SOME NEW RESULTS IN SUMMABILTY THEORY

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ABSTRACT. The main part of this paper deals with limitibility properties of weakly dependent uniformly bounded random variables with respect to weighted means and a uniform weighted limitation method. In the next part a weighted Le-Roy-limitation method is introduced and discussed. For example it is shown that every M_p-limitable sequence is limitable with respect to the weighted Le-Roy method.

1. INTRODUCTION

Let $(X_n)_{n=1}^{\infty}$ denote a sequence of uniformly by 1 bounded real-valued random variables (r.v.) with expectation $E(X_n) = 0$. We call (X_n) $s(n)$-weakly dependent ($s(n)$-w.d.) if there exists an increasing sequence $s(n) \leq n$ of positive real numbers such that $X_{i_1}, \dots, X_{i_r}$ are independent for all $i_1 \leq n, \dots, i_r \leq n$ with $\min_{1\leq k<j\leq r} |i_j - i_k| \geq s(n)$. The special case $s(n) = 1$ yields the independence of (X_n). The main result of this note is concerned to summability properties of such sequences of r.v. Let $P = (p_n)$ denote a positive weighted mean such that $P_N = \sum_{n=1}^{N} p_n$ tends to infinity (for $N \to \infty$). In section 2 we prove some auxiliary results and obtain a general theorem on the almost certainly convergence on (X_n) in the summation method P. As a corollary we obtain for example (in the case $p_n = 1$)

$$\frac{1}{N}\sum_{n=1}^{N} X_n \to 0 \qquad \text{almost surely (a.s.)} \tag{1.1}$$

if (X_n) is n^{β}-w.d. for $0 \leq \beta < 1$.

In section 3 we generalize several recent results of P. Schatte [11] to weakly dependent r.v. These results are concerned with the so-called uniform summation with respect to weighted means $P = (p_n)$.

Let $L(k,N)$ be positive integers ($k = 0,1,2,\dots; N = 1,2,3,\dots$) defined by

$$\sum_{n=k+1}^{L(k,N)} p_n \leq P_N < \sum_{n=k+1}^{L(k,N)+1} p_n. \tag{1.2}$$

An arbitrary sequence (x_n) of real numbers is called uniformly P-limitable to the limit x (i.e.P-$\lim^* x_n = x$), if

$$\lim_{N\to\infty} \sup_{k\geq 0} \left| \frac{1}{P_N} \sum_{n=k+1}^{L(k,N)} p_n x_n - x \right| = 0. \tag{1.3}$$

We prove extensions of results related to Theorem 1,2 in [11]. We note that the well-known H_∞-summation method (of [9], [10],[2]) is equivalent to uniform P-limitation with $p_n = 1/n$. We also give a generalization of the concept of H_∞-limitation to arbitrary weighted means.

Section 4 is devoted to a limitation method related to the summation method of Le Roy.

Remark. Our concept of $s(n)$-w.d. r.v. is motivated by a suggestion of D. Knuth ([5], see 3.5), for a possible definition of pseudo-random numbers. A sequence (x_n) of numbers in $[0,1)$ is $s(n)$-uniformly distributed (or "pseudo-random") if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i (h_1 x_n + h_2 x_n + 1 + \cdots + h_{[s(N)]} x_{n+[s(N)]-1})} = 0 \tag{1.4}$$

for all integral lattice points $(h_1, \ldots, h_{[s(N)]}) \neq (0, \ldots, 0)$. It should be remarked that every $s(n)$-uniformly distributed sequence is completely uniformly distributed. The growth of $s(n)$ measures the stage of complete distribution. Note that, for every N, the sequence

$$X_{n,N} = e^{2\pi i (h_1 x_n + h_2 x_n + 1 + \cdots + h_{[s(N)]} x_{n+[s(N)]-1})}, \quad n = 1, 2, \ldots, N$$

is $s(N)$-w.d. and that the sequel proofs can easily be adapted to get metric estimates about the means

$$\frac{1}{N} \sum_{n=1}^{N} X_{n,N}.$$

2. Auxiliary Results

Our basic tool is following lemma:

