CONTRIBUTIONS TO UNIFORMLY DISTRIBUTED FUNCTIONS II: COMPLETELY UNIFORMLY DISTRIBUTED FUNCTIONS

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Dedicated to Prof. Dr. E. Hlawka on the occasion of his 75^{th} birthday.

Abstract. Completely uniformly distributed (c.u.d.) functions can be defined similarly to c.u.d. sequences but there properties are significantly different. In the first part of this paper it is shown that special entire functions are c.u.d. and well distributed. Furthermore polynomials and the exponential function are discussed. In a final sections it is proved that almost no functions are c.u.d. with respect to Wiener measure but it is possible to find explicitely a distribution function different from the uniform distribution such that almost all functions are completely distributed with respect to these distribution functions.

1. Introduction

A measurable function $x:[0,\infty)\to\mathbf{R}^K$ has distribution function $F(y)$ $(y=(y_0,\dots,y_{K-1})\in I^K=[0,1]^K)$ modulo 1 if

$$\lim_{T\to\infty} F_T(y,x(t)) = F(y) \quad (y\in I^K), \tag{1}$$

where $F_T(y,x(t))$ is the empirical distribution function

$$F_T(y,x(t)) = \frac{1}{T}\int_0^T 1_{[0,y]}(\{x(t)\})\,dt \tag{2}$$

of the fractional part $\{x(t)\}$ of $x(t)$. ($\{z\}=z-[z]$, $[z]=\max\{n\in\mathbf{Z}:\ n\le z\}$, and $1_{[0,y]}$ is the characteristic function of the interval $[0,y]=\prod_{k=0}^{K-1}[0,y_k]$.) If $F(y)=\prod_{k=0}^{K-1}y_k$ and (1) holds then $x(t)$ is called *uniformly distributed modulo 1* (u.d.).

A measure for the convergence in (1) is the discrepancy

$$D_T(x(t),F) = \sup_{y\in I^K} |F_T(y,x(t)) - F(y)| \tag{3}$$

and it is easy to show that $x(t)$ has distribution function $F(y)$ (F-d.) if and only if

$$\lim_{T\to\infty} D_T(x(t),F) = 0. \tag{4}$$

The distribution funtion can be interpreted as a "measure for the dependence" of the components $(x_0(t),\dots,x_{K-1}(t))$ of $x(t)$. Of course a distribution fucntion need not exist. But it has been shown ([Hl],[Fl]) that almost all functions $x:[0,\infty)\to\mathbf{R}^K$ are u.d. (in the sense of Wiener measure).

Now let $x(t)$ be a one-dimensional function $x:[0,\infty)\to\mathbf{R}$. The object of this paper is to discuss the distribution behaviour of shifts of this function, namely the distribution of

$$x^{\mathcal{T}_K}(t) = (x(t), x(t+\tau_1),\dots,x(t+\tau_{K-1})), \tag{5}$$

where $\mathcal{T} = (\tau_k)_{k=1}^{\infty}$ is a given strictly monotone shifting sequence with $\tau_1 > 0$ and $\lim_{k\to\infty} \tau_k = \infty$. If there exist distribution funcions F_K such that $x^{\mathcal{T}_K}(t)$ is F_K-d. for all $K \geq 1$ then $x(t)$ is called *completely F_K-distributed with respect to $\mathcal{T}$* ($\mathcal{T}$-c.-F_K-d.) and if $x^{\mathcal{T}_K}(t)$ is u.d. for all $K \geq 1$ then $x(t)$ is called *completely uniformly distributed with respect to $\mathcal{T}$* ($\mathcal{T}$-c.u.d.). If $x(t)$ is $\mathcal{T}$-c.u.d. for all possible shifting sequences $\mathcal{T} = (\tau_k)_{k=1}^{\infty}$ then we will call $x(t)$ *completely uniformly distributed* (c.u.d.).

In Section 2 it is shown that there exist c.u.d. functions and some special exmples are discussed. In section 3 it is proved that almost no functions are $\mathcal{T}$-c.u.d. but for every shifting sequence $\mathcal{T} = (\tau_k)_{k=1}^{\infty}$ there exist distribution functions F_K such that almost all functions are $\mathcal{T}$-c.-F_K-d. (In fact a little bit more general result will be shown.)

