PRIME NUMBERS IN TWO BASES

MICHAEL DRMOTA, CHRISTIAN MAUDUIT, AND JOËL RIVAT

Abstract. If q_1 and q_2 are two coprime bases, f (resp. g) a strongly q_1-multiplicative (resp. strongly q_2-multiplicative) function of modulus 1 and ϑ a real number, we estimate the sums $\sum_{n\leq x}\Lambda(n)f(n)g(n)\exp(2i\pi\vartheta n)$ (and $\sum_{n\leq x}\mu(n)f(n)g(n)\exp(2i\pi\vartheta n)$), where Λ denotes the von Mangoldt function (and μ the Möbius function). The goal of this work is to introduce a new approach to study these sums involving simultaneously two different bases combining Fourier analysis, Diophantine approximation and combinatorial arguments. We deduce from these estimates a Prime Number Theorem (and Möbius orthogonality) for sequences of integers with digit properties in two coprime bases.

1. Introduction

We denote by $\mathbb{N}$ the set of non negative integers, by $\mathbb{U}$ the set of complex numbers of modulus 1, by $\mathcal{P}$ the set of prime numbers.

For $n \in \mathbb{N}$, $n \geq 1$, we denote by $\tau(n)$ the number of divisors of n, by $\omega(n)$ the number of distinct prime factors of n, by $\Lambda(n)$ the von Mangoldt function (defined by $\Lambda(n) = \log p$ if $n = p^k$ with $k \in \mathbb{N}$, $k \geq 1$ and $\Lambda(n) = 0$ otherwise) and by $\mu(n)$ the Möbius function (defined by $\mu(n) = (-1)^{\omega(n)}$ if n is squarefree and $\mu(n) = 0$ otherwise).

For $x \in \mathbb{R}$ we denote by $\pi(x)$ the number of prime numbers less or equal to x, by $\|x\|$ the distance of x to the nearest integer, and we set $\mathrm{e}(x) = \exp(2i\pi x)$. If f and g are two functions with g taking strictly positive values such that f/g is bounded, we write $f \ll g$ (or $f = O(g)$).

In all this paper q denotes an integer greater or equal to 2 and for any positive integer n,

$$n = \sum_{j\geq 0} \varepsilon_j(n)\, q^j \text{ with } \varepsilon_j(n) \in \{0,\ldots,q-1\} \text{ for all } j \in \mathbb{N} \tag{1}$$

is the representation of n in base q.

1.1. q-additive and q-multiplicative functions.

The notion of q-additive function has been introduced independently by Bellman and Shapiro in [2] and by Gelfond in [11].

Definition 1. *A function $h : \mathbb{N} \to \mathbb{R}$ is q-additive (resp. strongly q-additive) if for all $(a, b) \in \mathbb{N} \times \{0,\ldots,q-1\}$, we have*

$$h(aq + b) = h(aq) + h(b)$$

(resp. $h(aq + b) = h(a) + h(b)$).

It follows that any q-additive function h verifies $h(0) = 0$. If h is a strongly q-additive function then h is uniquely determined by the values $h(1),\ldots, h(q-1)$ and for any positive integer n written

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in base q as (1), we have

$$h\left(\sum_{j\geq 0}\varepsilon_j(n)\,q^j\right)=\sum_{j\geq 0}h(\varepsilon_j(n)).$$

The most classical example of q-additive function is the q-ary sum-of-digits function defined by $\mathrm{s}_q(n)=\sum_{j\geq 0}\varepsilon_j(n)$.

In a similar way we can define the notions of q-multiplicative function and strongly q-multiplicative function:

Definition 2. *A function $f:\mathbb{N}\to\mathbb{U}$ is q-multiplicative (resp. strongly q-multiplicative) if for all $(a,b)\in\mathbb{N}\times\{0,\ldots,q-1\}$, we have*

$$f(aq+b)=f(aq)\,f(b)$$

(resp. $f(aq+b)=f(a)\,f(b)$).

If h is a q-additive (resp. strongly q-additive) function then $f=\mathrm{e}(h)$ is q-multiplicative (resp. strongly q-multiplicative). Conversely if $f=\mathrm{e}(h)$ is a q-multiplicative (resp. strongly q-multiplicative) function from $\mathbb{N}$ to $\mathbb{U}$ then h is q-additive (resp. strongly q-additive) modulo 1.

Definition 3. *A strongly q-multiplicative function is called* **proper** *if it is not of the form $f(n)=\mathrm{e}(\vartheta n)$ with $(q-1)\vartheta\in\mathbb{Z}$.*

1.2. **q-additive functions and prime numbers.** Bassily and Katai studied in [17, 1] the limit distribution of q-additive functions along prime numbers. It follows in particular from their results that if h is a strongly q-additive function, then

$$\frac{1}{\pi(x)}\operatorname{card}\left\{p\leq x,\ p\in\mathcal{P},\ h(p)\leq\mu_h\log_q x+y\sqrt{\sigma_h^2\log_q x}\right\}=\Phi(y)+o(1),$$

where

$$\mu_h=\frac{1}{q}\sum_{j<q}h(j),\ \sigma_h^2=\frac{1}{q}\sum_{j<q}h(j)^2-\mu_h^2,$$

and Φ denotes the normal distribution function (see [6] for a generalization of this result to the case of two q-additive functions in coprime bases).

In [19, 20, 21, 22] Martin, Mauduit and Rivat studied the exponential sums associated to q-additive functions ([19] and [20] concern a more general class of arithmetic functions called digital functions, which include the function counting the number of occurences of the digit 0 in the q-ary representation). In particular they defined the notion of characteristic integer as follows:

Definition 4. *If h is a strongly q-additive integer valued function such that $\gcd(h(1),\ldots,h(q-1))=1$, the characteristic integer of h is*

$$d_h=\gcd\left(h(2)-2h(1),\ \ldots,\ h(q-1)-(q-1)h(1),q-1\right).$$

We have $(d_h,h(1))=1$ and, for any positive integer n, $h(n)\equiv h(1)n \bmod d_h$. It follows from Definition 3 that if h is a strongly q-additive integer valued function such that $\gcd(h(1),\ldots,h(q-1))=1$, then $f=\mathrm{e}(\alpha h)$ is proper if and only if $d_h\alpha\notin\mathbb{Z}$ (in particular $f=\mathrm{e}(\alpha\,\mathrm{s}_q)$ is proper if and only if $(q-1)\alpha\notin\mathbb{Z}$).

It follows from [19, 20] that

Theorem A. *If h is a strongly q-additive integer valued function such that $\gcd(h(1),\ldots,h(q-1))=1$, then for all $(\alpha,\beta)\in\mathbb{R}^2$ and $x\geq 2$ we have*

$$\sum_{n\leq x}\Lambda(n)\,\mathrm{e}\left(\alpha h(n)+\beta n\right)\ll(\log x)^4 x^{1-c_q(h)\|d_h\alpha\|^2},$$

where $c_q(h)>0$ is an explicit constant and the implicit constant depends only on q.

Theorem B. *If h is a strongly q-additive integer valued function such that* $\gcd(h(1),\dots,h(q-1)) = 1$, *then for any positive integer m such that* $\gcd(d_h,m)=1$, *we have for all integers a*

$$\operatorname{card}\{p \le x,\ p\in\mathcal{P},\ h(p)\equiv a \bmod m\} = \frac{\pi(x)}{m} + O\left((\log x)^3 x^{1-\frac{c_q(h)}{m^2}}\right),$$

where $c_q(h)$ is the constant from Theorem A.

Remark 1. *Without the coprimality condition* $\gcd(h(1),\dots,h(q-1))=1$, *it is still possible to get similar results to Theorem A and Theorem B but they are more complicated to formulate (see* [20, section 6.4]*).*

1.3. **q-additive functions in different bases.** The question of the statistical independence of sum-of-digits functions in pairwise coprime bases was first stated by Gelfond in his seminal paper [11].

By using a general method concerning pseudorandom sequences in the sense of Bertrandias (see [3, 4]) and generalizing previous results obtained by Mendès France (see [27]), Bésineau showed in [5] that, if $q_1,\dots,q_\ell$ are pairwise coprime bases and $a_1,\dots,a_\ell$, $m_1,\dots,m_\ell$ are integers such that $\gcd(m_i,q_i-1)=1$ for any $i\in\{1,\dots,\ell\}$, then

$$\operatorname{card}\{n\le x,\ \forall i\in\{1,\dots,\ell\},\ s_{q_i}(n)\equiv a_i \bmod m_i\} = \frac{x}{m_1\cdots m_\ell} + o(x).$$

Kamae obtained similar results when $\ell=2$ by studying the mutual singularity of the spectral measures associated to the sum-of-digits functions (see [14, 15, 16]). These results were extended in [29] to multiplicatively independent bases and finally in [23] to different bases by using a slightly different method involving the study of some class of Riesz products (see also [12] for a generalisation of Kamae's result to $\ell\ge 2$ q-additive functions in pairwise coprime bases by using ergodic methods).

By using a different approach based on exponential sums, Kim gave in [18] a full answer to Gelfond's question providing an explicit error term. It follows in particular from his result that if $q_1,\dots,q_\ell$ are pairwise coprime bases then, if for any $i\in\{1,\dots,\ell\}$ h_i is a strongly integer valued q_i-additive function such that $\gcd(h_i(1),\dots,h_i(q_i-1))=1$ and m_i is a positive integer such that $\gcd(d_{h_i},m_i)=1$, we have for all integers $a_1,\dots,a_\ell$,

$$\operatorname{card}\{n\le x,\ \forall i\in\{1,\dots,\ell\},\ h_{q_i}(n)\equiv a_i \bmod m_i\} = \frac{x}{m_1\cdots m_\ell} + O(x^{1-\delta}),$$

with $\delta = \frac{1}{120}\ell^{-2}\left(\max_{1\le i\le\ell} q_i\right)^{-3}\left(\max_{1\le i\le \ell} m_i\right)^{-2}$.

2. Statement of the results

Let q_1 and q_2 be coprime integers greater or equal to 2. The goal of this paper is to show a prime number theorem for sequences of integers defined by simultaneous strongly q_1-additive and q_2-additive conditions.

Theorem 1. *If f is a strongly q_1-multiplicative function and g a strongly q_2-multiplicative function such that* $\gcd(q_1,q_2)=1$ *and f or g is proper, then we have uniformly for $\vartheta\in\mathbb{R}$*

$$\left|\sum_{n\le x}\Lambda(n)f(n)g(n)\,\mathrm{e}(\vartheta n)\right| \ll x\exp\left(-c\,\frac{\log x}{\log\log x}\right)$$

for some positive constant c.

Remark 2. *If f and g are not proper then the sum above is of the kind $\sum_{n\le x}\Lambda(n)\,\mathrm{e}(\vartheta' n)$ for some $\vartheta'\in\mathbb{R}$ for which the best known upper bound (without the Riemann Hypothesis) is only of the size $x\exp\left(-c\sqrt{\log x}\right)$ for some positive constant c.*

It follows that under the conditions of Theorem 1 we have uniformly for $\vartheta \in \mathbb{R}$

$$\left| \sum_{\substack{p \le x \\ p \in \mathcal{P}}} f(p) g(p)\, \mathrm{e}(\vartheta p) \right| \ll x \exp\left(-c' \frac{\log x}{\log\log x} \right)$$

for some positive constant c', which means in particular (for $\vartheta = 0$) that strongly q-multiplicative functions in two coprime bases are statistically independent along prime numbers.

By the same method we show the following Theorem which implies that the product of two strongly q-multiplicative functions in coprime bases is orthogonal to the Möbius function.

Theorem 2. *If f is a strongly q_1-multiplicative function and g a strongly q_2-multiplicative function such that $\gcd(q_1, q_2) = 1$ and f or g is proper, then we have uniformly for $\vartheta \in \mathbb{R}$*

$$\left| \sum_{n \le x} \mu(n) f(n) g(n)\, \mathrm{e}(\vartheta n) \right| \ll x \exp\left(-c \frac{\log x}{\log\log x} \right)$$

for some positive constant c.

The sequence $(f(n)g(n))_{n\in\mathbb{N}}$ in Theorem 2 is produced by a zero entropy dynamical system, so that this result can be seen as a new class of sequences verifying Möbius orthogonality in connection with the Sarnak conjecture [30] (see [8] for a survey on the Sarnak conjecture).

As we will see in the proofs the upper bounds can be made more explicit if we restrict ourselves to special multiplicative functions (we only state Theorem 3 for the Λ-function but it also holds for the Möbius function).

Theorem 3. *If f_0 is an integer valued strongly q_1-additive function and g_0 is an integer valued strongly q_2-additive function such that $\gcd(q_1, q_2) = 1$, $\gcd(f_0(1), \ldots, f_0(q_1 - 1)) = 1$ and $\gcd(g_0(1), \ldots, g_0(q_2 - 1)) = 1$, then we have uniformly for $(\alpha, \beta, \vartheta) \in \mathbb{R}^3$ such that $d_{f_0}\alpha \notin \mathbb{Z}$ and $d_{g_0}\beta \notin \mathbb{Z}$*

$$\left| \sum_{n \le x} \Lambda(n)\, \mathrm{e}(\alpha f_0(n) + \beta g_0(n) + \vartheta n) \right|$$
$$\ll x \exp\left(-c \frac{\log x}{\log\log x} \right) + (\log x)^A x^{1 - c_1 \|d_{f_0}\alpha\|^2 / \log \|d_{f_0}\alpha\|^{-1} - c_2 \|d_{g_0}\beta\|^2 / \log \|d_{g_0}\beta\|^{-1}}$$

for some positive constants A, c, c_1, c_2.

Theorem 3 can be reformulated into a *prime number theorem* of the following kind.

Corollary 1. *If q_1, q_2, f_0 and g_0 are given as in Theorem 3, then for any positive integers m_1, m_2 such that $\gcd(d_{f_0}, m_1) = \gcd(d_{g_0}, m_2) = 1$ we have for all integers a_1, a_2,*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \mathrm{card}\{p \le x,\ p \in \mathcal{P},\ f_0(p) \equiv a_1 \bmod m_1,\ g_0(p) \equiv a_2 \bmod m_2\} = \frac{1}{m_1 m_2}.$$

In order to estimate sums of the form $\sum_n \Lambda(n) F(n)$ by using a combinatorial identity like Vaughan's identity (see (13.39) of [13]), it is sufficient to estimate bilinear sums of the form

$$\sum_m \sum_n a_m b_n F(mn)$$

(this method is described in details in [25]). These sums are said of type I if b_n is a smooth function of n. Otherwise they are said of type II. The key of this approach is that for type I sums the summation over the smooth variable n is of significant length, while for type II sums both summations have a significant length. It follows that in order to prove Theorem 1 it is enough to estimate these sums of type I and sums of type II, which will be done in section 7 (Proposition 4) and in section 8 (Proposition 5).

We introduce a new approach to study these sums involving simultaneously two different bases. In order to prove Proposition 4 and Proposition 5 we will separate the contribution of the bases q_1 and q_2 by a combination of several techniques including discrete Fourier analysis, Diophantine approximation and combinatorial arguments.

The study of type I sums leads (by carry properties) to consider periodic arithmetic functions with period $q_1^{\lambda_1} q_2^{\lambda_2}$. The first difficulty is to separate the contribution of the two bases and to combine arguments from [10, 9] and [25] with new Diophantine and Fourier arguments.

A second difficulty arises in the study of type II sums: the separation of the contributions coming from these two bases (by van der Corput and Cauchy-Schwarz inequalities) leads to much more difficult Fourier estimates than in the case of one base. In the proof of Proposition 5 we use new estimates on average of the Fourier transform of correlations of strongly q-multiplicative functions (analogue to the $U(2)$ Gowers norm) that are provided by combinatorial arguments in section 6 (Proposition 1 and Proposition 2).

A last difficulty appears in the non-diagonal terms of the sums of type II which leads to estimate a linear form of logarithms and which allows us to win a factor of the size $\exp(C\log x/\log\log x)$.

Sections 3–5 collect preliminary lemmas used in the rest of the paper.

3. Notations and preliminary lemmas

The following lemma is a classical method to detect real numbers in an interval modulo 1 by means of exponential sums. For $\alpha \in \mathbb{R}$ with $0 \le \alpha < 1$ we denote by χ_α the characteristic function of the interval $[0, \alpha)$ modulo 1:

$$\chi_\alpha(x) = \lfloor x \rfloor - \lfloor x - \alpha \rfloor . \tag{2}$$

Lemma 1. *For all $\alpha \in \mathbb{R}$ with $0 \le \alpha < 1$ and all integer $H \ge 1$ there exist real valued trigonometric polynomials $A_{\alpha,H}(x)$ and $B_{\alpha,H}(x)$ such that for all $x \in \mathbb{R}$*

$$|\chi_\alpha(x) - A_{\alpha,H}(x)| \le B_{\alpha,H}(x), \tag{3}$$

where

$$A_{\alpha,H}(x) = \sum_{|h|\le H} a_h(\alpha, H)\, \mathrm{e}(hx), \;\; B_{\alpha,H}(x) = \sum_{|h|\le H} b_h(\alpha, H)\, \mathrm{e}(hx), \tag{4}$$

with coefficients $a_h(\alpha, H)$ and $b_h(\alpha, H)$ satisfying

$$a_0(\alpha, H) = \alpha, \;\; |a_h(\alpha, H)| \le \min\left(\alpha, \tfrac{1}{\pi|h|}\right), \;\; |b_h(\alpha, H)| \le \tfrac{1}{H+1}\left(1 - \tfrac{|h|}{H+1}\right). \tag{5}$$

Proof. This is a consequence of Theorem 19 of [31] (see [26, Lemma 1 and (17)]). □

Similarly we can detect points in a d-dimensional box (modulo 1):

Lemma 2. *For all $(\alpha_1, \dots, \alpha_d) \in [0, 1)^d$ and $(H_1, \dots, H_d) \in \mathbb{N}^d$ with $H_1 \ge 1, \dots, H_d \ge 1$, we have for all $(x_1, \dots, x_d) \in \mathbb{R}^d$*

$$\left| \prod_{j=1}^{d} \chi_{\alpha_j}(x_j) - \prod_{j=1}^{d} A_{\alpha_j,H_j}(x_j) \right| \le \sum_{\emptyset \ne J \subseteq \{1,\dots,d\}} \prod_{j \notin J} \chi_{\alpha_j}(x_j) \prod_{j \in J} B_{\alpha_j,H_j}(x_j) \tag{6}$$

where $A_{\alpha,H}(.)$ and $B_{\alpha,H}(.)$ are the real valued trigonometric polynomials defined by (4).