Lemma 1. *Let (Y_n) be $s(n)$-w.d. with $|Y_n| \le p_n$ and let η be an arbitrary positive number. Then we have for the probability*

$$Pr\left(\left|\sum_{n=1}^{N} Y_n\right| > \eta\right) \le 2\sum_{n=1}^{s(N)} \exp\left(-\frac{\eta^2}{2\,s(N)^2 \sum_{\substack{j\le N\\ j\equiv n(s(N))}} p_j^2}\right).$$

Proof. Set $s = s(N)$. Since $(Y_{n+sj})_{j=1,2,\dots,[N/s]}$ ($[\cdot]$ denoting the integral part) are independent we have for fixed n

$$\Pr\left(\left|\sum_j Y_{n+sj}\right| > \frac{\eta}{s}\right) \le 2\,\exp\left(-\frac{\eta^2}{2\,s^2\sum_j p_{n+sj}^2}\right)$$

(cf. [8], [6]). The trivial estimate

$$\Pr\left(\left|\sum_{n=1}^{N} Y_n\right| > \eta\right) \le 2\sum_{n=1}^{s} \Pr\left(\left|\sum_j Y_{n+sj}\right| > \frac{\eta}{s}\right)$$

immediately yields the lemma.

Proposition 1. *Let $p_n = n^\alpha$, $\alpha \ge -1$, and let (X_n) be as above an $s(n)$-w.d. sequence of r.v. with $s(n) \le n^\beta$ $(0 \le \beta < 1)$. Then*

$$\left|\frac{1}{P_N}\sum_{n=1}^{N} p_n X_n\right| \le N^{-\frac{1-\beta}{2}+\varepsilon} \qquad a.s.$$

for every $\varepsilon > 0$ and all $N \ge N_0(\varepsilon)$.

Proof. First let us consider the case $\alpha = 0$. Since

$$\sum_{N=1}^{\infty} S(N)\,\exp\left(-\frac{N^{-1+\beta+2\varepsilon}N}{s(N)^2\left(\left[\frac{N}{s(N)}\right]+1\right)}\right) < \infty,$$

we obtain by Lemma 1 and the Borel—Cantelli-lemma

$$\Pr\left(\left|\frac{1}{N}\sum_{n=1}^{N} X_n\right| \le N^{-\frac{1-\beta}{2}+\varepsilon} \quad \text{for} \quad N \ge N_0\right) = 1. \tag{2.1}$$

The general case $\alpha \ge -1$ follows from (2.1) by Abel's partial summation:

$$\begin{aligned}
\left|\frac{1}{P_N}\sum_{n=1}^{N} p_n X_n\right| &= \left|\frac{N\,p_N}{P_N}\frac{1}{N}\sum_{n=1}^{N} x_n + \frac{1}{P_N}\sum_{n=1}^{N-1} n(p_n - p_{n-1})\frac{1}{n}\sum_{k=1}^{n} x_n\right| \\
&= O\left(\frac{1}{N^{\frac{1-\beta}{2}+\varepsilon}} + \frac{1}{P_N}\sum_{n=1}^{N-1} n(p_n - p_{n-1})\frac{1}{n^{\frac{1-\beta}{2}+\varepsilon}} + \frac{1}{P_N}\right) \\
&= O\left(\frac{1}{N^{\frac{1-\beta}{2}+\varepsilon}}\right),
\end{aligned}$$

and the proof is complete.

Remark. Obviously $s(n)$ has to be $\le n$. Hence the estimate of Proposition 1 works for a fairly large class of possible functions $s(n)$. By refining the above estimate one can replace the N^ε-powers by logarithmic factors. For example if $\alpha \ne -1$ and $\lim_{N\to\infty}(s(N)\,\log N)/N = 0$ then

$$\lim_{N\to\infty} \frac{1}{P_N} \sum_{n=1}^{N} p_n X_n = 0 \qquad \text{a.s.} \tag{2.2}$$

In the following we state two equivalent theorems for summation methoes; detailed proofs are given in a forthcoming paper [2] by the second author.

