SUBSEQUENCES OF AUTOMATIC SEQUENCES INDEXED BY $\lfloor n^c \rfloor$ AND CORRELATIONS

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Abstract. The main goal of this paper is to study the behavior of subsequences $\mathbf{u}_c = \{\mathbf{u}(\lfloor n^c \rfloor) : n \in \mathbb{N}\}$ of automatic sequences $\mathbf{u}$ that are indexed by $[n^c]$ for some $c > 1$. In particular we show that the densities of the letters of $\mathbf{u}_c$ are precisely the same as those of the original sequence (provided that $c < 7/5$). In this sense $\mathbf{u}_c$ and $\mathbf{u}$ behave in the same way. However, the pair correlation might be completely different as we will show in the special case of the Thue-Morse sequence. The proofs use exponential sum estimates like the double large sieve and a discrete Fourier analysis related to automatic sequences.

1. Introduction and main results

In this paper we consider relations between a sequence $\mathbf{u}$ with values in a finite set E and its subsequences of the shape $\{\mathbf{u}(\lfloor n^c \rfloor) : n \in \mathbb{N}\}$, as well as some correlations between such subsequences.

For a sequence $\mathbf{v}$ with values in E and for $a \in E$, we say that a is observed with the asymptotic (*resp.* logarithmic) density α if the quantity

$$\mathrm{dens}(\mathbf{v}, a) = \lim_{x\to\infty} \frac{1}{x} \#\{1 \leqslant n \leqslant x : \mathbf{v}(n) = a\} \tag{1}$$

resp.

$$\text{log-dens}(\mathbf{v}, a) = \lim_{x\to\infty} \frac{1}{\log n} \sum_{\substack{1\leqslant n \leqslant x \\ \mathbf{v}(n)=a}} \frac{1}{n} \tag{2}$$

exists and is equal to α. In other words, a is observed with the asymptotic (*resp.* logarithmic) density α if the sequence of these integers n for which $\mathbf{v}(n) = a$ has asymptotic (*resp.* logarithmic) density α.

We first compare the existence of the density with which a given value $a \in E$ is observed in $\mathbf{u}$ and in

$$\mathbf{u}_c := \{\mathbf{u}(\lfloor n^c \rfloor : n \in \mathbb{N}\}. \tag{3}$$

Since the sequence of integers $(\lfloor n^c \rfloor)_{n\in\mathbb{N}}$ has density zero, some "rigidity" is needed to be able to compare the two densities $\mathrm{dens}(\mathbf{u}, a)$ and $\mathrm{dens}(\mathbf{u}_c, a)$. One possibility is to consider

The first author was supported by the Agence Nationale de la Recherche, grant ANR-10-BLAN 0103 MUNUM.

The second and the third author were supported by the Austrian Science Foundation FWF, grant S9604, that is part of the National Research Network "Analytic Combinatorics and Probabilistic Number Theory".

the question for a family of values of c; in this vein, Harman and Rivat [7, Theorem 3] showed that if $\text{dens}(\mathbf{u}, a)$ exists, then $\text{dens}(\mathbf{u}_c, a)$ exists for almost all c in $(1, 2)$ in the sense of Lebesgue and is equal to $\text{dens}(\mathbf{u}, a)$.

For specific values of c, Mauduit and Rivat proved in [11, 12] the following result for the sum-of-digits function s_q in base q: For $m \geqslant 2$ and for any c in $(1, 7/5)$, the density with which a residue a modulo m is observed in the sequence $(s_q(\lfloor n^c \rfloor))_{n\in\mathbb{N}}$ exists and is equal to $1/m$. (Compare with [16], where the same result is shown for all $c \in \mathbb{R} \setminus \mathbb{N}$ provided that the base q is large enough (depending on c).)

We consider here the case when the sequence $\mathbf{u}$ is q-automatic; in this introduction, let us simply say that there exists a finite machine that produces the values $\mathbf{u}(n)$ by sequentially reading the digits of the integer n in base q. Thanks to a classical result of Cobham (cf. [3] or [1, Chapter 8]), when $\mathbf{u}$ is a q-automatic sequence with values in E, then, for any $a \in E$, the quantity $\text{log-dens}(\mathbf{u}, a)$ always exists; one should however notice that the quantity $\text{dens}(\mathbf{u}, a)$ does not always exist: consider for example the sequence which associates to n its most significant digit in base 10.

The Mauduit-Rivat above-mentioned result can be generalized in the following (here we use the notation (1)–(3)):

Theorem 1. *Let $q \geqslant 2$, $\mathbf{u}$ be a q-automatic sequence with values in a finite set E and $c \in (1, 7/5)$; let $a \in E$.*

(1) *The quantity* $\text{log-dens}(\mathbf{u}_c, a)$ *exists and is equal to* $\text{log-dens}(\mathbf{u}, a)$.
(2) *The quantity* $\text{dens}(\mathbf{u}_c, a)$ *exists if and only if* $\text{dens}(\mathbf{u}, a)$ *exists, and in this case, they are equal.*

For integer valued real numbers $c > 1$, it need not be the case any more, that the quantities $\text{dens}(\mathbf{u}_c, a)$ and $\text{dens}(\mathbf{u}, a)$ (if they exist) are equal. See for example [5], where it is proved that for a special family of q-automatic sequences $\mathbf{u}$ the asymptotic density of a in $(\mathbf{u}(n^2))_{n\in\mathbb{N}}$ always exists but that it is in general not equal to $\text{dens}(\mathbf{u}, a)$.

