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S8302-MAT Statistical Properties of Number Systems

Project 1 within the research area
"Numbertheoretical Algorithms"

Report on The First Two Years

Project Leader

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1 Short Description

Number systems naturally appear in number theoretical algorithms since every integer number is usually represented in the binary number system. However, the binary number system is not only used for representing numbers.

It is used in various algorithms. For example, the RSA enciphering algorithms (see [35]) needs the computation of powers $a^e \bmod m$, where a, e, m are large integers. By using the binary representation of e it is possible to compute $a^e \bmod m$ extremely fast. A completely different application of binary or q-ary representations of integers is the construction of (at the moment best known) low discrepancy sequences (see [14, 43]). Moreover, number systems appear in the theoretical analysis of various algorithms, for example, the sum-of-digits function plays an essential role in [17, 28, 29, 38, 34, 41, 46].

In all these applications the statistical behaviour of the digital expansions involved play an essential rôle in the (average case) analysis, e.g. the average value of the sum-of-digits function.

Therefore, the purpose of this project is to discuss statistical properties of number systems from various points of view.

Overall 10 papers [9, 10, 12, 13, 19, 20, 21, 22, 54, 55] have been written so far, one of them [12] is already published and three papers [9, 19, 54] are accepted for publication.

The main topics of the application for the first two year have been:

1. **Correlations between digits**
2. **The sum-of-digits function and prime numbers**
3. **Digital expansions and random sequences**

ad 1. We have proved that the q-adic number systems are asymptotically independent if the q-s are coprime [9]. This result has been extended to q-adic systems and the Zeckendorf expansion [13]. Furthermore, functional versions are given in [10].

In [19] the q-adic digits of the exponental sequence a^n $(n \le N)$ has been considered. It has been proved that the $(\log N)^{\frac{3}{2}-\varepsilon}$ first and the $\log N$ leadings digits of a^n $(n \le N)$ behave in the expected (uniformly distributed) way.

Finally the block distribution of G-adic digital expansions is discussed in [54].

ad 2. The joint distribution of q-additive and G-additive functions on polynomial values of primes has been determined in [13], functional versions are given in [10].

ad 3. In [12] it is proved that the d-dimensional sequences $(s_{q_1}(n)\alpha_1, \dots, s_{q_d}(n)\alpha_d)$ is uniformly distributed modulo 1 if and only if the numbers $\alpha_1, \dots, \alpha_k$ are irrational. Furthermore, we give estimates for the discrepancy of the one-dimensional sequence $(s_q(n)\alpha) \bmod 1$ in term of the continued fraction expansion of α.

We have also extended the original scale of topics to the field of

4. **Continued fraction expansions**

ad 4. It is a classial fact that Diophantine approximation properties of a real number α are (more or less) determined by the continued fraction expansion of α. Thus, *statistical* properties of the contiuned fraction expansion directly translate to *metric* properties of Diophantine approximaition. In [20] this has been worked out for formal Laurent series. Furthermore, in [21, 22] extensions of classical results of LeVeque and W. Philipp (concerning central limit laws and laws of the iterated logarithm) are given.

2 Scientific Background — Status of Research

Let $G = (G_j)_{j\geq 0}$ be a strictly increasing sequence of integers with $G_0 = 1$. Then every non-negative integer n has a (unique) proper G-ary digital expansion $n = \sum_{j\geq 0} \epsilon_j(n) G_j$ with integer digits $\epsilon_j(n) \geq 0$ provided that $\sum_{j<k} \epsilon_j(n) G_j < G_k$ for all $k \geq 0$. For example, for $G_j = q^j$ (with q a positive integer) we obtain the usual q-ary expansion. Further examples are Cantor representations, where G_j is of the form $G_j = b_0 b_1 \cdots b_j$ (with integers $b_j > 1$), and the Zeckendorf representations, where $G_j = F_{j+2}$ are the Fibonacci numbers. More generally, one also considers G_j which are solutions of a linear recurrence with constant coefficients.

A very significant functional of number systems is the sum-of-digits function $s_G(n)$ which is given by

$$s_G(n) = \sum_{j\geq 0} \epsilon_j(n).$$

It is convenient to consider a related sequence of (discrete) random variables X_N, $N \geq 0$, defined by

$$\mathbf{Pr}[X_N = k] = \frac{|\{n < N : s_G(n) = k\}|}{N}.$$

in order to state some results of the literature concerning statistical properties of number systems. For example, the expected value is exactly the average value of the sum-of-digits function: $\mathbf{E}X_N = \frac{1}{N}\sum_{n<N} s_G(n)$. There is a vast literature concerning asymptotic properties of $\mathbf{E}X_N$ and $\mathbf{V}X_N$ and on the distribution of X_N.

Asymptotic and exact formulas for $\mathbf{E}X_N$ are due to Bush [5], Bellman and Shapiro [4], Delange [7], and Trollope [56] for q-ary digital expansions and due to Pethö and Tichy [48] and Grabner and Tichy [26, 27] for G-ary digital expansions with respect to linear recurrences. Corresponding formulas for higher moments $\mathbf{E}X_N^d$ and for the variance $\mathbf{V}X_N$ can be found in Coquet [6], Kirschenhofer [32], Kennedy and Cooper [30], Grabner, Kirschenhofer, Prodinger, and Tichy [25], and in Dumont and Thomas [15]. All known results are of the form

$$\mathbf{E}X_N \sim C_1 \log N \quad \text{and} \quad \mathbf{V}X_N \sim C_2 \log N$$

with constants $C_1, C_2 > 0$ depending on the number system $G = (G_j)$.