It should be noticed that this concept of c.u.d. functions is similar to that known for sequences. A real sequence $(x_n)_{n=1}^{\infty}$ is c.u.d. mod 1 if the K-dimensional sequences $(x_n, x_{n+1}, \dots, x_{n+K-1})$ are u.d. mod 1 for all $K \geq 1$. It is interesting that the results for functions are significantly different from those known for sequencen (see [KN]).

2. Basic Results

First we will prove

Theorem 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function but not a polynomial with real Taylor coefficients a_n and*

$$\limsup_{R\to\infty} \frac{\log\log M(R)}{\log\log R} < \frac{3}{2}, \tag{6}$$

where $M(R) = \max_{|z|\leq R} |f(z)|$. Then $x(t) = f(t)$ is c.u.d.

Proof. By Weyl's criterion ([KN]) a function $x(t)$ is c.u.d. if and only if the one-dimensional functions

$$h_0 x(t) + h_1 x(t+\tau_1) + \cdots + h_{K-1} x(t+\tau_{K-1}) \tag{7}$$

are u.d. for all shifts τ_k and integral lattice points $h = (h_0, \dots, h_{K-1}) \in \mathbf{Z}^K \setminus \{(0, \dots, 0)\}$. But

$$g(z) = \sum_{k=0}^{K-1} h_k f(z+\tau_k) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{K-1} h_k \frac{f^{(n)}(\tau_k)}{n!} \right) z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{K-1} h_k \tau_k^n \right) \frac{f^{(n)}(z)}{n!} \tag{8}$$

is again a fucntion of type (6). Furthermore $g(z)$ is not constant because $\sum_{k=0}^{K-1} h_k \tau_k^n \neq 0$ for almost all n and $f^{(n)}(z) \neq$ *const.* for all n since $f(z)$ is not a polynomial. Thus by [DT] the function $y(t) = g(t)$ is u.d. modulo 1.

Theorem 1 has an interesting corollary, for it implies that there are c.u.d. functions that are well distributed, too. A function $x : [0,\infty) \to \mathbf{R}$ is well distributed if

$$\lim_{T\to\infty} \sup_{\tau\geq 0} D_T \left(x(t+\tau), \prod_{k=0}^{K-1} y_k \right) = 0. \tag{9}$$

It is known that a convex fucntion $x(t)$ is always well distributed [Dr]. Thus we have as a

Corollary. *Let $f(z)$ be an entire function as in Theorem 1 with non-negative Taylor coefficients a_n. Then $x(t) = f(t)$ is c.u.d. and well distributed.*

This corollary is interesting insofar because c.u.d. sequences and well distributed sequences are disjoint ([KN]).

Next we discuss polynomials and the exponential function.

Theorem 2. *Let $x(t) = p_N(t)$ be a polynomial of degree N. Then $x(t)$ is not c.u.d. But for every shifting sequence $\mathcal{T} = (\tau_k)_{k=1}^\infty$ $x(t)$ is $\mathcal{T}$-c.-F_K-d. for proper continuous distribution functions F_K. For example, if $N \geq 2$ and the numbers τ_k are linear independent over the rationals then $x(t)$ is $\mathcal{T}$-c.u.d.*

Proof. If $\tau_k = K$ then the $N+2$ functions $x(t), x(t+1), \ldots, x(t+N+1)$ are linear dependent over $\mathbf{Z}$ since the system of linear equations

$$\sum_{k=0}^{N+1} h_k k^n \quad (n = 0, 1, \ldots N+1) \tag{10}$$

has integral solutions different from $(0, \ldots, 0)$. Therefore $x(t)$ is not c.u.d. Now let $\mathcal{T} = (\tau_k)_{k=1}^\infty$ be any shifting sequence. If $x^{\mathcal{T}_K}(t)$ has a distribution function F_K then $(e(x) = e^{2\pi i x})$

$$\begin{aligned} c_h &= \int_{I^K} e\left(\sum_{k=0}^{K-1} h_k y_k\right) dF_K(y) = \lim_{T\to\infty} \frac{1}{T} \int_0^T e\left(\sum_{k=0}^{K-1} h_k x(t+\tau_k)\right) dt \\ &= \begin{cases} e\left(a_N \sum_{k=0}^{K-1} h_k \tau_k^N\right) & \text{if } \sum_{k=0}^{K-1} h_k \tau_k^n = 0 \ (n = 0, \ldots, N-1) \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{11}$$

