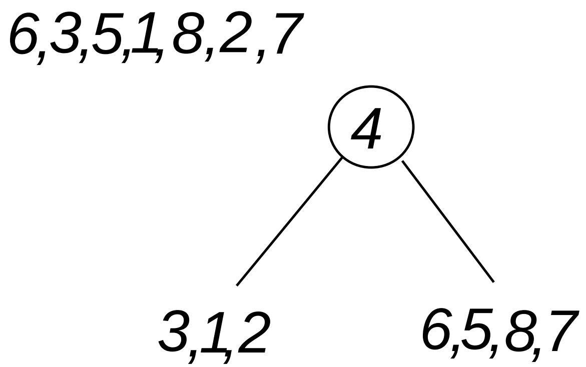
Quicksort

Sorting of data

4,6,3,5,1,8,2,7

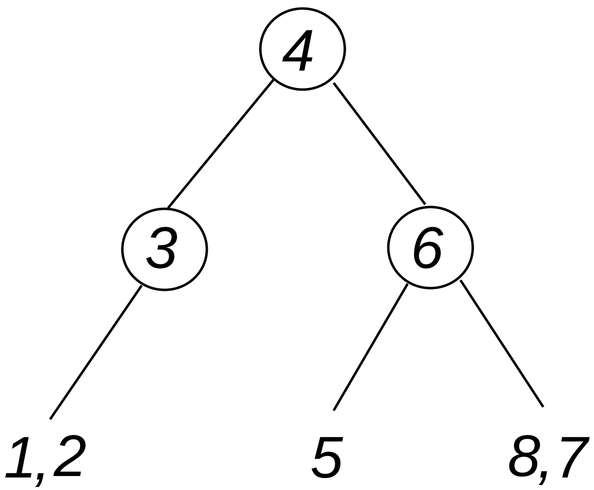
Quicksort

Sorting of data



Quicksort

Sorting of data



Quicksort

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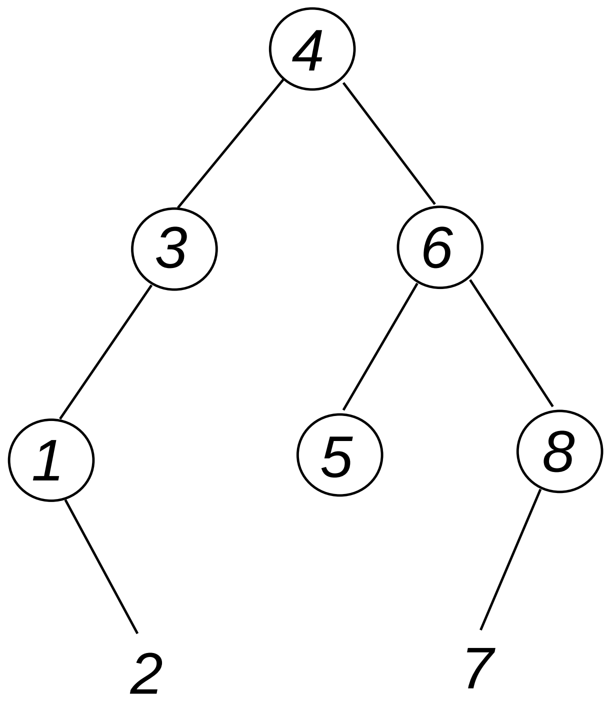

Quicksort

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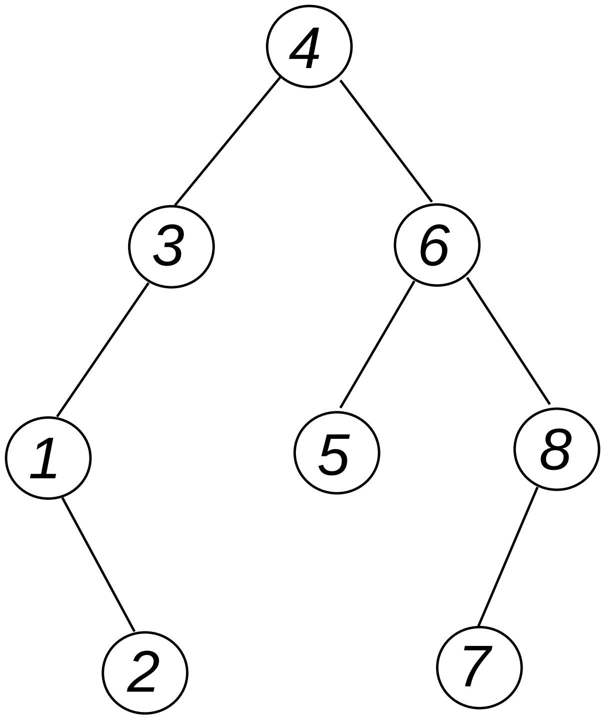

Binary Search Tree

Storing of data

4,6,3,5,1,8,2,7

Binary Search Tree

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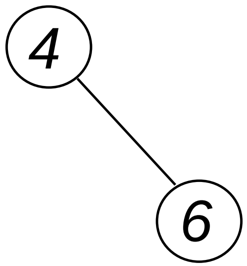
Binary Search Tree

Storing of data

3,5,1,8,2,7



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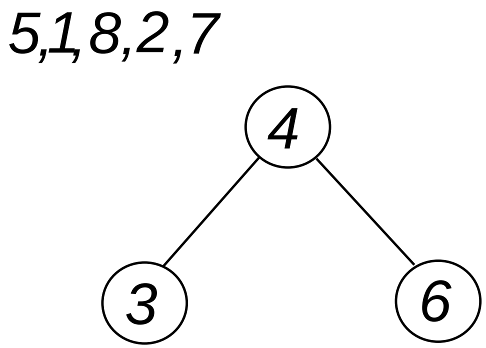

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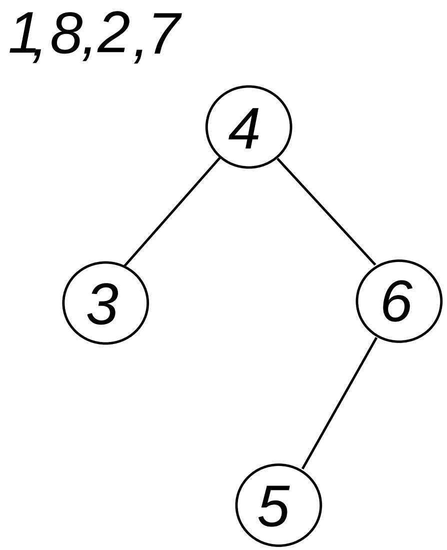

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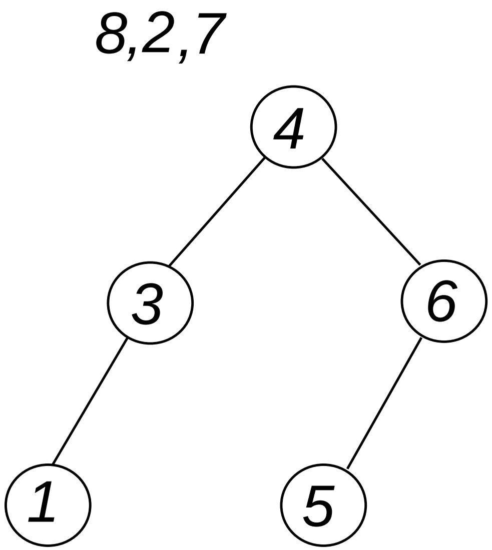

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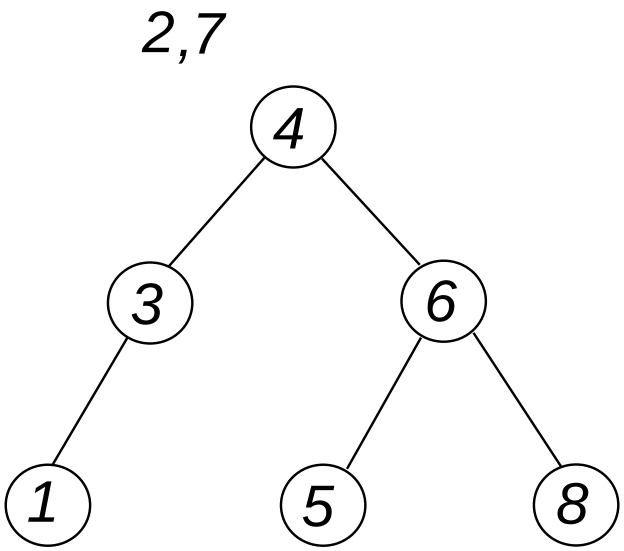