The asymptotic distribution of X_N and related problems are discussed in Schmidt [51], Schmid [50], Bassily and Katai [3], in Dumont and Thomas [16] and in Drmota and Gajdosik [11] (see also Drmota [8]), where a new appoach using generating functions is presented. Again we always have the same result, a central limit theorem of the form

$$\frac{X_N - \mathbf{E}X_N}{\sqrt{\mathbf{V}X_N}} \to N(0,1).$$

It is also very interesting to study not only the sum-of-digits function of all non-negative integers but to restrict to a subsequence, e.g. to an arithmetic sequence $an + b$ (see Schmid [50]) or to values of a polynomial $P(n)$ (see Bassily and Kátai [3] and Gittenberger and Thuswaldner [23] for b-additive functions on the Gaussian integers). Other sequences of interest are the sequence of primes p_n (or the values of a polynomial $P(p_n)$ with prime argument). The following general theorem of Bassily and Kátai [3] was an important starting point for several papers [9, 10, 13, 19, 55] which have been written within the first two years of this project. (Here and in the sequel $\Phi(y)$ denotes the distribution function of the normal law.)

Theorem 1 *Let f be a q-additive function, i.e. $f(n) = \sum_{j\geq 0} f(a_{q,j}(n) q^j)$ for $n = \sum_{j\geq 0} a_{q,j}(n) q^j$, such that $f(cq^j) = \mathcal{O}(1)$ as $j \to \infty$ and $c \in \{0, \ldots, q-1\}$.*

Assume that $\frac{D_q(N)}{(\log N)^\eta} \to \infty$ *as* $N \to \infty$ *for some* $\eta > 0$ *and let* $P(n)$ *be a polynomial with integer coefficients, degree* r *and positive leading term. Then, as* $N \to \infty$,

$$\frac{1}{N}\#\left\{n < N \,\middle|\, \frac{f(P(n)) - M_q(N^r)}{D_q(N^r)} < y\right\} \to \Phi(y)$$

and

$$\frac{1}{\pi(N)}\#\left\{p < N \,\middle|\, \frac{f(P(p)) - M_q(N^r)}{D_q(N^r)} < y\right\} \to \Phi(y),$$

where

$$M_q(N) := \sum_{k=0}^{[\log_q N]} m_{k,q}, \qquad D_q^2(N) = \sum_{k=0}^{[\log_q N]} \sigma_{k,q}^2$$

and

$$m_{k,q} := \frac{1}{q}\sum_{c=0}^{q-1} f(cq^k), \qquad \sigma_{k,q}^2 := \frac{1}{q}\sum_{c=0}^{q-1} f^2(cq^k) - m_{k,q}^2.$$

3 Results and Discussion

In what follows we describe the results of the 10 papers [9, 10, 12, 13, 19, 20, 21, 22, 54, 55] in detail.

[9] M. Drmota, The joint distribution of q-additive functions, *Acta Arith*, to appear.

The main result of [9] (which solves a major problem of the original application) reads as follows.

Theorem 2 *Let* $q_1, q_2 > 1$ *be coprime integers and let* f_ℓ *be* q_ℓ*-additive functions such that* $f_\ell(cq_\ell^j) = \mathcal{O}(1)$ *as* $j \to \infty$ *and* $c \in E_{q_\ell}$, $\ell = 1, 2$. *Assume that* $\frac{D_{q_\ell}(N)}{(\log N)^\eta} \to \infty$ *as* $N \to \infty$, $\ell = 1, 2$, *for some* $\eta > 0$. *Let* $P_\ell(x) = A_\ell x + B_\ell$, $\ell = 1, 2$, *be arbitrary linear polynomials with integer coefficients and positive leading terms* A_ℓ *coprime to* q_ℓ. *Then, as* $x \to \infty$,

$$\frac{1}{N}\#\left\{n < N \,\middle|\, \frac{f_\ell(P_\ell(n)) - M_{q_\ell}(N)}{D_{q_\ell}(N)} < y_\ell,\ \ell = 1, 2\right\} \to \Phi(y_1)\Phi(y_2).$$

This theorem especially applies to the sum-of-digits function $s_q(n) = \sum_{j \ge 0} a_{q,j}(n)$.

Corollary 1 *Let* $q_1, q_2 > 1$ *be coprime integers. Then, as* $N \to \infty$,

$$\frac{1}{N}\#\left\{n < N \,\middle|\, \frac{s_{q_\ell}(n) - \frac{q_\ell - 1}{2}\log_{q_\ell} N}{\sqrt{\frac{q_\ell^2 - 1}{12}\log_{q_\ell} N}} < y_\ell,\ \ell = 1, 2\right\} \to \Phi(y_1)\Phi(y_2)$$