Proposition 2. *Let (X_n) be a bounded sequence and $P = (p_n)$ a weighted mean such that* $\lim_{N\to\infty} p_N/P_N = 0$. *Then the following limitation properties are equivalent.*

$$(i) \qquad \lim_{N\to\infty} \frac{1}{P_N} \sum_{n=1}^{N} p_n x_n = x$$

$$(ii) \qquad \lim_{s\to 0+} s \sum_{n=1}^{\infty} p_n x_n e^{-sP_n} = x$$

Remark. The limitation method (ii) is a generalized Abel method with respect to P; $p_n = 1$ is the usual Abel limitation method (cf. [3], [2]).

Next we define a generalized H_∞-limitation method H_∞^P with respect to the weighted mean $P = (p_n)$. For a given sequence x_n of reals set

$$x_{n,0} = x_n, \quad x_{n,r+1} = \frac{1}{P_n} \sum_{k=1}^{n} p_k x_{k,r}. \tag{2.3}$$

Then H_∞^P-$\lim x_n = x$, if

$$\lim_{r\to\infty} \limsup_{n\to\infty} |x_{n,r} - x| = 0. \tag{2.4}$$

Proposition 3. *Let (x_n) be a bounded sequence and $P = (p_n)$ a weighted mean such that* $\lim_{N\to\infty} p_N/P_N = 0$ *and set $Q = (q_n) = (p_n/P_n)$. Then H_∞^P-$\lim x_n = x$ if and only if Q-$\lim^* x_n = x$.*

The proof is based on technical refinements of the special case of the harmonic means (cf. [2]).

3. Uniform summation

First we establish the following general theorem:

Theorem 1. *Let (X_n) be a sequence of $s(n)$-w.d. r.v., $|X_n| \le 1$, and let $P = (p_n)$ be a non-negative weighted mean. If there exits for every $\eta > 0$ an integer $N_0(\eta)$ such that*

$$\sum_{N=N_0(\eta)}^{\infty} \sum_{k=1}^{\infty} s(L(k,N)) \exp\left(-\frac{\eta\, P_N}{s(L(k,N))^2\, p_k}\right) < \infty, \tag{3.1}$$

then

$$P - \overset{*}{\lim} X_n = 0 \qquad a.s$$

Proof. Note that because of monotony (s=s(L(k,N)))

$$\sum_{k<ns+j\le L(k,N)} p_{nk+j}^2 < \sum_{m=k+1}^{\infty} p_n^2 \le p_{k+1} P_N$$

and that

$$\Pr\left(\sup_k \left|\frac{1}{P_N}\sum_{n=k+1}^{L(k,N)} p_n X_n\right| > \eta\right) \le \sum_{k=1}^{\infty} s \exp\left(-\frac{\eta^2 P_N}{2s^2 p_k}\right)$$

by Lemma 1. Thus the proof is complete by the Borel—Cantelli-Lemma.

For the special mean (n^α), $\alpha \ge -1$, we can get refined estimates. Set

$$d_N(p_n, X_n) = \sup_{k\ge 0}\left|\frac{1}{P_N}\sum_{n=k+1}^{L(k,N)} p_n X_n\right|. \tag{3.2}$$

Theorem 2. *Let (X_n) be a sequence of n^β-w.d. r.v., $|X_n| \le 1$, and $P = (n^\alpha)$, $-1 < \alpha < 0$, a weighted mean. If $\beta < \min(-\alpha, 1+\alpha)$ and $0 < \varepsilon < 1+\alpha-\beta$ then*

$$d_N(n^\alpha, X_n) \le N^{-(1+\alpha-\beta-\varepsilon)/2} \qquad a.s.$$

for sufficiently large N.