The second scope of this paper is to study some correlations of the sequences $\mathbf{u}_c$, when $\mathbf{u}$ is a q-automatic sequence. One can consider either the correlation of the sequence $\mathbf{u}_c$, seen as a subsequence of the sequence $\mathbf{u}$, i.e., study the distribution of the pairs $(\mathbf{u}(\lfloor n^c \rfloor), \mathbf{u}(\lfloor n^c \rfloor + k))$, or the correlation of the sequence $\mathbf{u}_c$, seen for itself, i.e., study the distribution of the pairs $(\mathbf{u}_c(n), \mathbf{u}_c(n + k))$.

Since the sequence $(\lfloor n^c \rfloor)_{n\in\mathbb{N}}$ is, in a way, quickly increasing, we may expect that there is no correlation in the second sense, whereas in the first sense, we are only modifying a few digits and thus a non-trivial correlation should be derived. We shall illustrate this phenomenon through the study of the specific Thue-Morse sequence $\mathbf{t}$ defined by

$$\mathbf{t}(n) \equiv s_2(n) \bmod 2,$$

where $s_2(n)$ denotes the sum of the digits of the integer n written in base 2; we shall show in Section 2.1 the well-known fact that the sequence $\mathbf{t}$ is 2-automatic.

The *correlation measure of order* 2 of the Thue-Morse sequence has been studied by Mauduit and Sárközy in [15], where they proved that for any $N \geqslant 5$ one has

$$\max_{M \leqslant N} \max_{\substack{0 \leqslant d_1 < d_2 \leqslant N \\ M + d_2 \leqslant N}} \left| \sum_{n=1}^{M} (-1)^{s_2(n+d_1)+s_2(n+d_2)} \right| \geqslant \frac{1}{12} N.$$

Mahler [9] showed that for any positive integer k, the function

$$x \mapsto \frac{1}{x} \sum_{n \leqslant x} (-1)^{s_2(n)+s_2(n+k)}$$

converges and has a non-zero limit for infinitely many k's: this pair correlation can be understood as a consequence of the unique ergodicity of the symbolic dynamical system associated to the Thue-Morse sequence, see [8, 10, 18]. Mahler's result implies that for every $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$ and $k > 0$ the density

$$\ell_k(\varepsilon_1, \varepsilon_2) := \lim_{x \to \infty} \frac{1}{x} \#\{1 \leqslant n \leqslant x : (\mathbf{t}(n), \mathbf{t}(n+k)) = (\varepsilon_1, \varepsilon_2)\}$$

exists and it is not equal to $1/4$ for infinitely many integers k. For the sequence $(\mathbf{t}(\lfloor n^c \rfloor))_{n \in \mathbb{N}}$ we have the following result:

Theorem 2. *Let $c \in (1, 7/5)$, ε_1 and ε_2 be in $\{0, 1\}$ and $k > 0$. We have*

$$\lim_{x \to \infty} \frac{1}{x} \#\Big\{1 \leqslant n \leqslant x : (\mathbf{t}(\lfloor n^c \rfloor), \mathbf{t}(\lfloor n^c \rfloor + k)) = (\varepsilon_1, \varepsilon_2)\Big\} = \ell_k(\varepsilon_1, \varepsilon_2).$$

For example, the quantity $\ell_1(0, 0)$ is equal to $1/6$. This corresponds to the fact

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leqslant x} (-1)^{s_2(n)+s_2(n+1)} = -\frac{1}{3},$$

see for example [10]. The next result illustrates our expectation that the values $s_2(\lfloor n^c \rfloor)$ and $s_2(\lfloor (n+1)^c \rfloor)$ are not correlated.

Theorem 3. *Let $c \in (1, 10/9)$. For any pair $(\varepsilon_1, \varepsilon_2) \in \{0, 1\}^2$, we have*

$$\lim_{x \to \infty} \frac{1}{x} \#\Big\{1 \leqslant n \leqslant x : (\mathbf{t}(\lfloor n^c \rfloor), \mathbf{t}(\lfloor (n+1)^c \rfloor)) = (\varepsilon_1, \varepsilon_2)\Big\} = \frac{1}{4}.$$

The key ingredient in the proof of Theorem 3 is indeed an upper bound for the relevant discrete Fourier series:

Proposition 1. *Let $c \in (1, 10/9)$; there exists a constant $\sigma_c > 0$ such that we have*

$$\sum_{1 \leqslant n \leqslant x} (-1)^{s_2(\lfloor n^c \rfloor)+s_2(\lfloor (n+1)^c \rfloor)} = O_c(x^{1-\sigma_c}).$$

Remark 1. The upper bound for c in the statement of Theorem 3 and Proposition 1 given by the proof is

$$\frac{4 \log 2 + 9 \log(2 + \sqrt{2})}{2 \log 2 + 9 \log(2 + \sqrt{2})} = 1.11145799\ldots.$$

As we expressed it, the pair correlation of the sequence $\mathbf{u}_c$, seen for itself, is zero because the sequence $(\lfloor n^c \rfloor)_{n\in\mathbb{N}}$ is quickly increasing. We justify this point of view by studying the slowly increasing sequence $\lfloor n \log n \rfloor$, a case where we observe a non-zero correlation:

Theorem 4. *The function*

$$x \mapsto \frac{1}{x} \sum_{1\leqslant n\leqslant x} (-1)^{s_2(\lfloor n\log n\rfloor)+s_2(\lfloor (n+1)\log(n+1)\rfloor)}$$

has no limit when x tends to infinity. Furthermore, for any $(\varepsilon_0, \varepsilon_1) \in \{0,1\}^2$, the function

$$x \mapsto \frac{1}{x}\#\Big\{1 \leqslant n \leqslant x : (\mathbf{t}(\lfloor n\log n\rfloor), \mathbf{t}(\lfloor (n+1)\log(n+1)\rfloor)) = (\varepsilon_1, \varepsilon_2)\Big\}$$

has no limit when x tends to infinity.