The method of the proof of Theorem 2 is closely related to the methods presented by Bassily and Katai [3]. It mainly relies on estimates for exponential sums of the form ($e(x) = e^{2\pi i x}$)

$$\frac{1}{N}\sum_{n<N} e\left((A_1\mathbf{m}_1 \cdot \mathbf{v}_1 + A_2\mathbf{m}_2 \cdot \mathbf{v}_2)n\right),$$

where $\mathbf{m}_\ell = (m_1^{(\ell)}, \ldots, m_{h_\ell}^{(\ell)})$ are integer vectors and $\mathbf{v}_\ell = \left(q_\ell^{-k_1^{(\ell)}-1}, \ldots, q_\ell^{-k_{h_\ell}^{(\ell)}-1}\right)$, $\ell = 1, 2$, where $k_j^{(\ell)}$ are integers in the range

$$(\log N)^\eta \le k_1^{(\ell)} < k_2^{(\ell)} < \cdots k_{h_\ell}^{(\ell)} \le \log_{q_\ell} N - (\log N)^\eta \quad (\ell = 1, 2).$$

Since we have

$$\sum_{n<N} e(\alpha n) \ll \frac{1}{|\alpha|}$$

for real numbers $|\alpha| \le \frac{1}{2}$ it remains to estimate $A_1\mathbf{m}_1 \cdot \mathbf{v}_1 + A_2\mathbf{m}_2 \cdot \mathbf{v}_2$ from below. (Upper estimates are trivial.) Interstingly this can be worked out with help of Baker's theorem on linear forms of logarithms.

Essentially Theorem 2 can be interpreted in the following way. If q_1, q_2 are coprime then specific functionals of the q_1-adic and q_2-adic digital expansions (e.g. the sum-of-digits functions) behave asymptotically independent.

[10] M. Drmota, M. Fuchs and E. Manstavicius, Functional limit theorems for digital expansions, *Acta. Math. Hung.*, submitted.

Let $s_{q,k}(n) := \sum_{0\le j\le k} a_{q,j}(n)$ denote the partial sums of sum-of-digits function. The sequence $(s_{q,k}(n))_{k\ge 0}$ may be considered as a an (increasing) *random walk* and really encodes the digital expansion of n.

We consider the probability space $(\mathbf{N}_0, \mathcal{P}(\mathbf{N}_0), \nu_N)$, where $\mathcal{P}(\mathbf{N}_0)$ denotes the powerset of $\mathbf{N}_0$ and ν_N is the probability measure defined by

$$\nu_N(A) := \frac{1}{N}\#\{0 \le n < N | n \in A\} \quad A \in \mathcal{P}(\mathbf{N}_0).$$

On this probability space we introduce the following stochastic process

$$X_N(t)(n) := \left(\frac{1}{\sigma_q \sqrt{L}} \left(s_{q,tL}(n) - tL\frac{q-1}{2} \right) \right) \quad t \in [0,1].$$

where $L := L_q := \log_q N$ and $\sigma_q := \sqrt{(q^2-1)/12}$.

Manstavicius [40] has proved that, $X_N(t) \to B(t)$ as $N \to \infty$, where $B(t)$ denotes the standard Brownian motion and weak convergence is considered in the space $\mathbf{D}[0,1]$.

Our first result is a direct generalization to polynomial sequences on integers and on primes. Therefore let $P(x)$ be a polynomial with integer coefficients, degree r, and positive leading term and define stochastic processes by

$$X_{N,P,q}(t)(n) := X_{N,P}(t)(n) := \frac{1}{\sigma_q\sqrt{rL}} \left(s_{q,trL}(P(n)) - trL\frac{q-1}{2} \right)$$

and

$$Y_{N,P,q}(t)(p) := Y_{N,P}(t)(p) := \frac{1}{\sigma_q\sqrt{rL}} \left(s_{q,trL}(P(p)) - trL\frac{q-1}{2} \right),$$

where $t \in [0,1]$. The only difference between $X_{N,P}$ and $Y_{N,P}$ is, that the second process is defined on primes. With this notation we have:

Theorem 3 *Let $q \ge 2$ and $P(x)$ be a integer polynomial of degree $r \ge 1$ with positive leading term. Then we have, as $N \to \infty$,*

$$X_{N,P}(t) \to B(t) \quad \textit{and} \quad Y_{N,P}(t) \to B(t)$$

The above properties even generalize to the joint distribution of two different digital espansions.

Theorem 4 *Let $q_1, q_2 \ge 2$ be coprime and $P_1(x), P_2(x)$ two integer polynomial of degrees $r_1, r_2 \ge 1$ with positive leading term. Then we have, as $N \to \infty$,*

$$(X_{N,P_1,q_1}(t_1), X_{N,P_1,q_2}(t_2)) \to (B_1(t_1), B_2(t_2))$$

and

$$(Y_{N,P_1,q_1}(t_1), Y_{N,P_1,q_2}(t_2)) \to (B_1(t_1), B_2(t_2))$$

where $(B_1(t_1), B_2(t_2))$ *denotes a Gaussian field consisting of two independent Brownian motions.*

Theorem 4 may be considered as a (better) theoretical justification to the statement that two q-ary digital expansions with coprime q are (asymptotically) independent.

Finally, in this paper several properties concerning Strassen's law (and the law of the iterated logarithm) are proved.

[13] M. Drmota and W. Steiner, The Zeckendorf Expansion of Polynomial Sequences, *J. Theor. Nombres Bordx.* submitted.

[55] W. Steiner, Parry expansions of polynomial sequences, *J. London Math. Soc.*, submitted.