Proof. For sufficintly large N we have for every $k > 0$

$$\sum_{k<n+sj\le L(k,N)} (n+sj)^{2\alpha} \le (k+1)^\alpha \sum_{k<sj\le L(k,N)} = O\left((k^\alpha+1)\frac{N^{1+\alpha}}{s}\right)$$

and $L(k,N) \le (k+1)N$. Thus we have by Lemma 1

$$\begin{aligned}
&\Pr\left(\left|\frac{1}{P_N}\sum_{n=k+1}^{L(k,N)} p_n X_n\right| > N^{-\frac{1+\alpha-\beta}{2}}\right)\\
&\le C_1((k+1)N)^\beta \exp\left(-C_2\frac{N^{-(1+\alpha-\beta-\varepsilon)}N^{1+\alpha}}{((k+1)N)^\beta (k+1)^\alpha}\right)\\
&= C_1((k+1)N)^\beta \exp\left(-C_2\, N^\varepsilon\, (k+1)^{-(\alpha+\beta)}\right)
\end{aligned}$$

Since

$$\sum_{N=N_0}^{\infty}\sum_{k=0}^{\infty} C_1(kN)^\beta \exp\left(-C_2\, N^\varepsilon\, (k+1)^{-(\alpha+\beta)}\right) < \infty$$

Borel—Cantelli-Lemma implies Theorem 2.

Note that Theorem 2 does not include the case $p_n = 1/n$. But with a different method we get

Theorem 3. *Let (X_n) be a sequence of n^β-w.d. r.v., $|X_n| \le 1$, and $P = (n^\alpha)$, $-1 \le \alpha < -1/2$, a weighted mean. If $\beta < -(1+2\alpha)$ then we have for $\alpha = -1$*

$$d_N(n^\alpha, X_n) \le \frac{c}{\log N} \qquad a.s$$

and for $\alpha \ge -1$

$$d_N(n^\alpha, X_n) \le \frac{c(\alpha)}{N^{1+\alpha}} \qquad a.s.$$

for sufficiently large N.

Proof. It is sufficient to prove

$$\sum_{n=1}^{N} n^\alpha X_n = O(1) \quad (N \to \infty) \quad \text{a.s.}$$

Let $\gamma = \frac{3}{4} + \frac{\alpha}{2} - \frac{\beta}{4}$. Then $\gamma - \alpha > 1$ and $\beta + 2\gamma < 1$. Since

$$\Pr\left(\left|\frac{1}{N}\sum_{n=1}^{N} x_n\right| > \frac{1}{\gamma N}\right) \le 2\,N^\beta \exp\left(-\frac{N^{-2\gamma}N}{4(N^\beta+1)}\right)$$

we get

$$\left|\sum_{n=1}^{N} X_n\right| \le N^{1-\gamma} \qquad \text{a.s.}$$

Thus we have by partial summation

$$\sum_{n=1}^{N} n^\alpha\, X_n = O\left(1 + \sum_{n=1}^{N} n^{\alpha-\gamma}\right) = O(1) \quad \text{a.s.}$$

4. The Weighted Le-Roy-Limitation Method

A series $\sum_{n=0}^{\infty} u_n$ is called summable with respect to Le-Roy-Method with limit u (see [12]) if

$$\lim_{\lambda \to 1-} \sum_{n=0}^{\infty} \frac{\Gamma(1+\lambda n)}{\Gamma(1+n)}\, u_n = u. \tag{4.1}$$

In this section we discuss a related limitation method. A sequence (x_n) is called limitable to Le-Roy (for short LR-lim $x_n = x$) if

$$\lim_{\lambda \to 1-} \frac{\sum_{n=0}^{\infty} \frac{\Gamma(1+\lambda n)}{\Gamma(1+n)}\, x_n}{\sum_{n=0}^{\infty} \frac{\Gamma(1+\lambda n)}{\Gamma(1+n)}} = x. \tag{4.2}$$

If $P = (p_n)$ $(n \ge 0)$ is a weighted mean then it is also possible to introdue a weighted Le-Roy limitation method by the definition

$$\lim_{\lambda \to 1-} \frac{\sum_{n=0}^{\infty} p_n\, \frac{\Gamma(1+\lambda P_n)}{\Gamma(1+P_n)}\, x_n}{\sum_{n=0}^{\infty} p_n\, \frac{\Gamma(1+\lambda P_n)}{\Gamma(1+P_n)}} = x \tag{4.3}$$

or for short LRP-lim $x_n = x$. It is obvious now to define a weighted Le-Roy-limitation method for a sequence (X_n) of r.v.