In Section 2 we give a precise definition and important properties of automatic sequences and we state some facts on the discrete Fourier transform of the sum-of-digits function. In Section 3 we state and prove some results on q-multiplicative functions which we use in order to show Theorem 1 and Theorem 2. In this part we use classical tools from analytic number theory such as certain properties of the Beurling-Selberg function and the double large sieve of Bombieri and Iwaniec. In Section 4 we study different exponential sums in detail in order to prove Proposition 1 and Theorem 3. Finally in Section 5 we outline the proof of Theorem 4.

2. Notation and auxiliary results

Let $q \geqslant 2$. Every integer $n > 0$ has a unique representation in base q (called the proper representation) of the form

$$n = \sum_{j=0}^{\nu} \varepsilon_j(n) q^j, \qquad \varepsilon_j(n) \in \{0,1,\ldots,q-1\}, \qquad \varepsilon_\nu(n) \neq 0.$$

The sum-of-digits function $s_q(n)$ is defined for $n > 0$ by $s_q(n) = \sum_{j=0}^{\nu} \varepsilon_j(n)$ and we let $s_q(0) = 0$. Throughout, we use the notation $\mathrm{e}(x)$ for the exponential function $e^{2\pi i x}$. If x is a real number then $\|x\|$ denotes the distance from x to its nearest integer and $\{x\}$ is the fractional part of x. The symbol $f \ll g$ means that $|f| = O(|g|)$.

2.1. **Automatic sequences.** We refer to the very complete monograph [1] of Allouche and Shallit for the definitions and properties of q-automata and q-automatic sequences. We just give here the minimal information for the reader who is not acquainted with those notions.

Definition 1. *Let $q \geqslant 2$. A q-*automaton *$\mathfrak{M}$ with values in a finite set E is given by:*

- *a finite non-empty set $\mathfrak{R} = \{r_1, \ldots, r_d\}$, the elements of which are called states,*
- *one element of $\mathfrak{R}$, which is singled out and called the* initial state*; we will use the notation r_1 for this element,*
- *a map $\delta : \mathfrak{R} \times \{0,1,\ldots,q-1\} \to \mathfrak{R}$,*

- *a map* $\tau : \mathfrak{R} \to E$.

Let us explain, how we associate to the q-automaton $\mathfrak{M}$ a sequence of elements of E, say $\mathbf{v}_{\mathfrak{M}}$, via a sequence $\mathbf{r}_{\mathfrak{M}}$ of elements of $\mathfrak{R}$.

(1) We let $\mathbf{r}_{\mathfrak{M}}(0) = r_1$ and $\mathbf{v}_{\mathfrak{M}}(0) = \tau(r_1)$.
(2) For $n \geqslant 1$, we consider the proper representation of n in base q and we let

$$\mathbf{r}_{\mathfrak{M}}(n) = \delta(\cdots\delta(\delta(r_1, \varepsilon_v(n)), \varepsilon_{\nu-1}(n)), \ldots, \varepsilon_0(n)),$$

and $\mathbf{v}_{\mathfrak{M}}(n) = \tau(\mathbf{r}_{\mathfrak{M}}(n))$.

Remark 2. One can consider the oriented graph where the vertices are $\mathfrak{R}$ and the oriented arrows are given by the map δ. To attain $\mathbf{v}_{\mathfrak{M}}(n)$, we start at r_1 and sequentially read the digits of n from the left to right, i.e. starting with $\varepsilon_\nu(n)$, going from one state to another on following the arrows numbered $\varepsilon_\nu, \varepsilon_{\nu-1}, \ldots, \varepsilon_0$; we thus arrive at a certain state $\mathbf{r}_{\mathfrak{M}}(n)$ and the value of $\mathbf{v}_{\mathfrak{M}}(n)$ is simply $\tau(\mathbf{r}_{\mathfrak{M}}(n))$ (see the example below).

Definition 2. *We say that a sequence* $\mathbf{u}$ *with values in* E *is* q-automatic*, if there exists a* q*-automaton* $\mathfrak{M}$ *with values in* E *such that we have* $\mathbf{u}(n) = \mathbf{v}_{\mathfrak{M}}(n)$ *for all* n*.*

Example 1 (Thue-Morse sequence). In the following we show that the Thue-Morse sequence $\mathbf{t}$, which we defined in the introduction, is a 2-automatic sequence. Let us consider the 2-automaton $\mathfrak{T}$ defined by:

- $E = \{0, 1\}$, $\quad \mathfrak{R} = \{r_1, r_2\}$,
- $\delta(r_1, 0) = \delta(r_2, 1) = r_1$, $\quad \delta(r_1, 1) = \delta(r_2, 0) = r_2$,
- $\tau(r_1) = 0$, $\quad \tau(r_2) = 1$.