We discuss the results of the two papers [13, 55] together.

The idea behind these two papers is to generalize the results of Bassily and Katai [3] and Drmota [9] to G-ary digital expansions, where the sequence $G = (G_j)_{j\geq 0}$ is defined by the linear recurrence

$$G_j = a_1 G_{j-1} + a_2 G_{j-2} + \ldots + a_d G_{j-d} \text{ for } j \geq d$$

and

$$G_j = a_1 G_{j-1} + a_2 G_{j-2} + \ldots + a_j G_0 \text{ for } 1 \leq j < d,\ G_0 = 1,$$

with non-negative integers a_i which satisfy the relations

$$(a_j, a_{j+1}, \ldots, a_d) \leq (a_1, a_2, \ldots, a_{d-j+1}) \text{ for } 2 \leq j \leq d$$

(where "$<$" denotes the lexicographical order) and $a_d > 0$.

Here the digits $\epsilon_j(n)$ of the G-ary expansion $n = \sum_{j\geq 0} \epsilon_j(n) G_j$ satisfy

$$(\epsilon_j(n), \epsilon_{j-1}(n), \ldots, \epsilon_{j-d+1}(n)) < (a_1, a_2, \ldots, a_d) \text{ for all } j \geq 0. \qquad (1)$$

These G-ary expansions of integers are strongly related to Parry's β-expansions of real numbers (with simple β-numbers) (cf. Parry [47], see Loraud [39] for details).

Furthermore let $\chi(x) = x^d - a_1 x^{d-1} - \ldots - a_{d-1}x - a_d$ be the characteristic polynomial of the linear recurrence and $\alpha \in \mathbf{R}^+$ its (unique) dominant root.

G-additive functions are defined similarly to q-additive functions, i.e.

$$f(n) = \sum_{j\geq 0} f(\epsilon_j(n) G_j) = \sum_{j\geq 0} f_j(\epsilon_j(n)) \text{ for all } n \in \mathbf{N},\ f(0) = 0.$$

Note that for the second part of the following theorem (the main result of [55]) we have to make some restrictions on the sequence G: we need $a_d = 1$, α has to be a Pisot number with minimal polynomial $\chi(x)$, i.e. $|\alpha_i| < 1$ for $2 \leq i \leq d$, and

$$\mathrm{Fin}(\alpha) = \mathbf{Z}[\alpha^{-1}] \cap \mathbf{R}^+, \qquad (2)$$

where $\mathrm{Fin}(\alpha)$ denotes the set of nonnegative real numbers with finite α-expansion, i.e. $\{x \in \mathbf{R}^+ : x = \sum_{j=-L}^{M} \epsilon_j \alpha^j \text{ with } (\epsilon_j, \ldots, \epsilon_{j-d+1}) < (a_1, \ldots, a_d) \text{ for all } j \leq M\}$.

Theorem 5 *Let G be as above, f a G-additive function such that $f_j(e) = \mathcal{O}(1)$ as $j \to \infty$ for all $e \in \{0, \ldots, a_1\}$. Then, for all $\varepsilon > 0$, the expected value of $f(n)$, $0 \le n < N$, is given by*

$$\frac{1}{N} \sum_{n<N} f(n) = M(N) + \mathcal{O}\left((\log N)^{\varepsilon}\right), \tag{3}$$

where $M(N) = \sum_{k=0}^{[\log_\alpha N]} \mu_k$ with $\mu_k = \sum_{e=1}^{a_1} p_e f_k(e)$. (with some explicitly computable constants p_e). Set $D(N)^2 = \sum_{j,k=0}^{[\log_\alpha N]} \sigma_{j,k}^{(2)}$ with

$$\sigma_{j,k}^{(2)} = \begin{cases} \sum\limits_{e=1}^{a_1} p_e f_k(e)^2 - \mu_k^2 & \text{if } j = k \\ \sum\limits_{i=2}^{d} \left(\frac{\alpha_i}{\alpha}\right)^{|j-k|} \sum\limits_{e=1}^{a_1} \sum\limits_{e'=1}^{a_1} p_{e,e'}^{(i)} f_{\min(j,k)}(e) f_{\max(j,k)}(e') & \text{if } j \ne k \end{cases}$$

(and some explicitly computable constants $p_{e,e'}^{(i)}$).

Furthermore assume that there exists a constant $c > 0$ such that $\sigma_{k,k}^{(2)} \ge c$ for all $k \ge 0$. Then we have

$$\frac{1}{N} \#\{n < N : \frac{f(n) - M(N)}{D(N)} < x\} \to \Phi(x)$$

and

$$\frac{1}{N} \sum_{n<N} \left(\frac{f(n) - M(N)}{D(N)}\right)^L \to \int_{-\infty}^{\infty} x^L \, d\Phi(x)$$

as $N \to \infty$.

Let G be as above with $a_d = 1$, irreducible characteristic polynomial $\chi(x)$ and its dominant root α a Pisot number which satisfies (2). Let $P(n)$ be a polynomial of degree r with integer coefficients and positive leading term. Then, as $N \to \infty$,

$$\frac{1}{N} \#\{n < N : \frac{f(P(n)) - M(N^r)}{D(N^r)} < x\} \to \Phi(x)$$

and for every integer $L \ge 0$

$$\frac{1}{\pi(N)} \#\{p \in \mathbf{P}, p < N : \frac{f(P(p)) - M(N^r)}{D(N^r)} < x\} \to \Phi(x)$$

The special case $d = 2$ was proved in [13], the general case in [55].