First we want to discuss the sum

$$S(\lambda, P) = \sum_{n=0}^{\infty} p_n \frac{\Gamma(1+\lambda P_n)}{\Gamma(1+P_n)}. \tag{4.4}$$

We have

$$\begin{aligned} S(\lambda, P) &= \sum_{n=0}^{\infty} \frac{p_n}{\Gamma(1+P_n)} \int_0^\infty e^{-t}\, t^{\lambda P_n}\, dt \\ &= \int_0^\infty \sum_{n=0}^{\infty} \frac{p_n}{\Gamma(1+P_n)}\, t^{\lambda P_n}\, e^{-t}\, dt. \end{aligned} \tag{4.5}$$

For $P = (p_n) = (1)$ this integral simplifies to

$$S(\lambda, P) = S(\lambda) = \int_0^\infty e^{t^\lambda - t}\, dt. \tag{4.6}$$

This integral can be evaluated asymptotically.

Lemma 2. *For $\lambda \to 1-$ we have*

$$\int_0^\infty e^{t^\lambda - t}\, dt = \frac{1}{1-\lambda} \frac{1}{\log \frac{1}{1-\lambda}} + O\left(\frac{1}{1-\lambda} \left(\frac{\log\log \frac{1}{1-a}}{\log \frac{1}{1-\lambda}} \right)^2 \right). \tag{4.7}$$

Proof. Substituting $u = 1/(1-\lambda)$ we get

$$S(\lambda) = \int_0^\infty e^{-x(1-x^{-1/u})}\, dx.$$

Now it is easy to see that

$$\left(\int_0^{\frac{u}{\log^2 u}} + \int_{3\frac{u}{\log u}\log\log u}^{\infty} \right) e^{-x(1-x^{-1/u})}\, dx = O\left(\frac{u}{\log^2 u} \right)$$

and that

$$e^{-x(1-x^{-1/u})} = e^{-\frac{x\log x}{u}} \left(1 + O\left(\frac{\log u\, \log\log u}{u} \right) \right)$$

in the remaining interval $[u/\log^2 u, 3u \log\log u/\log u]$. Since

$$\int_{\frac{u}{\log^2 u}}^{3\frac{u}{\log u}\log\log u} e^{-\frac{x\log x}{u}} = \frac{u}{\log u} \int_{1/\log u}^{3\log\log u} e^{-y\left(1+\frac{\log(y/\log u)}{\log u}\right)},$$

$$e^{-y\left(1+\frac{\log(y/\log u)}{\log u}\right)} = e^{-y} \left(1 + O\left(\frac{(\log\log u)^2}{\log u} \right) \right),$$

and

$$\int_{1/\log u}^{3\log\log u} e^{-y} = 1 + O\left(\frac{1}{\log u} \right)$$

$S(\lambda)$ can evaluated asymptotically by

$$\begin{aligned} S(\lambda) &= \frac{u}{\log u}\left(1+O\left(\frac{1}{\log u}\right)\right)\left(1+O\left(\frac{(\log\log u)^2}{\log u}\right)\right)\times \\ &\quad\times\left(1+O\left(\frac{\log u\,\log\log u}{u}\right)\right)+O\left(\frac{u}{\log^2 u}\right) \\ &= \frac{u}{\log u}\left(1+O\left(\frac{(\log\log u)^2}{\log u}\right)\right), \end{aligned}$$

which implies Lemma 2.

In general we have

Proposition 4. *If $P=(1)$ or if $P=(p_n)$ with $p_n = o(1/\log P_n)$ then*

$$S(\lambda, P) \sim \frac{1}{1-\lambda}\frac{1}{\log\frac{1}{1-\lambda}} \qquad (\lambda\to 1-). \tag{4.8}$$

Proposition 4 is a direct consequence of the following lemmata.