Proof. See [7, Lemma 3]. □

Lemma 3. *If $\mathcal{N}$ be a finite set, $f_1 : \mathcal{N} \to \mathbb{R}, \dots, f_d : \mathcal{N} \to \mathbb{R}$, $U_1, \dots, U_d$ are positive integers and*

$$g : \mathcal{N} \times \{0, \dots, U_1 - 1\} \times \cdots \times \{0, \dots, U_d - 1\} \to \mathbb{C}$$

such that $|g| \leq 1$, then the sum

$$S = \sum_{n \in \mathcal{N}} \sum_{0 \leq u_1 < U_1} \cdots \sum_{0 \leq u_d < U_d} g(n, u_1, \dots, u_d) \prod_{j=1}^{d} \chi_{U_j^{-1}} \left(f_j(n) - \frac{u_j}{U_j} \right)$$

can be approximated, for any positive integers $H_1, \dots, H_d$, by

$$\widetilde{S} = \sum_{\substack{|h_1| \leq H_1 \\ \cdots \\ |h_d| \leq H_d}} a_{h_1}(U_1^{-1}, H_1) \cdots a_{h_d}(U_d^{-1}, H_d) \sum_{\substack{0 \leq u_1 < U_1 \\ \cdots \\ 0 \leq u_d < U_d}} \mathrm{e} \left(-\frac{h_1 u_1}{U_1} - \cdots - \frac{h_d u_d}{U_d} \right)$$
$$\sum_{n \in \mathcal{N}} g(n, u_1, \dots, u_d) \, \mathrm{e} \left(h_1 f_1(n) + \cdots + h_d f_d(n) \right)$$

with the error estimate:

$$\left| S - \widetilde{S} \right| \leq \frac{U_1 \cdots U_d}{(H_1 + 1) \cdots (H_d + 1)} \sum_{|h_1| \leq H_1 / U_1} \left(1 - \frac{|h_1| \, U_1}{H_1 + 1} \right) \cdots \sum_{|h_d| \leq H_d / U_d} \left(1 - \frac{|h_d| \, U_d}{H_d + 1} \right)$$
$$\sum_{\substack{(\delta_1, \dots, \delta_d) \in \{0,1\}^d \\ (\delta_1, \dots, \delta_d) \neq (0, \dots, 0)}} \left| \sum_{n \in \mathcal{N}} \mathrm{e} \left(\delta_1 h_1 U_1 f_1(n) + \cdots + \delta_d h_d U_d f_d(n) \right) \right|. \tag{7}$$

Proof. From the proof of Lemma 3 in [7], using the bound of $|b_h(\alpha, H)|$ given by (5) we get

$$\left| S - \widetilde{S} \right| \leq \sum_{\ell=1}^{d} \sum_{1 \leq j_1 < \cdots < j_\ell \leq d} \frac{U_{j_1} \cdots U_{j_\ell}}{(H_{j_1} + 1) \cdots (H_{j_\ell} + 1)}$$
$$\sum_{|h_{j_1}| \leq H_{j_1} / U_{j_1}} \left(1 - \frac{|h_{j_1}| \, U_{j_1}}{H_{j_1} + 1} \right) \cdots \sum_{|h_{j_\ell}| \leq H_{j_\ell} / U_{j_\ell}} \left(1 - \frac{|h_{j_\ell}| \, U_{j_\ell}}{H_{j_\ell} + 1} \right)$$
$$\left| \sum_{n \in \mathcal{N}} \mathrm{e} \left(h_{j_1} U_{j_1} f_{j_1}(n) + \cdots + h_{j_\ell} U_{j_\ell} f_{j_\ell}(n) \right) \right|.$$

For all positive integers H and U we can write

$$\sum_{|h| \leq H/U} \left(1 - \frac{|h| \, U}{H + 1} \right) = \frac{1}{U} \sum_{0 \leq k < U} \sum_{|h| \leq H} \left(1 - \frac{|h|}{H + 1} \right) \mathrm{e} \left(\frac{kh}{U} \right),$$

and, since Fejer's periodic kernel $x \mapsto \sum_{|h| \leq H} \left(1 - \frac{|h|}{H+1} \right) \mathrm{e}(hx)$ is non negative, picking just $k = 0$ we get

$$1 \leq \frac{U}{H + 1} \sum_{|h| \leq H/U} \left(1 - \frac{|h| \, U}{H + 1} \right).$$

Inserting this inequality in the bound of $\left| S - \widetilde{S} \right|$ above for all pairs (H_j, U_j) such that $j \in \{1, \dots, d\} \setminus \{j_1, \dots, j_\ell\}$ we obtain (7). $\square$

The following two lemmas are useful generalizations of van der Corput's inequality.

Lemma 4. *For all complex numbers $z_1, \dots, z_N$ and all integers $k \geq 1$ and $R \geq 1$ we have*

$$\left|\sum_{1\leq n\leq N} z_n\right|^2 \leq \frac{N+kR-k}{R}\left(\sum_{1\leq n\leq N}|z_n|^2 + 2\sum_{1\leq r<R}\left(1-\frac{r}{R}\right)\sum_{1\leq n\leq N-kr}\Re\left(z_{n+kr}\overline{z_n}\right)\right), \tag{8}$$

where $\Re(z)$ denotes the real part of z.

Proof. See for example Lemma 17 of [24]. □

Lemma 5. *For integers $1 \leq A \leq B \leq N$ and complex numbers $z_1, \dots, z_N$ of modulus ≤ 1, we have for any integer $R \geq 1$,*

$$\left|\sum_{n=A}^{B} z_n\right| \leq \left(\frac{B-A+1}{R}\sum_{|r|<R}\left(1-\frac{|r|}{R}\right)\sum_{\substack{1\leq n\leq N\\ 1\leq n+r\leq N}} z_{n+r}\overline{z_n}\right)^{1/2} + \frac{R}{2}.$$

Proof. This is Lemma 15 of [24]. □

We will often make use of the following upper bound of geometric series of ratio $\mathrm{e}(\xi)$ for $(L_1, L_2) \in \mathbb{Z}^2$, $L_1 \leq L_2$ and $\xi \in \mathbb{R}$:

$$\left|\sum_{L_1<\ell\leq L_2}\mathrm{e}(\ell\xi)\right| \leq \min(L_2-L_1, |\sin\pi\xi|^{-1}). \tag{9}$$

Lemma 6. *For all real numbers $U > 0$, $\xi \in \mathbb{R}$ with $\xi \neq 0$, $\varphi \in \mathbb{R}$, $(M_1, M_2) \in \mathbb{Z}^2$ with $M_1 \leq M_2$ we have*

$$\sum_{M_1<m\leq M_2}\min\left(U, |\sin\pi(m\xi+\varphi)|^{-1}\right) \ll \left(3+\lfloor (M_2-M_1)\,\|\xi\|\rfloor\right)\left(3U + \|\xi\|^{-1}\log\|\xi\|^{-1}\right). \tag{10}$$

Proof. If $\|\xi\| > 1/3$ the result follows from the choice of an appropriate implied constant. Otherwise, by periodicity and parity we may assume that $0 < \xi \leq 1/3$. The number of integers in the interval $(M_1\xi+\varphi, M_2\xi+\varphi]$ is at most $1 + \lfloor (M_2-M_1)\xi\rfloor$. It follows that there is at most k values of m, with $k \leq 3 + \lfloor (M_2-M_1)\xi\rfloor$, say $m_1 < \cdots < m_k$ such that $\|m\xi+\varphi\| < \xi/2$. For the values $m_j - 1$, m_j and $m_j + 1$ we take the trivial bound U. Furthermore by convexity we have

$$\begin{aligned}\sum_{m_j+1<m<m_{j+1}-1}|\sin\pi(m\xi+\varphi)|^{-1} &\leq \int_{m_j+3/2}^{m_{j+1}-3/2}|\sin\pi\,\|t\xi+\varphi\||^{-1}\,dt\\ &= \xi^{-1}\int_{m_j\xi+\varphi+3\xi/2}^{m_{j+1}\xi+\varphi-3\xi/2}|\sin\pi\,\|u\||^{-1}\,du\\ &\leq \xi^{-1}\int_{\xi}^{1-\xi}|\sin\pi u|^{-1}\,du \ll \xi^{-1}\log\xi^{-1},\end{aligned}$$

which gives the result. □

The lemmas 7, 8 and 9 allow to estimate on average the minimums arising from (9).

Lemma 7. *Let $(a, m) \in \mathbb{Z}^2$ with $m \geq 1$, $\delta = \gcd(a, m)$ and $b \in \mathbb{R}$. For all positive real numbers U we have*

$$\sum_{0\leq n\leq m-1}\min\left(U, \left|\sin\pi\tfrac{an+b}{m}\right|^{-1}\right) \leq \delta\min\left(U, \left|\sin\pi\tfrac{\delta\,\|b/\delta\|}{m}\right|^{-1}\right) + \frac{2\,m}{\pi}\log(2\,m). \tag{11}$$

Proof. The result is trivial for $m=1$. For $m\geq 2$ after using Lemma 6 of [25] it suffice to observe that
$$\frac{\delta}{\sin\frac{\pi\delta}{2m}}+\frac{2m}{\pi}\log\frac{2m}{\pi\delta}\leq\frac{1}{\sin\frac{\pi}{2m}}+\frac{2m}{\pi}\log\frac{2m}{\pi}\leq\frac{2\,m}{\pi}\log(2\,m).$$
□

Lemma 8. *Let d, m_1, …, m_d be positive integers with $\gcd(m_i,m_j)=1$ for all $i\neq j$. Let $(a_1,\dots,a_d)\in\mathbb{Z}^d$ and $\delta_j=\gcd(a_j,m_j)$ for $j=1,\dots,d$. For all positive real numbers U and all real numbers φ we have*

$$\frac{1}{m_1\cdots m_d}\sum_{0\leq n_1<m_1}\cdots\sum_{0\leq n_d<m_d}\min\left(U,\left|\sin\pi\left(\tfrac{a_1n_1}{m_1}+\cdots+\tfrac{a_dn_d}{m_d}+\varphi\right)\right|^{-1}\right)$$
$$\leq\min\left(\frac{\delta_1\cdots\delta_d}{m_1\cdots m_d}U,\left|\sin\pi\left(\frac{m_1\cdots m_d}{\delta_1\cdots\delta_d}\varphi\right)\right|^{-1}\right)+\frac{2}{\pi}\log\left(2\frac{m_1\cdots m_d}{\delta_1\cdots\delta_d}\right), \tag{12}$$

Proof. Writing $m'_j=m_j/\delta_j$ and $a'_j=a_j/\delta_j$ for $j=1,\dots,d$, and using periodicity, the left hand side of (12) is equal to
$$\frac{1}{m'_1\cdots m'_d}\sum_{0\leq n_1<m'_1}\cdots\sum_{0\leq n_d<m'_d}\min\left(U,\left|\sin\pi\left(\tfrac{a'_1n_1}{m'_1}+\cdots+\tfrac{a'_dn_d}{m'_d}+\varphi\right)\right|^{-1}\right)$$
Let $m=m'_1\cdots m'_d$. If $m=1$ then inequality (12) is trivially satisfied, so we may assume that $m\geq 2$. Since $\gcd(m'_i,m'_j)=1$ for all $i\neq j$, by the chinese remainder theorem this is equal to
$$\frac{1}{m}\sum_{0\leq n\leq m-1}\min\left(U,\left|\sin\pi\left(\tfrac{a'_1n}{m'_1}+\cdots+\tfrac{a'_dn}{m'_d}+\varphi\right)\right|^{-1}\right).$$
Observing that $\frac{a'_1}{m'_1}+\cdots+\frac{a'_d}{m'_d}=\frac{a'}{m}$ for some $a'\in\mathbb{Z}$ such that $\gcd(a',m)=1$, it follows by Lemma 6 of [25] that the left hand side of (12) is at most
$$\frac{1}{m}\min\left(U,\frac{1}{\sin\left(\pi\frac{1}{m}\left\|m\varphi\right\|\right)}\right)+\frac{1}{m\sin\frac{\pi}{2m}}+\frac{2}{\pi}\log\frac{2\,m}{\pi}.$$
Since sinus is concave over $[0,\pi/2]$ we have
$$m\sin\left(\pi\frac{1}{m}\left\|m\varphi\right\|\right)\geq\sin\left(\pi\left\|m\varphi\right\|\right)=\left|\sin\left(\pi m\varphi\right)\right|$$
and for $m\geq 2$,
$$\frac{1}{m\sin\frac{\pi}{2m}}\leq\frac{1}{2\sin\frac{\pi}{4}}<\frac{2}{\pi}\log\pi,$$
which completes the proof of (12). □

Lemma 9. *Let m and A be positive integers and $b\in\mathbb{R}$. For all real numbers $U>0$ we have*
$$\frac{1}{A}\sum_{1\leq a\leq A}\sum_{0\leq n<m}\min\left(U,\left|\sin\pi\tfrac{an+b}{m}\right|^{-1}\right)\ll\tau(m)\;U+m\log m \tag{13}$$
and if $|b|\leq\frac{1}{2}$ we have the sharper bound
$$\frac{1}{A}\sum_{1\leq a\leq A}\sum_{0\leq n<m}\min\left(U,\left|\sin\pi\tfrac{an+b}{m}\right|^{-1}\right)\ll\tau(m)\min\left(U,\left|\sin\pi\tfrac{b}{m}\right|^{-1}\right)+m\log m, \tag{14}$$
where $\tau(m)$ denotes the number of divisors of m.

Proof. Using (11) we have for all $b \in \mathbb{R}$:

$$\sum_{0\le n<m} \min\left(U, \left|\sin \pi \tfrac{an+b}{m}\right|^{-1}\right) \ll \gcd(a,m)\, U + m\log m$$

while for $|b| \le \frac{1}{2}$, since $\gcd(a,m)\,\|b/\gcd(a,m)\| = |b|$ this can be sharpened using (11) to

$$\sum_{0\le n<m} \min\left(U, \left|\sin \pi \tfrac{an+b}{m}\right|^{-1}\right) \ll \gcd(a,m) \min\left(U, \left|\sin \pi \tfrac{b}{m}\right|^{-1}\right) + m\log m.$$

Now

$$\sum_{1\le a\le A} \gcd(a,m) = \sum_{\substack{d\,|\,m\\ d\le A}} d \sum_{\substack{1\le a\le A\\ \gcd(a,m)=d}} 1 \le \sum_{\substack{d\,|\,m\\ d\le A}} d \sum_{\substack{1\le a\le A\\ d\,|\,a}} 1 = \sum_{\substack{d\,|\,m\\ d\le A}} d \left\lfloor \frac{A}{d} \right\rfloor \le A\, \tau(m) \tag{15}$$

which implies (13) and (14) when $|b| \le \frac{1}{2}$. ☐

The following lemma is a classical application of the large sieve inequality.

Lemma 10. *For all $(z_1,\dots,z_N) \in \mathbb{C}^N$ and all positive integers Q we have*

$$\sum_{q\le Q} \sum_{\substack{a=1\\ (a,q)=1}}^{q} \left| \sum_{n=1}^{N} z_n \,\mathrm{e}\left(\frac{an}{q}\right) \right|^2 \le (N-1+Q^2) \sum_{n=1}^{N} |z_n|^2 . \tag{16}$$

Proof. See Theorem 3 and Section 8 of [28]. ☐

The following lemma gather some well known useful properties of Fejer's Kernel.

Lemma 11. *Let K denote Fejer's (non periodic) kernel and $\widehat{K}$ its Fourier transform:*

$$K(t) = \left(\frac{\sin \pi t}{\pi t}\right)^2, \quad \widehat{K}(t) = \int_{\mathbb{R}} K(u)\,\mathrm{e}(-ut)\,du = \max(0, 1-|t|), \tag{17}$$

for all $t \in \mathbb{R}$ we have

$$\widehat{K}(t) \le K\left(\frac{t}{2}\right) \tag{18}$$

and for all integers $N \ge 2$, we have [1]

$$\frac{1}{N} \sum_{n\in\mathbb{Z}} K\left(\frac{n}{N}\right) \mathrm{e}(nt) = \widehat{K}(N\,\|t\|). \tag{19}$$

Proof. We have $\cos x \le 1 - \frac{x^2}{2} + \frac{x^4}{24}$ for all $x \in \mathbb{R}$ hence for $|t| \le 1$ we have

$$K\left(\frac{t}{2}\right) = \frac{2(1-\cos \pi t)}{(\pi t)^2} \ge 1 - \frac{\pi^2 t^2}{12} \ge 1 - \frac{\pi^2 |t|}{12} \ge 1 - |t| .$$

Observing that both sides of (19) are 1-periodic even functions we may assume that $0 \le t \le \frac{1}{2}$, so that $t = \|t\|$. By Poisson's summation,

$$\frac{1}{N} \sum_{n\in\mathbb{Z}} K\left(\frac{n}{N}\right) \mathrm{e}(n\,\|t\|) = \sum_{s\in\mathbb{Z}} \widehat{K}(N(\|t\| - s)) = \widehat{K}(N\,\|t\|),$$

since for $|s| \ge 1$ and $N \ge 2$ we have $|N(\|t\|-s)| \ge 2(|s| - \|t\|) \ge 1$, thus $\widehat{K}(N(\|t\|-s)) = 0$. ☐

[1] (19) does not hold for $N = 1$

4. Correlations and discrete Fourier transforms

For any function $\psi : \mathbb{Z} \to \mathbb{C}$ and any $d \in \mathbb{Z}$ we denote by $\psi^{[d]}$ the function defined by

$$\forall n \in \mathbb{Z},\ \psi^{[d]}(n) = \psi(n)\overline{\psi(n+d)} \tag{20}$$

and for any function $\psi : \mathbb{Z}^2 \to \mathbb{C}$ and any $(d_1, d_2) \in \mathbb{Z}^2$ we denote by $\psi^{<d_1,d_2>}$ the function defined by

$$\forall (n_1, n_2) \in \mathbb{Z}^2,\ \psi^{<d_1,d_2>}(n_1, n_2) = \psi(n_1, n_2)\overline{\psi(n_1 + d_1, n_2)\psi(n_1, n_2 + d_2)}\psi(n_1 + d_1, n_2 + d_2). \tag{21}$$

For any function $f : \mathbb{N} \to \mathbb{C}$ and any $\lambda \in \mathbb{N}$, let us denote by f_λ the q^λ-periodic function defined by

$$\forall n \in \{0, \ldots, q^\lambda - 1\},\ \forall k \in \mathbb{Z},\ f_\lambda(n + kq^\lambda) = f(n). \tag{22}$$

The Discrete Fourier Transform of f_λ is defined for $t \in \mathbb{R}$ by

$$\widehat{f_\lambda}(t) = \frac{1}{q^\lambda} \sum_{0 \le u < q^\lambda} f_\lambda(u) \operatorname{e}\left(-\frac{ut}{q^\lambda}\right) = \frac{1}{q^\lambda} \sum_{0 \le u < q^\lambda} f(u) \operatorname{e}\left(-\frac{ut}{q^\lambda}\right). \tag{23}$$

With this definition, the Fourier inversion formula gives for any $n \in \mathbb{Z}$:

$$f_\lambda(n) = \sum_{0 \le h < q^\lambda} \widehat{f_\lambda}(h) \operatorname{e}\left(\frac{hn}{q^\lambda}\right), \tag{24}$$

and by Parseval's formula for any $\lambda \in \mathbb{N}$ and $t \in \mathbb{R}$ we have

$$\sum_{0 \le h < q^\lambda} \left|\widehat{f_\lambda}(h + t)\right|^2 = \frac{1}{q^\lambda} \sum_{0 \le u < q^\lambda} \left|f_\lambda(u)\right|^2. \tag{25}$$

For $\lambda \in \mathbb{N}$ and $d \in \mathbb{Z}$ the function $f_\lambda^{[d]}$ defined by (20) is q^λ-periodic function so that for any $t \in \mathbb{R}$ we have

$$\widehat{f_\lambda^{[d]}}(t) = \frac{1}{q^\lambda} \sum_{0 \le u < q^\lambda} f_\lambda(u)\overline{f_\lambda}(u + d) \operatorname{e}\left(-\frac{ut}{q^\lambda}\right).$$

Applying the Fourier inversion formula (24) with $n = u + d$ we get for any $t \in \mathbb{R}$

$$\widehat{f_\lambda^{[d]}}(t) = \sum_{0 \le h < q^\lambda} \widehat{f_\lambda}(h + t)\overline{\widehat{f_\lambda}(h)} \operatorname{e}\left(-\frac{hd}{q^\lambda}\right),$$

which permits to interpret $d \mapsto q^{-\lambda}\widehat{f_\lambda^{[d]}}(t)$ as the Discrete Fourier Transform of $h \to \widehat{f_\lambda}(h+t)\overline{\widehat{f_\lambda}(h)}$. Apply (25) to the summation over d we obtain for any $t \in \mathbb{R}$

$$\frac{1}{q^\lambda} \sum_{0 \le d < q^\lambda} \left|\widehat{f_\lambda^{[d]}}(t)\right|^2 = \sum_{0 \le h < q^\lambda} \left|\widehat{f_\lambda}(h + t)\right|^2 \left|\widehat{f_\lambda}(h)\right|^2. \tag{26}$$

By the Cauchy-Schwarz inequality

$$\frac{1}{q^\lambda} \sum_{0 \le d < q^\lambda} \left|\widehat{f_\lambda^{[d]}}(t)\right|^2 \le \left(\sum_{0 \le h < q^\lambda} \left|\widehat{f_\lambda}(h + t)\right|^4\right)^{1/2} \left(\sum_{0 \le h < q^\lambda} \left|\widehat{f_\lambda}(h)\right|^4\right)^{1/2}.$$

As for any $t \in \mathbb{Z}$, in the summation above $h + t$ reach exactly once each residue class modulo q^λ, thus by periodicity we get

$$\frac{1}{q^\lambda} \sum_{0 \le d < q^\lambda} \left|\widehat{f_\lambda^{[d]}}(t)\right|^2 \le \sum_{0 \le h < q^\lambda} \left|\widehat{f_\lambda}(h)\right|^4. \tag{27}$$

Remark 3. *By* (26) *we observe that the upper bound in* (27) *is attained for* $t = 0$.