Theorem 5 especially applies if $a_1 \ge a_2 \ge \ldots \ge a_d = 1$. Here it is well known that α is a Pisot number with minimal polynomial $\chi(x)$. Furthermore, (2) has been shown in this case by Frougny and Solomyak [18]. For $d = 3, a_2 = 0, a_3 = 1$, α_2 and α_3 are complex numbers and have therefore absolute value $1/\sqrt{\alpha}$. From Akiyama [1] we know that (2) holds in this case. Hence Theorem 5 holds for these sequences too and the only restriction for $d = 3$ is $a_3 = 1$.

The proof of Theorem 5 is again related to the method of Bassily and Katai [3] but further ingrediences are needed. One crucial point is to consider proper tilings of the torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$ in order to determine the value of a digit $\epsilon_k(n)$ without using the greedy algorithm. Interestingly one can use Markov partitions of a toral automorphism induced by the linear recurrence of G. (In general there is only a toral endomorphism which is 1-1 if and only if $a_d = 1$.) For $d = 2$ this Markov partition consists of rectangles modulo 1 and is, thus, much more easy to handle, compare with [13]. For $d > 2$, the involved sets have fractal boundary and are related to Rauzy fractals. (Fortunately these fractal sets have been studied by Praggastis [49].)

In the first paper [13] it is also proved that the joint distribution of different q-ary and G-ary expansions is asymptotically normal. Especially, Theorem 2 of [9] is generalized to arbitrary polynomials and to polynomials over primes.

[54] W. STEINER, Generalized De Bruijn digraphs and the distribution of G-additive functions, *Discrete Math.*, to appear.

The purpose of this paper is twofold. First it calculates explicitly the characteristic polynomial of specific graphs which are called *generalized De Bruijn graphs* and second it is shown that these results can be applied to characterize the distribution of patterns in G-ary digital expansions, where $G = (G_j)_{j\geq 0}$ satisfies as linear recurrence as in the previous paper [55].

Considers an arbitrary sequence $a_1, a_2, \ldots, a_r$ of positive integers such that

$$(a_n, a_{n+1}, \ldots, a_r, 0, 0, \ldots) < (a_1, a_2, \ldots, a_r, 0, 0, \ldots)$$

for all $n \geq 2$, where $<$ denotes the lexicographic order. Then the vertex set of the generalized De Bruijn graph $\mathbf{A}_L$ is the set $\mathcal{B}_L$ consisting of blocks $(\eta_1, \ldots, \eta_L)$ of non-negative integers such that

$$(\eta_n, \eta_{n+1}, \ldots, \eta_L, 0, 0, \ldots) < (a_1, a_2, \ldots, a_r, 0, 0, \ldots)$$

for all $n \geq 2$. Two blocks $(\eta_1, \ldots, \eta_L)$, $(\theta_1, \ldots, \theta_L)$ constitute an directed edge if $(\eta_2, \ldots, \eta_L) = (\theta_1, \ldots, \theta_{L-1})$ and $(\eta_1, \ldots, \eta_L, \theta_L) \in \mathcal{B}_{L+1}$. For $r = 1$ and $a_1 = q$ this graph reduces to the usual De Bruijn graph.

The first result of [54] says that the characteristic polynomial of $\mathbf{A}_L$ is given by

$$\chi(\mathbf{A}_L)(x) x^{|\mathcal{B}_L|-r}(x^r - a_1 x^{r-1} - \cdots - a_r).$$

This is a remarkable fact and seems to be completely new. The proof of this property relies on a reduction algorithm on the adjacency matrix of $\mathbf{A}_L$.

The above property is then applied to prove a distributional result for blocks in digital expansions based on the sequence $(G_j)_{j\geq 0}$ (defined by a linear recurrence as above). They number of occurances of a certain digital block satisfies a central limit theorem. (Here the methods of [8] are applied.) It turns out that a specific *transition matrix* of these digital expansions is exactly $\mathbf{A}_L$ and thus, the above formula for the characteristic polynomial can be applied.

[19] M. FUCHS, Digital expansions of exponential sequences, *J. Theor. Nombres Bordx.*, to appear.

In this paper we consider the q-ary digital expansion of an exponential sequence a^n. Especially, we investigate the following number

$$B_{q,a,b}(a^n; e_s e_{s-1} \cdots e_0) := \#\{a \leq i \leq b \mid d_{i-s+j}(a^n) = e_j, 0 \leq j \leq s\}$$

(where $a \leq b$ are integers, $e_s e_{s-1} \cdots e_0$ is a digital block, and $d_i(a^n)$ denotes the i-th digit in the q-ary digital expansion of a^n) which is the number of occurences of the digital block $e_s e_{s-1} \cdots e_0$ in the digital expansion of a^n between a and b.