Its graph (as described in the previous remark) is given in Figure 1. It is readily seen that

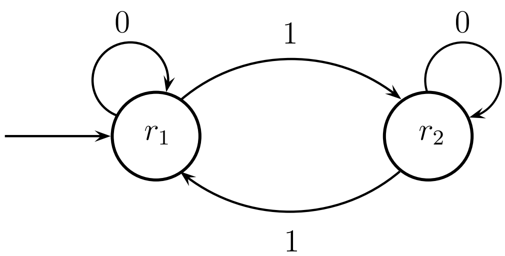

FIGURE 1. The graph of the automaton $\mathfrak{T}$ of the Thue-Morse sequence

the state denoted by $\mathbf{r}_{\mathfrak{T}}(n)$ is r_1 if we have read an even number of 1's in the expansion of n in base 2, and is r_2 otherwise. Thus, due to the definition of τ we have

$$\mathbf{v}_{\mathfrak{T}}(n) = \begin{cases} 0, & \text{if } n \text{ contains an even number of 1's} \\ 1, & \text{if } n \text{ contains an odd number of 1's,} \end{cases}$$

so that $\mathbf{v}_{\mathfrak{T}}(n) = \mathbf{t}(n)$ for all n, which proves that $\mathbf{t}$ is a 2-automatic sequence.

Transition matrices: To any given $k \in \{0, 1, \dots, q-1\}$ we associate the $d \times d$ matrix $M(k) = (m_{ij}(k))$, such that

$$m_{ij}(k) = \begin{cases} 1, & \text{if } \delta(r_j, k) = r_i, \\ 0, & \text{otherwise.} \end{cases}$$

For example, for the Thue-Morse sequence, we have

$$M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad M(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The dynamics of the automaton $\mathfrak{M}$, i.e. the sequence $\mathbf{r}_{\mathfrak{M}}$, can be obtained in the following way:[1] For $n \geqslant 1$, with the proper representation $n = \sum_{i=0}^{\nu} \varepsilon_i(n) q^i$ in base q, we have

$$M(\varepsilon_0(n)) M(\varepsilon_1(n)) \cdots M(\varepsilon_\nu(n)) e_1 = e_{\mathbf{r}_{\mathfrak{M}}(n)}.$$

In the sequel, it will turn out to be convenient to introduce the notation

$$S(n) = M(\varepsilon_0(n)) M(\varepsilon_1(n)) \cdots M(\varepsilon_\nu(n)) \text{ for } n \geqslant 1 \text{ and } S(0) = \mathrm{Id}_d, \tag{4}$$

where Id_d is the identity element in $\mathbb{C}^{d \times d}$. In order to check whether $\mathbf{v}_{\mathfrak{M}}(n)$ is equal to a given value $a \in E$, we simply have to compute the product

$$\mathfrak{z}_a^T e_{\mathbf{r}_{\mathfrak{M}}(n)} = \mathfrak{z}_a^T S(n) e_1,$$

where the vector $\mathfrak{z}_a$ is defined by

$$(\mathfrak{z}_a)_i = \begin{cases} 1, & \text{if } \tau(r_i) = a, \\ 0, & \text{otherwise.} \end{cases}$$

It is equal to 1 if $\mathbf{v}_{\mathfrak{M}}(n) = a$ and 0 otherwise. The advantage of this matrix representation is that, as shown by Peter [17], it permits to give a criterion for the existence of the asymptotic density with which the element a in E is recognized by $\mathbf{v}_{\mathfrak{M}}$. We consider the matrix $M = (M(0) + \cdots + M(q-1))/q$; it is a stochastic matrix and thus there exists a positive integer m such that the sequence $(M^{mk})_{k \in \mathbb{N}}$ converges. In particular, Peter showed that $\mathrm{dens}(\mathbf{v}_{\mathfrak{M}}, a)$ exists and is equal to α if and only if for all $1 \leqslant j \leqslant d$,

$$\lim_{k \to \infty} \mathfrak{z}_a^T M^{mk} e_j$$

exists and is equal to α. A special case of importance is the positive regular case, where M admits a power all the entries of which are positive: in this case $(M^k)_k$ tends towards a matrix all the columns of which are equal and Peter's criterion is trivially satisfied.

[1]For $1 \leqslant j \leqslant d$, we denote by e_j (and for notational convenience also by e_{r_j}) the d dimensional unit vector defined by $(e_j)_i = \delta_{ij}$ (the Kronecker symbol).

2.2. **Fourier transform.** Let $q \geqslant 2$, $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{N}$. The discrete Fourier transform $F_\lambda(.,\alpha)$ of the function $u \mapsto \mathrm{e}(\alpha s_q(\,\cdot\,))$ is defined for all $h \in \mathbb{Z}$ by

$$F_\lambda(h,\alpha) = \frac{1}{q^\lambda} \sum_{0 \leqslant u < q^\lambda} \mathrm{e}\left(\alpha s_q(u) - huq^{-\lambda}\right).$$

Proofs of the following properties of the Fourier transform can be found in [13, 14].

Lemma 1. *Let $q \geqslant 2$, $\alpha \in \mathbb{R}$, $h \in \mathbb{Z}$, and $\lambda \geqslant 1$; set $c_q = \frac{\pi^2}{12 \log q}\left(1 - \frac{2}{q+1}\right)$. Then we have*

$$|F_\lambda(h,\alpha)| \leqslant e^{\pi^2/48} q^{-c_q ||(q-1)\alpha||^2 \lambda},$$

and

$$\sum_{0 \leqslant h < 2^\lambda} |F_\lambda(h,\alpha)| \leqslant \sqrt{2}(2+\sqrt{2})^{\frac{\lambda}{4}},$$

as well as

$$\sum_{0 \leqslant h < q^\lambda} |F_\lambda(h,\alpha) F_\lambda(-h,\alpha)| \leqslant 1.$$