Iterating (20) we consider now

$$f_\lambda^{[a,b]}(n) = \left(f_\lambda^{[b]}\right)^{[a]}(n) = f_\lambda(n)\overline{f_\lambda}(n+b)\overline{f_\lambda}(n+a)f_\lambda(n+a+b). \tag{28}$$

Applying (27) with f_λ replaced by $f_\lambda^{[b]}$ we get for any $b \in \mathbb{Z}$ and $t \in \mathbb{Z}$

$$\frac{1}{q^\lambda} \sum_{0\le a<q^\lambda} \left|\widehat{f_\lambda^{[a,b]}}(t)\right|^2 \le \frac{1}{q^\lambda} \sum_{0\le a<q^\lambda} \left|\widehat{f_\lambda^{[a,b]}}(0)\right|^2 = \sum_{0\le h<q^\lambda} \left|\widehat{f_\lambda^{[b]}}(h)\right|^4,$$

so that for any $t \in \mathbb{Z}$

$$\frac{1}{q^{2\lambda}} \sum_{0\le a<q^\lambda} \sum_{0\le b<q^\lambda} \left|\widehat{f_\lambda^{[a,b]}}(t)\right|^2 \le \frac{1}{q^{2\lambda}} \sum_{0\le a<q^\lambda} \sum_{0\le b<q^\lambda} \left|\widehat{f_\lambda^{[a,b]}}(0)\right|^2 = \frac{1}{q^\lambda} \sum_{0\le b<q^\lambda} \sum_{0\le h<q^\lambda} \left|\widehat{f_\lambda^{[b]}}(h)\right|^4. \tag{29}$$

5. Carry property

We recall Definition 1 of [26].

Definition 5. *A function* $f : \mathbb{N} \to \mathbb{U}$ *has the* carry property *if, uniformly for* $(\lambda, \kappa, \rho) \in \mathbb{N}^3$ *with* $\rho < \lambda$, *the number of integers* $0 \le \ell < q^\lambda$ *such that there exists* $(k_1, k_2) \in \{0, \dots, q^\kappa - 1\}^2$ *with*

$$f(\ell q^\kappa + k_1 + k_2)\,\overline{f(\ell q^\kappa + k_1)} \ne f_{\kappa+\rho}(\ell q^\kappa + k_1 + k_2)\,\overline{f_{\kappa+\rho}(\ell q^\kappa + k_1)} \tag{30}$$

is at most $O(q^{\lambda-\rho})$, *where the implied constant may depend only on* q *and* f.

Lemma 12. *If* $f : \mathbb{N} \to \mathbb{U}$ *satisfies Definition 5, then for* $(\mu, \nu, \rho) \in \mathbb{N}^3$ *with* $2\rho < \nu$ *the set* $\mathcal{E}$ *of* $(m, n) \in \{q^{\mu-1}, \dots, q^\mu - 1\} \times \{q^{\nu-1}, \dots, q^\nu - 1\}$ *such that there exists* $k < q^{\mu+\rho}$ *with* $f(mn + k)\,\overline{f(mn)} \ne f_{\mu+2\rho}(mn + k)\,\overline{f_{\mu+2\rho}(mn)}$ *satisfies*

$$\operatorname{card} \mathcal{E} \ll (\log q)\, q^{\mu+\nu-\rho}, \tag{31}$$

where the implied constant may depend only on q *and* f.

Proof. This is Lemma 8 of [26]. □

Lemma 13. *Any strongly* q*-multiplicative function has the carry property (see Definition* (5)*).*

Proof. Let f be a strongly q-multiplicative function and $(\lambda, \kappa, \rho) \in \mathbb{N}^3$ with $\rho < \lambda$. Considering $f_{\kappa+\rho}$ in (30), the inequality may occur only by carry propagation when the digits of $\ell q^\kappa + k_1$ of indexes $\kappa, \dots, \kappa + \rho - 1$ are equal to $q - 1$, *i.e.* for integers ℓ with $\gg q^\rho$ least significant digits equal to $q - 1$. It follows that f has the carry property. □

6. Fourier Transforms of strongly q-multiplicative functions

The main purpose of this section is to prove Proposition 1 and Proposition 2.

Proposition 1. *If* f *is a proper strongly* q*-multiplicative function, then there exist constants* $c_1 > 0$, $c_2 > 0$ *such that for all* $\lambda \in \mathbb{N}$ *we have*

$$\frac{1}{q^\lambda} \sum_{0\le b<q^\lambda} \sum_{0\le h<q^\lambda} \left|\widehat{f_\lambda^{[b]}}(h)\right|^4 \le c_1 q^{-c_2\lambda}.$$

By (29) and Proposition 1 we obtain

Proposition 2. *If f is a proper strongly q-multiplicative function, then there exist constants $c_1 > 0$, $c_2 > 0$ such that for all $t \in \mathbb{Z}$ and $\lambda \in \mathbb{N}$ we have*

$$\frac{1}{q^{2\lambda}} \sum_{0\le a<q^\lambda} \sum_{0\le b<q^\lambda} \left| \widehat{f_\lambda^{[a,b]}}(t) \right|^2 \le c_1 q^{-c_2\lambda}. \tag{32}$$

The proof of Proposition 1 is divided into several Lemmas.

Lemma 14. *Suppose that $b \equiv j \bmod q$ (with $0 \le j < q$). Then*

$$\begin{aligned}\widehat{f_\lambda^{[b]}}(h) = \frac{1}{q} &\sum_{0\le \ell\le q-1-j} f(\ell)\overline{f(\ell+j)}\ \mathrm{e}(-h\ell q^{-\lambda})\ \widehat{f_{\lambda-1}^{[\lfloor b/q\rfloor]}}(h) \\ &+ \frac{1}{q} \sum_{q-j\le \ell\le q-1} f(\ell)\overline{f(\ell+j-q)}\ \mathrm{e}(-h\ell q^{-\lambda})\ \widehat{f_{\lambda-1}^{[\lfloor b/q\rfloor+1]}}(h).\end{aligned}$$

Proof. We write $b = qb' + j$ and split up the sum over $0 \le u < q^\lambda$ into q according to the residue class of u: $u = qu' + \ell$, $0 \le \ell < q-1$, $0 \le u' < q^{\lambda-1}$, and use the property $f_\lambda(qm+r) = f_{\lambda-1}(m)f(r)$ (for $0 \le r < q$):

$$\begin{aligned}\widehat{f_\lambda^{[b]}}(h) &= \frac{1}{q^\lambda} \sum_{\ell=0}^{q-1} \sum_{0\le u'<q^{\lambda-1}} f_\lambda(qu'+\ell)\overline{f_\lambda(qu'+qb'+\ell+j)}\, \mathrm{e}\left(-hu'q^{-(\lambda-1)} - \ell h q^{-\lambda}\right) \\ &= \frac{1}{q^\lambda} \sum_{\ell=0}^{q-1-j} \sum_{0\le u'<q^{\lambda-1}} f_{\lambda-1}(u')\overline{f_{\lambda-1}(u'+b')} f(\ell)\overline{f(\ell+j)}\, \mathrm{e}\left(-hu'q^{-(\lambda-1)}\right) \mathrm{e}\left(-\ell h q^{-\lambda}\right) \\ &+ \frac{1}{q^\lambda} \sum_{\ell=q-j}^{q-1} \sum_{0\le u'<q^{\lambda-1}} f_{\lambda-1}(u')\overline{f_{\lambda-1}(u'+b'+1)} f(\ell)\overline{f(\ell+j-q)}\, \mathrm{e}\left(-hu'q^{-(\lambda-1)}\right) \mathrm{e}\left(-\ell h q^{-\lambda}\right) \\ &= \frac{1}{q} \sum_{0\le\ell\le q-1-j} f(\ell)\overline{f(\ell+j)}\ \mathrm{e}(-h\ell q^{-\lambda})\ \widehat{f_{\lambda-1}^{[b']}}(h) \\ &+ \frac{1}{q} \sum_{q-j\le\ell\le q-1} f(\ell)\overline{f(\ell+j-q)}\ \mathrm{e}(-h\ell q^{-\lambda})\ \widehat{f_{\lambda-1}^{[b'+1]}}(h).\end{aligned}$$

□

For any $(\lambda, h) \in \mathbb{N}^2$ let us consider

$$\Gamma_\lambda(h) = \max_{0\le b<q^\lambda} \left| \widehat{f_\lambda^{[b]}}(h) \right|.$$

We have $\Gamma_\lambda(0) = 1$ and it follows from Lemma 14 that for all $h \ge 1$ we have

$$\Gamma_\lambda(h) \le \Gamma_{\lambda-1}(h). \tag{33}$$

Lemma 15 will give a better estimate of $\Gamma_\lambda(h)$ in terms of $\Gamma_{\lambda-1}(h)$ (or $\Gamma_{\lambda-2}(h)$ in the case $q = 2$).

Lemma 15. *If $q = 2$ then we have*

$$\Gamma_\lambda(h) \le \max\left(\left|\cos\frac{\pi h}{2^\lambda}\right|, \sin^2\frac{\pi h}{2^\lambda} \right) \Gamma_{\lambda-2}(h) \tag{34}$$

whereas for $q \ge 3$ we have

$$\Gamma_\lambda(h) \le \max\left\{ \max_{2\le j<q} \frac{|1+f(q-j+1)\overline{f(1)f(q-j)}\,\mathrm{e}(-hq^{-\lambda})|+q-2}{q}, \right.$$
$$\left. \max_{0\le j<q-1} \frac{|1+f(j)f(1)\overline{f(j+1)}\,\mathrm{e}(-hq^{-\lambda})|+q-2}{q} \right\} \Gamma_{\lambda-1}(h).$$

Proof. Suppose that $q=2$. This implies that $f(h)=\zeta^{s_2(n)}$, where $|\zeta|=1$ and $s_2(n)$ denotes the binary sum-of-digits function. In particular this simplifies the recurrence relation in Lemma 14: $f(\ell)\overline{f(\ell+j)}=\zeta^{-j}$ and $f(\ell)\overline{f(\ell+j-2)}=\zeta^{2-j}$.

If b is even, that is $j=0$, then we get

$$|\widehat{f_\lambda^{[b]}}(h)| \le \left|\frac{1+\mathrm{e}(-h2^{-\lambda})}{2}\right| \Gamma_{\lambda-1}(h) = \left|\cos\frac{\pi h}{2^\lambda}\right| \Gamma_{\lambda-1}(h) \le \left|\cos\frac{\pi h}{2^\lambda}\right| \Gamma_{\lambda-2}(h).$$

If b is odd then either $\lfloor b/2\rfloor$ or $\lfloor b/2\rfloor+1$ is even. In both case

$$\begin{aligned}
|\widehat{f_\lambda^{[b]}}(h))| &\le \frac{|\widehat{f_{\lambda-1}^{[\lfloor b/2\rfloor]}}(h)|+|\widehat{f_{\lambda-1}^{[\lfloor b/2\rfloor+1]}}(h)|}{2}\\
&\le \left(\frac{1}{2}+\frac{1}{2}\left|\cos\frac{\pi h}{2^{\lambda-1}}\right|\right)\Gamma_{\lambda-2}(h)\\
&\le \max\left(\cos^2\frac{\pi h}{2^\lambda}, \sin^2\frac{\pi h}{2^\lambda}\right)\Gamma_{\lambda-2}(h).
\end{aligned}$$

Putting these two estimates together we derive

$$\Gamma_\lambda(h) \le \max\left(\left|\cos\frac{\pi h}{2^\lambda}\right|, \cos^2\frac{\pi h}{2^\lambda}, \sin^2\frac{\pi h}{2^\lambda}\right)\Gamma_{\lambda-2}(h)$$

which completes the proof of (34).

If $q\ge 3$ and $0\le j\le q-1$ then we either have $j\ge 2$ or $q-j\ge 2$. Suppose first that $q-j\ge 2$. Then by Lemma 14

$$\begin{aligned}
\left|\widehat{f_\lambda^{[b]}}(h)\right| &\le \frac{|1+f(j)f(1)\overline{f(j+1)}\,\mathrm{e}(-hq^{-\lambda})|+q-j-2}{q}\left|\widehat{f_{\lambda-1}^{[\lfloor b/q\rfloor]}}(h)\right| + \frac{j}{q}\left|\widehat{f_{\lambda-1}^{[\lfloor b/q\rfloor+1]}}(h)\right|\\
&\le \frac{|1+f(j)f(1)\overline{f(j+1)}\,\mathrm{e}(-hq^{-\lambda})|+q-2}{q}\Gamma_{\lambda-1}(h).
\end{aligned}$$

Similarly if $j\ge 2$ we obtain

$$\begin{aligned}
\left|\widehat{f_\lambda^{[b]}}(h)\right| &\le \frac{q-j}{q}\left|\widehat{f_{\lambda-1}^{[\lfloor b/q\rfloor]}}(h)\right| + \frac{|1+f(q-j+1)\overline{f(1)f(q-j)}\,\mathrm{e}(-hq^{-\lambda})|+q-j-2}{q}\left|\widehat{f_{\lambda-1}^{[\lfloor b/q\rfloor+1]}}(h)\right|\\
&\le \frac{|1+f(q-j+1)\overline{f(1)f(q-j)}\,\mathrm{e}(-hq^{-\lambda})|+q-2}{q}
\end{aligned}$$

and consequently,

$$\Gamma_\lambda(h) \le \max\left\{ \max_{2\le j<q} \frac{|1+f(q-j+1)\overline{f(1)f(q-j)}\,\mathrm{e}(-hq^{-\lambda})|+q-2}{q}, \right.$$
$$\left. \max_{0\le j<q-1} \frac{|1+f(j)f(1)\overline{f(j+1)}\,\mathrm{e}(-hq^{-\lambda})|+q-2}{q} \right\} \Gamma_{\lambda-1}(h)$$

which completes the proof of the lemma. □

Lemma 16. *Suppose that $q \ge 2$. Then for every strongly q-multiplicative function f there exist $L \ge 2$, a proper subset $S \subset \{0,1,\dots,q-1\}^L$ and a constant $c' = c'(q,f,L,S) > 0$ such that*

$$\left|\widehat{f_\lambda^{[b]}}(h)\right| \le q^{-c' J(h)},$$

where $J(h)$ denotes the number of sub-blocks of length L in the q-ary expansion of $h = \sum_{j=0}^{\lambda-1} \varepsilon_j(h)$ that are not contained in S.

Moreover, the set S can be chosen in a way that all prefixes of length $L-1$ from the elements of S are different.

Proof. If $q=2$ let us consider $S = \{(0,0,0),(0,1,1),(1,0,0),(1,1,1)\} \subset \{0,1\}^3$. For any $j \in \{3,\dots,\lambda\}$ we have

$$\frac{h}{2^j} = \frac{\varepsilon_{j-1}(h)}{2} + \frac{\varepsilon_{j-2}(h)}{4} + \dots + \frac{\varepsilon_0(h)}{2^j},$$

so that $\frac{h}{2^j} \in \left[\frac18,\frac38\right] \cup \left[\frac58,\frac78\right]$ if and only if $(\varepsilon_{j-1}(h),\varepsilon_{j-2}(h),\varepsilon_{j-3}(h)) \notin S$ and in this case we have

$$\max\left(\left|\cos\frac{\pi h}{2^\lambda}\right|, \left|\sin\frac{\pi h}{2^\lambda}\right|\right) \le \cos\frac{\pi}{8}.$$

This gives Lemma 16 in the case $q=2$ with $c' = -\log(\cos\frac{\pi}{8})/\log 2 > 0$.

In the case $q \ge 3$ the situation is slightly different. In some sense it simplifies because $\Gamma_\lambda(h)$ is directly related with $\Gamma_{\lambda-1}(h)$ (and not with $\Gamma_{\lambda-2}(h)$, see Lemma 15) but on the other hand we have to be more careful with the values $hq^{-\lambda}$.

For $2 \le j < q$ let $\alpha_j \in [0,1)$ be defined by $\mathrm{e}(\alpha_j) = f(q-j+1)\overline{f(1)f(q-j)}$. Similarly for $0 \le j < q-1$ let $\beta_j \in [0,1)$ be given by $\mathrm{e}(\beta_j) = f(j)f(1)\overline{f(j+1)}$. Set $T = \{\alpha_j : 2 \le j < q\} \cup \{\beta_j : 0 \le j < q-1\}$.

By Lemma 15 we have to specify conditions for h that ensure that $hq^{-\lambda}$ is different from (and even not too close to) α_j and β_j. The idea is to cover T with q-adic intervals $[mq^{-L},(m+1)q^{-L})$ that we encode with the q-adic digits of m. It is clear that $hq^{-\lambda} \bmod 1 \in [mq^{-L},(m+1)q^{-L})$ if and only if the digits of $m = m_0 + m_1 q + \dots + m_{L-1}q^{L-1}$ coincide with the last L digits of h: $m_{L-j} = \varepsilon_{\lambda-j}(h)$, $1 \le j \le L$. In particular we can find a collection $\mathcal{I}$ of q-adic intervals with a sufficiently large (and common) length q^{-L} with the following two properties:

(1) T is contained in the interior of the union of all intervals of $\mathcal{I}$, where we assume that we work on the torus, that is, 0 and 1 are identified.
(2) The digit blocks $(m_1,m_2,\dots,m_{L-1})$ of length $L-1$ corresponding to those $m = m_0 + m_1 q + \dots + m_{L-1}q^{L-1}$ for which the interval $[mq^{-L},(m+1)q^{-L})$ is contained in $\mathcal{I}$ are all different.

Both conditions are very easy to satisfy if the elements of T are not q-adic rational numbers. In the case of q-adic rational numbers we can increase the value of L in order to satisfy the conditions.

Let τ be the minimal distance of an element of T to the boundary of the union of all intervals of $\mathcal{I}$. By the first property it follows that $\tau > 0$. Furthermore we let S be the set of all q-ary digit blocks of m for which $[mq^{-L},(m+1)q^{-L})$ is contained in $\mathcal{I}$. Now suppose that $h < q^\lambda$ has the property that the digit block $(\varepsilon_{\lambda-L}(h),\dots,\varepsilon_{\lambda-1}(h))$ is not contained in S. Then $hq^{-\lambda} \bmod 1$ is not contained in the union of intervals of $\mathcal{I}$ which implies that

$$\|hq^{-\lambda} - \alpha_j\| \ge \tau \quad \text{and} \quad \|hq^{-\lambda} - \beta_{j'}\| \ge \tau \tag{35}$$

for all $(j,j') \in \{2,\dots,q-1\} \times \{0,\dots,q-2\}$. Consequently we have

$$\Gamma_\lambda(h) \le \frac{|1+\mathrm{e}(\tau)|+q-2}{q}\Gamma_{\lambda-1}(h).$$

Of course this implies by induction that

$$\Gamma_\lambda(h) \le q^{-c' J(h)},$$

with

$$c' = \frac{1}{\log q}\log\frac{q}{|1+\mathrm{e}(\tau)|+q-2} > 0.$$

$\square$

Remark 4. *Note that replacing τ by $\tau/2$ we can keep the same L and S in order to obtain* (35) *for any slight pertubation of α_j and β_j (slight pertubation of the strongly q-multiplicative function f).*

Lemma 17. *Let $J(h)$, $0 \le h < q^\lambda$ be a function of the form given in Lemma 16, that is, it counts the number of blocks of length L in the q-ary expansion of h that are not contained in S, where all prefixes of length $L-1$ from the elements of S are different.*

Then for every $\varepsilon > 0$ there exist constants $\eta = \eta(\varepsilon,q,L,S) \ge 0$ and $c = c(\varepsilon,q,L,S) > 0$ such that

$$|h < q^\lambda : J(h) \le \varepsilon\lambda\}| \le c\, q^{\eta\,\lambda}.$$

Moreovever we have

$$\lim_{\varepsilon\to 0}\eta(\varepsilon) = 0.$$

Proof. Let $\varepsilon_j(h)$ denote the j-th binary digit of h. Furthermore for every block $B \in \{0,1,\dots,q-1\}^{L-1}$ let

$$a_\lambda^{(B)}(x) = \sum_{h<q^\lambda,\, (\varepsilon_{\lambda-L-1}(h),\dots,\varepsilon_{\lambda-1}(h)=B} x^{J(h)}.$$

Then we have $a_1^{(B)}(x) = 1$ for all $B \in \{0,1,\dots,q-1\}^{L-1}$. Now for every block B of length $L-1$ let B' be the prefix of B of length $L-2$, that is, $B = (B'\delta(B))$, where $\delta(B)$ denotes the last digit of B and denote by B'' the suffix of B of length $L-2$. With the help of this notation we get the recurrence relations

$$a_{\lambda+1}^{(B)}(x) = \sum_{C\colon C''=B',\,(C,\delta(B))\in S} a_\lambda^{(C)}(x) + x \sum_{C\colon C''=B',\,(C,\delta(B))\notin S} a_\lambda^{(C)}(x)$$

Iterating this recurrence we obtain

$$\left(a_\lambda^{(B)}(x)\right)_{B\in\{0,1,\dots,q-1\}^{L-1}} = A(x)^{\lambda-1}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix},$$

where the matrix $A(x) = (a_{B,C}(x))_{B,C\in\{0,1,\dots,q-1\}^{L-1}}$ is given by

$$a_{B,C}(x) = \begin{cases} 1 & \text{if } C''=B' \text{ and } (C,\delta(B)) \in S,\\ x & \text{if } C''=B' \text{ and } (C,\delta(B)) \notin S,\\ 0 & \text{otherwise.}\end{cases}$$

If $x > 0$ then $A(x)$ is a positive irreducible matrix. By Perron-Frobenius theory there exists a unique dominating eigenvalue $\rho(x)$ of $A(x)$ so that

$$\begin{aligned}\sum_{h<q^\lambda,} x^{J(h)} &= \sum_B a_\lambda^{(B)}(x) \\ &= \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} A(x)^{\lambda-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= D(x)\rho(x)^{\lambda-1} + O\left(\rho(x)^{(1-\gamma)\lambda}\right) \\ &\le E(x)\rho(x)^{\lambda-1}\end{aligned}$$

for some real number $\gamma = \gamma(x) > 0$ and certain positive functions $D(x)$ and $E(x)$.