By using a result due to Kiss and Tichy [33], we prove that the average number of occurences of an arbitrary digital block in the most significant digits is, except of a bounded error term, asymptotically equal to the expected value

Theorem 6 *Let a, q be integer both ≥ 2 such that $\log_a q$ is irrational. We consider a digital block $e_s e_{s-1} \cdots e_0$ with $s \geq 0, 0 \leq e_i \leq q-1, 0 \leq i \leq s$. There exists a positive real constant γ, such that we have, as $N \longrightarrow \infty$,*

$$\frac{1}{N} \sum_{n=1}^{N} B_{q,u(n),v(n)}(a^n; e_s e_{s-1} \cdots e_0) = \frac{\gamma}{q^{s+1}} \log_q N + O(1),$$

with $u(n) = \left[n \log_q a - \gamma \log_q N\right]$ *and* $v(n) = \left[n \log_q a\right]$.

A similar result is true for the least significant digits if q is a prime number:

Theorem 7 *Let $a \geq 2$ be an integer and $p \in \mathbf{P}$ a prime with $(a,p) = 1$. We consider a digital block $e_s e_{s-1} \cdots e_0$ with $s \geq 0$ and $0 \leq e_i \leq p-1, 0 \leq i \leq s$. Furthermore, let ϵ, η be arbitrary positive real numbers and $A_1(N), A_2(N)$ positive integer-valued functions with*

$$[(\log_p N)^\eta] \leq A_1(N) < A_2(N) \leq [(\log_p N)^{\frac{3}{2}-\epsilon}].$$

Then we have for a positive real number λ, as $N \longrightarrow \infty$,

$$\frac{1}{N} \sum_{n=1}^{N} B_{p,A_1(N)+1,A_2(N)}(a^n; e_s e_{s-1} \cdots e_0)$$
$$= \frac{1}{p^{s+1}}(A_2(N) - A_1(N)) + O\left(\frac{1}{\log^\lambda N}\right).$$

The proofs of both results heavily depend on estimations of exponential sums and the concept of discrepancy of real sequences modulo one plays an important rôle.

As a corollary we get lower bounds for the mean value of the sum-of-digits function as well as for the number of digit changes of the first N terms of an exponential sequence.

[12] M. Drmota and G. Larcher, The sum-of-digits-function and uniform distribution modulo 1, *J. Number Theory* **89** (2001), 65–96.

This paper is devoted to the sequence $(s_q(n)\alpha) \bmod 1$, where $s_q(n)$ denotes the q-ary sum-of-digits function and α is a real number. It is well known (see [14]) that this sequence is uniformly distributed modulo 1 if and only if α is irrational. The following extension of this property has been conjectured in the application of this project:

Theorem 8 *Let $q_1, q_2, \ldots, q_d$ be pairwisely comprime integers > 1. Then the d-dimensional sequence $(\alpha_1 s_{q_1}(n), \alpha_2 s_{q_2}(n), \ldots, \alpha_d s_{q_d}(n))$ $(n \geq 0)$ is uniformly distributed modulo 1 if and only if $\alpha_1, \alpha_2, \ldots, \alpha_d$ are irrational.*

It is possible to prove a little more, namely that the d-dimensional sequence $(\alpha_1 s_{q_1}(n), \alpha_2 s_{q_2}(n), \ldots, \alpha_d s_{q_d}(n))$ is well distributed modulo 1, too (see [14]). This fact was also observed by Grabner, Liardet and Tichy [24] via ergodic means, even for Cantor representations. However, our method works with exponential sums. In fact, by generalizing a method of Kim [31] we proved that

$$\left|\frac{1}{N} \sum_{n<N} e(h_1 \alpha_1 s_{q_1}(n) + h_2 \alpha_2 s_{q_2}(n) + \cdots + h_d \alpha_d s_{q_d}(n))\right|$$
$$\ll \exp\left(-c \log N \sum_{j=1}^{d} \|(q_j - 1) h_j \alpha_j\|^2\right),$$

(where $e(x) = e^{2\pi i x}$ and $\|x\| = \min_{k \in \mathbf{Z}} |x - k|$ denotes the distance to the integers). With help of this estimate we also proved estimates for the discrepancy of the d-dimensional sequence.

Furthermore, we discussed the one-dimensional sequence $(s_q(n)\alpha) \bmod 1$ and derived estimates for the discrepancy in terms of the continued fraction expansion of α.

Theorem 9 *Let $q > 1$ be an integer and let α be an irrational number with continued fraction expansion $\alpha = [a_0; a_1, \ldots]$ and best approximation denominators $1 = q_0 \le q_1 < q_2 \ldots$.*

Then there is a positive constant $d(q)$ and for every $\gamma \ge 0$ there is a constant $c(q,\gamma) > 0$ such that the discrepancy D_N of $(\{\alpha \cdot s_q(n)\})$ can be estimated by

$$D_N \le \frac{c(q,\gamma)}{\sqrt{\log N}} \left(\sum_{i_0 - \gamma \log\log\log N \le l \le i_0} a_l + \frac{a_{i_0+1}}{(\log\log N)^{\gamma \frac{\log 2}{2} - \frac{1}{2}}} + \frac{(\log\log\log N)^{3/2}}{(\log\log N)^{\gamma \frac{\log 2}{2}}} \cdot \max_{l \le i_0 - \gamma \log\log\log N} a_l \right),$$

where i_0 is defined by $q_{i_0} \le d(q)\sqrt{\log N \cdot \log\log N} < q_{i_0+1}$.