3. Generalized q-multiplicative functions and automatic sequences

In this section we prove Theorem 1 and Theorem 2. These two results will directly follow from Theorem 5, which is a generalization of a result of Mauduit and Rivat [12, Theorem 1] (see also [11]). They have shown that for all q-multiplicative functions f the following result holds true: If $c \in (1, 7/5)$, $\gamma = 1/c$ and $q \geqslant 2$, then for all $\delta \in (0, (7-5c)/9)$

$$\left| \sum_{1 \leqslant n \leqslant x} f(\lfloor n^c \rfloor) - \sum_{1 \leqslant m \leqslant x^c} \gamma m^{\gamma - 1} f(m) \right| \ll x^{1-\delta},$$

where the implied constant depends at most on c, δ and q. Recall that a q-multiplicative function $f : \mathbb{N} \to \mathbb{C}$ is defined by the property that for every triple $(a,b,k) \in \mathbb{N}^3$ with $b < q^k$ we have

$$f(q^k a + b) = f(q^k a) f(b).$$

The following definition is a generalization of this property to matrix valued functions:

Definition 3. *Let $d \geqslant 1$. We call a function $F : \mathbb{N} \to \mathbb{C}^{d\times d}$ a generalized q-multiplicative function in $\mathbb{C}^{d\times d}$, if there exists a constant $L \geqslant 0$ such that for all $k \geqslant 0$ there exist functions $G_k^{(j)} : \mathbb{N} \to \mathbb{C}^{d\times d}$, $j = 1, 2$, such that for every triple $(a,b,k) \in \mathbb{N}^3$ with $a > 0$ and $b < q^k - L$ we have*

$$F(q^k a + b) = G_k^{(1)}(b)\, G_k^{(2)}(a).$$

Theorem 5. *Let $c \in (1, 7/5)$, $q \geqslant 2$, $d \geqslant 1$ and assume that F is a generalized q-multiplicative function in $\mathbb{C}^{d\times d}$ and there exists a submultiplicative norm $\|\cdot\|_s$ such that* *we have* $\|F(n)\|_s \leqslant 1$, $\|G_k^{(1)}(n)\|_s \leqslant 1$ *and* $\|G_k^{(2)}(n)\|_s \leqslant 1$ *for all* $k \geqslant 0$ *and* $n \geqslant 0$. *Set* $\gamma = 1/c$. *Then we have for all* $\delta \in (0, (7-5c)/9)$ *that*

$$\left\| \sum_{1\leqslant n\leqslant x} F(\lfloor n^c \rfloor) - \sum_{1\leqslant m\leqslant x^c} \gamma m^{\gamma-1} F(m) \right\|_s \ll x^{1-\delta},$$

where the implied constant depends at most on c, δ, q, d, *the norm* $\|\cdot\|_s$ *and* L.

Remark 3. By transposition, this result also holds true if the generalized q-multiplicative function F in $\mathbb{C}^{d\times d}$ satisfies

$$F(q^k a + b) = G_k^{(2)}(a)\, G_k^{(1)}(b),$$

instead of $F(q^k a + b) = G_k^{(1)}(b)\, G_k^{(2)}(a)$. In terms of q-automatic sequences (see Section 2.1 and the proof of Theorem 1), the definition of generalized q-multiplicativity as given in Definition 3 corresponds to the fact that the automaton reads the input digits from left to right. Contrary, if the automaton read the digits from right to left, this would yield a generalized q-multiplicative function satisfying the relation stated in this remark.

3.1. **Proof of Theorem 5.** The proof of this theorem goes along the line of Mauduit's and Rivat's proof of [12, Theorem 1]. Let $\|\cdot\|_s$ be the norm mentioned in Theorem 5. We denote by $\|\cdot\|_{\max}$ the maximum norm (*i.e.*, if $A = (a_{ij}) \in \mathbb{C}^{d\times d}$, then $\|A\|_{\max} = \max_{i,j} |a_{ij}|$). Recall that this norm is not submultiplicative.

The first steps of the proof are analog to [12]. The only difference is the fact that we use the triangle inequality for arbitrary norms in $\mathbb{C}^{d\times d}$ instead of the triangle inequality for the absolute value in $\mathbb{C}$. Recall that $c > 1$. A short calculation shows that m has the form $m = \lfloor n^c \rfloor$ if and only if

$$\lfloor -m^\gamma \rfloor - \lfloor -(m+1)^\gamma \rfloor = 1,$$

where $\gamma = 1/c$ (otherwise, this difference is zero). If we set $\Psi(u) = u - \lfloor u \rfloor - 1/2$, then we obtain

$$\begin{aligned} \sum_{1\leqslant n\leqslant x} F(\lfloor n^c \rfloor) &= \sum_{1\leqslant m\leqslant x^c} F(m)\left(\lfloor -m^\gamma \rfloor - \lfloor -(m+1)^\gamma \rfloor\right) \\ &= \sum_{1\leqslant m\leqslant x^c} F(m)\left((m+1)^\gamma - m^\gamma\right) \\ &\quad + \sum_{1\leqslant m\leqslant x^c} F(m)\left(\Psi(-(m+1)^\gamma) - \Psi(-m^\gamma)\right). \end{aligned} \tag{5}$$

Next, we recall a result which can be found in [12, Lemma 2]. If $\theta \in [0,1]$, then

$$\sum_{m\geqslant 1} \left|(m+1)^\theta - m^\theta - \theta m^{\theta-1}\right| \leqslant \frac{1}{4}.$$