By assumption all prefixes of length $L-1$ of elements of S are different. Hence, every row of $A(x)$ contains at most one 1. This implies that the largest eigenvalue of $A(0)$ is at most 1. Since the largest eigenvalue is a continuous function in the entries of a matrix it follows that

$$\lim_{x\to 0+} \rho(x) \le 1. \tag{36}$$

Now suppose that $0 < x < 1$. Then we have

$$\begin{aligned}\sum_{h<q^\lambda} x^{J(h)} &\ge \sum_{h<q^\lambda,\, J(h)\le\varepsilon\lambda} x^{J(h)} \\ &\ge x^{\varepsilon\lambda}\, |\{h < q^\lambda : J(h) \le \varepsilon\lambda\}|.\end{aligned}$$

Hence, be choosing $x = \varepsilon$ we obtain

$$\begin{aligned}|\{h < q^\lambda : J(h) \le \varepsilon\lambda\}| &\le \frac{E(x)\rho(x)^{\lambda-1}}{x^{\varepsilon\lambda}} \\ &\le \frac{E(\varepsilon)}{\rho(\varepsilon)} \left(\frac{\rho(\varepsilon)}{\varepsilon^\varepsilon}\right)^\lambda.\end{aligned}$$

This proves the lemma with

$$c(\varepsilon) = \frac{E(\varepsilon)}{\rho(\varepsilon)} \quad \text{and} \quad \eta(\varepsilon) = \max\left\{0, \frac{1}{\log q} \log \frac{\rho(\varepsilon)}{\varepsilon^\varepsilon}\right\},$$

since (36) implies $\lim_{\varepsilon\to 0} \eta(\varepsilon) = 0$. □

Lemma 18. *Suppose that f is a proper strongly q-multiplicative function. Then there exist constants $C_1 = C_1(q) > 0$ and*

$$0 < C_2 = C_2(f,q) \le \frac{4}{q\log q} \tag{37}$$

such that

$$|\widehat{f}_\lambda(t)| \le C_1 q^{-C_2\lambda} \tag{38}$$

uniformly for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{N}$.

Proof. As f is strongly q-multiplicative, for any $t \in \mathbb{R}$ we have

$$\widehat{f}_\lambda(t) = \prod_{\ell=1}^{\lambda} \left(\frac{1}{q} \sum_{0\le j<q} f(j)\, \mathrm{e}\left(\frac{-jt}{q^\ell}\right)\right). \tag{39}$$

If we put $(\alpha_0,\dots,\alpha_{q-1})\in\mathbb{R}^q$ such that $f(j)=\mathrm{e}(\alpha_j)$ for $j\in\{0,\dots,q-1\}$ and

$$\varphi(t)=\left|\sum_{0\le j<q}\mathrm{e}(\alpha_j-jt)\right|$$

and if we define $\gamma_q(f)$ by

$$q^{\gamma_q(f)}=\max_{t\in\mathbb{R}}\sqrt{\varphi(t)\varphi(qt)}$$

it follows from (39) that for any $t\in\mathbb{R}$ we have

$$\left|\widehat{f_\lambda}(t)\right|\le q^{-\lambda}\prod_{\ell=1}^{\lambda}\varphi(tq^{-\ell})\le q^{1-\lambda}\prod_{1\le\ell\le\lambda/2}\varphi(tq^{-(2\ell-1)})\varphi(tq^{-2\ell})\le q^{1-\lambda+2\lfloor\lambda/2\rfloor\gamma_q(f)}$$

so that for any $t\in\mathbb{R}$,

$$\left|\widehat{f_\lambda}(t)\right|\le q^{\lambda(\gamma_q(f)-1)+1}.$$

As in [19, (8)], let $\sigma_q(f)$ be defined by

$$\sigma_q(f)=\min_{t\in\mathbb{R}}\sum_{0\le j<i<q}\|\alpha_i-\alpha_j-(i-j)t\|^2.$$

If $\sigma_q(f)>0$ it follows from [19, Lemme 8] that $\gamma_q(f)\le 1-\frac{16}{q^2(q-1)\log q}\sigma_q(f)$ so that (38) holds with $C_1=q$ and

$$C_2=\frac{16}{q^2(q-1)\log q}\sigma_q(f)$$

which satisfies (37) by observing that $\sigma_q(f)\le q(q-1)/4$.

If $\sigma_q(f)=0$ it follows from [19, Lemme 1] that $\alpha_0,\alpha_1,\dots,\alpha_{q-1}$ form an arithmetic progression modulo 1 (note that $\alpha_0\equiv 0 \bmod 1$) thus for any integer n we have $f(n)=\mathrm{e}(\alpha_1 s_q(n))$. Since f is proper and $\mathrm{s}_q(n)\equiv n \bmod q-1$ for any integer n, it follows that $(q-1)\alpha_1\notin\mathbb{Z}$ so that we can apply [19, Lemme 11] to obtain (38) with $C_1=q$ and

$$C_2=\frac{4\|(q-1)\alpha_1\|^2}{q(q+\sqrt{2}-1)^2\log q}$$

which satisfies (37). □

Proof of Proposition 1. By applying (29) Lemma 16 and (26) we have

$$\begin{aligned}\frac{1}{q^{2\lambda}}\sum_{0\le a<q^\lambda}\sum_{0\le b<q^\lambda}\left|\widehat{f_\lambda^{[a,b]}}(t)\right|^2&\le\frac{1}{q^\lambda}\sum_{0\le b<q^\lambda}\sum_{0\le h<q^\lambda}\left|\widehat{f_\lambda^{[b]}}(h)\right|^4\\&\le\frac{1}{q^\lambda}\sum_{0\le b<q^\lambda}\sum_{0\le h<q^\lambda}q^{-2c'J(h)}\left|\widehat{f_\lambda^{[b]}}(h)\right|^2\\&=\sum_{0\le h<q^\lambda}q^{-2c'J(h)}\sum_{0\le k<q^\lambda}\left|\widehat{f_\lambda}(h+k)\right|^2\left|\widehat{f_\lambda}(k)\right|^2\\&=S_1+S_2\end{aligned}$$

with

$$S_1=\sum_{0\le h<q^\lambda,\,J(h)>\varepsilon\lambda}q^{-2c'J(h)}\sum_{0\le k<q^\lambda}\left|\widehat{f_\lambda}(h+k)\right|^2\left|\widehat{f_\lambda}(k)\right|^2$$

and

$$S_2 = \sum_{0 \le h < q^\lambda,\, J(h) \le \varepsilon\lambda} q^{-2c' J(h)} \sum_{0 \le k < q^\lambda} \left|\widehat{f}_\lambda(h+k)\right|^2 \left|\widehat{f}_\lambda(k)\right|^2.$$

The first sum can be directly estimated:

$$S_1 \le q^{-2c'\varepsilon\lambda} \sum_{0 \le h,k < q^\lambda} \left|\widehat{f}_\lambda(h+k)\right|^2 \left|\widehat{f}_\lambda(k)\right|^2 = q^{-2c'\varepsilon\lambda}.$$

For the second sum we apply Lemma 18 for the term $\left|\widehat{f}_\lambda(h+k)\right|^2$ and obtain with the help of Lemma 17

$$\begin{aligned} S_2 &\le C_1 q^{-2C_2\lambda} \sum_{0 \le h < q^\lambda,\, J(h) \le \varepsilon\lambda} \sum_{0 \le k < q^\lambda} \left|\widehat{f}_\lambda(k)\right|^2 \\ &= C_1 q^{-2C_2\lambda} |\{h < q^\lambda : J(h) \le \varepsilon\lambda\}| \\ &\le C_1 c\, q^{-(2C_2-\eta)\lambda}. \end{aligned}$$

Finally we obtain

$$\frac{1}{q^\lambda} \sum_{0 \le b < q^\lambda} \sum_{0 \le h < q^\lambda} \left|\widehat{f_\lambda^{[b]}}(h)\right|^4 \le q^{-2c'\varepsilon\lambda} + C_1 c\, q^{-(2C_2-\eta)\lambda} \tag{40}$$

and if we choose $\varepsilon > 0$ such that $\eta(\varepsilon) \le C_2$,

$$c_1 = c_1(q, f, L, S) = 1 + C_2 c \tag{41}$$

and

$$c_2 = c_2(q, f, L, S) = \min\{2c', C_2\}, \tag{42}$$

this completes the proof of Proposition 1.

In special cases we can be more precise, and in particular if $f = \mathrm{e}(\alpha f_0)$, where $\alpha \in \mathbb{R}$ and f_0 is an integer valued strongly q-additive function with $\gcd(f_0(1), \ldots, f_0(q-1)) = 1$ we have the following Proposition.

Proposition 3. *If f_0 is an integer valued strongly q-additive function such that* $\gcd(f_0(1), \ldots, f_0(q_1 - 1)) = 1$ *and $f = \mathrm{e}(\alpha f_0)$, then uniformly for $\alpha \in \mathbb{R}$ such that $d_{f_0}\alpha \notin \mathbb{Z}$ we have*

$$\frac{1}{q^{2\lambda}} \sum_{0 \le a < q^\lambda} \sum_{0 \le b < q^\lambda} \left|\widehat{f_\lambda^{[a,b]}}(t)\right|^2 \le c_{10} q^{-c_9 \|d_{f_0}\alpha\|^2 / \log(1/\|d_{f_0}\alpha\|)\,\lambda}. \tag{43}$$

First it follows from the proof of Lemma 18 that

$$|\widehat{f}_\lambda(h)| \le q \cdot q^{-c_3(\sigma_q(\alpha f_0) + \|(q-1)f_0(1)\alpha\|^2)\lambda}$$

for some constant $c_3 = c_3(f_0, q) > 0$. However, by (19) and (20) from [20] we have

$$\sigma_q(\alpha f_0) + \|(q-1)f_0(1)\alpha\|^2 \ge c_4 \|d_{f_0}\alpha\|^2$$

for some constant $c_4 = c_4(f_0, q) > 0$. Hence we have

$$|\widehat{f}_\lambda(h)| \le q \cdot q^{-c_5 \|d_{f_0}\alpha\|^2 \lambda} \tag{44}$$

for some constant $c_5 = c_5(f_0, q) > 0$.

Due to periodicity and symmetry we only have to consider the interval $0 \le \alpha \le 1/(2d_{f_0})$. Let us put

$$A = \frac{1}{2q^2 \max_j\{|f_0(q-j+1) - f_0(1) - f_0(q-j)|, |f_0(j) + f_0(1) - f_0(j+1)|\}}$$

and start with the case $0 \le \alpha \le A$, which means that $\alpha_j = \alpha(f_0(q-j+1) - f_0(1) - f_0(q-j))$ and $\beta_j = \alpha_j(f_0(j) + f_0(1) - f_0(j+1))$ satisfy $|\alpha_j| \le 1/(2q^2)$ and $|\beta_j| \le 1/(2q^2)$. We choose $L = 2$, $S = \{(00), (11), \ldots, (q-1, q-1)\}$ and obtain by Lemma 16

$$\left|\widehat{f_\lambda^{[b]}}(h)\right| \le q^{-c'\,J(h)}.$$

We have

$$\sum_{h<q^\lambda} x^{J(h)} = q\left(1 + (q-1)x\right)^{\lambda-1},$$

so that, we can take in the proof of Lemma 17

$$E(\varepsilon) = q,\ \ \rho(\varepsilon) = 1 + (q-1)\varepsilon \text{ and } \eta(\varepsilon) = \frac{1}{\log q}\log\frac{1+(q-1)\varepsilon}{\varepsilon^\varepsilon}.$$

By choosing $\varepsilon = c_6\|d_{f_0}\alpha\|^2/\log(1/\|d_{f_0}\alpha\|)$ for some $c_6 = c_6(q,f)$ such that $\eta(\varepsilon) \le C_2 = c_5\|d_{f_0}\alpha\|^2$, we obtain by (40) the upper bound

$$\frac{1}{q^\lambda}\sum_{0\le b<q^\lambda}\sum_{0\le h<2^\lambda}\left|\widehat{f_\lambda^{[b]}}(h)\right|^4 \le c_8 q^{-c_7\|d_{f_0}\alpha\|^2/\log(1/\|d_{f_0}\alpha\|)\,\lambda} \tag{45}$$

with some positive constant $c_8 = c_8(q, f_0)$.

If $A \le \alpha \le 1/(2d_{f_0})$, then it follows from (44) that we have uniformly

$$|\widehat{f_\lambda}(h)| \le q \cdot q^{-c_5\|d_{f_0}A\|^2\lambda}.$$

When α varies in the compact set $[A, 1/(2d_{f_0})]$ it is enough by Remark 4 to consider only finitely many different L and S in the construction used in the proof of Lemma 16. If we take for $\widetilde{c_1}$ the maximum over L and S of all the $c_1(q, \alpha f_0, L, S)$ defined by (41) and for $\widetilde{c_2}$ the minimum over L and S of all the $c_2(q, \alpha f_0, L, S)$ defined by (42) then we obtain

$$\frac{1}{q^\lambda}\sum_{0\le b<q^\lambda}\sum_{0\le h<2^\lambda}\left|\widehat{f_\lambda^{[b]}}(h)\right|^4 \le \widetilde{c_1} q^{-\widetilde{c_2}\lambda} \tag{46}$$

uniformly for $\alpha \in [A, 1/(2d_{f_0})]$.

Putting together (45) and (46) it follows that

$$\frac{1}{q^\lambda}\sum_{0\le b<q^\lambda}\sum_{0\le h<2^\lambda}\left|\widehat{f_\lambda^{[b]}}(h)\right|^4 \le c_{10} q^{-c_9\|d_{f_0}\alpha\|^2/\log(1/\|d_{f_0}\alpha\|)\,\lambda}$$

with some positive constants $c_9 = c_9(q, f_0)$ and $c_{10} = c_{10}(q, f_0)$ holds uniformly for all α and Proposition 3 follows from (29).

7. Sums of type I

Let

$$1 \le M \le N \tag{47}$$

be integers. Let μ_1 and ν_1 be the unique integers such that

$$q_1^{\mu_1-1} \le M < q_1^{\mu_1} \quad \text{and} \quad q_1^{\nu_1-1} \le N < q_1^{\nu_1} \tag{48}$$

and assume that

$$\mu_1 \le \xi_1(\mu_1+\nu_1) \tag{49}$$

where

$$0 < \xi_1 < 1/3 \tag{50}$$

will be fixed in (66).

Similarly let μ_2 and ν_2 be the unique integers such that

$$q_2^{\mu_2-1} \le M < q_2^{\mu_2} \quad \text{and} \quad q_2^{\nu_2-1} \le N < q_2^{\nu_2} \tag{51}$$

and assume that

$$\mu_2 \le \xi_1(\mu_2+\nu_2). \tag{52}$$

For any $\vartheta \in \mathbb{R}$, any interval $I(M,N) \subseteq [\frac{MN}{4}, MN]$, f and g two strongly q-multiplicative functions we consider

$$S_I(\vartheta) = \sum_{\frac{M}{2}<m\le M} \left| \sum_{\substack{n \\ mn\in I(M,N)}} f(mn)g(mn)\,\mathrm{e}(\vartheta mn) \right|.$$

Proposition 4. *For any integers M and N satisfying* (47), (48), (49), (51) *and* (52) *we have uniformly for $\vartheta \in \mathbb{R}$*

$$S_I(\vartheta) \ll (\log MN)^{\frac12\omega(q_1)+\frac12\omega(q_2)}\ (MN)^{1-\sigma(q_1,q_2)} \tag{53}$$

for some explicit $\sigma(q_1,q_2) > 0$.

Proof. By Cauchy-Schwarz

$$S_I^2(\vartheta) \le M \sum_{\frac{M}{2}<m\le M} \left| \sum_{\substack{n \\ mn\in I(M,N)}} f(mn)g(mn)\,\mathrm{e}(\vartheta mn) \right|^2,$$

and by Lemma 5, for any integers

$$R \ge 1 \tag{54}$$

and

$$L \ge \frac{7N}{4},$$

we have

$$S_I^2(\vartheta) \ll \frac{MN}{R} \sum_{r\in\mathbb{Z}} \widehat{K}\left(\frac{r}{R}\right) \sum_{\frac{M}{2}<m\le M} S_{I,1}\left(r,m,\left\lfloor \tfrac{N}{4}\right\rfloor+1, \left\lfloor \tfrac{N}{4}\right\rfloor+L\right) \mathrm{e}(\vartheta mr) + M^2R^2$$

where $\widehat{K}$ is the Fourier transform of the Fejer kernel defined by (17) and

$$S_{I,1}(r,m,A,B) = \sum_{A\le n\le B} f(mn+mr)\overline{f(mn)}g(mn+mr)\overline{g(mn)}.$$

Let $D > 0$ be a parameter to be chosen later (in (62)) which will represent a margin in the carry propagation and let δ_1 and δ_2 be the unique integers such that

$$q_1^{\delta_1 - 1} \le D < q_1^{\delta_1} \text{ and } q_2^{\delta_2 - 1} \le D < q_2^{\delta_2}.$$

Combining Lemma 13 and Lemma 12, applied to f (with (μ_1, ν_1, δ_1) in place of (μ, ν, ρ)) and to g (with (μ_2, ν_2, δ_2) in place of (μ, ν, ρ)), for

$$q_1^{\lambda_1 - 1} < MRD \le q_1^{\lambda_1} \tag{55}$$

and

$$q_2^{\lambda_2 - 1} < MRD \le q_2^{\lambda_2} \tag{56}$$

we can replace f by f_{λ_1} and g by g_{λ_2} in $S_{I,1}$ introducing an admissible error term $O(M^2N^2/D)$. We obtain

$$S_I^2(\vartheta) \ll \frac{MN}{R} \sum_{r \in \mathbb{Z}} \widehat{K}\left(\frac{r}{R}\right) \sum_{\frac{M}{2} < m \le M} S_{I,2}\left(r, m, \left\lfloor \tfrac{N}{4} \right\rfloor + 1, \left\lfloor \tfrac{N}{4} \right\rfloor + L\right) \mathrm{e}(\vartheta m r) + M^2R^2 + \frac{M^2N^2}{D} \tag{57}$$

with

$$S_{I,2}(r, m, A, B) = \sum_{A \le n \le B} f_{\lambda_1}(mn + mr)\overline{f_{\lambda_1}(mn)} g_{\lambda_2}(mn + mr)\overline{g_{\lambda_2}(mn)}.$$

Observing that $n \mapsto f_{\lambda_1}(mn + mr)\overline{f_{\lambda_1}(mn)} g_{\lambda_2}(mn + mr)\overline{g_{\lambda_2}(mn)}$ is periodic of period $q_1^{\lambda_1} q_2^{\lambda_2}$ we choose

$$L = \left\lceil \frac{7N}{4\, q_1^{\lambda_1} q_2^{\lambda_2}} \right\rceil q_1^{\lambda_1} q_2^{\lambda_2},$$

so that

$$S_{I,2}\left(r, m, \left\lfloor \tfrac{N}{4} \right\rfloor + 1, \left\lfloor \tfrac{N}{4} \right\rfloor + L\right) = \left\lceil \frac{7N}{4\, q_1^{\lambda_1} q_2^{\lambda_2}} \right\rceil S_{I,2}(r, m, 0, q_1^{\lambda_1} q_2^{\lambda_2} - 1)$$

Since n runs over all residue classes modulo $q_1^{\lambda_1} q_2^{\lambda_2}$ and $\gcd(q_1, q_2) = 1$, we may replace n by $n_1 q_2^{\lambda_2} + n_2 q_1^{\lambda_1}$ with $0 \le n_1 < q_1^{\lambda_1}$ and $0 \le n_2 < q_2^{\lambda_2}$. Thus by periodicity

$$S_{I,2}(r, m, 0, q_1^{\lambda_1} q_2^{\lambda_2} - 1) = \sum_{0 \le n_1 < q_1^{\lambda_1}} f_{\lambda_1}(mn_1 q_2^{\lambda_2} + mr)\overline{f_{\lambda_1}(mn_1 q_2^{\lambda_2})} \sum_{0 \le n_2 < q_2^{\lambda_2}} g_{\lambda_2}(mn_2 q_1^{\lambda_1} + mr)\overline{g_{\lambda_2}(mn_2 q_1^{\lambda_1})},$$

and again since $\gcd(q_1, q_2) = 1$,

$$S_{I,2}(r, m, 0, q_1^{\lambda_1} q_2^{\lambda_2} - 1) = \sum_{0 \le n_1 < q_1^{\lambda_1}} f_{\lambda_1}(mn_1 + mr)\overline{f_{\lambda_1}(mn_1)} \sum_{0 \le n_2 < q_2^{\lambda_2}} g_{\lambda_2}(mn_2 + mr)\overline{g_{\lambda_2}(mn_2)}.$$

We apply Cauchy-Schwarz as follows:

$$\sum_r \sum_m \left(\left| A_r^{1/2} \sum_{n_1} \right| \cdot \left| A_r^{1/2} \sum_{n_2} \right| \right) \le \left(\sum_r \sum_m A_r \left| \sum_{n_1} \right|^2 \right)^{1/2} \left(\sum_r \sum_m A_r \left| \sum_{n_2} \right|^2 \right)^{1/2}$$

with $A_r = R^{-1} \widehat{K}(r/R)$. Observing that $\left\lceil \frac{7N}{4\, q_1^{\lambda_1} q_2^{\lambda_2}} \right\rceil \ll \left(1 + \frac{N}{q_1^{2\lambda_1}}\right)^{1/2} \left(1 + \frac{N}{q_2^{2\lambda_2}}\right)^{1/2}$ it suffice to estimate

$$\left(1 + \frac{N}{q_1^{2\lambda_1}}\right) \frac{1}{R} \sum_{r \in \mathbb{Z}} \widehat{K}\left(\frac{r}{R}\right) \sum_m \left| \sum_{0 \le n_1 < q_1^{\lambda_1}} f_{\lambda_1}(mn_1 + mr)\overline{f_{\lambda_1}(mn_1)} \right|^2 \tag{58}$$

and

$$\left(1+\frac{N}{q_2^{2\lambda_2}}\right)\frac{1}{R}\sum_{r\in\mathbb{Z}}\widehat{K}\left(\frac{r}{R}\right)\sum_{m}\left|\sum_{0\le n_2<q_2^{\lambda_2}} g_{\lambda_2}(mn_2+mr)\overline{g_{\lambda_2}(mn_2)}\right|^2. \tag{59}$$

We will focus on (58) (the case of (59) is similar).