For expamle, if α has bounded continued fraction expansion then

$$D_N \le c(q,\alpha) \frac{\log\log\log N}{\sqrt{\log N}}.$$

Conversely we only know that $D_N > c'(q)(\log N)^{-\frac{1}{2}}$. It is an unsolved problem to decide whether there are α such that $D_N \ll (\log N)^{-\frac{1}{2}}$.

[20] M. Fuchs, On metric diophantine approximation in the field of formal laurent series, *Finite Fields Appl.*, submitted.

Consider the field of formal laurent series $\mathbf{F}_q((T^{-1}))$ which is the set of all formal sums

$$\alpha = \sum_{k \le n} a_k T^k, \quad a_k \in \mathbf{F}_q, a_n \ne 0.$$

Endow $\mathbf{F}_q((T^{-1}))$ with the valuation defined by $|\alpha| = q^n$ if $\alpha \ne 0$ and $|0| = 0$.

There is a continued fraction theory in $\mathbf{F}_q((T^{-1}))$ (for an introduction see [52]) with polynomials playing the rôle of integers. For the metric theory of this continued fraction expansion consider the following subset $H = \{\alpha \in H : |\alpha| < 1\}$ of $\mathbf{F}_q((T^{-1}))$ which can be seen as an analogue of the interval $[0,1]$. Observe, that H together with the restriction of the valuation on H is a compact abelian group and therefore we have an unique, translation invariant probability measure which we are going to denote by h.

In this paper we prove an analogue in the Laurent series case of a central limit theorem in metric diophantine approximation due to LeVeque (see [36]). Moreover, we add an iterated logarithm law for the number of solutions of the diophantine approximation problem.

In order to state the result let f be a positive, non-increasing, real-valued function defined on the non-negativ real numbers satisfying the following conditions

$$f(x) = \mathrm{O}(x^{-1}), f'(x) = \mathrm{O}(x^{-2}), \text{ as } x \longrightarrow \infty; \quad \sum_{k=0}^{\infty} f(k) = \infty.$$

Consider the diophantine approximation problem

$$\left| \alpha - \frac{P}{Q} \right| < \frac{f(\deg Q)}{|Q|^2} \tag{4}$$

where $\alpha \in H$ and P, Q are polynomials and define a sequence of random variables by $W_n(\alpha) := \#\{(P,Q) | 0 \le \deg Q \le n, (P,Q) = 1, P/Q \text{ is a solution of } (4)\}$. Then we have the following result

Theorem 10 *Set* $A(n) = \frac{(q-1)^2}{q} \sum_{k=0}^{n} q^{\lceil \log_q f(k) \rceil}$.
I. For fixed real number ω *we have*

$$\lim_{n \longrightarrow \infty} h\left[W_n - A(n) < \omega((q-1)A(n))^{1/2}\right] = \Phi(\omega)$$

II. Furthermore

$$\limsup_{n \longrightarrow \infty} \frac{1}{(2(q-1)A(n) \log\log A(n))^{1/2}} (W_n(\alpha) - A(n)) = 1 \quad a.s.,$$

$$\liminf_{n \longrightarrow \infty} \frac{1}{(2(q-1)A(n) \log\log A(n))^{1/2}} (W_n(\alpha) - A(n)) = -1 \quad a.s.$$

Especially, we have $W_n(\alpha) \sim A(n) \quad a.s.$

The proof makes use of several results from the metric theory of continued fractions in the formal Laurent series case proved by Niederreiter [42]. In particular, the fact that the sequence of partial quotients is an independent and identically distributed sequence of random variables (in contrary to the classical case) plays an important rôle. This is used to approximate the sequence W_n by an independent sequence of random variables and then the central limit theorem resp. the law of iterated logarithm for a sequence of independent random variables is applied.

[21] M. Fuchs, On a problem of W. J. LeVeque, *manuscript.*

Let f be a positive, real-valued function defined on the non-negative real numbers satisfying the following conditions

$$f \downarrow 0, \quad \sum_{k=1}^{\infty} f(k) = \infty,$$
$$\sum_{k=1}^{n} f(k) k^{-\delta} \ll \left(\sum_{k=1}^{n} f(k)\right)^{1/2},$$
$$\sum_{k=1}^{n} f(k)^2 \ll \left(\sum_{k=1}^{n} f(k)\right)^{1/2},$$

where $0 < \delta < 1/2$.

We are concerned with the classical diophantine approximation problem

$$\left| x - \frac{p}{q} \right| \leq \frac{f(\log q)}{q^2}, \tag{5}$$

where $x \in [0, 1]$.

Fix integers s, r with $r \geq 1$ and define the following sequence of random variables

$$X_n(x) := \#\{(p,q) | 1 \leq q \leq n, q \equiv s \bmod r, p/q \text{ is a solution of (5)}\}. \tag{6}$$

In [37], LeVeque conjectured a central limit theorem for the above sequence of random variables (actually, LeVeque considered a more restrictive class of functions f and defined the sequence of random variables without the restriction that the denominators have to be in an arithmetic progression). We managed to settle this conjecture in this paper by proving the following theorem.