Since $\|F(n)\|_s \leqslant 1$ for all $n \in \mathbb{N}$, we get

$$\left\| \sum_{1 \leqslant m \leqslant x^c} F(m)\left((m+1)^\gamma - m^\gamma\right) - \sum_{1 \leqslant m \leqslant x^c} \gamma m^{\gamma-1} F(m) \right\|_s \leqslant \frac{1}{4}.$$

Together with (5), this implies that in order to prove Theorem 5 it suffices to show that for all $\delta \in (0, (7-5c)/9)$ and for all $M \gg 1$ we have

$$S_M := \left\| \sum_{M < m \leqslant 2M} F(m)\left(\Psi(-(m+1)^\gamma) - \Psi(-m^\gamma)\right) \right\|_s \ll M^{\gamma(1-\delta)}, \tag{6}$$

where the implied constant depends on c, δ, d and the norm $\|\cdot\|_s$. Indeed, this follows from a standard argument using geometric series. If we set $M_k = x^c/2^k$, we have

$$\left\| \sum_{1 \leqslant n \leqslant x} F(\lfloor n^c \rfloor) - \sum_{1 \leqslant m \leqslant x^c} \gamma m^{\gamma-1} F(m) \right\|_s \ll \sum_{k \geqslant 0} M_{k+1}^{\gamma(1-\delta)} \ll x^{1-\delta}.$$

We start with approximating the function Ψ by trigonometric polynomials. Let $H \geqslant 1$ be an integer. Then it follows from Vaaler's approximation method using the Beurling-Selberg function (see [19, Theorem 19]) that the following holds true: There exist coefficients $a_H(h)$ with $0 \leqslant a_H(h) \leqslant 1$ such that the trigonometric polynomials

$$\Psi_H(t) = -\frac{1}{2i\pi} \sum_{1 \leqslant |h| \leqslant H} \frac{a_H(h)}{h} \mathrm{e}(ht)$$

and

$$\kappa_H(t) = \sum_{|h| \leqslant H} \left(1 - \frac{|h|}{H+1}\right) \mathrm{e}(ht) \tag{7}$$

verify

$$|\Psi(t) - \Psi_H(t)| \leqslant \frac{1}{2H+2} \kappa_H(t).$$

The function $\kappa_H(t)$ is the Fejer kernel and we have

$$\frac{1}{2H+2} \sum_{M \leqslant m \leqslant 2M} \kappa_H(m^\theta) \ll_\theta H^{-1}M + H^{1/2}M^{\theta/2} + H^{-1/2}M^{1-\theta/2}, \tag{8}$$

for every $0 < \theta < 1$ and for every $M \geqslant 1$ (this is [12, Lemma 5] and follows easily from [6, Theorem 2.2]). We set $H_0 := \left\lfloor M^{1-\gamma(1-\delta)} \right\rfloor$, where δ is a constant satisfying $0 < \delta < (7-5c)/9$, and we get

$$\begin{aligned} S_M \leqslant &\left\| \sum_{M < m \leqslant 2M} F(m)\left(\Psi_{H_0}(-(m+1)^\gamma) - \Psi_{H_0}(-m^\gamma)\right) \right\|_s \\ &+ \frac{1}{2H_0+2} \sum_{M < m \leqslant 2M} \kappa_{H_0}\left(-(m+1)^\gamma\right) + \frac{1}{2H_0+2} \sum_{M < m \leqslant 2M} \kappa_{H_0}\left(-m^\gamma\right). \end{aligned}$$

The last two sums can be handled by (8). This yields

$$S_M \ll \left\| \sum_{M<m\leqslant 2M} F(m)\left(\Psi_{H_0}(-(m+1)^\gamma) - \Psi_{H_0}(-m^\gamma)\right)\right\|_s + H_0^{-1}M + H_0^{1/2}M^{\gamma/2} + H_0^{-1/2}M^{1-\gamma/2}.$$

For our special choice of H_0 we have that

$$H_0^{1/2}M^{\gamma/2} = M^{1/2+\gamma\delta/2} \geqslant M^{1/2-\gamma\delta/2} = H_0^{-1/2}M^{1-\gamma/2}.$$

Thus we get

$$S_M \ll \left\| \sum_{M<m\leqslant 2M} F(m)\left(\Psi_{H_0}(-(m+1)^\gamma) - \Psi_{H_0}(-m^\gamma)\right)\right\|_s + M^{\gamma(1-\delta)} + M^{1/2+\gamma\delta/2}. \tag{9}$$

Next, we treat the sum that arises in (9). Replacing Ψ_{H_0} by its expression, this sum is bounded above by

$$\begin{aligned}&\sum_{1\leqslant |h|\leqslant H_0} \frac{1}{|h|}\left\| \sum_{M<m\leqslant 2M} F(m)\left(\mathrm{e}\left(h(m+1)^\gamma\right) - \mathrm{e}\left(hm^\gamma\right)\right)\right\|_s \\ &= \sum_{\ell\geqslant 0}\ \sum_{H_{\ell+1}<|h|\leqslant H_\ell} \frac{1}{|h|}\left\| \sum_{M<m\leqslant 2M} F(m)\left(\mathrm{e}\left(h(m+1)^\gamma\right) - \mathrm{e}\left(hm^\gamma\right)\right)\right\|_s, \end{aligned}\tag{10}$$

where $H_\ell = H_0/2^\ell$. Putting $\varphi_h(t) = \mathrm{e}(h(t+1)^\gamma - ht^\gamma) - 1$, we get by partial summation[2]

$$\begin{aligned}&\sum_{M<m\leqslant 2M} F(m)\left(\mathrm{e}\left(h(m+1)^\gamma\right) - \mathrm{e}\left(hm^\gamma\right)\right) \\ &\quad= \varphi_h(2M)\sum_{M<m\leqslant 2M} F(m)\,\mathrm{e}\left(hm^\gamma\right) - \int_M^{2M} \varphi_h'(t)\sum_{M<m\leqslant t} F(m)\,\mathrm{e}\left(hm^\gamma\right)\mathrm{d}t.\end{aligned}$$