Writing $m=d_1m'$ with $\gcd(m',q_1^{\lambda_1})=1$ and observing that for any $u\in\mathbb{Z}$, $n_1\mapsto f_{\lambda_1}(d_1m'n_1+u)$ is $\frac{q_1^{\lambda_1}}{d_1}$-periodic, we get

$$\left(1+\frac{N}{q_1^{2\lambda_1}}\right)\sum_{\substack{d_1\mid q_1^{\lambda_1}\\ d_1\le M}}\frac{d_1^2}{R}\sum_{r\in\mathbb{Z}}\widehat{K}\left(\frac{r}{R}\right)\sum_{\substack{m'\le M/d_1\\ \gcd(m',q_1^{\lambda_1})=1}}\left|\sum_{0\le n_1<\frac{q_1^{\lambda_1}}{d_1}} f_{\lambda_1}(d_1m'n_1+d_1m'r)\overline{f_{\lambda_1}(d_1m'n_1)}\right|^2.$$

By Fourier inversion the inner sum over n_1 is equal to

$$\sum_{0\le n_1<\frac{q_1^{\lambda_1}}{d_1}}\sum_{0\le h<q_1^{\lambda_1}}\sum_{0\le k<q_1^{\lambda_1}}\widehat{f_{\lambda_1}}(h)\overline{\widehat{f_{\lambda_1}}(h-k)}\,\mathrm{e}\left(\frac{kd_1m'n_1+hd_1m'r}{q_1^{\lambda_1}}\right).$$

Since $\gcd(m',q_1^{\lambda_1})=1$ we may replace h by $h\widetilde{m'}$ and k by $k\widetilde{m'}$ where $m'\,\widetilde{m'}\equiv 1 \bmod q_1^{\lambda_1}$, and this gives

$$\sum_{0\le n_1<\frac{q_1^{\lambda_1}}{d_1}}\sum_{0\le h<q_1^{\lambda_1}}\sum_{0\le k<q_1^{\lambda_1}}\widehat{f_{\lambda_1}}(h\widetilde{m'})\overline{\widehat{f_{\lambda_1}}((h-k)\widetilde{m'})}\,\mathrm{e}\left(\frac{kd_1n_1+hd_1r}{q_1^{\lambda_1}}\right),$$

which is equal to

$$\frac{q_1^{\lambda_1}}{d_1}\sum_{0\le h<q_1^{\lambda_1}}\sum_{k\equiv 0 \bmod \frac{q_1^{\lambda_1}}{d_1}}\widehat{f_{\lambda_1}}(h\widetilde{m'})\overline{\widehat{f_{\lambda_1}}((h-k)\widetilde{m'})}\,\mathrm{e}\left(\frac{hd_1r}{q_1^{\lambda_1}}\right).$$

Writing $h=h'+\ell\frac{q_1^{\lambda_1}}{d_1}$ we obtain

$$\frac{q_1^{\lambda_1}}{d_1}\sum_{0\le h'<\frac{q_1^{\lambda_1}}{d_1}}\sum_{0\le\ell<d_1}\sum_{0\le k'<d_1}\widehat{f_{\lambda_1}}\left(h'\widetilde{m'}+\ell\widetilde{m'}\frac{q_1^{\lambda_1}}{d_1}\right)\overline{\widehat{f_{\lambda_1}}\left(h'\widetilde{m'}+\ell\widetilde{m'}\frac{q_1^{\lambda_1}}{d_1}-k'\widetilde{m'}\frac{q_1^{\lambda_1}}{d_1}\right)}\,\mathrm{e}\left(\frac{h'd_1r}{q_1^{\lambda_1}}\right),$$

and if we put $\ell'=\ell-k'$ by periodicity we get for (58) the estimate

$$\left(1+\frac{N}{q_1^{2\lambda_1}}\right)q_1^{2\lambda_1}\sum_{\substack{d_1\mid q_1^{\lambda_1}\\ d_1\le M}}\sum_{\substack{m'\le M/d_1\\ \gcd(m',q_1^{\lambda_1})=1}}$$
$$\frac{1}{R}\sum_{r\in\mathbb{Z}}\widehat{K}\left(\frac{r}{R}\right)\left|\sum_{0\le h'<\frac{q_1^{\lambda_1}}{d_1}}\left|\sum_{0\le\ell'<d_1}\widehat{f_{\lambda_1}}\left(h'\widetilde{m'}+\ell'\widetilde{m'}\frac{q_1^{\lambda_1}}{d_1}\right)\right|^2\mathrm{e}\left(\frac{h'd_1r}{q_1^{\lambda_1}}\right)\right|^2.$$

By (18) we have $\widehat{K}(r/R) \le K(r/2R)$ and expanding the square, (19) permits to obtain

$$\left(1+\frac{N}{q_1^{2\lambda_1}}\right) q_1^{2\lambda_1} \sum_{\substack{d_1 \mid q_1^{\lambda_1}\\ d_1\le M}} \sum_{\substack{m'\le M/d_1\\ \gcd(m',q_1^{\lambda_1})=1}} \sum_{0\le h'<\frac{q_1^{\lambda_1}}{d_1}} \left| \sum_{0\le \ell'<d_1} \widehat{f_{\lambda_1}}\left(h'\widetilde{m'}+\ell'\widetilde{m'}\frac{q_1^{\lambda_1}}{d_1}\right)\right|^2$$
$$\sum_{0\le h''<\frac{q_1^{\lambda_1}}{d_1}} \widehat{K}\left(2R\left\|(h''-h')\frac{d_1}{q_1^{\lambda_1}}\right\|\right)\left|\sum_{0\le \ell''<d_1} \widehat{f_{\lambda_1}}\left(h''\widetilde{m'}+\ell''\widetilde{m'}\frac{q_1^{\lambda_1}}{d_1}\right)\right|^2.$$

The last sum over h'' is equal to

$$\sum_{-h'\le\widetilde{h''}<-h'+\frac{q_1^{\lambda_1}}{d_1}} \widehat{K}\left(2R\left\|\widetilde{h''}\frac{d_1}{q_1^{\lambda_1}}\right\|\right)\left|\sum_{0\le \ell''<d_1} \widehat{f_{\lambda_1}}\left(h'\widetilde{m'}+\widetilde{h''}\widetilde{m'}+\ell''\widetilde{m'}\frac{q_1^{\lambda_1}}{d_1}\right)\right|^2.$$

The function $\widetilde{h''}\mapsto \left\|\widetilde{h''}\frac{d_1}{q_1^{\lambda_1}}\right\|$ is $\frac{q_1^{\lambda_1}}{d_1}$-periodic, and since $\widehat{f_{\lambda_1}}$ is $q_1^{\lambda_1}$-periodic, we observe that

$$\begin{aligned}\sum_{0\le \ell''<d_1} \widehat{f_{\lambda_1}}\left(h'\widetilde{m'}+\left(\widetilde{h''}+\frac{q_1^{\lambda_1}}{d_1}\right)\widetilde{m'}+\ell''\widetilde{m'}\frac{q_1^{\lambda_1}}{d_1}\right) &= \sum_{0\le \ell''<d_1} \widehat{f_{\lambda_1}}\left(h'\widetilde{m'}+\widetilde{h''}\widetilde{m'}+(\ell''+1)\widetilde{m'}\frac{q_1^{\lambda_1}}{d_1}\right)\\ &= \sum_{0\le \ell''<d_1} \widehat{f_{\lambda_1}}\left(h'\widetilde{m'}+\widetilde{h''}\widetilde{m'}+\ell''\widetilde{m'}\frac{q_1^{\lambda_1}}{d_1}\right)\end{aligned}$$

so that this quantity is $\frac{q_1^{\lambda_1}}{d_1}$-periodic in $\widetilde{h''}$. We deduce that the sum over $\widetilde{h''}$ may be written

$$\sum_{-\frac{q_1^{\lambda_1}}{2d_1}<\widetilde{h''}\le\frac{q_1^{\lambda_1}}{2d_1}} \widehat{K}\left(2R\widetilde{h''}\frac{d_1}{q_1^{\lambda_1}}\right)\left|\sum_{0\le \ell''<d_1} \widehat{f_{\lambda_1}}\left(h'\widetilde{m'}+\widetilde{h''}\widetilde{m'}+\ell''\widetilde{m'}\frac{q_1^{\lambda_1}}{d_1}\right)\right|^2,$$

which by Cauchy-Schwarz (for the sum over ℓ'') and interverting the summations is at most

$$d_1\sum_{0\le \ell''<d_1}\sum_{-\frac{q_1^{\lambda_1}}{2d_1}<\widetilde{h''}\le\frac{q_1^{\lambda_1}}{2d_1}} \widehat{K}\left(2R\widetilde{h''}\frac{d_1}{q_1^{\lambda_1}}\right)\left|\widehat{f_{\lambda_1}}\left(h'\widetilde{m'}+\widetilde{h''}\widetilde{m'}+\ell''\widetilde{m'}\frac{q_1^{\lambda_1}}{d_1}\right)\right|^2. \tag{60}$$

As the support of $\widehat{K}$ is $[-1,1]$ (see (17)) the length of the summation over $\widetilde{h''}$ is $\le 1+\frac{q_1^{\lambda_1}}{Rd_1}\le q_1^{\lambda_1-\rho'}$ where $\rho'=\rho-2$ and $\rho\in\mathbb{N}$ is defined by

$$q_1^{\rho-1}<R\le q_1^{\rho}. \tag{61}$$

As f is strongly q_1-multiplicative, by (39) and Lemma 18 we obtain for any $t\in\mathbb{R}$,

$$\left|\widehat{f_{\lambda_1}}(t)\right| = \left|\widehat{f_{\lambda_1-\rho'}}(t)\right|\left|\widehat{f_{\rho'}}(t\ q_1^{-(\lambda_1-\rho')})\right| \ll q_1^{-C_2\rho'}\left|\widehat{f_{\lambda_1-\rho'}}(t)\right|.$$

Since $0\le\widehat{K}\le 1$, the sum (60) is at most

$$d_1^2\max_{k\in\mathbb{Z}}\sum_{0\le h<q_1^{\lambda_1-\rho'}}\left|\widehat{f_{\lambda_1}}\left((h+k)\widetilde{m'}\right)\right|^2 \ll d_1^2 q_1^{-2C_2\rho'}\max_{k\in\mathbb{Z}}\sum_{0\le h<q_1^{\lambda_1-\rho'}}\left|\widehat{f_{\lambda_1-\rho'}}\left((h+k)\widetilde{m'}\right)\right|^2 = d_1^2 q_1^{-2C_2\rho'},$$

where the last equality is obtained by (25), observing that since $\gcd(\widetilde{m'},q_1^{\lambda_1-\rho'})=1$, in the last sum $h\widetilde{m'}$ runs over a complete set of residues modulo $q_1^{\lambda_1-\rho'}$.

Therefore again by Cauchy-Schwarz (for the sum over ℓ') we obtain that (58) is bounded by

$$q_1^{-2C_2\rho'}\left(1+\frac{N}{q_1^{2\lambda_1}}\right)q_1^{2\lambda_1}\sum_{\substack{d_1\mid q_1^{\lambda_1}\\ d_1\le M}} d_1^2\,\frac{M}{d_1}\max_{\substack{\widetilde{m'}\in\mathbb{Z}\\ \gcd(\widetilde{m'},q_1^{\lambda_1})=1}}\sum_{0\le h'<\frac{q_1^{\lambda_1}}{d_1}} d_1\sum_{0\le \ell'<d_1}\left|\widehat{f_{\lambda_1}}\left(h'\widetilde{m'}+\ell'\widetilde{m'}\frac{q_1^{\lambda_1}}{d_1}\right)\right|^2$$

$$\ll q_1^{-2C_2\rho'}M\left(q_1^{2\lambda_1}+N\right)\sum_{\substack{d_1\mid q_1^{\lambda_1}\\ d_1\le M}} d_1^2\sum_{0\le h<q_1^{\lambda_1}}\left|\widehat{f_{\lambda_1}}\left(h\right)\right|^2,$$

and finally by (55) and (56) is

$$\ll R^{-2C_2}M^3\left(M^2R^2D^2+N\right)\tau(q_1^{\lambda_1}).$$

We choose

$$D=N^{\delta} \tag{62}$$

with $0<\delta<1/6$ and

$$R=\left\lfloor N^{1/2}M^{-1}D^{-1}\right\rfloor \tag{63}$$

with

$$M\le N^{\frac{1}{2}-\delta}, \tag{64}$$

so that (54) is satisfied. Assuming also

$$M\le (N^{1/2}M^{-1}D^{-1})^{C_2/2}, \tag{65}$$

we obtain that (58) is

$$\ll R^{-C_2}MN\,\tau(q_1^{\lambda_1})\ll R^{-C_2}MN\,\lambda_1^{\omega(q_1)}\,\tau(q_1).$$

The condition (65) is ensured if

$$\mu_1\left(1+\left(\frac{3}{2}-\delta\right)\frac{C_2}{2}\right)<(\mu_1+\nu_1)\left(\frac{1}{2}-\delta\right)\frac{C_2}{2},$$

which is the case if (50) is restricted to

$$0<\xi_1<\frac{(1-2\delta)C_2}{4+(3-2\delta)C_2}. \tag{66}$$

It follows from (37) and (66) that $\xi_1<\frac{1}{4}$ (which implies (50)), so that $\mu_1<\frac{1}{4}(\mu_1+\nu_1)$ which implies that $3\mu_1<\nu_1$ and $M\le N^{1/3}\le N^{\frac{1}{2}-\delta}$, so that (64) holds. Since $\frac{1}{2}-\delta>0$, $M\ll (MN)^{1/4}$ and $N\gg (MN)^{3/4}$ we have

$$R^{-C_2}\ll (MN)^{\frac{C_2}{4}-\frac{3C_2}{4}\left(\frac{1}{2}-\delta\right)}=(MN)^{-\frac{C_2}{8}+\frac{3\delta C_2}{4}}$$

so that finally (58) is

$$\ll (MN)^{1-\frac{C_2(f,q_1)}{8}(1-6\delta)}\,(\log MN)^{\omega(q_1)}\,\tau(q_1).$$

In the same way (59) is

$$\ll (MN)^{1-\frac{C_2(g,q_2)}{8}(1-6\delta)}\,(\log MN)^{\omega(q_2)}\,\tau(q_2).$$

It follows from (57) that

$$S_I^2(\vartheta)\ll (MN)^{2-\frac{C_2(f,q_1)+C_2(g,q_2)}{16}(1-6\delta)}\,(\log MN)^{\frac{1}{2}\omega(q_1)+\frac{1}{2}\omega(q_2)}+M^2R^2+\frac{M^2N^2}{D}.$$

By (63), $M^2R^2 \le ND^{-2} = N^{1-2\delta} \le (MN)^{1-2\delta}$ and $\frac{M^2N^2}{D} = \frac{M^2N^2}{N^\delta} \le (MN)^{2-\frac{3\delta}{4}}$ Choosing

$$\delta = \frac{C_2(f,q_1) + C_2(g,q_2)}{6(2 + C_2(f,q_1) + C_2(g,q_2))}$$

we obtain (53) with $\sigma(q_1,q_2) = \frac{3}{4}\delta$. □

8. Sums of type II

Let

$$1 \le M \le N \tag{67}$$

be integers. Let μ_1 and ν_1 be the unique integers such that

$$q_1^{\mu_1 - 1} \le M < q_1^{\mu_1} \quad \text{and} \quad q_1^{\nu_1 - 1} \le N < q_1^{\nu_1} \tag{68}$$

and assume that

$$\xi_1(\mu_1 + \nu_1) \le \mu_1 \le \tfrac{1}{2}(\mu_1 + \nu_1) \tag{69}$$

where ξ_1 satisfies (66).

Similarly let μ_2 and ν_2 be the unique integers such that

$$q_2^{\mu_2 - 1} \le M < q_2^{\mu_2} \quad \text{and} \quad q_2^{\nu_2 - 1} \le N < q_2^{\nu_2} \tag{70}$$

and assume that

$$\xi_1(\mu_2 + \nu_2) \le \mu_2 \le \tfrac{1}{2}(\mu_2 + \nu_2). \tag{71}$$

We assume also that the multiplicative dependence of the variables in the type II sums has been removed by the classical method described (for example) in section 5 of [25].

For any $\vartheta \in \mathbb{R}$, $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ two sequences of complex numbers of modulus at most 1 and f and g two strongly q-multiplicative functions we consider

$$S_{II}(\vartheta) = \sum_{\frac{M}{2} < m \le M} \sum_{\frac{N}{2} < n \le N} a_m b_n f(mn) g(mn) \, \mathrm{e}(\vartheta mn).$$

Proposition 5. *For any integers M and N satisfying* (68), (69), (70), (71), *we have uniformly for $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ two sequences of complex numbers of modulus at most* 1 *and $\vartheta \in \mathbb{R}$*

$$|S_{II}(\vartheta)| \ll MN \exp\left(-C \frac{\log(MN)}{\log\log(MN)}\right) \tag{72}$$

for some explicit constant $C = C(f, g, \xi_1, \xi_2) > 0$.