Binary Search Tree

Storing of data

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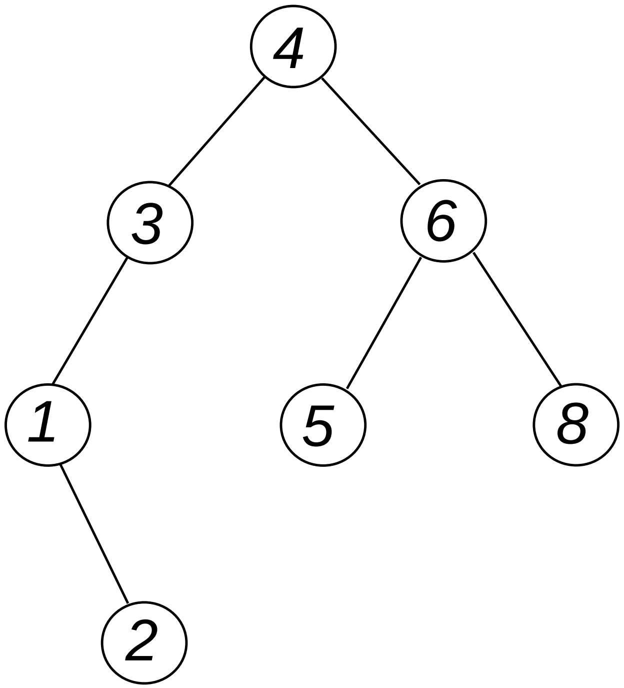
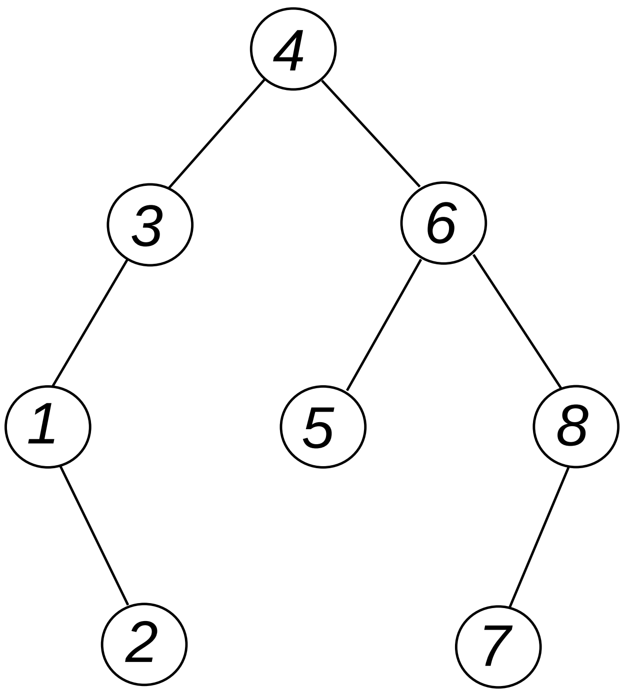


Binary Search Tree

Storing of data



Quicksort

Median of 3

4,6,3,5,1,8,2,7

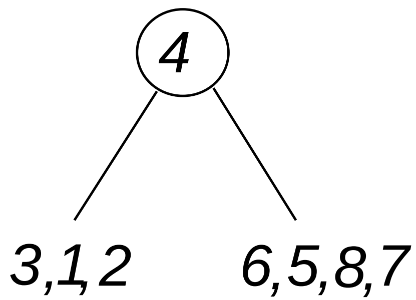
Quicksort

Median of 3

4,6,3,5,1,8,2,7

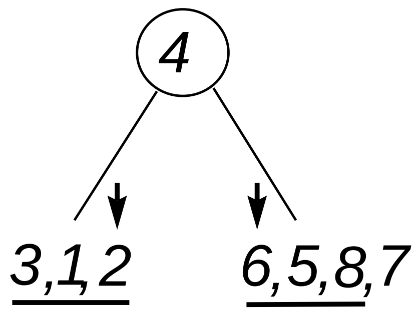
Quicksort

Median of 3



Quicksort

Median of 3



Quicksort

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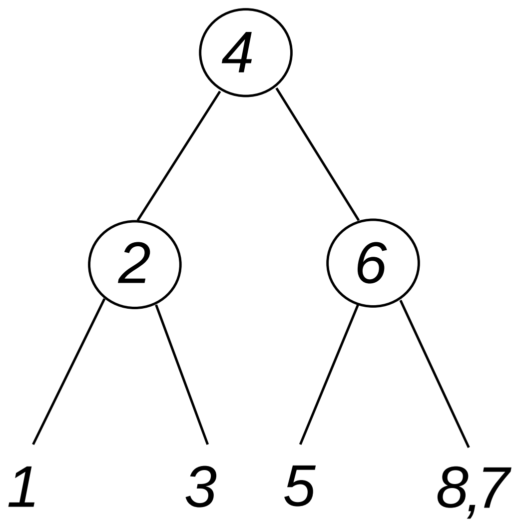
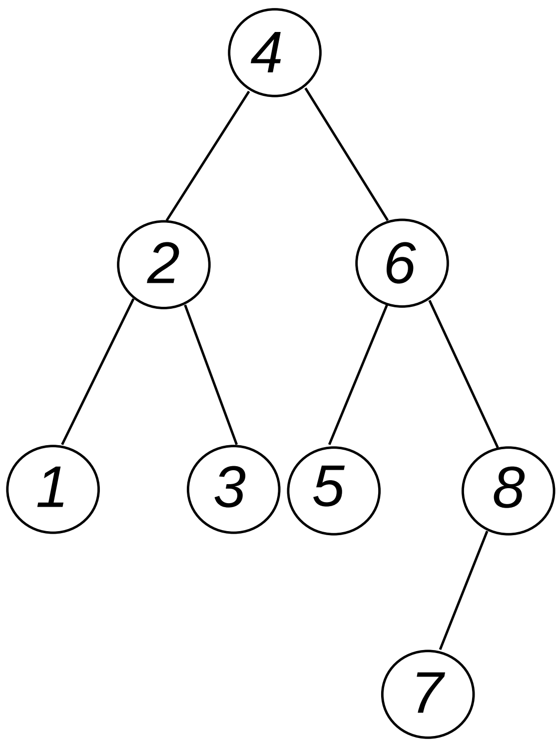


Quicksort

Median of 3



Probabilistic Model

Every permutation on the data $\{1, 2, \ldots, n\}$ ist equally likely

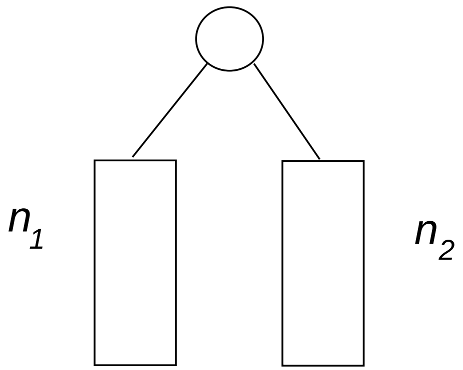
$\longrightarrow$ probability distribution on binary (m-ary) trees of size n

$\longrightarrow$ all tree parameters are **random variables**

Probabilistic Model

Recursive structure

Subtrees have the same structure:
$(n = n_1 + n_2 + 1)$.



Splitting probabilities: p_{n_1,n_2}

Quicksort: $$p_{n_1,n_2} = \frac{1}{n}$$

Median of 3: $$p_{n_1,n_2} = \frac{n_1 n_2}{\binom{n}{3}}$$

Probabilistic Model

General Model

$m \geq 2$, $t \geq 0$ ... given integers $\qquad$ n keys (data)

- If $n \geq m$, we **randomly select** $m-1$ **pivots** $\boxed{x_1 < x_2 < \cdots < x_{m-1}}$.
- The pivots are stored in the **root**.
- The remaining $n-m+1$ keys are divided into m **subsets** $I_1, \ldots, I_m$:

  $I_1 := \{x_i : x_i < x_1\}$, $I_2 := \{x_i : x_1 < x_i < x_2\}$, $\ldots, I_m := \{x_i : x_{m-1} < x_i\}$.
- Apply this procedure **recursively** to $I_1, I_2, \ldots, I_m$.