If $|h| \leqslant M^{1-\gamma}$ we have $\varphi_h(t) \ll |h|M^{\gamma-1}$ and $\varphi_h'(t) \ll |h|M^{\gamma-2}$ on the interval $[M, 2M]$. Thus we obtain for $H_\ell \leqslant M^{1-\gamma}$ (note, that $\|\int A(t)\,\mathrm{d}t\|_{\max} \leqslant \int \|A(t)\|_{\max}\mathrm{d}t$ which implies $\|\int A(t)\,\mathrm{d}t\|_s \ll \int \|A(t)\|_s\mathrm{d}t$)

$$\begin{aligned}&\sum_{H_{\ell+1}<|h|\leqslant H_\ell} \frac{1}{|h|}\left\| \sum_{M<m\leqslant 2M} F(m)\left(\mathrm{e}\left(h(m+1)^\gamma\right) - \mathrm{e}\left(hm^\gamma\right)\right)\right\|_s \\ &\qquad \ll_d \max_{M'\in[M,2M]} M^{\gamma-1}\sum_{H_{\ell+1}<|h|\leqslant H_\ell}\left\|\sum_{M<m\leqslant M'} F(m)\,\mathrm{e}\left(hm^\gamma\right)\right\|_s.\end{aligned}$$

[2]If $A(t) \in \mathbb{C}^{d\times d}$, we denote by $\int A(t)\mathrm{d}t$ the matrix (B_{ij}) with $B_{ij} = \int A_{ij}(t)\mathrm{d}t$.

Moreover, we trivially get for $\ell \geqslant 0$,

$$\sum_{H_{\ell+1}<|h|\leqslant H_\ell} \frac{1}{|h|} \left\| \sum_{M<m\leqslant 2M} F(m) \left(\mathrm{e}\left(h(m+1)^\gamma\right) - \mathrm{e}\left(hm^\gamma\right)\right) \right\|_s \ll \max_{u\in\{0,1\}} \frac{1}{H_{\ell+1}} \sum_{H_{\ell+1}<|h|\leqslant H_\ell} \left\| \sum_{M<m\leqslant 2M} F(m)\, \mathrm{e}\left(h(m+u)^\gamma\right) \right\|_s .$$

Since the sum over ℓ in (10) has $\ll \log(H_0)$ summands, we obtain

$$\left\| \sum_{M<m\leqslant 2M} F(m) \left(\Psi_{H_0}(-(m+1)^\gamma) - \Psi_{H_0}(-m^\gamma)\right) \right\|_s \ll_{c,d} (\log H_0) \max_{0<H\leqslant H_0} \max_{u\in\{0,1\}} \max_{\tilde{M}\in[M,2M]} \min\left(M^{1-\gamma}, H^{-1}\right) S_{H,M,M',u}, \tag{11}$$

where $S_{H,M,M',u}$ is defined by

$$S_{H,M,M',u} = \sum_{H<h\leqslant 2H} \left\| \sum_{M<m\leqslant M'} F(m)\, \mathrm{e}\left(h(m+u)^\gamma\right) \right\|_s . \tag{12}$$

Proposition 2. *Let $\gamma \in (1/2,1)$, $q \geqslant 2$, $d \geqslant 1$ and F be given as in Theorem 5. Then we have for all $1/2 \leqslant H \leqslant M \leqslant M' \leqslant 2M$ and $u \in [0,1]$ that*

$$S_{H,M,M',u} \ll H^{9/8} M^{(2+\gamma)/4} (1 + H^{-1/2} M^{(1-\gamma)/2}) \sqrt{\log(3M)},$$

where the implied constant depends on γ, d the norm $\|\cdot\|_s$ and L.

As in [12, page 195] one can now show that this result implies (6). This in turn (as already noted) proves Theorem 5. Thus, we omit the details and continue with proving Proposition 2. Since the next few steps are of particular importance, we treat them in detail. The final steps are again as in [12, Section 4], see the comments at the end of the proof of Proposition 2.

Proof of Proposition 2. Throughout this proof we write S instead of $S_{H,M,M',u}$. Let $k \in \mathbb{N}$ such that $B \ll H^{-1/4} M^{1-\gamma/2} \ll B$ with $B = q^k$. We can assume that $k \geqslant 1$ (otherwise, the statement holds trivially). Then there exist integers A, R, A' and R' such that

$$M = AB + R \text{ with } 0 \leqslant R < B, \quad \text{and} \quad M' = A'B + R' \text{ with } 0 \leqslant R' < B.$$

We have $A \leqslant A' \leqslant 2A+1$ and $AB \ll M \ll AB$. This allows us to write

$$S = \sum_{H<h\leqslant 2H} \left\| \sum_{A\leqslant a<A'} \sum_{0\leqslant b<B} F(Ba+b)\, \mathrm{e}\left(h(Ba+b+u)^\gamma\right) \right\|_s + O(HB).$$