As often in this approach, getting an upper bound for the sums of type II is the most difficult part. The proof is quite long and complicated and will be developped over several sections and completed at formula (97). By the Cauchy-Schwarz inequality we have

$$|S_{II}(\vartheta)|^2 \ll M \sum_{\frac{M}{2} < m \le M} \left| \sum_{\frac{N}{2} < n \le N} b_n f(mn) g(mn) \, \mathrm{e}(\vartheta mn) \right|^2.$$

Let R_0 be an integer to be defined later (by (92)) such that

$$1 \le R_0 \le N. \tag{73}$$

Applying Lemma 4 to the summation over n with $k = 1$ and then summing over m we get

$$|S_{II}(\vartheta)|^2 \ll \frac{M^2N^2}{R_0} + \frac{MN}{R_0} \sum_{1 \le r_0 < R_0} \left(1 - \frac{r_0}{R_0}\right) \Re(S_1(r_0)),$$

with

$$S_1(r_0) = \sum_{\frac{M}{2}<m\le M} \sum_{\frac{N}{2}<n\le N-r_0} b_{n+r_0}\overline{b_n} f(mn+mr_0)\overline{f(mn)}g(mn+mr_0)\overline{g(mn)}\,\mathrm{e}(\vartheta m r_0).$$

Let ρ_1 and ρ_2 be the unique integers such that

$$q_1^{\rho_1-1} < R_0 \le q_1^{\rho_1} \quad \text{and} \quad q_2^{\rho_2-1} < R_0 \le q_2^{\rho_2}, \tag{74}$$

and

$$\mu_{12} = \mu_1 + 2\rho_1 \quad \text{and} \quad \mu_{22} = \mu_2 + 2\rho_2. \tag{75}$$

If f and g have the carry property explained in Definition 5, then by Lemma 12 the number of (m,n) for which $f(mn+mr_0)\overline{f(mn)} \neq f_{\mu_{12}}(mn+mr_0)\overline{f_{\mu_{12}}(mn)}$ is $O(MN/R_0)$, and similarly the number of (m,n) for which $g(mn+mr_0)\overline{g(mn)} \neq g_{\mu_{22}}(mn+mr_0)\overline{g_{\mu_{22}}(mn)}$ is also $O(MN/R_0)$. Hence

$$S_1(r_0) = S_1'(r_0) + O(MN/R_0),$$

where

$$S_1'(r_0) = \sum_{\frac{M}{2}<m\le M} \sum_{\frac{N}{2}<n\le N-r_0} b_{n+r_0}\overline{b_n} f_{\mu_{12}}(mn+mr_0)\overline{f_{\mu_{12}}(mn)}g_{\mu_{22}}(mn+mr_0)\overline{g_{\mu_{22}}(mn)}\,\mathrm{e}(\vartheta m r_0).$$

Using the Cauchy-Schwarz inequality, this leads to

$$|S_{II}(\vartheta)|^4 \ll \frac{M^4N^4}{R_0^2} + \frac{M^2N^2}{R_0^2} R_0 \sum_{1\le r_0<R_0} |S_1'(r_0)|^2. \tag{76}$$

We reverse the order of summation in $S_1'(r_0)$ and obtain:

$$|S_1'(r_0)| \le \sum_{\frac{N}{2}<n\le N-r_0} \left| \sum_{\frac{M}{2}<m\le M} f_{\mu_{12}}(mn+mr_0)\overline{f_{\mu_{12}}(mn)}g_{\mu_{22}}(mn+mr_0)\overline{g_{\mu_{22}}(mn)}\,\mathrm{e}(\vartheta m r_0)\right|.$$

We may extend the summation over n to $(N/2,N]$ and apply the Cauchy-Schwarz inequality:

$$|S_1'(r_0)|^2 \ll N \sum_{\frac{N}{2}<n\le N} \left| \sum_{\frac{M}{2}<m\le M} f_{\mu_{12}}(mn+mr_0)\overline{f_{\mu_{12}}(mn)}g_{\mu_{22}}(mn+mr_0)\overline{g_{\mu_{22}}(mn)}\,\mathrm{e}(\vartheta m r_0)\right|^2.$$

Applying to the summation over m the Lemma 4 with positive integers $k = q_1^{\mu_{11}}$ and R_1 such that

$$M \ll q_1^{\mu_{11}} R_1 \ll M \tag{77}$$

and then summing over n and r_0 we get

$$\frac{1}{R_0} \sum_{1\le r_0<R_0} |S_1'(r_0)|^2 \ll \frac{M^2N^2}{R_1} + MN\ \Re(S_2), \tag{78}$$

with

$$S_2 = \frac{1}{R_0R_1} \sum_{1\le r_0<R_0} \sum_{1\le r_1<R_1} \left(1-\frac{r_1}{R_1}\right)\ \mathrm{e}(q_1^{\mu_{11}} r_0 r_1 \vartheta)\ S_2'(r_0,r_1)$$

and

$$S_2'(r_0,r_1) = \sum_{\frac{N}{2}<n\le N} \sum_{M/2<m\le M-q_1^{\mu_{11}}r_1} \psi_1^{<q_1^{\mu_{11}}r_1, r_0>}(m,n)$$

using notation (21) with

$$\psi_1(m,n) = f_{\mu_{12}}(mn)\ g_{\mu_{22}}(mn).$$

Using (76) and (78) we obtain uniformly for $\vartheta \in \mathbb{R}$:

$$|S_{II}(\vartheta)|^4 \ll \frac{M^4N^4}{R_0^2} + \frac{M^4N^4}{R_1} + \frac{M^3N^3}{R_0R_1} \sum_{1\le r_0<R_0} \sum_{1\le r_1<R_1} |S_2'(r_0,r_1)|\,. \tag{79}$$

Using the Cauchy-Schwarz inequality we get

$$|S_{II}(\vartheta)|^8 \ll \frac{M^8N^8}{R_0^4} + \frac{M^8N^8}{R_1^2} + \frac{M^6N^6}{R_0^2R_1^2} R_0R_1 \sum_{1\le r_0<R_0} \sum_{1\le r_1<R_1} |S_2'(r_0,r_1)|^2\,. \tag{80}$$

We have

$$|S_2'(r_0,r_1)|^2 \le N \sum_{\frac{N}{2}<n\le N} \left| \sum_{M/2<m\le M-q_1^{\mu_{11}}r_1} \psi_1^{<q_1^{\mu_{11}}r_1,r_0>}(m,n) \right|^2$$

and applying to the summation over m the Lemma 4 with positive integers $k = q_2^{\mu_{21}}$ and R_2 such that

$$M \ll q_2^{\mu_{21}} R_2 \ll M \tag{81}$$

we obtain

$$\begin{aligned} |S_2'(r_0,r_1)|^2 \ll & \frac{M^2N^2}{R_2} + \frac{MN}{R_2} \Re \sum_{1\le r_2<R_2} \left(1-\frac{r_2}{R_2}\right) \\ & \sum_{\frac{N}{2}<n\le N} \sum_{M/2<m\le M-q_1^{\mu_{11}}r_1-q_2^{\mu_{21}}r_2} \psi_1^{<q_1^{\mu_{11}}r_1,r_0>}(m+q_2^{\mu_{21}}r_2,n) \overline{\psi_1^{<q_1^{\mu_{11}}r_1,r_0>}(m,n)}. \end{aligned}$$

Writing $f_{\mu_{12}} = f_{\mu_{11}} f_{\mu_{11},\mu_{12}}$ and $g_{\mu_{22}} = g_{\mu_{21}} g_{\mu_{21},\mu_{22}}$, using the periodicity and then summing over r_0 and r_1 we get

$$\frac{1}{R_0R_1} \sum_{1\le r_0<R_0} \sum_{1\le r_1<R_1} |S_2'(r_0,r_1)|^2 \ll \frac{M^2N^2}{R_2} + \frac{MN}{R_0R_1R_2} \sum_{1\le r_0<R_0} \sum_{1\le r_1<R_1} \sum_{1\le r_2<R_2} |S_3(r_0,r_1,r_2)| \tag{82}$$

with

$$S_3(r_0,r_1,r_2) = \sum_{\frac{N}{2}<n\le N} \sum_{M/2<m\le M-q_1^{\mu_{11}}r_1-q_2^{\mu_{21}}r_2} \psi_2^{<q_1^{\mu_{11}}r_1,r_0>}(m+q_2^{\mu_{21}}r_2,n) \overline{\psi_2^{<q_1^{\mu_{11}}r_1,r_0>}(m,n)},$$

using notation (21) with

$$\psi_2(m,n) = f_{\mu_{11},\mu_{12}}(mn)\; g_{\mu_{21},\mu_{22}}(mn).$$

For $i \in \{1,2\}$, we filter the variables in the expressions $f_{\mu_{11},\mu_{12}}(.)$ (for $i=1$) and $g_{\mu_{21},\mu_{22}}(.)$ (for $i=2$) in terms of variables v_{i1}, v_{i2}, u_{i1}, u_{i2}, u_{i3}, u_{i4} (we denote by τ the permutation on $\{1,2\}$ exchanging 1 and 2) as follows:

$$\begin{aligned} r_in &\equiv v_{i1} \bmod q_i^{\mu_{i2}-\mu_{i1}},\ 0\le v_{i1} < q_i^{\mu_{i2}-\mu_{i1}}, \\ r_ir_0 &\equiv v_{i2} \bmod q_i^{\mu_{i2}-\mu_{i1}},\ 0\le v_{i2} < q_i^{\mu_{i2}-\mu_{i1}}, \\ mn &\equiv u_{i1}q_i^{\mu_{i1}} + w_{i1} \bmod q_i^{\mu_{i2}},\ 0\le u_{i1} < q_i^{\mu_{i2}-\mu_{i1}},\ 0\le w_{i1} < q_i^{\mu_{i1}}, \\ m(n+r_0) &\equiv u_{i2}q_i^{\mu_{i1}} + w_{i2} \bmod q_i^{\mu_{i2}},\ 0\le u_{i2} < q_i^{\mu_{i2}-\mu_{i1}},\ 0\le w_{i2} < q_i^{\mu_{i1}}, \\ r_{\tau(i)}r_0q_{\tau(i)}^{\mu_{\tau(i)1}} &\equiv u_{i3}q_i^{\mu_{i1}} + w_{i3} \bmod q_i^{\mu_{i2}},\ 0\le u_{i3} < q_i^{\mu_{i2}-\mu_{i1}},\ 0\le w_{i3} < q_i^{\mu_{i1}}, \\ r_{\tau(i)}nq_{\tau(i)}^{\mu_{\tau(i)1}} &\equiv u_{i4}q_i^{\mu_{i1}} + w_{i4} \bmod q_i^{\mu_{i2}},\ 0\le u_{i4} < q_i^{\mu_{i2}-\mu_{i1}},\ 0\le w_{i4} < q_i^{\mu_{i1}}, \end{aligned}$$

with auxiliary variables w_{i1}, w_{i2}, w_{i3}, w_{i4} (which do not influence the corresponding values of $f_{\mu_{11},\mu_{12}}(.)$ and $g_{\mu_{21},\mu_{22}}(.)$). Using (2) with

$$\alpha_1 = q_1^{\mu_{11}-\mu_{12}}, \quad \alpha_2 = q_2^{\mu_{21}-\mu_{22}}, \tag{83}$$

since f is strongly q_1-multiplicative and g is strongly q_2-multiplicative, using notation (28) we can write

$$\begin{aligned} S_3(r_0,r_1,r_2) = &\sum_{u_{11},u_{12},u_{13},u_{14}} \sum_{v_{11},v_{12}} \sum_{u_{21},u_{22},u_{23},u_{24}} \sum_{v_{21},v_{22}} \\ & f_{\mu_{12}-\mu_{11}}^{[v_{11}+v_{12},u_{13}+u_{14}]}(u_{12})\, \overline{f_{\mu_{12}-\mu_{11}}^{[v_{11},u_{14}]}(u_{11})}\, g_{\mu_{22}-\mu_{21}}^{[u_{23}+u_{24},v_{21}+v_{22}]}(u_{22})\, \overline{g_{\mu_{22}-\mu_{21}}^{[u_{24},v_{21}]}(u_{21})} \\ & \sum_{\frac{N}{2}<n\le N} \sum_{M/2<m\le M-q_1^{\mu_{11}}r_1-q_2^{\mu_{21}}r_2} \mathbf{1}_{r_1n\equiv v_{11} \bmod q_1^{\mu_{12}-\mu_{11}}}\, \mathbf{1}_{r_1r_0\equiv v_{12} \bmod q_1^{\mu_{12}-\mu_{11}}} \\ & \chi_{\alpha_1}\left(\frac{mn}{q_1^{\mu_{12}}} - \frac{u_{11}}{q_1^{\mu_{12}-\mu_{11}}}\right) \chi_{\alpha_1}\left(\frac{m(n+r_0)}{q_1^{\mu_{12}}} - \frac{u_{12}}{q_1^{\mu_{12}-\mu_{11}}}\right) \\ & \chi_{\alpha_1}\left(\frac{r_2r_0q_2^{\mu_{21}}}{q_1^{\mu_{12}}} - \frac{u_{13}}{q_1^{\mu_{12}-\mu_{11}}}\right) \chi_{\alpha_1}\left(\frac{r_2nq_2^{\mu_{21}}}{q_1^{\mu_{12}}} - \frac{u_{14}}{q_1^{\mu_{12}-\mu_{11}}}\right) \\ & \mathbf{1}_{r_2n\equiv v_{21} \bmod q_2^{\mu_{22}-\mu_{21}}}\, \mathbf{1}_{r_2r_0\equiv v_{22} \bmod q_2^{\mu_{22}-\mu_{21}}} \\ & \chi_{\alpha_2}\left(\frac{mn}{q_2^{\mu_{22}}} - \frac{u_{21}}{q_2^{\mu_{22}-\mu_{21}}}\right) \chi_{\alpha_2}\left(\frac{m(n+r_0)}{q_2^{\mu_{22}}} - \frac{u_{22}}{q_2^{\mu_{22}-\mu_{21}}}\right) \\ & \chi_{\alpha_2}\left(\frac{r_1r_0q_1^{\mu_{11}}}{q_2^{\mu_{22}}} - \frac{u_{23}}{q_2^{\mu_{22}-\mu_{21}}}\right) \chi_{\alpha_2}\left(\frac{r_1nq_1^{\mu_{11}}}{q_2^{\mu_{22}}} - \frac{u_{24}}{q_2^{\mu_{22}-\mu_{21}}}\right). \end{aligned}$$

Let

$$F(h,a,b) = \widehat{f_{\mu_{12}-\mu_{11}}^{[a,b]}}(h), \quad G(h,a,b) = \widehat{g_{\mu_{22}-\mu_{21}}^{[a,b]}}(h).$$

We observe that F and G satisfy for all $(a,b)\in\mathbb{Z}^2$:

$$\sum_{0\le h<q_1^{\mu_{12}-\mu_{11}}} |F(h,a,b)|^2 = 1, \qquad \sum_{0\le h<q_2^{\mu_{22}-\mu_{21}}} |G(h,a,b)|^2 = 1,$$

and for all $(h,a,b)\in\mathbb{Z}^3$:

$$F(h,a,0) = F(h,0,b) = \begin{cases} 1 & \text{if } h\equiv 0 \bmod q_1^{\mu_{12}-\mu_{11}}, \\ 0 & \text{otherwise,} \end{cases}$$
$$G(h,a,0) = G(h,0,b) = \begin{cases} 1 & \text{if } h\equiv 0 \bmod q_2^{\mu_{22}-\mu_{21}}, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Lemma 3 to $S_3(r_0,r_1,r_2)$ (we have $d=6$) we obtain by (7)

$$S_3(r_0,r_1,r_2) = S_4(r_0,r_1,r_2) + O(E_4(r_0,r_1,r_2)) \tag{84}$$

with, for any integer $H\ge 1$,

$$
\begin{aligned}
S_4(r_0,r_1,r_2) = &\sum_{h_{11},h_{12},h_{14}} a_{h_{11}}(\alpha_1,H)a_{h_{12}}(\alpha_1,H)a_{h_{14}}(\alpha_1,H)\\
&\sum_{h_{21},h_{22},h_{24}} a_{h_{21}}(\alpha_2,H)a_{h_{22}}(\alpha_2,H)a_{h_{24}}(\alpha_2,H)\\
&q_1^{\mu_{11}-\mu_{12}} \sum_{0\le k_{11}<q_1^{\mu_{12}-\mu_{11}}} \sum_{v_{11}} \mathrm{e}\left(\frac{-k_{11}v_{11}}{q_1^{\mu_{12}-\mu_{11}}}\right) \sum_{v_{12}\equiv r_1r_0 \bmod q_1^{\mu_{12}-\mu_{11}}}\\
&\sum_{u_{13}} \chi_{\alpha_1}\left(\frac{r_2r_0q_2^{\mu_{21}}}{q_1^{\mu_{12}}} - \frac{u_{13}}{q_1^{\mu_{12}-\mu_{11}}}\right) \sum_{u_{14}} \mathrm{e}\left(-\frac{h_{14}u_{14}}{q_1^{\mu_{12}-\mu_{11}}}\right)\\
&q_1^{2(\mu_{12}-\mu_{11})} F(h_{12}, v_{11}+v_{12}, u_{13}+u_{14})\ \overline{F(-h_{11},v_{11},u_{14})}\\
&q_2^{\mu_{21}-\mu_{22}} \sum_{0\le k_{21}<q_2^{\mu_{22}-\mu_{21}}} \sum_{v_{21}} \mathrm{e}\left(\frac{-k_{21}v_{21}}{q_2^{\mu_{22}-\mu_{21}}}\right) \sum_{v_{22}\equiv r_2r_0 \bmod q_2^{\mu_{22}-\mu_{21}}}\\
&\sum_{u_{23}} \chi_{\alpha_2}\left(\frac{r_1r_0q_1^{\mu_{11}}}{q_2^{\mu_{22}}} - \frac{u_{23}}{q_2^{\mu_{22}-\mu_{21}}}\right) \sum_{u_{24}} \mathrm{e}\left(-\frac{h_{24}u_{24}}{q_2^{\mu_{22}-\mu_{21}}}\right)\\
&q_2^{2(\mu_{22}-\mu_{21})} G(h_{22}, v_{21}+v_{22}, u_{23}+u_{24})\ \overline{G(-h_{21},v_{21},u_{24})}\\
&\sum_{M/2<m\le M-q_1^{\mu_{11}}r_1-q_2^{\mu_{21}}r_2} \ \sum_{\frac{N}{2}<n\le N}\\
&\mathrm{e}\left(\frac{(h_{11}+h_{12})mn + h_{12}mr_0 + h_{14}r_2nq_2^{\mu_{21}} + k_{11}r_1nq_1^{\mu_{11}}}{q_1^{\mu_{12}}}\right)\\
&\mathrm{e}\left(\frac{(h_{21}+h_{22})mn + h_{22}mr_0 + h_{24}r_1nq_1^{\mu_{11}} + k_{21}r_2nq_2^{\mu_{21}}}{q_2^{\mu_{22}}}\right).
\end{aligned}
$$

and

$$
\begin{aligned}
E_4(r_0,r_1,r_2) = &\frac{q_1^{3(\mu_{12}-\mu_{11})}q_2^{3(\mu_{22}-\mu_{21})}}{H^6} \sum_{|h_{11}|,|h_{12}|,|h_{14}|\le Hq_1^{\mu_{11}-\mu_{12}}} \ \sum_{|h_{21}|,|h_{22}|,|h_{24}|\le Hq_2^{\mu_{21}-\mu_{22}}}\\
&\sum_{v_{12}\equiv r_1r_0 \bmod q_1^{\mu_{12}-\mu_{11}}} \sum_{u_{13}} \chi_{\alpha_1}\left(\frac{r_2r_0q_2^{\mu_{21}}}{q_1^{\mu_{12}}} - \frac{u_{13}}{q_1^{\mu_{12}-\mu_{11}}}\right)\\
&\sum_{v_{22}\equiv r_2r_0 \bmod q_2^{\mu_{22}-\mu_{21}}} \sum_{u_{23}} \chi_{\alpha_2}\left(\frac{r_1r_0q_1^{\mu_{11}}}{q_2^{\mu_{22}}} - \frac{u_{23}}{q_2^{\mu_{22}-\mu_{21}}}\right)\\
&\sum_{\substack{(\delta_{11},\delta_{12},\delta_{14},\delta_{21},\delta_{22},\delta_{24})\in\{0,1\}^6\\ (\delta_{11},\delta_{12},\delta_{14},\delta_{21},\delta_{22},\delta_{24})\ne(0,\ldots,0)}} \ \sum_{M/2<m\le M-q_1^{\mu_{11}}r_1-q_2^{\mu_{21}}r_2} \ \sum_{\frac{N}{2}<n\le N}\\
&\sum_{v_{11}} \mathbf{1}_{r_2n\equiv v_{21} \bmod q_2^{\mu_{22}-\mu_{21}}} \sum_{v_{21}} \mathbf{1}_{r_2r_0\equiv v_{22} \bmod q_2^{\mu_{22}-\mu_{21}}}\\
&\mathrm{e}\left(\frac{(\delta_{11}h_{11}+\delta_{12}h_{12})q_1^{\mu_{12}-\mu_{11}}mn + \delta_{12}h_{12}q_1^{\mu_{12}-\mu_{11}}mr_0 + \delta_{14}h_{14}q_1^{\mu_{12}-\mu_{11}}r_2nq_2^{\mu_{21}}}{q_1^{\mu_{12}}}\right)\\
&\mathrm{e}\left(\frac{(\delta_{21}h_{21}+\delta_{22}h_{22})q_1^{\mu_{22}-\mu_{21}}mn + \delta_{22}h_{22}q_2^{\mu_{22}-\mu_{21}}mr_0 + \delta_{24}h_{24}q_2^{\mu_{22}-\mu_{21}}r_1nq_1^{\mu_{11}}}{q_2^{\mu_{22}}}\right).
\end{aligned}
$$

8.1. **Estimate of** $E_4(r_0,r_1,r_2)$**.** Observing that in $E_4(r_0,r_1,r_2)$ there is only one contributing value for v_{12}, v_{22}, v_{11}, v_{21}, u_{13} and u_{23} we can write:

$$\begin{aligned} E_4(r_0,r_1,r_2) =& \frac{q_1^{3(\mu_{12}-\mu_{11})} q_2^{3(\mu_{22}-\mu_{21})}}{H^6} \sum_{|h_{11}|,|h_{12}|,|h_{14}|\le H q_1^{\mu_{11}-\mu_{12}}} \sum_{|h_{21}|,|h_{22}|,|h_{24}|\le H q_2^{\mu_{21}-\mu_{22}}} \\ & \sum_{\substack{(\delta_{11},\delta_{12},\delta_{14},\delta_{21},\delta_{22},\delta_{24})\in\{0,1\}^6 \\ (\delta_{11},\delta_{12},\delta_{14},\delta_{21},\delta_{22},\delta_{24})\neq(0,\dots,0)}} \sum_{M/2<m\le M-q_1^{\mu_{11}}r_1-q_2^{\mu_{21}}r_2} \sum_{\frac{N}{2}<n\le N} \\ & \mathrm{e}\left(\frac{(\delta_{11}h_{11}+\delta_{12}h_{12})mn+\delta_{12}h_{12}mr_0+\delta_{14}h_{14}r_2nq_2^{\mu_{21}}}{q_1^{\mu_{11}}}\right) \\ & \mathrm{e}\left(\frac{(\delta_{21}h_{21}+\delta_{22}h_{22})mn+\delta_{22}h_{22}mr_0+\delta_{24}h_{24}r_1nq_1^{\mu_{11}}}{q_2^{\mu_{21}}}\right). \end{aligned}$$

Let $E_{40}(r_0,r_1,r_2)$ be the contribution of the terms for which $(\delta_{11},\delta_{12},\delta_{21},\delta_{22})=(0,0,0,0)$, and $E_{41}(r_0,r_1,r_2)$ be the contribution of the terms for which $(\delta_{11},\delta_{12},\delta_{21},\delta_{22})\neq(0,0,0,0)$.