Lemma 3. *If $p_n = o(1/\log P_n)$ or if $\lim_{n\to\infty} p_n = c > 0$ then*

$$\sum_{n=0}^{\infty}\frac{p_n}{\Gamma(1+P_n)}\, y^{P_n} \sim \int_1^\infty \frac{y^t}{\Gamma(1+t)}\,dt \qquad (y\to\infty). \tag{4.9}$$

Lemma 4. *For $y\ge 1$ we have uniformly*

$$\int_1^\infty \frac{y^t}{\Gamma(1+t)}\,dt = e^y + O\left(\frac{y}{\log(y+1)}\right). \tag{4.10}$$

Proof of Lemma 3. Set $f_y(t) = y^t/\Gamma(1+t)$ and let $t(y)$ chosen such that $f_y(t) = \max_{t\ge 0} f_y(t)$. Using the asymptotic expansion for $\Gamma(1+t)$ and $\Gamma'(1+t)$ it is easy to show that $t(y)\sim y$, $(y\to\infty)$ and that $f_y(t(y)) = o(e^y)$, $(y\to\infty)$. Therefore we have

$$\int_1^\infty f_y(t)\,dt \le \sum_{P_n\le t(y)} p_n f(P_n) + \sum_{P_n\ge t(y)} p_n f(P_{n-1}) + o(e^y).$$

In the fist case $p_n = o(1/\log P_n)$ implies that that

$$f(P_{n-1}) = f(P_n)(1+o(1))$$

and in the second case we have

$$p_n = p_{n-1}(1+o(1)).$$

Thus

$$\int_1^\infty f_y(t)\,dt \le \sum_{n=0}^\infty p_n f_y(P_n)\cdot(1+o(1)) \qquad (y\to\infty).$$

On the other hand we get

$$\int_1^\infty f_y(t)\,dt \geq \sum_{P_n \leq t(y)} p_n f(P_{n-1}) + \sum_{P_n \geq t(y)} p_n f(P_n) + o(e^y),$$

where we have to use

$$f(P_{n-1}) = f(P_n)(1 - O(p_n \log y)) \qquad (P_n \leq t(y))$$

and $p_n = o(1/\log P_n)$ to get

$$\sum_{P_n \leq t(y)} p_n^2 f(P_n) = o\left(\frac{e^y}{\log y}\right) \qquad (y \to \infty),$$

which completes the first case. The second case is trivial again. This proves Lemma 3.

Proof of Lemma 4. As in the proof of Lemma 3 it can be proved that

$$F(y) = \int_1^\infty \frac{y^t}{\Gamma(1+t)}\,dt \sim \sum_{n=0}^\infty \frac{y^n}{n!} = e^y \quad (y \to \infty).$$

But in addition we have

$$\begin{aligned} F'(y) &= \int_1^\infty \frac{t\,y^{t-1}}{\Gamma(1+t)}\,dt = \int_1^\infty \frac{y^{t-1}}{\Gamma(1+t-1)}\,dt \\ &= \int_0^1 y^t\,dt + F(y). \end{aligned}$$

Since

$$G(y) = \int_0^1 \frac{y^t}{\Gamma(1+t)}\,dt = O\left(\int_0^1 y^t\,dt\right) = O\left(\frac{y-1}{\log y}\right),$$

a solution of the differential equation $F' - F = G$ has the form

$$F(y) = C\,e^y - e^y \int_y^\infty e^{-\eta} G(\eta)\,d\eta = C\,e^y + O\left(\frac{y}{\log y}\right),$$

but in this case C must be equal to 1.

Therefore we can reformulate the definition of the weighted Le-Roy-limitation method in the important cases $p_n = o(1/\log P_n)$ and $\lim_{n\to\infty} p_n = c > 0$ to

$$\lim_{\lambda \to 1-} (1-\lambda)\,\log\frac{1}{1-\lambda} \sum_{n=0}^\infty p_n \frac{\Gamma(1+\lambda P_n)}{\Gamma(1+P_n)}\,x_n = x. \tag{4.11}$$

The reason why we have considered the weighted Le-Roy-limitation method is the following

Theorem 4. *If P-$\lim x_n = x$ then LRP-$\lim x_n = x$, too.*

Therefore all metric theorems concerning the limitability of weakly dependent r.v. are valid for the Le-Roy-limitation method.