By Wiener-Schoenberg-theorem it sufficies to show that

$$\lim_{H\to\infty} \frac{1}{H^K} \sum_{||h||_\infty \leq H} |c_h|^2 = 0. \tag{12}$$

But this sum can be estimated by $O(H^{-N-1})$. Therefore there exists a continuous distribution function F_K.

If the numbers τ_k are linear independent over the rationals then $c_h = 0$ if and only if $h = (0, \ldots, 0)$. So $x(t)$ is $\mathcal{T}$-c.u.d. in this case

The exponential function can be treated in a similar way. You only have to discuss the linear combinations

$$\sum_{k=0}^{K-1} h_k x(t+\tau_k) = e^t \sum_{k=0}^{K-1} h_k e^{\tau_k} \tag{13}$$

to get

Theorem 3. *Let $x(t) = e^t$. Then $x(t)$ is not c.u.d. If $\mathcal{T} = (\tau_k)_{k=1}^\infty$ is any shifting sequence then $x(t)$ is $\mathcal{T}$-c.-F_K-d. for proper continuous distribution functions F_K and is $\mathcal{T}$-c.u.d. if and only if the numbers e^{τ_k} are linear independent over the rationals.*

Remark. It is not very difficult to get asymptotic extimates $(T \to \infty)$ for the discrepancy for every K in these examples but it seems to be very difficult to get uniform estimates in K and T.

3. Metric Results

We want to discuss metric properties with respect to Wiener measure μ_W that is defined on proper subsets of the space C of continuous functions $x : [0, \infty) \to \mathbf{R}$ with $x(0) = 0$. The transition densities of the stochastic process that is related to Wiener measure is given by

$$p(t; x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/(2t)}. \tag{14}$$

If $f : \mathbf{R}^K \to \mathbf{C}$ is a measurable function and $0 < t_1 < \cdots < t_K$ then

$$\int_C f(x(t_1), \ldots, x(t_K))\, d\mu_W(x) = \int_{\mathbf{R}^K} f(x_1, \ldots, x_K) \prod_{k=1}^{K} p(t_k - t_{k-1}; x_{k-1}, x_k)\, dx_1 \cdots dx_K. \tag{15}$$

$(t_0 = 0,\, x_0 = 0)$. So, if $(x(t), x(t+\tau_1), \ldots, x(t+\tau_{K-1}))$ has a distribution function F_K almost surely (a.s.) then the Fourier coefficients of the density function f_K can be evaluated by

$$\int_C \lim_{T\to\infty} \frac{1}{T} \int_0^T e\left(h_0 x(t) + \cdots + h_{K-1} x(t+\tau_{K-1})\right) dt\, d\mu_W(x) = \begin{cases} e^{-2\pi^2\left[(h_1+\cdots+h_{K-1})^2\tau_1 + (h_2+\cdots+h_{K-1})^2(\tau_2-\tau_1) + \cdots + h_{K-1}^2(\tau_{K-1}-\tau_{K-2})\right]} \\ \qquad\qquad \text{if } \; h_0 + \cdots + h_{K-1} = 0 \\ 0 \qquad\qquad \text{otherwise.} \end{cases} \tag{16}$$