In $E_{40}(r_0,r_1,r_2)$ we have $(\delta_{14},\delta_{24})\in\{(1,1),(1,0),(0,1)\}$ so that

$$E_{40}(r_0,r_1,r_2) \ll M\frac{q_1^{\mu_{12}-\mu_{11}}q_2^{\mu_{22}-\mu_{21}}}{H^2}E_{401}(r_1,r_2) + M\frac{q_1^{\mu_{12}-\mu_{11}}}{H}E_{402}(r_1,r_2) + M\frac{q_2^{\mu_{22}-\mu_{21}}}{H}E_{403}(r_1,r_2),$$

where $E_{401}(r_1,r_2)$, $E_{402}(r_1,r_2)$ and $E_{403}(r_1,r_2)$, defined below, can be estimated by Lemma 7:

$$\begin{aligned} E_{401}(r_1,r_2) &= \sum_{|h_{14}|\le H q_1^{\mu_{11}-\mu_{12}}} \sum_{|h_{24}|\le H q_2^{\mu_{21}-\mu_{22}}} \min\left(N, \left|\sin\pi\left(\frac{h_{14}r_2q_2^{\mu_{21}}}{q_1^{\mu_{11}}}+\frac{h_{24}r_1q_1^{\mu_{11}}}{q_2^{\mu_{21}}}\right)\right|^{-1}\right) \\ &\ll \sum_{|h_{14}|\le H q_1^{\mu_{11}-\mu_{12}}} \left(\gcd(r_1q_1^{\mu_{11}},q_2^{\mu_{21}})N + q_2^{\mu_{21}}\log q_2^{\mu_{21}}\right) \\ &\ll H q_1^{\mu_{11}-\mu_{12}}\left(\gcd(r_1q_1^{\mu_{11}},q_2^{\mu_{21}})N + q_2^{\mu_{21}}\log q_2^{\mu_{21}}\right), \end{aligned}$$

$$\begin{aligned} E_{402}(r_1,r_2) &= \sum_{|h_{14}|\le H q_1^{\mu_{11}-\mu_{12}}} \min\left(N, \left|\sin\frac{\pi h_{14}r_2q_2^{\mu_{21}}}{q_1^{\mu_{11}}}\right|^{-1}\right) \\ &\ll \left(\gcd(r_2q_2^{\mu_{21}},q_1^{\mu_{11}})N + q_1^{\mu_{11}}\log q_1^{\mu_{11}}\right), \end{aligned}$$

and

$$\begin{aligned} E_{403}(r_1,r_2) &= \sum_{|h_{24}|\le H q_2^{\mu_{21}-\mu_{22}}} \min\left(N, \left|\sin\frac{\pi h_{24}r_1q_1^{\mu_{11}}}{q_2^{\mu_{21}}}\right|^{-1}\right) \\ &\ll \left(\gcd(r_1q_1^{\mu_{11}},q_2^{\mu_{21}})N + q_2^{\mu_{21}}\log q_2^{\mu_{21}}\right). \end{aligned}$$

Since $\gcd(q_1,q_2)=1$, $q_1^{\mu_{11}}\le M\le N$ and $q_2^{\mu_{21}}\le M\le N$ (by (67)), using (15) we get uniformly for $r_0\in\{1,\dots,R_0-1\}$:

$$\begin{aligned} & R_1^{-1}R_2^{-1}\sum_{1\le r_1<R_1}\sum_{1\le r_2<R_2} E_{40}(r_0,r_1,r_2) \\ & \ll MN\frac{q_1^{\mu_{12}-\mu_{11}}}{H}\left(\tau(q_1^{\mu_{11}})+\log q_1^{\mu_{11}}\right) + MN\frac{q_2^{\mu_{22}-\mu_{21}}}{H}\left(\tau(q_2^{\mu_{21}})+\log q_2^{\mu_{21}}\right). \end{aligned}$$

We may estimate $E_{41}(r_0, r_1, r_2)$ by

$$\frac{q_1^{3(\mu_{12}-\mu_{11})} q_2^{3(\mu_{22}-\mu_{21})}}{H^6} \sum_{|h_{11}|,|h_{12}|,|h_{14}|\leq Hq_1^{\mu_{11}-\mu_{12}}} \sum_{|h_{21}|,|h_{22}|,|h_{24}|\leq Hq_2^{\mu_{21}-\mu_{22}}}$$
$$\sum_{\substack{(\delta_{11},\delta_{12},\delta_{14},\delta_{21},\delta_{22},\delta_{24})\in\{0,1\}^6 \\ (\delta_{11},\delta_{12},\delta_{21},\delta_{22})\neq(0,0,0,0)}} \sum_{M/2<m\leq M-q_1^{\mu_{11}}r_1-q_2^{\mu_{21}}r_2}$$
$$\min\left(N, \left|\sin\pi\left(\frac{(\delta_{11}h_{11}+\delta_{12}h_{12})m+\delta_{14}h_{14}r_2q_2^{\mu_{21}}}{q_1^{\mu_{11}}} + \frac{(\delta_{21}h_{21}+\delta_{22}h_{22})m+\delta_{24}h_{24}r_1q_1^{\mu_{11}}}{q_2^{\mu_{21}}}\right)\right|^{-1}\right).$$

The contribution of the terms for which $\delta_{11} = 1$ is estimated by

$$\frac{q_1^{\mu_{12}-\mu_{11}}}{H} \sum_{M/2<m\leq M-q_1^{\mu_{11}}r_1-q_2^{\mu_{21}}r_2} \max_{\varphi\in\mathbb{R}} \left(\sum_{|h_{11}|\leq Hq_1^{\mu_{11}-\mu_{12}}} \min\left(N, \left|\sin\pi\left(\frac{h_{11}m}{q_1^{\mu_{11}}}+\varphi\right)\right|^{-1}\right)\right).$$

which by Lemma 7 is at most

$$\frac{q_1^{\mu_{12}-\mu_{11}}}{H} \sum_{M/2<m\leq M-q_1^{\mu_{11}}r_1-q_2^{\mu_{21}}r_2} (\gcd(m, q_1^{\mu_{11}})N + q_1^{\mu_{11}}\log q_1^{\mu_{11}})$$

and by (15), since by (67) we have $q_1^{\mu_{11}} \leq M \leq N$ and $q_2^{\mu_{21}} \leq M \leq N$, we obtain uniformly over r_0, r_1, r_2 that the contribution of the terms for which $\delta_{11} = 1$ is estimated by

$$\ll MN\frac{q_1^{\mu_{12}-\mu_{11}}}{H}(\tau(q_1^{\mu_{11}}) + \log q_1^{\mu_{11}}).$$

We may argue similarly if $\delta_{12} = 1$, if $\delta_{21} = 1$ and if $\delta_{22} = 1$. Therefore we obtain

$$E_{41}(r_0, r_1, r_2) \ll MN\frac{q_1^{\mu_{12}-\mu_{11}}}{H}(\tau(q_1^{\mu_{11}}) + \log q_1^{\mu_{11}}) + MN\frac{q_2^{\mu_{22}-\mu_{21}}}{H}(\tau(q_2^{\mu_{21}}) + \log q_2^{\mu_{21}}),$$

and finally, uniformly for $r_0 \in \{1, \ldots, R_0 - 1\}$:

$$R_1^{-1}R_2^{-1}\sum_{1\leq r_1<R_1}\sum_{1\leq r_2<R_2} E_4(r_0, r_1, r_2)$$
$$\ll MN\frac{q_1^{\mu_{12}-\mu_{11}}}{H}(\tau(q_1^{\mu_{11}}) + \log q_1^{\mu_{11}}) + MN\frac{q_2^{\mu_{22}-\mu_{21}}}{H}(\tau(q_2^{\mu_{21}}) + \log q_2^{\mu_{21}})$$

which, if we choose

(85) $$H = R_0^3\max(R_1, R_2),$$

by (75), (77), (81) gives

(86) $$R_1^{-1}R_2^{-1}\sum_{1\leq r_1<R_1}\sum_{1\leq r_2<R_2} E_4(r_0, r_1, r_2)$$
$$\ll MNR_0^{-1}\left((1+\mu_{11})^{\omega(q_1)} + \log q_1^{\mu_{11}} + (1+\mu_{21})^{\omega(q_2)} + \log q_2^{\mu_{21}}\right).$$

8.2. **Estimate of** $S_4(r_0,r_1,r_2)$**.**

$$\begin{aligned}
S_4(r_0,r_1,r_2) \ll\ & q_1^{\mu_{12}-\mu_{11}} \sum_{h_{11},h_{12},h_{14}} \min(\alpha_1,|h_{11}|^{-1}) \min(\alpha_1,|h_{12}|^{-1}) \min(\alpha_1,|h_{14}|^{-1}) \\
& q_2^{\mu_{22}-\mu_{21}} \sum_{h_{21},h_{22},h_{24}} \min(\alpha_2,|h_{21}|^{-1}) \min(\alpha_2,|h_{22}|^{-1}) \min(\alpha_2,|h_{24}|^{-1}) \\
& \sum_{u_{13}} \chi_{\alpha_1}\left(\frac{r_2 r_0 q_2^{\mu_{21}}}{q_1^{\mu_{12}}} - \frac{u_{13}}{q_1^{\mu_{12}-\mu_{11}}}\right) \sum_{v_{12}\equiv r_1 r_0 \bmod q_1^{\mu_{12}-\mu_{11}}} \\
& \sum_{u_{14}} \sum_{v_{11}} |F(h_{12}, v_{11}+v_{12}, u_{13}+u_{14})\ F(-h_{11}, v_{11}, u_{14})| \\
& \sum_{u_{23}} \chi_{\alpha_2}\left(\frac{r_1 r_0 q_1^{\mu_{11}}}{q_2^{\mu_{22}}} - \frac{u_{23}}{q_2^{\mu_{22}-\mu_{21}}}\right) \sum_{v_{22}\equiv r_2 r_0 \bmod q_2^{\mu_{22}-\mu_{21}}} \\
& \sum_{u_{24}} \sum_{v_{21}} |G(h_{22}, v_{21}+v_{22}, u_{23}+u_{24})\ G(-h_{21}, v_{21}, u_{24})| \\
& \sum_{0\le k_{11}<q_1^{\mu_{12}-\mu_{11}}} \sum_{0\le k_{21}<q_2^{\mu_{22}-\mu_{21}}} \Bigg| \sum_{M/2<m\le M-q_1^{\mu_{11}}r_1-q_2^{\mu_{21}}r_2} \sum_{\frac{N}{2}<n\le N} \\
& \mathrm{e}\left(\frac{(h_{11}+h_{12})mn + h_{12}mr_0 + h_{14}r_2 n q_2^{\mu_{21}} + k_{11} r_1 n q_1^{\mu_{11}}}{q_1^{\mu_{12}}}\right) \\
& \mathrm{e}\left(\frac{(h_{21}+h_{22})mn + h_{22}mr_0 + h_{24}r_1 n q_1^{\mu_{11}} + k_{21} r_2 n q_2^{\mu_{21}}}{q_2^{\mu_{22}}}\right) \Bigg|.
\end{aligned}$$

By Cauchy-Schwarz and Proposition 2 we have uniformly for u_{13} and v_{12}:

$$\begin{aligned}
&\sum_{v_{11}} \sum_{u_{14}} |F(h_{12}, v_{11}+v_{12}, u_{13}+u_{14})\ F(h_{12}, v_{11}, u_{14})| \\
&\quad \le \left(\sum_{v_{11}} \sum_{u_{14}} |F(h_{12}, v_{11}+v_{12}, u_{13}+u_{14})|^2\right)^{1/2} \left(\sum_{v_{11}} \sum_{u_{14}} |F(h_{12}, v_{11}, u_{14})|^2\right)^{1/2} \\
&\quad \le c_1(f)\, q_1^{-c_2(f)(\mu_{12}-\mu_{11})},
\end{aligned} \tag{87}$$

and uniformly for u_{23} and v_{22}, similarly:

$$\sum_{v_{21}} \sum_{u_{24}} |G(h_{22}, v_{21}+v_{22}, u_{23}+u_{24})\ G(h_{22}, v_{21}, u_{24})| \le c_1(g)\, q_2^{-c_2(g)(\mu_{22}-\mu_{21})}.$$

Furthermore by (83)

$$\sum_{u_{13}} \chi_{\alpha_1}\left(\frac{r_2 r_0 q_2^{\mu_{21}}}{q_1^{\mu_{12}}} - \frac{u_{13}}{q_1^{\mu_{12}-\mu_{11}}}\right) = 1 = \sum_{u_{23}} \chi_{\alpha_2}\left(\frac{r_1 r_0 q_1^{\mu_{11}}}{q_2^{\mu_{22}}} - \frac{u_{23}}{q_2^{\mu_{22}-\mu_{21}}}\right).$$

Observing that the summations over v_{12} and v_{22} take each only one value, this gives

$$
\begin{aligned}
S_4(r_0,r_1,r_2) \ll\ & c_1(f) q_1^{(1-c_2(f))(\mu_{12}-\mu_{11})} \sum_{h_{11},h_{12},h_{14}} \min(\alpha_1,|h_{11}|^{-1}) \min(\alpha_1,|h_{12}|^{-1}) \min(\alpha_1,|h_{14}|^{-1}) \\
& c_1(g) q_2^{(1-c_2(g))(\mu_{22}-\mu_{21})} \sum_{h_{21},h_{22},h_{24}} \min(\alpha_2,|h_{21}|^{-1}) \min(\alpha_2,|h_{22}|^{-1}) \min(\alpha_2,|h_{24}|^{-1}) \\
& \sum_{0\le k_{11}<q_1^{\mu_{12}-\mu_{11}}} \sum_{0\le k_{21}<q_2^{\mu_{22}-\mu_{21}}} \\
& \Bigg| \sum_{M/2<m\le M-q_1^{\mu_{11}}r_1-q_2^{\mu_{21}}r_2} \sum_{\frac{N}{2}<n\le N} \mathrm{e}\left(\frac{(h_{11}+h_{12})mn + h_{12}mr_0 + h_{14}r_2nq_2^{\mu_{21}} + k_{11}r_1nq_1^{\mu_{11}}}{q_1^{\mu_{12}}} \right) \\
& \qquad \mathrm{e}\left(\frac{(h_{21}+h_{22})mn + h_{22}mr_0 + h_{24}r_1nq_1^{\mu_{11}} + k_{21}r_2nq_2^{\mu_{21}}}{q_2^{\mu_{22}}} \right) \Bigg|,
\end{aligned}
$$

and by (9) we get

$$
\begin{aligned}
S_4(r_0,r_1,r_2) \ll\ & c_1(f)\ q_1^{(1-c_2(f))(\mu_{12}-\mu_{11})} \sum_{h_{11},h_{12},h_{14}} \min(\alpha_1,|h_{11}|^{-1}) \min(\alpha_1,|h_{12}|^{-1}) \min(\alpha_1,|h_{14}|^{-1}) \\
& c_1(g)\ q_2^{(1-c_2(g))(\mu_{22}-\mu_{21})} \sum_{h_{21},h_{22},h_{24}} \min(\alpha_2,|h_{21}|^{-1}) \min(\alpha_2,|h_{22}|^{-1}) \min(\alpha_2,|h_{24}|^{-1}) \\
& \sum_{0\le k_{11}<q_1^{\mu_{12}-\mu_{11}}} \sum_{0\le k_{21}<q_2^{\mu_{22}-\mu_{21}}} \sum_{M/2<m\le M-q_1^{\mu_{11}}r_1-q_2^{\mu_{21}}r_2} \\
& \min\left(N, \left| \sin\pi\left(\left(\frac{h_{11}+h_{12}}{q_1^{\mu_{12}}} + \frac{h_{21}+h_{22}}{q_2^{\mu_{22}}} \right) m + \frac{h_{14}r_2q_2^{\mu_{21}}}{q_1^{\mu_{12}}} + \frac{h_{24}r_1q_1^{\mu_{11}}}{q_2^{\mu_{22}}} \right.\right.\right. \\
& \qquad \left.\left.\left. + \frac{k_{11}r_1}{q_1^{\mu_{12}-\mu_{11}}} + \frac{k_{21}r_2}{q_2^{\mu_{22}-\mu_{21}}} \right) \right|^{-1} \right).
\end{aligned}
$$

Let us write

$$
S_4(r_0,r_1,r_2) \ll S_{40}(r_0,r_1,r_2) + S_{41}(r_0,r_1,r_2) \tag{88}
$$

where $S_{40}(r_0,r_1,r_2)$ denotes the contribution in the right hand side above of the terms for which $h_{11}+h_{12} = h_{21}+h_{22} = 0$ and $S_{41}(r_0,r_1,r_2)$ denotes the contribution of the remaining terms.