Probabilistic Model

General Splitting Probabilities

$\mathbf{V}_n = (V_{n,1}, V_{n,2}, \ldots, V_{n,m})$.. random splitting vector

$V_{n,k} := |I_k|$ ... number of keys in the kth subset
(= the number of nodes in the kth subtree of the root)

$$V_{n,1} + V_{n,2} + \cdots + V_{n,m} = n - (m-1) = n + 1 - m$$

$$\boxed{\mathbb{P}\{\mathbf{V}_n = (n_1, \ldots, n_m)\} = \frac{\binom{n_1}{t} \cdots \binom{n_m}{t}}{\binom{n}{mt+m-1}}}$$

$$(n_1 + n_2 + \cdots + n_m = n - m + 1)$$

Quicksort: $m = 2$, $t = 0$ **Median of 3:** $m = 2$, $t = 1$

Probabilistic Model

One-dimensional projection

$$\mathbb{P}\{V_{n,j} = \ell\} = \frac{\binom{\ell}{t}\binom{n-\ell-1}{(m-1)t+m-2}}{\binom{n}{mt+m-1}}$$

Profile of Trees

Parameters of interest:

- **Depth** of a random node: D_n
- **Internal path length**: I_n (sum of all distances to the root)
- **Height** H_n
- **Profile** $X_{n,k}$ (number of nodes at depth k)

Remark:

Number of comparisions in Quicksort = internal path length I_n

Profile of Trees

Relations to the profile $X_{n,k}$:

- $\mathbf{Pr}\{D_n = k\} = \frac{1}{n}\,\mathbf{E}\,X_{n,k}$

- $I_n = \sum_{k\ge 0} k X_{n,k}$

- $H_n = \max\{k \ge 0 : X_{n,k} > 0\}$

- The profile describes the **shape** of the tree.

Profile of Trees

Recursive relation:

$$\boxed{X_{n,k} \stackrel{\mathsf{d}}{=} X^{(1)}_{V_{n,1},k-1} + X^{(2)}_{V_{n,2},k-1} + \cdots + X^{(m)}_{V_{n,m},k-1}}$$

$(X^{(j)}_{n,k})_{k\geq 0}$, $j = 1, \ldots, m$ ... independent copies of $X_{n,k}$

Profile of Trees

Expected Profile

$$F(\theta) := \frac{t!}{m(mt+m-1)!}(\theta+t)(\theta+t+1)\cdots(\theta+mt+m-2),$$

$\boxed{\lambda_1(z)}$, $\lambda_2(z)$, …, $\lambda_{(m-1)(t+1)}(z)$ ... roots of $\boxed{F(\theta) = z}$:
$\boxed{\Re(\lambda_1(z))} \geq \Re(\lambda_2(z)) \geq \ldots .$

$\beta(\alpha) > 0$ ... defined by $\boxed{\beta(\alpha)\lambda_1'(\beta(\alpha)) = \alpha}$.

$$\alpha = \alpha_0 := \left(\frac{1}{t+1} + \frac{1}{t+2} + \cdots + \frac{1}{(t+1)m-1}\right)^{-1}$$

Profile of Trees

Expected Profile $\boxed{k = \alpha \log n}$

- $0 < \alpha = k/\log n < \alpha_0$:

$$\boxed{\mathbf{E}\, X_{n,k} \sim (m-1)m^k}.$$

- $\alpha = k/\log n > \alpha_0$:

$$\boxed{\mathbf{E}\, X_{n,k} \sim \frac{E(\beta(\alpha))n^{\lambda_1(\beta(\alpha))-\alpha\log(\beta(\alpha))-1}}{\sqrt{2\pi(\alpha+\beta(\alpha)^2\lambda_1''(\beta(\alpha)))\log n}}}$$

  for some continuous function $E(z)$

  Note: $m^k = n^{\alpha \log m}$

Profile of Trees

Expected Profile

$$\alpha_{\mathsf{max}} := \left(\frac{1}{t+2} + \frac{1}{t+3} + \cdots + \frac{1}{(t+1)m}\right)^{-1}$$

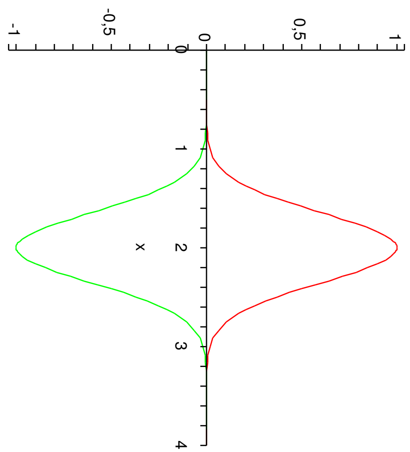
$$\boxed{\mathbf{E}\, X_{n,k} \sim \frac{n}{\sqrt{2\pi(\alpha_{\mathsf{max}} + \lambda_1''(1))\log n}} \exp\left(-\frac{(k-\alpha_{\mathsf{max}}\log n)^2}{2(\alpha_{\mathsf{max}} + \lambda_1''(1))\log n}\right)}$$

($\Longrightarrow$ CLT for depth D_n)

Profile of Trees

The average profile: $m = 2$, $t = 0$ (special case)

$$\boxed{\mathbf{E}\, X_{n,k} = \frac{n}{\sqrt{4\pi \log n}} \left(e^{-\frac{(k-2\log n)^2}{4\log n}} + \mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right)\right)}.$$



Profile of Trees

Theorem 1 $m \ge 2$, $t \ge 0$ ... *given integers*
$(X_{n,k})_{k\ge 0}$ ... *random profile*

$I = \{\beta > 0 : 1 < \lambda_1(\beta^2) < 2\lambda_1(\beta) - 1\}$,
$I' = \{\beta\lambda_1'(\beta) : \beta \in I\}$

$$\boxed{\beta(\alpha)\lambda_1'(\beta(\alpha)) = \alpha}.$$

$\Longrightarrow$

$$\boxed{\left(\frac{X_{n,\lfloor \alpha \log n \rfloor}}{\mathbf{E}\, X_{n,\lfloor \alpha \log n \rfloor}}, \alpha \in I'\right) \xrightarrow{\mathrm{d}} \Big(Y(\beta(\alpha)), \alpha \in I'\Big)}$$

in $D(I')$ *(Skorohod topology).*

Profile of Trees

Random analytic functions

$B \subseteq \mathbb{C}$, $(I \subseteq B)$ $Y(z)$ ... random analytic function on B

$$\boxed{Y(z) \stackrel{\mathrm{d}}{=} zV_1^{\lambda_1(z)-1}Y^{(1)}(z) + zV_2^{\lambda_1(z)-1}Y^{(2)}(z) \cdots + zV_m^{\lambda_1(z)-1}Y^{(m)}(z)}$$

$Y^{(j)}(z)$ ... independent copies of $Y(z)$

$\mathbf{V} = (V_1, V_2, \ldots, V_m)$ ... random vector supported on the simplex $\Delta = \{(s_1, \ldots, s_m) : s_j \geq 0, s_1 + \cdots + s_m = 1\}$ with density

$$f(s_1, \ldots, s_m) = \frac{((t+1)m-1)!}{(t!)^m} (s_1 \cdots s_m)^t.$$

$\mathbf{V}, Y^{(1)}(z), \ldots, Y^{(m)}(z)$ ... independent.

Profile of Trees

Profile Polynomials

$$\boxed{W_n(z) := \sum_k X_{n,k} z^k}$$

$$X_{n,k} \stackrel{\mathrm{d}}{=} X^{(1)}_{V_{n,1},k-1} + X^{(2)}_{V_{n,2},k-1} + \cdots + X^{(m)}_{V_{n,m},k-1},$$

$\Longrightarrow$

$$\boxed{W_n(z) \stackrel{\mathrm{d}}{=} zW^{(1)}_{V_{n,1}}(z) + zW^{(2)}_{V_{n,2}}(z) + \cdots + zW^{(m)}_{V_{n,m}}(z) + m - 1}$$

for $n \geq m$

Profile of Trees

Profile Polynomials

Theorem 2 B ... *complex region,* $(1/m, \beta(\alpha_+)) \in B$,

$$\lambda_1(\beta(\alpha_+)) - \alpha_+ \log(\beta(\alpha_+)) - 1 = 0.$$

$\Longrightarrow$

$$\boxed{\left(\frac{W_n(z)}{\mathbf{E}\, W_n(z)}, z \in B\right) \xrightarrow{\mathrm{d}} (Y(z), z \in B)}$$

in $\mathcal{H}(B)$.