Proof of Theorem 4. Since

$$\sum_{m=0}^{M} \frac{\Gamma(1-\lambda P_m)}{\Gamma(1+P_m)} p_m x_m = \frac{\Gamma(1+\lambda P_M)\, P_M}{\Gamma(1+P_M)} \frac{1}{P_M} \sum_{m=0}^{M} p_m x_m$$
$$+ \sum_{m=0}^{M-1} P_m \left(\frac{\Gamma(1-\lambda P_m)}{\Gamma(1+P_m)} - \frac{\Gamma(1-\lambda P_{m+1})}{\Gamma(1+P_{m+1})} \right) \frac{1}{P_m} \sum_{\mu=0}^{m} p_\mu x_\mu$$

it sufficies to show that

$$\lim_{M\to\infty} \frac{\Gamma(1+\lambda P_M)\, P_M}{\Gamma(1+P_M)} = 0 \qquad (\lambda < 1)$$

which is trivial and that the continuous matrix summation method $(a_{\lambda m})$ with

$$a_{\lambda m} = \frac{P_m}{S(\lambda, P)} \left(\frac{\Gamma(1-\lambda P_m)}{\Gamma(1+P_m)} - \frac{\Gamma(1-\lambda P_{m+1})}{\Gamma(1+P_{m+1})} \right)$$

is regular. It is easy to see that $\frac{d^2}{dx^2} \log \Gamma(1+x) > 0$ $(x \geq 1)$ implies that

$$\frac{d}{dx} \left(\frac{\Gamma(1+\lambda x)}{\Gamma(1+x)} \right) < 0 \quad (1 > \lambda > \lambda_0, x \geq x_0)$$

and therefore $a_{\lambda m} > 0$ for $1 > \lambda > \lambda_0$ and $m \geq m_0$. Using the fact that $\lim_{\lambda\to 1-} a_{\lambda m} = 0$ for all m an application of the continuous analogon of Toeplitz's theorem completes the proof.

Final Remark. The limitation method of Le-Roy can also be applied in the theory of uniform distribution. A sequence $x_n \in [0,1)$ is called uniformly distributed with respect to Le-Roy if

$$\lim_{\lambda\to 1-} (1-\lambda) \log \frac{1}{1-\lambda} \sum_{n=0}^{\infty} \frac{\Gamma(1+\lambda n)}{\Gamma(1+n)} 1_{[0,\alpha)}(x_n) = \alpha \tag{4.12}$$

for all subintervals $[0,\alpha)$ of $[0,1)$; 1_I denotes the characteristic function of I. (For a general survey of the theory of uniformly distributed sequences see [4], [6].) For instance our Theorem 4 yields that almost all sequences are Le-Roy-uniformly distributed.

As a quantitative measure of the distribution behaviour of a sequence the so-called Le-Roy discrepancy $D_\lambda^{LR}(x_n)$ can be introduced by

$$D_\lambda^{LR}(x_n) = \sup_{0\leq\alpha\leq 1} \left| (1-\lambda) \log \frac{1}{1-\lambda} \sum_{n=0}^{\infty} \frac{\Gamma(1+\lambda n)}{\Gamma(1+n)} 1_{[0,\alpha)}(x_n) - \alpha \right|. \tag{4.13}$$

As in the classical case, (x_n) is Le-Roy uniformly distributed if and only if

$$\lim_{\lambda\to 1-} D_\lambda^{LR}(x_n) = 0.$$

The Le-Roy discrepancy can be estimated in terms of exponential sums (cf. [1]):

$$D_\lambda^{LR}(x_n) \le \frac{4}{m+1} + \frac{4}{\pi}(1-\lambda)\log\frac{1}{1-\lambda}\left|\sum_{n=0}^{\infty}\frac{\Gamma(1+\lambda n)}{\Gamma(1+n)}\,e^{2\pi i h x_n}\right|. \tag{4.14}$$

This is a direct generalization of the well known Erdös—Turan-inequality (cf. [7]). This inequality yields to estimates for the discrepancy of special sequences: For example let α be an irrational of approximation type $\eta = 1$, then for every $\varepsilon > 0$:

$$D_\lambda^{LR}(n\alpha) = O\left((1-\lambda)\left(\log\frac{1}{1-\lambda}\right)^{1-\varepsilon}\right). \tag{4.15}$$

All these results can be generalized to the weighted Le-Roy method.

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