8.3. **Diagonal part.**

We handle here $S_{40}(r_0,r_1,r_2)$ for which $h_{11}+h_{12} = h_{21}+h_{22} = 0$. We have

$$
\begin{aligned}
S_{40}(r_0,r_1,r_2) \ll\ & c_1(f)\ q_1^{(1-c_2(f))(\mu_{12}-\mu_{11})} \sum_{h_{12},h_{14}} \min(\alpha_1,|h_{12}|^{-1})^2 \min(\alpha_1,|h_{14}|^{-1}) \\
& c_1(g)\ q_2^{(1-c_2(g))(\mu_{22}-\mu_{21})} \sum_{h_{22},h_{24}} \min(\alpha_2,|h_{22}|^{-1})^2 \min(\alpha_2,|h_{24}|^{-1}) \\
& M\ S_{401}(r_1,r_2,h_{14},h_{24}),
\end{aligned}
$$

where $S_{401}(r_1,r_2,h_{14},h_{24})$ is equal to

$$
\sum_{0\le k_{11}<q_1^{\mu_{12}-\mu_{11}}} \sum_{0\le k_{21}<q_2^{\mu_{22}-\mu_{21}}} \min\left(N, \left|\sin\pi\left(\frac{h_{14}r_2q_2^{\mu_{21}} + k_{11}r_1q_1^{\mu_{11}}}{q_1^{\mu_{12}}} + \frac{h_{24}r_1q_1^{\mu_{11}} + k_{21}r_2q_2^{\mu_{21}}}{q_2^{\mu_{22}}} \right)\right|^{-1}\right)
$$

Since $\gcd(q_1,q_2)=1$, by Lemma 8 we have uniformly in h_{14} and h_{24}:

$$S_{401}(r_1,r_2,h_{14},h_{24}) \ll N\gcd(r_1,q_1^{\mu_{12}-\mu_{11}})\gcd(r_2,q_2^{\mu_{22}-\mu_{21}}) + q_1^{\mu_{12}-\mu_{11}}q_2^{\mu_{22}-\mu_{21}}\log\left(q_1^{\mu_{12}-\mu_{11}}q_2^{\mu_{22}-\mu_{21}}\right)$$

By (77), (81), (75),(74) we have $q_1^{\mu_{12}-\mu_{11}} \asymp R_0^2R_1$ and $q_2^{\mu_{22}-\mu_{21}} \asymp R_0^2R_2$ and if we assume that

$$R_0^4R_1R_2 \ll N^{\frac{4}{5}} \tag{89}$$

then

$$S_{401}(r_1,r_2,h_{14},h_{24}) \ll N\gcd(r_1,q_1^{\mu_{12}-\mu_{11}})\gcd(r_2,q_2^{\mu_{22}-\mu_{21}}).$$

Since

$$\sum_{h_{12}}\left(\min(\alpha_1,|h_{12}|^{-1})\right)^2 \ll \alpha_1; \quad \sum_{h_{14}}\min(\alpha_1,|h_{14}|^{-1}) \ll 1+\log H,$$

$$\sum_{h_{22}}\left(\min(\alpha_2,|h_{22}|^{-1})\right)^2 \ll \alpha_2; \quad \sum_{h_{24}}\min(\alpha_2,|h_{24}|^{-1}) \ll 1+\log H,$$

this leads to

$$\begin{aligned}S_{40}(r_0,r_1,r_2) \ll q_1^{(1-c_2(f))(\mu_{12}-\mu_{11})}q_2^{(1-c_2(g))(\mu_{22}-\mu_{21})}\alpha_1\alpha_2(1+\log H)^2\\ MN\gcd(r_1,q_1^{\mu_{12}-\mu_{11}})\gcd(r_2,q_2^{\mu_{22}-\mu_{21}}),\end{aligned}$$

which by (15) and (83) gives uniformly for $r_0\in\{1,\dots,R_0-1\}$:

$$\begin{aligned}&R_1^{-1}R_2^{-1}\sum_{1\le r_1<R_1}\sum_{1\le r_2<R_2}S_{40}(r_0,r_1,r_2)\\ &\ll MNq_1^{-c_2(f)(\mu_{12}-\mu_{11})}q_2^{-c_2(g)(\mu_{22}-\mu_{21})}(1+\log H)^2\tau(q_1^{\mu_{12}-\mu_{11}})\tau(q_2^{\mu_{22}-\mu_{21}}).\end{aligned} \tag{90}$$

Remark 5. *The existence of two conditions ($h_{11}+h_{12}=0$ and $h_{21}+h_{22}=0$) in this diagonal part provided the factor $\alpha_1\alpha_2$ which was crucial to permit to eliminate the factor $q_1^{\mu_{12}-\mu_{11}}q_2^{\mu_{22}-\mu_{21}}$ and get a satisfactory upper bound.*

8.4. **Non diagonal part.**

We handle here $S_{41}(r_0,r_1,r_2)$ for which $h_{11}+h_{12}\neq 0$ or $h_{21}+h_{22}\neq 0$. After making the summation over n and extending the summation over m (in order to remove its dependence on r_1 and r_2), we apply Lemma 7. We distinguish three cases:

- if $h_{11}+h_{12}\neq 0$ and $h_{21}+h_{22}=0$, by Lemma 7, remembering that $MR_0^2\asymp q_1^{\mu_{12}}$, uniformly for $\varphi\in\mathbb{R}$ we get

$$\begin{aligned}\sum_{M/2<m\le M}\min\left(N,\left|\sin\pi\left(\frac{(h_{11}+h_{12})m}{q_1^{\mu_{12}}}+\varphi\right)\right|^{-1}\right)\\ \ll N\gcd\left(h_{11}+h_{12},q_1^{\mu_{12}}\right)+q_1^{\mu_{12}}\log q_1^{\mu_{12}}\\ \ll HN+MR_0^2\log(MN).\end{aligned}$$

- if $h_{11}+h_{12}=0$ and $h_{21}+h_{22}\neq 0$, by Lemma 7, remembering that $MR_0^2\asymp q_2^{\mu_{22}}$, uniformly for $\varphi\in\mathbb{R}$, we get

$$\begin{aligned}\sum_{M/2<m\le M}\min\left(N,\left|\sin\pi\left(\frac{(h_{21}+h_{22})m}{q_2^{\mu_{22}}}+\varphi\right)\right|^{-1}\right)\\ \ll N\gcd\left(h_{21}+h_{22},q_2^{\mu_{22}}\right)+q_2^{\mu_{22}}\log q_2^{\mu_{22}}\\ \ll HN+MR_0^2\log(MN).\end{aligned}$$

- if $h_{11}+h_{12} \neq 0$ and $h_{21}+h_{22} \neq 0$, in order to apply Lemma 6 we introduce
$$\xi = \frac{h_{11}+h_{12}}{q_1^{\mu_{12}}} + \frac{h_{21}+h_{22}}{q_2^{\mu_{22}}}.$$
By (10), uniformly for $\varphi \in \mathbb{R}$, we get the estimate
$$\sum_{M/2<m\leq M} \min\left(N, \left|\sin\pi\left(\left(\frac{h_{11}+h_{12}}{q_1^{\mu_{12}}} + \frac{h_{21}+h_{22}}{q_2^{\mu_{22}}}\right)m+\varphi\right)\right|^{-1}\right)$$
$$\ll \left\|\xi\right\|^{-1}\log\left\|\xi\right\|^{-1} + N + MN\left\|\xi\right\| + M\log\left\|\xi\right\|^{-1}.$$
By [32, Corollary 9.22] (see also [6, p. 30]) there exist an absolute constant $C > 0$ such that
$$\begin{aligned}\left|\frac{-h_{11}-h_{12}}{h_{21}+h_{22}}q_1^{-\mu_{12}}q_2^{\mu_{22}}-1\right| & \\ &\geq \exp\left(-C\log q_1\log q_2\log\max(\mu_{12},\mu_{22})\log\max(\left|h_{11}+h_{12}\right|, \left|h_{21}+h_{22}\right|)\right)\\ &\geq \exp\left(-C_1\log\log(MN)\log(2H)\right)\end{aligned}$$
for some $C_1 > 0$ depending only on q_1 and q_2, so that by (75)
$$\begin{aligned}\left\|\xi\right\| &\geq \left|h_{21}+h_{22}\right| q_2^{-\mu_{22}}\exp\left(-C_1\log\log(MN)\log(2H)\right)\\ &\geq M^{-1}R_0^{-2}\exp\left(-C_1\log\log(MN)\log(2H)\right).\end{aligned}$$
Since $\left\|\xi\right\| \leq 4H/M$, this leads to
$$\sum_{M/2<m\leq M} \min\left(N, \left|\sin\pi\left(\left(\frac{h_{11}+h_{12}}{q_1^{\mu_{12}}} + \frac{h_{21}+h_{22}}{q_2^{\mu_{22}}}\right)m+\varphi\right)\right|^{-1}\right)$$
$$\ll MR_0^2\exp\left(C_1\log\log(MN)\log(2H)\right)\log(MN) + HN + M\log(MN).$$

It follows that
$$\begin{aligned}S_{41}(r_0,r_1,r_2) \ll & c_1(f)\ q_1^{(2-c_2(f))(\mu_{12}-\mu_{11})} \sum_{h_{11},h_{12},h_{14}} \min(\alpha_1, \left|h_{11}\right|^{-1})\min(\alpha_1, \left|h_{12}\right|^{-1})\min(\alpha_1, \left|h_{14}\right|^{-1})\\ & c_1(g)\ q_2^{(2-c_2(g))(\mu_{22}-\mu_{21})} \sum_{h_{21},h_{22},h_{24}} \min(\alpha_2, \left|h_{21}\right|^{-1})\min(\alpha_2, \left|h_{22}\right|^{-1})\min(\alpha_2, \left|h_{24}\right|^{-1})\\ & \big(MR_0^2\exp\left(C_1\log\log(MN)\log(2H)\right)\log(MN) + HN + M\log(MN)\big),\end{aligned}$$
and by (75), (77), (81),
$$\begin{aligned}S_{41}(r_0,r_1,r_2) \ll & c_1(f)\ c_1(g)\ R_0^4\ R_1\ R_2\ (1+\log H)^6\\ & \big(MR_0^2\exp\left(C_1\log\log(MN)\log(2H)\right)\log(MN) + HN + M\log(MN)\big).\end{aligned} \tag{91}$$

If we choose
$$R_0 = \exp\left(\frac{\log(MN)}{21\,C_1\log\log(MN)}\right), \tag{92}$$
$$R_1 = R_0^2, \tag{93}$$
and
$$R_2 = R_0^4, \tag{94}$$
the conditions (73), (77), (81) and (89) are satisfied and by (85) we have
$$H = R_0^7,$$

and

$$\log H = \frac{\log(MN)}{3C_1 \log\log(MN)}.$$

It follows from (91) that

$$S_{41}(r_0,r_1,r_2) \ll c_1(f)\ c_1(g) \exp\left(\frac{\log(MN)}{C_1 \log\log(MN)}\right) \left(M(MN)^{1/3} + N + M\right). \tag{95}$$

8.5. **Completion of the estimate of the sums of type II.**

By (88), (90) and (95) we conclude that

$$\begin{aligned} & R_1^{-1} R_2^{-1} \sum_{1\le r_1<R_1} \sum_{1\le r_2<R_2} S_4(r_0,r_1,r_2) \\ & \ll MN q_1^{-c_2(f)(\mu_{12}-\mu_{11})} q_2^{-c_2(g)(\mu_{22}-\mu_{21})} (1+\log H)^2 \tau(q_1^{\mu_{12}-\mu_{11}}) \tau(q_2^{\mu_{22}-\mu_{21}}) \\ & \quad + c_1(f)\, c_1(g)\, \exp\left(\frac{\log(MN)}{C_1\log\log(MN)}\right) \left(M(MN)^{1/3} + N + M\right). \end{aligned} \tag{96}$$

By (80), (82) we have

$$\begin{aligned} |S_{II}(\vartheta)|^8 \ll_\varepsilon & \frac{M^8N^8}{R_0^4} + \frac{M^8N^8}{R_1^2} + \frac{M^8N^8}{R_2} \\ & + \frac{M^7N^7}{R_0R_1R_2} \sum_{1\le r_0<R_0} \sum_{1\le r_1<R_1} \sum_{1\le r_2<R_2} |S_3(r_0,r_1,r_2)|, \end{aligned}$$

and it follows from (84) and (96) that

$$\begin{aligned} |S_{II}(\vartheta)|^8 \ll & \frac{M^8N^8}{R_0^4} + \frac{M^8N^8}{R_1^2} + \frac{M^8N^8}{R_2} \\ & + M^8N^8 q_1^{-c_2(f)(\mu_{12}-\mu_{11})} q_2^{-c_2(g)(\mu_{22}-\mu_{21})} (1+\log H)^2 \tau(q_1^{\mu_{12}-\mu_{11}}) \tau(q_2^{\mu_{22}-\mu_{21}}) \\ & + c_1(f)\, c_1(g)\, \exp\left(\frac{\log(MN)}{C_1\log\log(MN)}\right) M^8N^8 \left(M(MN)^{-2/3} + M^{-1} + N^{-1}\right). \end{aligned}$$

observing by (67) and (69) that $(MN)^{\xi_1} \le M \le (MN)^{\frac{1}{2}}$ we obtain

$$\begin{aligned} |S_{II}(\vartheta)|^8 \ll & \frac{M^8N^8}{R_0^4} + \frac{M^8N^8}{R_1^2} + \frac{M^8N^8}{R_2} \\ & + M^8N^8 q_1^{-c_2(f)(\mu_{12}-\mu_{11})} q_2^{-c_2(g)(\mu_{22}-\mu_{21})} (1+\log H)^2 \tau(q_1^{\mu_{12}-\mu_{11}}) \tau(q_2^{\mu_{22}-\mu_{21}}) \\ & + c_1(f)\, c_1(g) \exp\left(\frac{\log(MN)}{C_1\log\log(MN)}\right) (MN)^{8-\min(\frac{1}{6},\xi_1)}, \end{aligned} \tag{97}$$

with ξ_1 satisfying (66), which gives (72).

9. Proof of Theorems 1, 2 and 3

By Proposition 4 we have uniformly for $\vartheta \in \mathbb{R}$

$$S_I(\vartheta) \ll (\log MN)^{\frac{1}{2}\omega(q_1)+\frac{1}{2}\omega(q_2)}\ (MN)^{1-\sigma(q_1,q_2)}$$

and by Proposition 5 we have uniformly for $\vartheta \in \mathbb{R}$

$$|S_{II}(\vartheta)| \ll MN \exp\left(-C\,\frac{\log(MN)}{\log\log(MN)}\right)$$

so that applying [25, Lemma 1], or its analogue in the case of μ obtained using (13.40) instead of (13.39) from [13] we obtain uniformly for $\vartheta \in \mathbb{R}$

$$\left| \sum_{x/4<n\leq x} \Lambda(n) f(n) g(n) \,\mathrm{e}(\vartheta n) \right| \ll x \exp\left(-c \frac{\log x}{\log\log x}\right)$$

and

$$\left| \sum_{x/4<n\leq x} \mu(n) f(n) g(n) \,\mathrm{e}(\vartheta n) \right| \ll x \exp\left(-c \frac{\log x}{\log\log x}\right).$$

Applying these two inequalities with x replaced by $x/4^k$ for $0 \leq k \leq K = \left\lfloor \frac{\log x}{2\log 4} \right\rfloor$ and observing that $\frac{\log(x/4^k)}{\log\log(x/4^k)} \geq \frac{\log x}{2\log\log x}$ we have

$$\begin{aligned}
\left| \sum_{n\leq x} \Lambda(n) f(n) g(n) \,\mathrm{e}(\vartheta n) \right| &\leq \sum_{k=0}^{K} \left| \sum_{x/4^{k+1}<n\leq x^k} \Lambda(n) f(n) g(n) \,\mathrm{e}(\vartheta n) \right| \\
&\ll \sum_{k=0}^{K} \frac{x}{4^k} \exp\left(-c \frac{\log(x/4^k)}{\log\log(x/4^k)}\right) \\
&\ll x \exp\left(-c \frac{\log x}{2\log\log x}\right) \sum_{k=0}^{K} \frac{1}{4^k} \\
&\ll x \exp\left(-c \frac{\log x}{2\log\log x}\right)
\end{aligned}$$

and similarly for μ, which completes the proofs of Theorems 1 and 2.

Theorem 3 follows from the proof of Theorem 1 where by Cauchy-Schwarz and Proposition 3 we have to replace (87) by

$$\begin{aligned}
&\sum_{v_{11}} \sum_{u_{14}} \left| F(h_{12}, v_{11}+v_{12}, u_{13}+u_{14}) \; F(h_{12}, v_{11}, u_{14}) \right| \\
&\qquad \leq c_{10}(q_1, f_0) \, q_1^{-c_9(q_1,f_0)\|d_{f_0}\alpha\|^2/\log(1/\|d_{f_0}\alpha\|)\,(\mu_{12}-\mu_{11})},
\end{aligned}$$

and similarly for q_2 and g_0.

Finally Corollary 1 can be deduced from Theorem 3 by an argument similar to the proof of Theorem 3 of [25] (see Section 11).

References

[1] N. L. Bassily and I. Kátai, *Distribution of the values of q-additive functions on polynomial sequences*, Acta Math. Hungar., 68 (1995), pp. 353–361.

[2] R. Bellman and H. N. Shapiro, *On a problem in additive number theory*, Ann. of Math. (2), 49 (1948), pp. 333–340.

[3] J.-P. Bertrandias, *Suites pseudo-aléatoires et critères d'équirépartition modulo un*, Compositio Math., 16 (1964), pp. 23–28 (1964).

[4] ——, *Espaces de fonctions bornées et continues en moyenne asymptotique d'ordre p*, Bull. Soc. Math. France Mém., 5 (1966), p. 106.

[5] J. Bésineau, *Indépendance statistique d'ensembles liés à la fonction "somme des chiffres"*, Acta Arith., 20 (1972), pp. 401–416.

[6] M. Drmota, *The joint distribution of q-additive functions*, Acta Arith., 100 (2001), pp. 17–39.

[7] M. Drmota, C. Mauduit, and J. Rivat, *Normality along squares*, J. Eur. Math. Soc. (JEMS), (to appear).

[8] S. Ferenczi, J. Kułaga-Przymus, and M. Lemańczyk, *Sarnak's conjecture – what's new*, arXiv:1710.04039, (to appear).
[9] E. Fouvry and C. Mauduit, *Méthodes de crible et fonctions sommes des chiffres*, Acta Arithmetica, 77 (1996), pp. 339–351.
[10] ——, *Sommes des chiffres et nombres presques premiers*, Mathematische Annalen, 305 (1996), pp. 571–599.
[11] A. O. Gelfond, *Sur les nombres qui ont des propriétés additives et multiplicatives données*, Acta Arith., 13 (1967/1968), pp. 259–265.
[12] P. J. Grabner, P. Liardet, and R. F. Tichy, *Spectral disjointness of dynamical systems related to some arithmetic functions*, Publ. Math. Debrecen, 66 (2005), pp. 213–243.
[13] H. Iwaniec and E. Kowalski, *Analytic number theory*, vol. 53 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 2004.
[14] T. Kamae, *Mutual singularity of spectra of dynamical systems given by "sums of digits" to different bases*, in Dynamical systems, Vol. I—Warsaw, Soc. Math. France, Paris, 1977, pp. 109–114. Astérisque, No. 49.
[15] ——, *Sum of digits to different bases and mutual singularity of their spectral measures*, Osaka J. Math., 15 (1978), pp. 569–574.
[16] ——, *Cyclic extensions of odometer transformations and spectral disjointness*, Israel J. Math., 59 (1987), pp. 41–63.
[17] I. Kátai, *Distribution of digits of primes in q-ary canonical form*, Acta Math. Hungar., 47 (1986), pp. 341–359.
[18] D.-H. Kim, *On the joint distribution of q-additive functions in residue classes*, J. Number Theory, 74 (1999), pp. 307–336.
[19] B. Martin, C. Mauduit, and J. Rivat, *Théorème des nombres premiers pour les fonctions digitales*, Acta Arith., 165 (2014), pp. 11–45.
[20] ——, *Fonctions digitales le long des nombres premiers*, Acta Arith., 170 (2015), pp. 175–197.
[21] ——, *Propriétés locales des chiffres des nombres premiers*, Journal de l'Institut Mathématique de Jussieu, (2017).
[22] ——, *Nombres premiers avec contraintes digitales multiples*, Bulletin de la Société Mathématique de France, (to appear).
[23] C. Mauduit and B. Mossé, *Suites de G_q-orbite finie*, Acta Arith., 57 (1991), pp. 69–82.
[24] C. Mauduit and J. Rivat, *La somme des chiffres des carrés*, Acta Math., 203 (2009), pp. 107–148.
[25] ——, *Sur un problème de Gelfond: la somme des chiffres des nombres premiers*, Ann. of Math. (2), 171 (2010), pp. 1591–1646.
[26] ——, *Prime numbers along Rudin-Shapiro sequences*, J. Eur. Math. Soc. (JEMS), 17 (2015), pp. 2595–2642.
[27] M. Mendès France, *Nombres normaux. Applications aux fonctions pseudo-aléatoires*, J. Analyse Math., 20 (1967), pp. 1–56.
[28] H. L. Montgomery, *The analytic principle of the large sieve*, Bull. Amer. Math. Soc., 84 (1978), pp. 547–567.
[29] M. Queffelec, *Sur la singularité des produits de Riesz et des mesures spectrales associées à la somme des chiffres*, Israel J. Math., 34 (1979), pp. 337–342 (1980).
[30] P. Sarnak, *Three Lectures on the Mobius function randomness and dynamics*, http://publications.ias.edu/sarnak/paper/512, (2011).
[31] J. Vaaler, *Some extremal functions in Fourier analysis*, Bull. Amer. Math. Soc., 12 (1985), pp. 183–216.
[32] M. Waldschmidt, *Diophantine approximation on linear algebraic groups*, vol. 326 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2000. Transcendence properties of the exponential function in several variables.

E-mail address: michael.drmota@tuwien.ac.at

Institut für Diskrete Mathematik und Geometrie TU Wien, Wiedner Hauptstr. 8–10, 1040 Wien, Austria

E-mail address: mauduit@iml.univ-mrs.fr

Université d'Aix-Marseille and Institut Universitaire de France, Institut de Mathématiques de Marseille, CNRS UMR 7373, 163, avenue de Luminy, Case 907, 13288 MARSEILLE Cedex 9, France

E-mail address: joel.rivat@univ-amu.fr

Université d'Aix-Marseille, Institut de Mathématiques de Marseille, CNRS UMR 7373, 163, avenue de Luminy, Case 907, 13288 MARSEILLE Cedex 9, France