Remark Th. 2 $\Longrightarrow$ Th. 1

Profile of Trees

Profile Polynomials

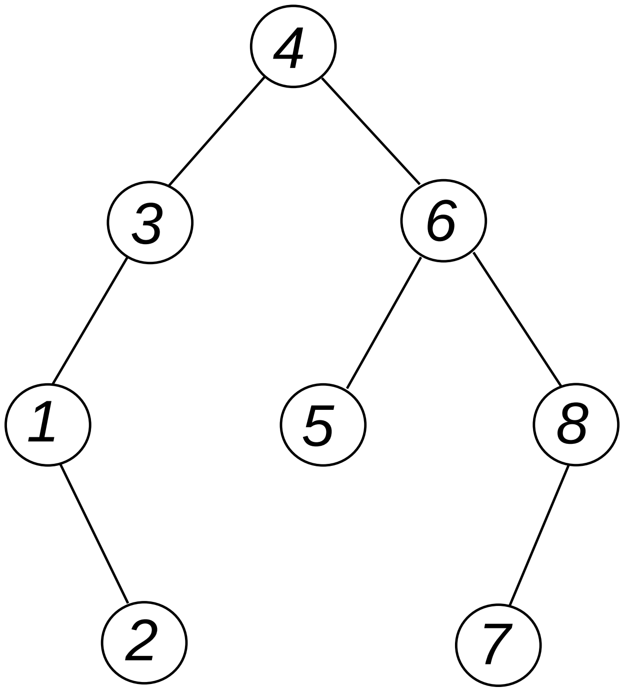
$(X_{n,k})$ ... random profile

$\Longrightarrow W_n(z) := \sum_{k\geq 0} X_{n,k} z^k$ ... random analytic function

$\Longrightarrow \dfrac{W_n(z)}{\mathbf{E}\, W_n(z)}$ ... random analytic function

Profile of Trees

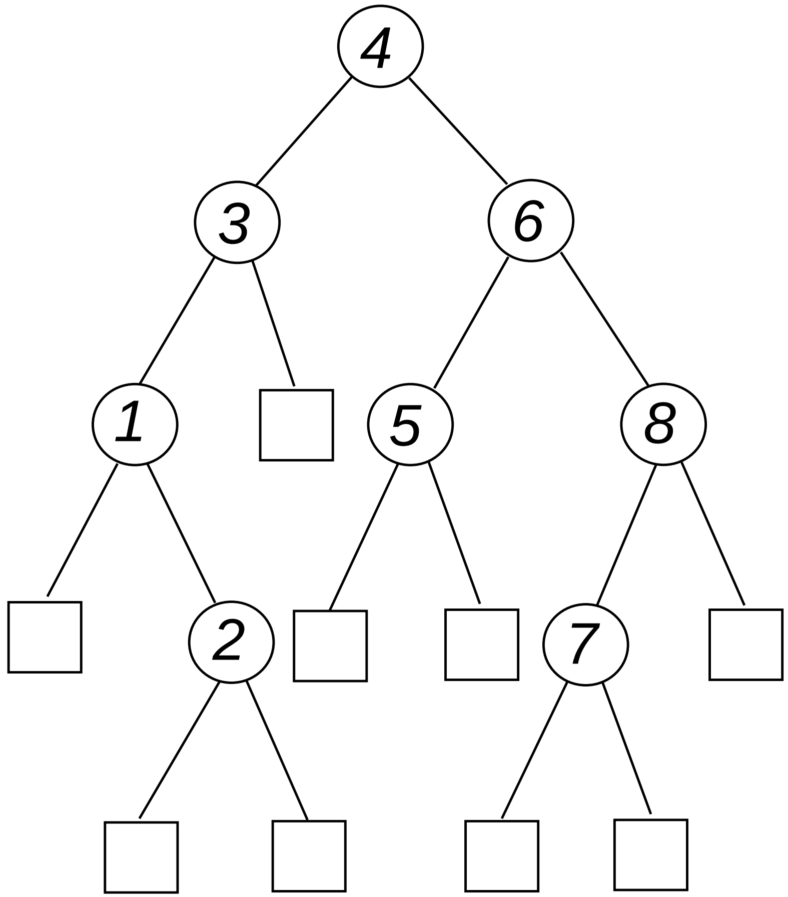
External profile of binary search trees



adding free places

Profile of Trees

External profile of binary search trees



□ ... free place

Profile of Trees

External profile of binary search trees

$Y_{n,k}$ = number of free (= external) nodes at level k,
$X_{n,k}$ = number of (internal) nodes at level k.

$$\boxed{X_{n,k} = \sum_{j>k} 2^{k-j}\, Y_{n,j}}$$

Profile of Trees

External profile of binary search trees

$$\boxed{W_n^{\mathsf{ext}}(z) = \sum_{k \ge 0} Y_{n,k}\, z^k}$$

Lemma 1 *The normalized external profile polynomials*

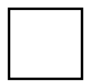
$$M_n(z) = \frac{W_n^{\mathsf{ext}}(z)}{\mathbf{E}\, W_n^{\mathsf{ext}}(z)}$$

constitute a **martingale** *with respect to the natural filtration that is induced by the growing tree process* (T_n)*.*

Remark $\mathbf{E}\, W_n^{\mathsf{ext}}(z) = \binom{2z+n-1}{n}$

Profile of Trees

Growing tree process



Profile of Trees

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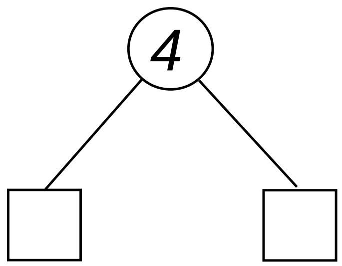

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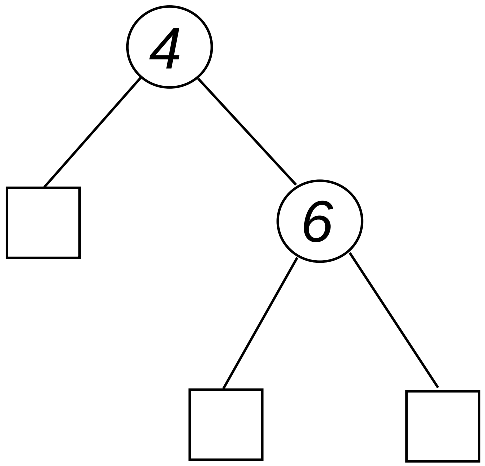

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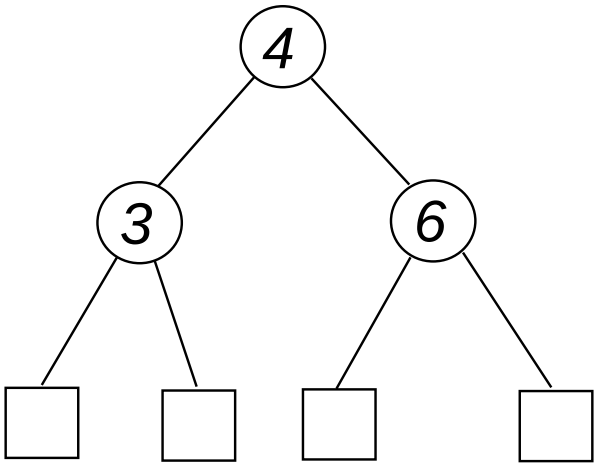

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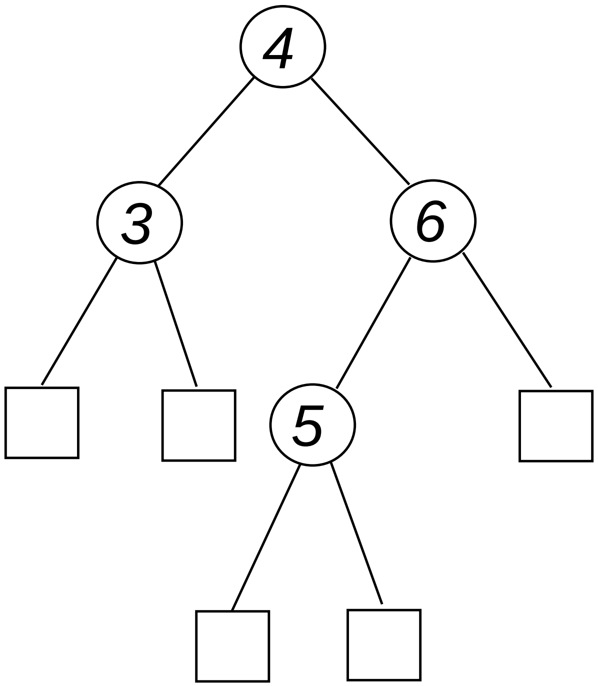