So it might be expected that for almost all functions $x(t)$ the distributions of $(x(t), x(t+\tau_1), \ldots, x(t+\tau_{K-1}))$ have density functions

$$\begin{aligned} f_K(y_0, \ldots, y_{K-1}) &= \sum_{h_1,\ldots,h_{K-1}\in\mathbf{Z}} e^{-2\pi^2\left[\sum\limits_{k=1}^{K-1}\left(\sum\limits_{i=k}^{K-1} h_i\right)^2 (\tau_k - \tau_{k-1})\right]} e\left(\sum_{k=1}^{K-1} h_k (y_k - y_0)\right) \\ &= \sum_{l_1,\ldots,l_{K-1}\in\mathbf{Z}} \prod_{k=1}^{K-1} p(\tau_k - \tau_{k-1}; y_0, y_k + l_k) \end{aligned} \tag{17}$$

First we state a law of the iterated logarithm without proof since it is not very difficult, uses standard methods, but is rather technical (see [BDT]).

Theorem 4. *Let $\mathcal{T} = (\tau_k)_{k=1}^\infty$ be a fixed shifting sequence and F_K distribution functions with density (17). Then, for every $K > 0$ there exist positive constants $c_{\mathcal{T}_K} > 0$ such that*

$$\limsup_{T\to\infty} \frac{T\, D_T\left(x^{\mathcal{T}_K}(t), F_K\right)}{\sqrt{2T \log\log T}} = c_{\mathcal{T}_K} \quad a.s. \tag{18}$$

Corollary. *Let $\mathcal{T} = (\tau_k)_{k=1}^\infty$ and F_K be as in Theorem 4. Then almost all functions $x(t)$ are $\mathcal{T}$-c.-F_K-d. and therefore almost no functions are $\mathcal{T}$-c.u.d.*

We will prove this corollary in a little bit more general context. Let $K(T)$ be a positive monotone function with $\lim_{T\to\infty} K(T) = \infty$ and $\mathcal{T} = (\tau_k)_{k=1}^\infty$ a shifting sequence. Then a function $x(t)$ is called $K(T)$-$\mathcal{T}$-c.-F_K-d. if

$$\lim_{T\to\infty} D_T\left(x^{\mathcal{T}_{[K(T)]}}(t), F_{[K(T)]}\right) = 0. \tag{19}$$

Trivially every $K(T)$-$\mathcal{T}$-c.-F_K-d. function is $\mathcal{T}$-c.-F_K-d. and on the other hand if $x(t)$ is $\mathcal{T}$-c.-F_K-d. then there exists a proper function $K(T)$ such that $x(t)$ is $K(T)$-$\mathcal{T}$-c.-F_K-d. Next we will prove

Theorem 5. *Let $\mathcal{T}=(\tau_k)_{k=1}^{\infty}$ be a shifting sequence satisfying*

$$\tau_k \le k^{\gamma k^2} \quad \text{and} \quad \tau_k-\tau_{k-1}\ge k^{-\sigma} \qquad (k\ge k_0) \tag{20}$$

with $\gamma,\sigma\ge 0$ and $K(T)$ a positive monotone function with

$$K(T)\le \delta\sqrt{\frac{\log T}{\log\log T}} \qquad (T\ge T_0) \tag{21}$$

such that $\delta^2(1+\gamma/2+\sigma/4)<1$. Then almost all functions $x(t)$ are $K(T)$-$\mathcal{T}$-c.-F_K-d. with respect to Wiener measure and density functions f_k defined in (17).

Proof. Set

$$S_{T,h}(x)=\frac{1}{T}\int_0^T e\left(\sum_{k=0}^{K-1} h_k x(t+\tau_k)\right)dt. \tag{22}$$