Theorem 11 *Set* $F(n) := \sum_{k=1}^{n} f(k)$. *Then we have*

$$\lim_{n \longrightarrow \infty} \lambda\left[X_n \leq \sigma_1 F(n) + \omega\left(\sigma_2 F(n) \log F(n)\right)^{1/2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega} e^{-u^2/2} du,$$

where

$$\sigma_1 = \frac{2}{r}, \quad \sigma_2 = \frac{12(s,r)\varphi(r)}{\pi^2 r C(s,r)},$$

and

$$C(s,r) = r^2 \prod_{p|r} \left(1 - \frac{1}{p}\right) \prod_{p|\frac{r}{(s,r)}} \left(1 + \frac{1}{p}\right).$$

For the proof continued fraction theory was used together with central limit theorems for double arrays of weakly dependent random variables.

[22] M. Fuchs, Invariance principles in metric diophantine approximation, *manuscript.*

Next, let f be a positive real-valued function defined on the non-negative real numbers satisfying the following conditions

$$f \downarrow 0, \quad \sum_{k=1}^{\infty} f(k) = \infty,$$
$$\sum_{k=1}^{n} f(k) k^{-\delta_1} \ll \left(\sum_{k=1}^{n} f(k)\right)^{1/2-\delta_2},$$

where $0 < \delta_1, \delta_2 < 1/2$.

The situation is getting much easier if we consider instead of (6) the following sequence of random variables

$$X_n(x) := \#\{(p,q) | 1 \leq q \leq n, q \equiv s(r), (p,q) \leq d, p/q \text{ is a solution of (5)}\}$$

where s, r, d are integers with $r, d \geq 1$.

The statistical behaviour of the above sequence of random variables (again under more restrictions on the function f and without the condition that the denominators have to be in an arithmetic progression) was already treated by LeVeque (see [36] and [37]). LeVeque's results where improved by Philipp a few years later (see [44]). In this paper we in turn improve Philipp's result by proving almost sure and distribution type invariance principles.

In order to state the results put $F(n) = \sum_{k=1}^{n} f(k)$ and define two suitable constants σ and τ (we avoid the exact definition). Moreover, set

$$n_t = \begin{cases} \max\{n | \tau^2 F(n) \leq t\} & \text{if } t \geq \tau^2 F(1) \\ 0 & \text{otherwise,} \end{cases}$$

where $t \geq 0$. Using this notation we define a stochastic process as

$$X(t) = X_{n_t} - \sigma F(n_t)$$

and the result of the paper reads as follows

Theorem 12 *Without changing the distribution of $\{X(t), t \geq 0\}$ we can redefine the process $\{X(t), t \geq 0\}$ on a richer probability space together with standar Brownian motion $\{B(t), t \geq 0\}$ such that, as $t \longrightarrow \infty$,*

$$X(t) - B(t) = o((t \log\log t)^{1/2}) \ a.s. \tag{7}$$

and

$$\lambda \left[\frac{1}{\sqrt{t}} \sup_{s \leq t} |X(s) - B(s)| \geq \epsilon \right] \longrightarrow 0 \tag{8}$$

for all $\epsilon > 0$.

Moreover, by using the additional assumption $\sum_{k=1}^{n} f^2(k) \ll F(n)^{1-\delta_3}$ (where $0 < \delta_3 < 1$), we can reduce the error term in the almost sure invariance principle to $t^{1/2-\lambda}$ for a suitable constant λ.

The above result was obtained by using continued fraction theory and applying an extension of a theorem due to Philipp and Stout [45] about the almost sure invariance principle of weakly dependent random variables. It entails several consequences, for instance, a functional central limit theorem and a Strassen's type version of the iterated logarithm law. With the additional assumption we even get a functional iterated logarithm law for the maximum.

4 Cooperations

4.1 Internal Cooperations

The most intensive cooperations within the FSP have been to *G. Larcher* (Analysis of Digital Point Sets) and to *R. F. Tichy* and *P. Kirschenhofer* (Algorithmic Diophantine Problems).

The paper [13] was written together with Larcher and is devoted to the distribution of the sequence $(s_q(n)\alpha) \bmod 1$ (compare with the above description). Especially a problem of the original application (concerning the multi-dimensional sequences) could be completely solved in this paper. It is planed to continue this cooperation in the future. We also want to consider low discrepancy sequences.

There have been several contacts to Tichy and Kirschenhofer and their groups (in Graz and at conferences in Budapest and Marseille). There are several common interests, e.g. in Baker's method which has been used in papers [9, 19].

4.2 External Cooperations

Within the first two years several conferences has been attended and several foreign guest have been invited. So it was easy to start cooperations to persons and institutions outside the FSP.

For example, E. Manstavicius (University of Vilnius, Lithuania) was invited and a joint paper [10] was written on functional limit theorems related to digital expansions.

Furthermore, Drmota was invited by P. Liardet (Université de Provence, Marseille, Febr. 2000) to stay in Marseille for one month. We started to work on the distribution of co-cycles of odometers, a problem which can be translated to the distribution of the sequence $s_q(n+k) - s_q(n)$.

Drmota was also invited by C. Mauduit (Université de Marseille) to present the results of [9, 13] (Sept. 2000). There is a further connection to the Université de Marseille. In [55] Steiner (who is employed by the present project) uses properties of the Rauzy fractal which has been recently studied by A. Siegel in her thesis [53].

Two further coopertations have been started with Austrian mathematicians who are not related to the FSP. It is planed to work together with *J. Schoissengeier* (University of Vienna) on digital expansions with respect to different bases and with *P.J. Grabner* (TU Graz) on the relation between the binary and the Zeckendorf representations of integers.

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