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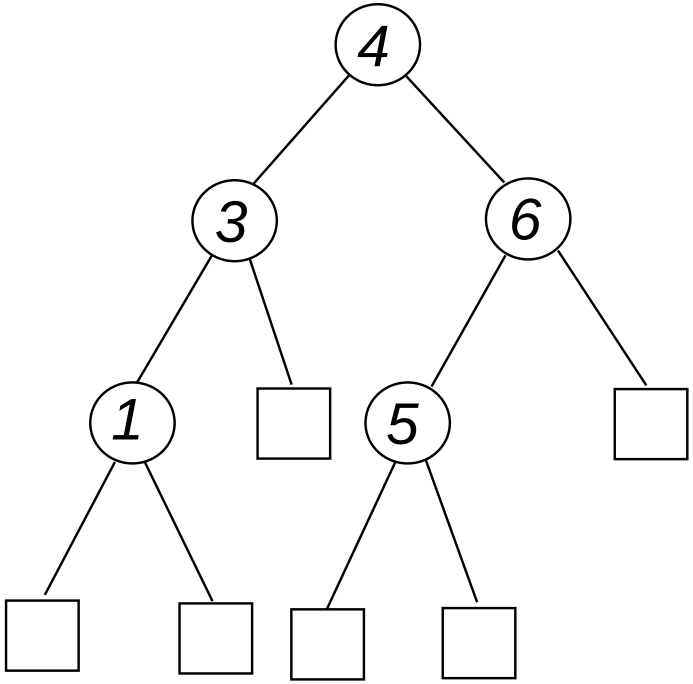

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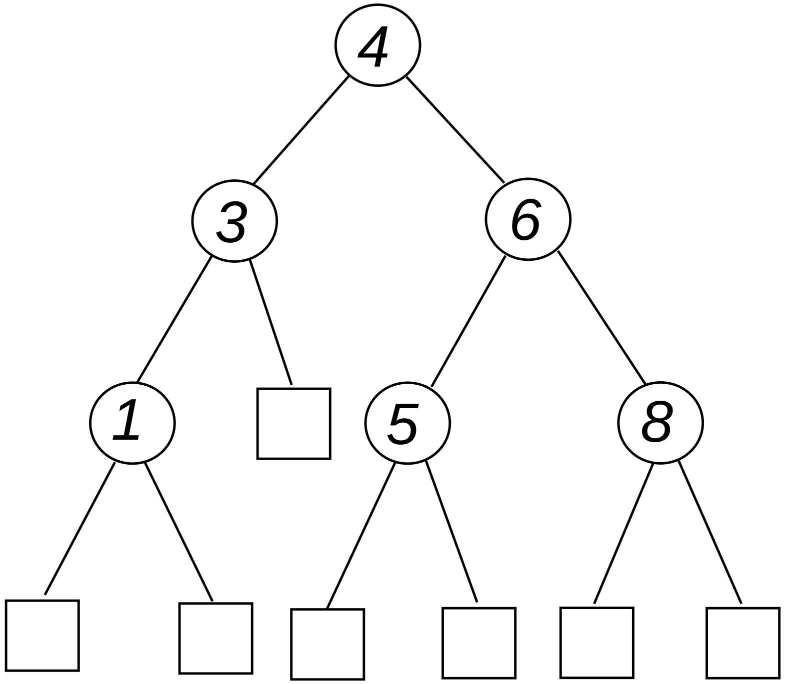

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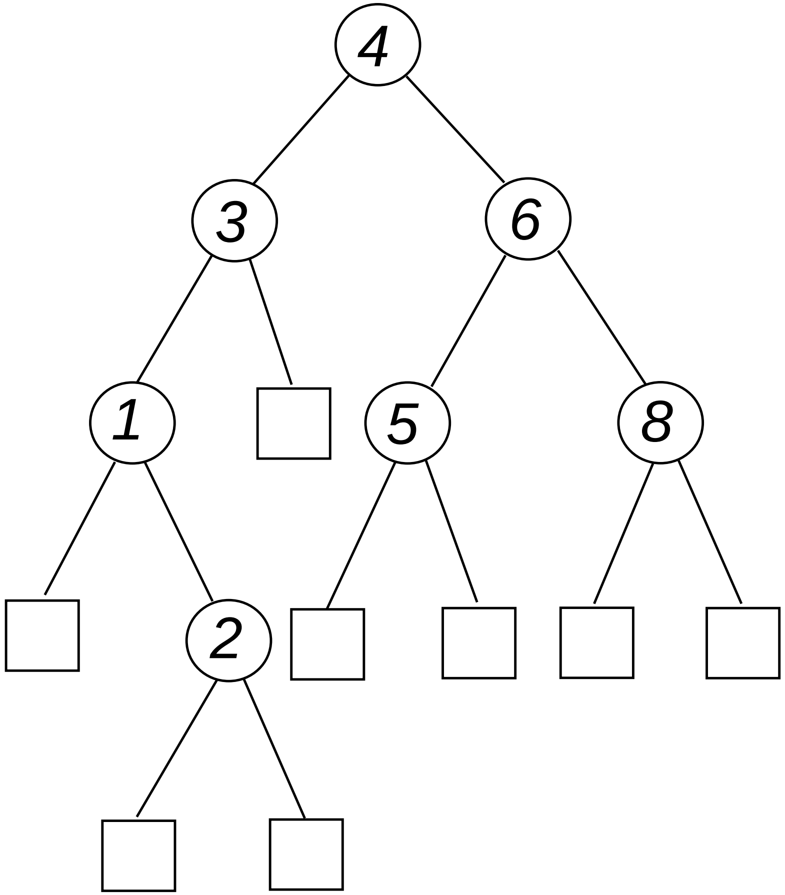
Growing tree process



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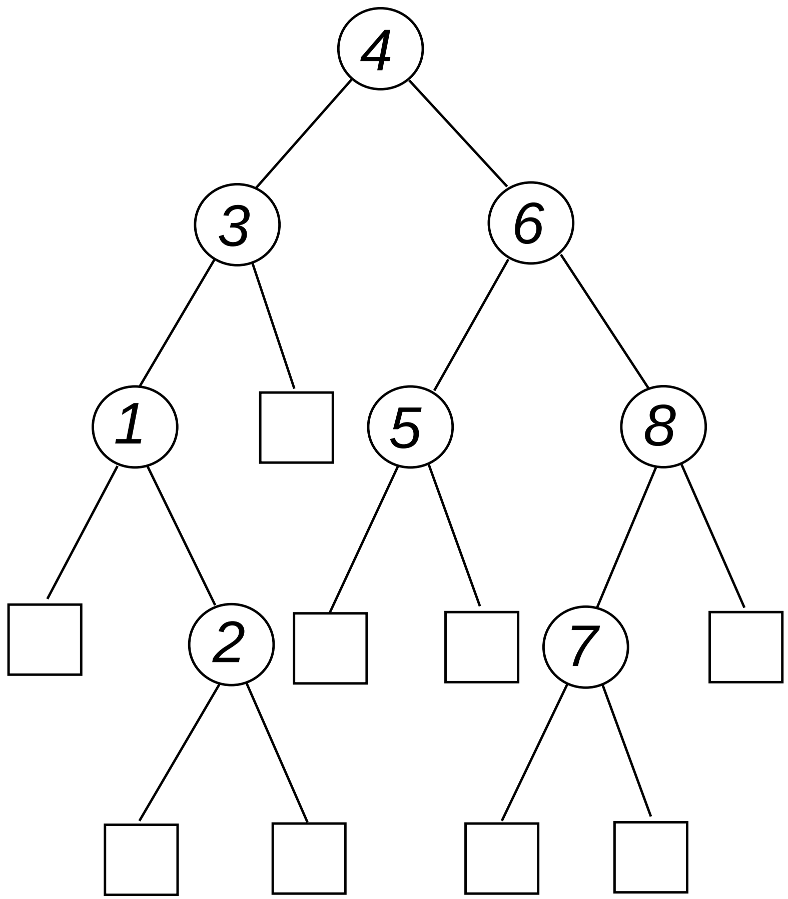
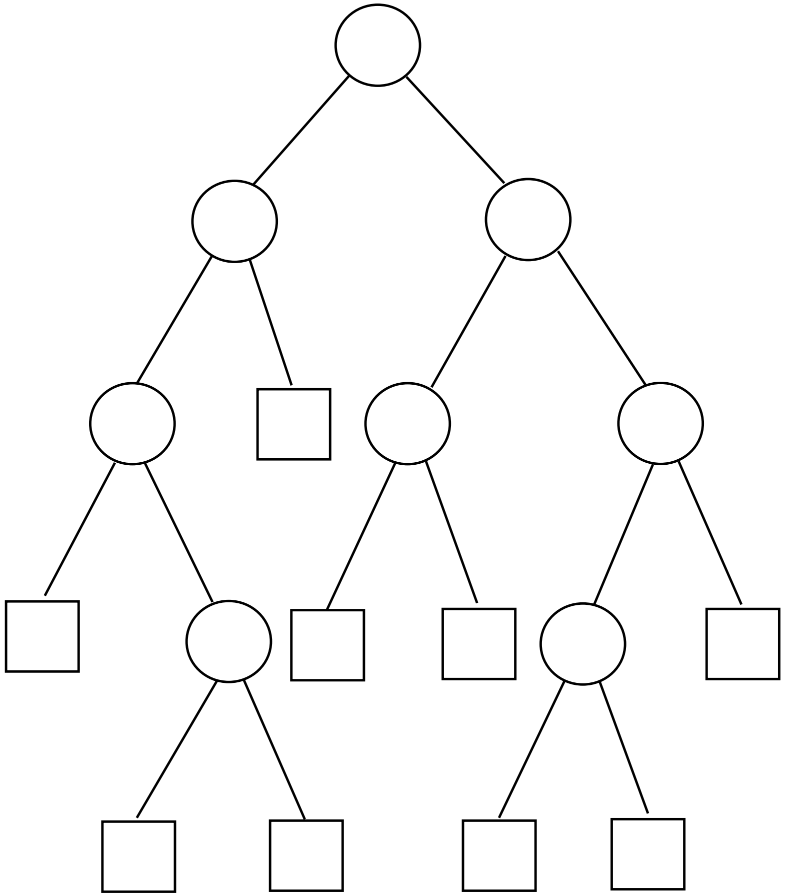