Then

$$\begin{aligned}
&\int_C \left|S_{T,h}(x)\right|^2 \, d\mu_W(x)\\
&=\frac{2}{T^2}\int_0^T\int_0^t\int_C e\left(\sum_{k=0}^{K-1} h_k(x(t+\tau_k)-x(s+\tau_k))\right)d\mu_W(x)\,ds\,dt\\
&=\frac{2}{T^2}\int_{\tau_{K-1}}^T\int_0^{t-\tau_{K-1}}\int_C e\left(\sum_{k=0}^{K-1} h_k(x(t+\tau_k)-x(s+\tau_k))\right)d\mu_W(x)\,ds\,dt+O\left(\frac{\tau_{K-1}}{T}\right)\\
&=\begin{cases}
\dfrac{2}{T^2}e^{-2\pi^2\sum\limits_{k=1}^{K-1}(\tau_k-\tau_{k-1})\left(\left(\sum\limits_{i=0}^{k-1}h_i\right)^2\left(\sum\limits_{j=k}^{K-1}h_j\right)^2\right)}\times\\
\quad\times\left[\dfrac{T}{2\pi^2\left(\sum\limits_{k=0}^{K-1}h_k\right)^2}-\dfrac{1-e^{-2\pi^2T\left(\sum\limits_{k=0}^{K-1}h_k\right)^2}}{4\pi^4\left(\sum\limits_{k=0}^{K-1}h_k\right)^4}\right]+O\left(\frac{\tau_{K-1}}{T}\right) & \text{if } \sum\limits_{k=0}^{K-1}h_k\neq 0\\
c_h^2+O\left(\frac{\tau_{K-1}}{T}\right) & \text{if } \sum\limits_{k=0}^{K-1}h_k=0.
\end{cases}
\end{aligned} \tag{23}$$

So we have in general

$$\int_C \left|S_{T,h}(x)-c_h\right|^2 \, d\mu_W(x)=O\left(\frac{\tau_{K-1}}{T}\right) \tag{24}$$

which implies that

$$\mu_W\left(\{x\in C:\ |S_{T,h}(x)-c_h|\ge\eta\}\right)=O\left(\frac{\tau_{K-1}}{\eta^2 T}\right). \tag{25}$$

Using a generalized Erdös-Turan-inequality ([Gr])

$$D_T(x^{\mathcal{T}_K}(t),F_K)\le\left(\frac{3}{2}\right)^K\left(\frac{2||f_K||_\infty}{H}+\sum_{0<||h||_\infty\le H}\frac{1}{r(h)}\left|S_{T,h}-c_h\right|\right) \tag{26}$$

($r(h) = \prod_{k=0}^{K-1} \max(1, |h_k|)$) and the Borel-Cantelli-lemma it sufficies to find sequences T_i, η_i, H_i ($i \ge i_0$) such that

$$\begin{aligned}
&(i) \quad \lim_{i\to\infty} \frac{T_{i+1}}{T_i} = 1 \\
&(ii) \quad \lim_{i\to\infty} \left(\frac{3}{2}\right)^{K(T_i)} \eta_i \left(\log H_i\right)^{K(T_i)} = 0 \\
&(iii) \quad \lim_{i\to\infty} \left(\frac{3}{2}\right)^{K(T_i)} \frac{1}{H_i} \prod_{k=1}^{K(T_i)} \left(1 + \frac{1}{\sqrt{2\pi(\tau_k - \tau_{k-1})}}\right) = 0 \\
&(iv) \quad \sum_{i\ge i_0} \frac{\tau_{K(T_i)-1}}{\eta_i^{2+K(T_i)} T_i} \left(\log \frac{1}{\eta_i}\right)^{K(T_i)^2} < \infty.
\end{aligned} \tag{27}$$

Namely, (ii), (iii), and (iv) imply that almost all functions satisfy

$$\lim_{i\to\infty} D_{T_i}\left(x^{\mathcal{T}_{[K(T_i)]}}(t), F_{[K(T_i)]}\right) = 0. \tag{28}$$

Now (28) and (i) ensure that (19) holds a.s.

For the purpose to construct such sequences choose some $\beta > \delta/2 + \delta\sigma/4$ such that $\gamma\delta^2/2 + \delta^2/2 + \beta\delta = 1 - \varepsilon$ for some $\varepsilon > 0$ and set $T_i = i^{2/\varepsilon}$, $\eta_i = e^{-\beta\sqrt{\log T_i \log\log T_i}}$, and $H_i = (\eta_i(\log 1/\eta_i)^{K(T_i)})^{-1}$. Then it is easy to show that the four conditions of (27) are satisfied. So the proof of Theorem 5 is complete.

Remark. The corollary of Theorem 4 is not a direct consequence of Theorem 5 but it is easy to derive it from the preceding proof.

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