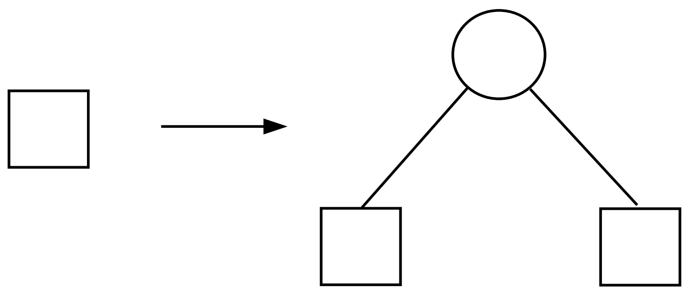
Profile of Trees

Growing tree process



Profile of Trees

Growing tree process



Unfortunately this procedure only works for $m = 2$ and $t = 0$.
In all other cases there is **no** corresponding **martingale**.

Function Spaces

$D \subseteq \mathbb{C}$ ... complex domain

- $\mathcal{H}(D)$ ... space of **all analytic functions on** D with the topology of uniform convergence on compact sets.
  This topology can be defined by the family of seminorms $f \mapsto \sup_K |f|$, where K ranges over the compact subsets of D.
  $\mathcal{H}(D)$ is a **Fréchet space**, i.e. a locally convex space with a topology that can be defined by a complete metric, and it has (by Montel's theorem on normal families) the property that every closed bounded subset is compact.
  The topology is separable (for example, by regarding $\mathcal{H}(D)$ as a subspace of $C_0^\infty(D)$).

- $\mathcal{B}(D)$ ... **Bergman space** of all **square integrable analytic functions on** D, equipped with the norm given by $\|f\|^2_{\mathcal{B}(D)} = \int_D |f(z)|^2 \, dm(z)$, where m is the two-dimensional Lebesgue measure.
  $\mathcal{B}(D)$ is a **separable Hilbert space** since it can be regarded as a closed subspace of $L^2(\mathbb{R}^2)$.

Function Spaces

Lemma 2 *The embedding $\mathcal{B}(D) \to \mathcal{H}(D)$ is continuous.*

$\Longrightarrow$
Convergence in distribution in $\mathcal{B}(D)$ implies convergence in $\mathcal{H}(D)$.

Lemma 3 *$D' \subset D$ subdomain of D.*
Then the restriction mappings $\mathcal{H}(D) \to \mathcal{H}(D')$ and $\mathcal{B}(D) \to \mathcal{B}(D')$ are continuous.

$\Longrightarrow$
Convergence in distribution in $\mathcal{H}(D)$ or $\mathcal{B}(D)$ implies convergence (of the restrictions) in $\mathcal{H}(D')$ or $\mathcal{B}(D')$, respectively.

Function Spaces

Local – global convergence

Theorem 3 $D \subseteq \mathbb{C}$ ... *complex domain,*
(W_n) ... *sequence of random analytic functions on* D*.*

For all $x \in D$*, there is an open subdomain* D_x *with* $x \in D_x \subset D$ *and a random analytic function* Z_x *on* D_x *such that, as* $n \to \infty$*,*

$$W_n \xrightarrow{\mathsf{d}} Z_x \quad \textit{in } \mathcal{H}(D_x).$$

$\Longrightarrow$ *There exists a random analytic function* Z *on* D *such that*

$$W_n \xrightarrow{\mathsf{d}} Z \quad \textit{in } \mathcal{H}(D)$$

and the restriction $Z|_{D_x} \stackrel{\mathsf{d}}{=} Z_x$ *for every* x*.*

Function Spaces

Zolotarev metric

B ... Banach space, $s = m + \alpha > 0$ ... real number, $m \in \mathbb{Z}$, $0 < \alpha \leq 1$

$$\boxed{\mathcal{F}_s := \{f \in C^m(B, \mathbb{R}) : \|D^m f(x) - D^m f(y)\| \leq \|x - y\|^\alpha, \quad x, y \in B\}}.$$

X, Y ... random variables with values in B,
$\mathcal{L}(X)$, $\mathcal{L}(Y)$ ... laws of X, Y

Zolotarev metric ζ_s:

$$\boxed{\zeta_s(X, Y) := \zeta_s(\mathcal{L}(X), \mathcal{L}(Y)) := \sup_{f \in \mathcal{F}_s} |\mathbf{E}\,(f(X) - f(Y))|}.$$

Remark: $\mathbf{E}\,\|X\|^s < \infty$, $\mathbf{E}\,\|Y\|^s < \infty$, and $\mathbf{E}\,X^{\otimes k} = \mathbf{E}\,Y^{\otimes k}$ for all $k \leq m$
$\Longrightarrow \ \zeta_s(X, Y) < \infty.$ $\qquad$ ($\langle g, \mathbf{E}\,X^{\otimes k}\rangle = \mathbf{E}\,g(X, X, \ldots, X)$, g multilinear)

Function Spaces

Zolotarev metric

$\mathbf{z} = (z_1, \ldots, z_m)$, $z_k \in B^{\otimes k}$, $k = 1, \ldots, m$,

$$\boxed{\mathcal{P}_{s,\mathbf{z}}(B) := \{\mathcal{L}(X) : \mathbf{E}\,\|X\|^s < \infty,\, \mathbf{E}\,X^{\otimes k} = z_k,\, k = 1, \ldots, m\}},$$

ζ_s is finite on each $\mathcal{P}_{s,\mathbf{z}}(B)$, and is also a **semi-metric** on $\mathcal{P}_{s,\mathbf{z}}(B)$.

Function Spaces

Zolotarev metric

Theorem 4 *H ... separable Hilbert space,* $s > 0$
$\mathbf{z} = (z_1, \ldots, z_m)$, $z_k \in B^{\otimes k}$, $k = 1, \ldots, m$,

$\Longrightarrow$ ζ_s *is a* **complete metric** *on the set* $\mathcal{P}_{s,\mathbf{z}}(H)$

X_n, X *... H-valued random variables with* $\mathcal{L}(X), \mathcal{L}(Y) \in \mathcal{P}_{s,\mathbf{z}}(H)$*:*

$$\boxed{\zeta_s(X_n, X) \to 0 \quad \Longrightarrow \quad X_n \xrightarrow{\text{d}} X}.$$

Remark: X_n is tight

Function Spaces

Minimal L^s-metric ℓ_s

X, Y ... random variables with values in a Banach space B

$\mathbf{E}\,\|X\|^s$, $\mathbf{E}\,\|Y\|^s < \infty$

$$\boxed{\ell_s(X,Y) := \inf\{(\mathbf{E}\,\|X' - Y'\|^s)^{(1/s)\wedge 1} \, : \, \mathcal{L}(X') = \mathcal{L}(X), \mathcal{L}(Y') = \mathcal{L}(Y)\}}.$$

Function Spaces

Minimal L^s-metric ℓ_s

Lemma 4 $\mathcal{L}(X), \mathcal{L}(Y) \in \mathcal{P}_{s,\mathbf{z}}(B)$

$s > 1 \implies$

$$\boxed{\zeta_s(X,Y) \leq \left((\mathbf{E}\, \|X\|^s)^{1-1/s} + (\mathbf{E}\, \|Y\|^s)^{1-1/s} \right) \ell_s(X,Y)}.$$

$0 < s \leq 1 \implies$

$$\boxed{\zeta_s(X,Y) \leq \ell_s(X,Y)}.$$

Contraction Method

Fixed point equation

$\mathcal{P}(H)$ ... set of all probability distributions on Hilbert space H.

$A_1^*, \ldots, A_m^*$ ... random linear operators in H,
b^* ... random variable in H

$$T : \mathcal{P}(H) \to \mathcal{P}(H)$$
$$\mathcal{L}(Z) \mapsto \mathcal{L}\left(\sum_{r=1}^{m} A_r^*(Z^{(r)}) + b^* \right),$$

$\mathcal{L}(Z^{(r)}) = \mathcal{L}(Z)$ for $r = 1, \ldots, m$,
$(A_1^*, \ldots, A_m^*, b^*)$, $Z^{(1)}, \ldots, Z^{(m)}$ independent.

Contraction Method

Fixed point equation

A ... linear operator A in H

$$\|A\|_{\text{op}} := \sup_{\|x\|=1} \|Ax\|$$

s-integrable means $\mathbf{E}\,\|A\|_{\text{op}}^s < \infty$.

Lemma 5 $A_1^*, \ldots, A_m^*, b^*$ *be as above and* s*-integrable for some* $0 < s \leq 2$.

$0 < s \leq 1 \Longrightarrow T(\mathcal{P}_s) \subseteq \mathcal{P}_s$.

$1 < s \leq 2$ *and* $\mathbf{E}\, b^* = 0 \Longrightarrow T(\mathcal{P}_{s,0}) \subseteq \mathcal{P}_{s,0}$.

Contraction Method

Contraction

Lemma 6 *Let* $A_1^*, \ldots, A_m^*, b^*$ *be and* s*-integrable for some* $0 < s \leq 2$*:* $\mathbf{E}\,\|A_r^*\|_{\mathsf{op}}^s < \infty$*.*

Further assume that

$$\boxed{\mathbf{E}\,\sum_{r=1}^{m} \|A_r^*\|_{\mathsf{op}}^s < 1}.$$

$0 < s \leq 1 \implies$ *the restriction of* T *to* $\mathcal{P}_s$ *is a* **strict contraction***.*

$1 < s \leq 2$*,* $\mathbf{E}\,b^* = 0 \implies$ *the restriction of* T *to* $\mathcal{P}_{s,0}$ *is a* **strict contraction***.*

Remark: $\zeta_s(A(X), A(Y)) \leq \|A\|_{\mathsf{op}}^s\,\zeta_s(X, Y)$

Contraction Method

A recurrence:

$(X_n)_{n\geq 0}$ ... sequence of random variables in H

$$\boxed{X_n \stackrel{d}{=} \sum_{r=1}^{m} A_r^{(n)}\left(X^{(r)}_{I_r^{(n)}}\right) + b^{(n)}}, \quad n \geq n_0,$$

$A_r^{(n)}$ ... random linear operators in H,
$b^{(n)}$ ... random variable in H,
$I^{(n)} = (I_1^{(n)}, \dots, I_m^{(n)})$.. random integer vector ($I_r^{(n)} \in \{0, \dots, n\}$),
$(X_j^{(1)}), \dots, (X_j^{(m)})$, $(A_1^{(n)}, \dots, A_m^{(n)}, b^{(n)}, I^{(n)})$ ... independent,
$\mathcal{L}(X_j^{(r)}) = \mathcal{L}(X_j)$ for all r and j.

Contraction Method

Theorem 5 $(X_n)_{n\geq 0}$ ... *sequence of random variables in H (as above)*
All quantities being s-integrable for some $0 < s \leq 2$.
(For $1 < s \leq 2$ assume further $\mathbf{E}\, X_n = 0$)

There exists $A_1^*, \ldots, A_m^*, b^*$ with

$$\mathbf{E}\, \|A_r^{(n)} - A_r^*\|_{\text{op}}^s \to 0, \quad \mathbf{E}\, \|b^{(n)} - b^*\|^s \to 0,$$

$$\mathbf{E} \sum_{r=1}^m \|A_r^*\|_{\text{op}}^s < 1,$$

$$\mathbf{E}\left[\mathbf{1}_{\{I_r^{(n)} \leq \ell\}} \|A_r^{(n)}\|_{\text{op}}^s\right] \to 0 \quad (r, \ell \geq 1 \text{ integers})$$

Let $\mathcal{L}(X)$ be the unique fixed point of T in $\mathcal{P}_s(H)$ for $0 < s \leq 1$ and in $\mathcal{P}_{s,0}(H)$ for $1 < s \leq 2$.

$$\Longrightarrow \quad \boxed{\zeta_s(X_n, X) \to 0} \quad n \to \infty.$$

Contraction Method

Analytic functions

$(X_n)_{n\geq 0}$ ... sequence of random analytic functions in a domain $D \subseteq \mathbf{C}$

$$\boxed{X_n \stackrel{d}{=} \sum_{r=1}^{m} A_r^{(n)} \cdot X^{(r)}_{I_r^{(n)}} + b^{(n)}}, \quad n \geq n_0, \tag{1}$$

$A_1^{(n)}, \ldots, A_m^{(n)}$, $b^{(n)}$ ... random analytic functions in D,
$I^{(n)} = (I_1^{(n)}, \ldots, I_m^{(n)})$ ... random integers vector ($I_r^{(n)} \in \{0, \ldots, n\}$);
$\mathcal{L}(X_j^{(r)}) = \mathcal{L}(X_j)$ for all r and j,
$(A_1^{(n)}, \ldots, A_m^{(n)}, b^{(n)}, I^{(n)}), (X_j^{(1)}), \ldots, (X_j^{(m)})$ ... independent.

Contraction Method

Contraction for random analytic functions

$$T : \mathcal{P}(\mathcal{H}(\tilde{D})) \to \mathcal{P}(\mathcal{H}(\tilde{D}))$$
$$\mathcal{L}(Z) \mapsto \mathcal{L}\left(\sum_{r=1}^{m} A_r^* \cdot Z^{(r)} + b^*\right),$$

$A_1^*, \ldots, A_m^*$, b^* ... random analytic functions in $\tilde{D}$,
$\mathcal{L}(Z^{(r)}) = \mathcal{L}(Z)$ for $r = 1, \ldots, m$,
$(A_1^*, \ldots, A_m^*, b^*)$, $Z^{(1)}, \ldots, Z^{(m)}$ ... independent.

Contraction Method

Theorem 6 $0 < s \leq 2$, (X_n) *as above with* $\mathbf{E}\, X_n(z) = 0$
$A_r^{(n)}$, $b^{(n)}$ ... *analytic funct. of* $z \in D$, $\mathbf{E}\,|A_r^{(n)}(z)|^s < \infty$, $\mathbf{E}\,|b^{(n)}(z)|^s < \infty$

Suppose that there exist $A_1^*, \ldots, A_m^*$ *and* b^* *in* D *and a connected subset* $\Delta \subseteq D$ *such that for each* $x \in \Delta$ *there exists a neighbourhood* $U_x \subseteq D$ *of* x *and a number* $s(x) \leq s$ *with*

$$\sup_{z \in U_x} \mathbf{E}\,|A_r^{(n)}(z) - A_r^*(z)|^{s(x)} \to 0, \qquad \sup_{z \in U_x} \mathbf{E}\,|b^{(n)}(z) - b^*(z)|^{s(x)} \to 0,$$

$$\sup_{z \in U_x} \mathbf{E}\,|A_r^*(z)|^{s(x)} < \infty,$$

$$\mathbf{E}\,\sum_{r=1}^{m} |A_r^*(x)|^{s(x)} < 1,$$

$$\sup_{z \in U_x} \mathbf{E}\left[\mathbf{1}_{\{I_r^{(n)} \leq \ell\}} |A_r^{(n)}(z)|^{s(x)}\right] \to 0,$$

Let $\mathcal{L}(X)$ *bet the fixed point of the map* T.

$$\Longrightarrow \quad \boxed{X_n \xrightarrow{d} X} \quad \text{in} \quad \mathcal{H}(\tilde{D})$$

for some domain $\tilde{D} \subseteq D$ *with* $\Delta \subseteq \tilde{D}$

Contraction Method

Proof of Th. 2

$W_n(z)$ ... random profile polynomials

$$\boxed{X_n(z) := \frac{W_n(z) - \mathbf{E}\, W_n(z)}{\mathbf{E}\, W_n(z)} = \frac{W_n(z)}{\mathbf{E}\, W_n(z)} - 1}$$

$$\Longrightarrow \boxed{X_n(z) \stackrel{d}{=} \sum_{r=1}^{m} z \frac{G_{V_{n,r}}(z)}{G_n(z)} X^{(r)}_{V_{n,r}} + \frac{1}{G_n(z)} \left(m - 1 - G_n(z) + z \sum_{r=1}^{m} G_{V_{n,r}}(z) \right)}.$$

Notation: $G_n(z) := \mathbf{E}\, W_n(z)$,

Contraction Method

Proof of Th. 2

$$I_r^{(n)} = V_{n,r}$$

$$A_r^{(n)} = z\frac{G_{V_{n,r}}(z)}{G_n(z)},$$

$$b^{(n)} = \frac{1}{G_n(z)}\left(m - 1 - G_n(z) + z\sum_{r=1}^{m} G_{V_{n,r}}(z)\right)$$

Cauchy Integral

Th. 2 $\Longrightarrow$ Th. 1

Idea:

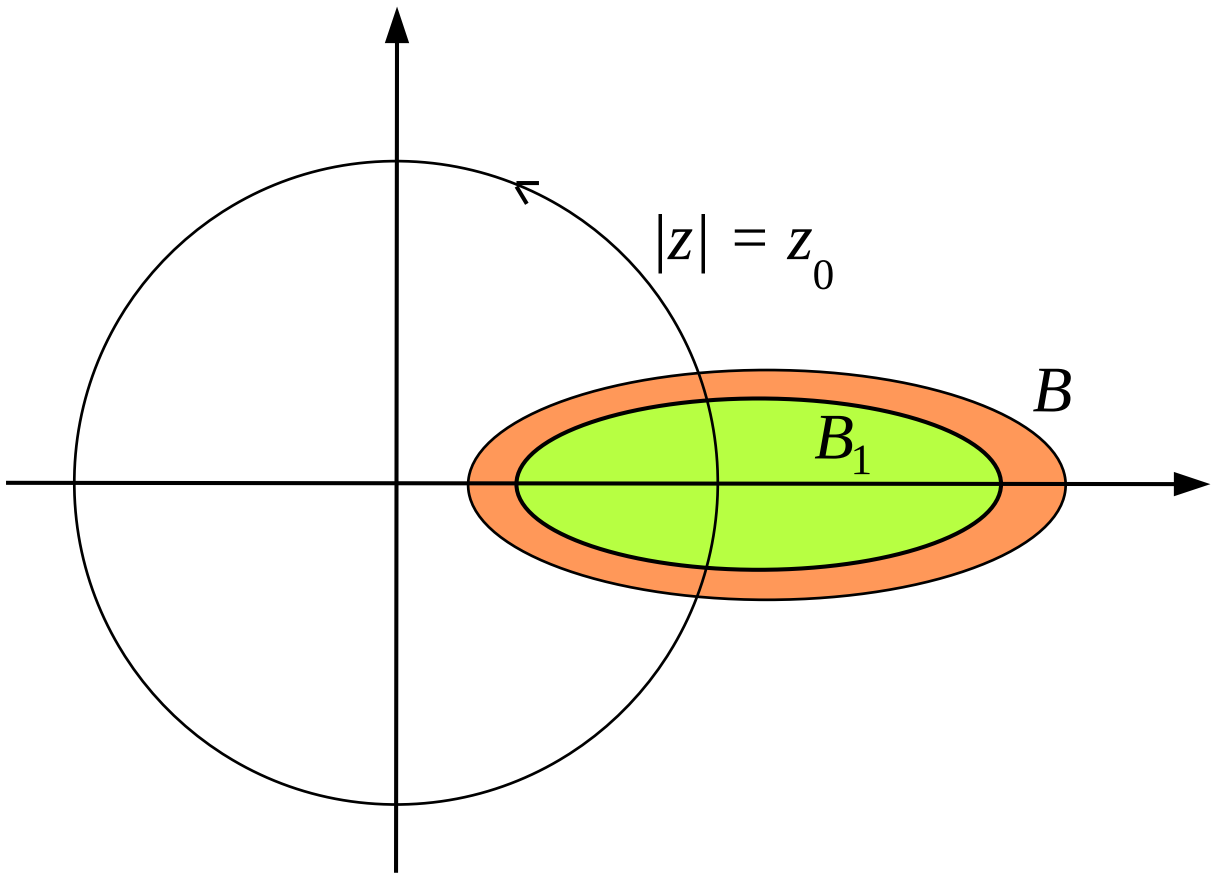
$$X_{nk} = \frac{1}{2\pi \mathrm{i}} \int_{|z|=z_0} W_n(z)\, z^{-k-1}\, dz$$

$$\Longrightarrow \quad \boxed{\frac{X_{nk}}{\mathbf{E}\, X_{nk}}} = \frac{1}{\mathbf{E}\, X_{nk}} \frac{1}{2\pi \mathrm{i}} \int\limits_{|z|=z_0} \frac{W_n(z)}{\mathbf{E}\, W_n(z)} \mathbf{E}\, W_n(z)\, z^{-k-1}\, dz$$

$$= \frac{1}{\mathbf{E}\, X_{nk}} \frac{1}{2\pi \mathrm{i}} \int\limits_{|z|=\beta(\alpha),\, z\in B_1} \boxed{\frac{W_n(z)}{\mathbf{E}\, W_n(z)}} \mathbf{E}\, W_n(z)\, z^{-k-1}\, dz$$

$$+ \frac{1}{\mathbf{E}\, X_{nk}} \frac{1}{2\pi \mathrm{i}} \int\limits_{|z|=z_0,\, z\notin B_1} W_n(z) z^{-k-1}\, dz.$$

Cauchy Integral



Cauchy Integral

Continuous operators

$(k = \lfloor \alpha \log n \rfloor)$

$$\boxed{T_n(G)(\alpha) = \frac{1}{\mathbf{E}\, X_{n,\lfloor \alpha \log n \rfloor}} \frac{1}{2\pi \mathrm{i}} \int_{|z|=\beta(\alpha),\, z\in B_1} G(z)\, \mathbf{E}\, W_n(z) z^{-\lfloor \alpha \log n \rfloor - 1}\, dz}, \alpha \in I_c'.$$

$\Longrightarrow$ $T_n(W_n(z)/\mathbf{E}\, W_n(z))$ is an approximation to

$$\left(X_{n,\lfloor \alpha \log n \rfloor}/\mathbf{E}\, X_{n,\lfloor \alpha \log n \rfloor},\ \alpha \in I_c'\right).$$

Cauchy Integral

Notation: $\|f\|_E := \sup_E |f|$

Lemma 7

1. *The operators T_n are uniformly continuous with respect to the supremum norm:*

$$\|T_n(F) - T_n(G)\|_{I'_c} \leq C \cdot \|F - G\|_{B_1}.$$

(for some constant $C > 0$ depending on I_c and B_1).

2. *$F_n \to F$ uniformly on $B_1 \Longrightarrow T_n(F_n) \to F$ uniformly on I'_c.*

Proof method: Saddle point method

Cauchy Integral

Application of Lemma 7:

$$(W_n(z)/\mathbf{E}\,W_n(z),\, z \in B_1) \to (Y(z),\, z \in B_1)$$

$$\Longrightarrow \quad T_n(W_n(z)/\mathbf{E}\,W_n(z)) \to Y(\beta(\alpha)),\ \alpha \in I_c'$$

Cauchy Integral

Lemma 8 *$I_c \subseteq I = \{\beta > 0 : 1 < \lambda_1(\beta^2) < 2\lambda_1(\beta) - 1\}$ a compact interval.*

$$\Longrightarrow \quad \sup_{\alpha \in I_c'} \left| \frac{\frac{1}{2\pi} \int_{|z|=\beta(\alpha), z \notin B_1} W_n(z) z^{-\lfloor \alpha \log n \rfloor - 1}\, dz}{\mathbf{E}\, X_{n, \lfloor \alpha \log n \rfloor}} \right| \xrightarrow{\mathsf{p}} 0$$

as $n \to \infty$.

Proof method: Second moment estimates.

Cauchy Integral

Th. 2 $\Longrightarrow$ Th. 1:

$$\left(\frac{X_{n,\lfloor\alpha \log n\rfloor}}{\mathbf{E}\, X_{n,\lfloor\alpha \log n\rfloor}},\ \alpha \in I_c'\right) = T_n(W_n(z)/\mathbf{E}\, W_n(z)) + o_{\mathsf{p}}(1)$$

$$\xrightarrow{\mathsf{d}} \Big(Y(\beta(\alpha))\,\alpha \in I_c'\Big)$$

Thank You!