Next, Taylor's theorem implies that

$$e(h(Ba+b+u)^\gamma) = \mathrm{e}(hB^\gamma a^\gamma)\, \mathrm{e}(\mathbf{x}(a,h)\cdot \mathbf{y}(b)) + O_\gamma(HB^4 M^{\gamma-4}),$$

where

$$\mathbf{x}(a,h) = (ha^{\gamma-1}, ha^{\gamma-2}, ha^{\gamma-3}),$$
$$\mathbf{y}(b) = (\gamma_1 B^{\gamma-1}(b+u), \gamma_2 B^{\gamma-2}(b+u)^2, \gamma_3 B^{\gamma-3}(b+u)^3),$$

with $\gamma_1 = \gamma$, $\gamma_2 = \gamma(\gamma-1)/2$ and $\gamma_3 = \gamma(\gamma-1)\gamma-2)/6$. Thus, we have

$$S = \sum_{H<h\leqslant 2H} \left\| \sum_{A\leqslant a<A'} \sum_{0\leqslant b<B} F(Ba+b)\,\mathrm{e}(hB^\gamma a^\gamma)\,\mathrm{e}(\mathbf{x}(a,h)\cdot\mathbf{y}(b)) \right\|_s + O(HB + H^2B^4M^{\gamma-3}).$$

The generalized q-multiplicity of F implies that there exist functions $G_k^{(1)}$ and $G_k^{(2)}$ such that we have for all $A \leqslant a < A'$ and $0 \leqslant b < B - L$ that

$$F(Ba+b) = G_k^{(1)}(b)\; G_k^{(2)}(a).$$

Using this property, we obtain (if $B - L \leqslant b < B$, we use the trivial estimate)

$$\begin{aligned} S &= \sum_{H<h\leqslant 2H} \left\| \sum_{A\leqslant a<A'} \sum_{0\leqslant b<B} \tilde{G}_k^{(1)}(b)\; G_k^{(2)}(a)\,\mathrm{e}(hB^\gamma a^\gamma)\,\mathrm{e}(\mathbf{x}(a,h)\cdot\mathbf{y}(b)) \right\|_s \\ &\qquad + O(HA + HB + H^2B^4M^{\gamma-3}) \\ &\ll \sum_{H<h\leqslant 2H} \sum_{A\leqslant a<A'} \left\| \sum_{0\leqslant b<B} G_k^{(1)}(b)\,\mathrm{e}(\mathbf{x}(a,h)\cdot\mathbf{y}(b)) \right\|_s \\ &\qquad + HA + HB + H^2B^4M^{\gamma-3}, \end{aligned}$$

where we used the submultiplicity of $\|\cdot\|_s$. Note, that $\|A\|_s \ll_d \sum_{1\leqslant i,j\leqslant d} |a_{ij}|$ for any matrix $A = (a_{ij})$. Hence we get

$$S \ll_d \sum_{1\leqslant i,j\leqslant d} \sum_{H<h\leqslant 2H} \sum_{A\leqslant a<A'} \left| \sum_{0\leqslant b<B} G_{ij}(b)\,\mathrm{e}(\mathbf{x}(a,h)\cdot\mathbf{y}(b)) \right| + HA + HB + H^2B^4M^{\gamma-3},$$

where $G_k^{(1)}(b) = (G_{ij}(b))_{1\leqslant i,j\leqslant d}$. Set $\mathcal{X} = \{\mathbf{x}(a,h) : A \leqslant a < A', H < h \leqslant 2H\}$ and $\mathcal{Y} = \{\mathbf{y}(b) : 0 \leqslant b < B\}$. Note, that $\mathbf{x}(a,h) \neq \mathbf{x}(a',h')$ if $(a,h) \neq (a',h')$ and $\mathbf{y}(b) \neq \mathbf{y}(b')$ if $b \neq b'$. We obtain that there exist complex numbers $\alpha_{ij}(\mathbf{x}(a,h))$ and $\beta_{ij}(\mathbf{y}(b))$ with $|\alpha_{ij}(\mathbf{x}(a,h))| = 1$ and $|\beta_{ij}(\mathbf{y}(b))| \ll_d 1$ such that

$$\sum_{H<h\leqslant 2H} \sum_{A\leqslant a<A'} \left| \sum_{0\leqslant b<B} G_{ij}(b)\,\mathrm{e}(\mathbf{x}(a,h)\cdot\mathbf{y}(b)) \right| = \sum_{\substack{\mathbf{x}(a,h)\in\mathcal{X} \\ \mathbf{y}(b)\in\mathcal{Y}}} \alpha_{ij}(\mathbf{x}(a,h))\beta_{ij}(\mathbf{y}(b))\,\mathrm{e}(\mathbf{x}(a,h)\cdot\mathbf{y}(b)).$$

(Note, that $|G_{ij}(b)| \leqslant \|G_k^{(1)}(b)\|_{\max} \ll \|G_k^{(1)}(b)\|_s \ll 1$.) We set

$$\Delta_k^{-1} = \gamma_k H B^k M^{\gamma-k}, \qquad X_k = \gamma_k^{-1}\Delta_k^{-1} B^{-\gamma}, \qquad Y_k = \Delta_k^{-1} H^{-1} A^{k-\gamma}$$

for $k = 1, \dots, 3$. Then we have that the k-th component of $\mathbf{x}(a,h) \in \mathcal{X}$ has absolute value less than or equal to X_k. A similar result holds for the points in $\mathcal{Y}$ (with X_k replaced by Y_k). Hence, we can apply [2, Lemma 2.4] (the double large sieve of Bombieri and Iwaniec) in dimension 3 and we obtain

$$\left(\sum_{H<h\leqslant 2H} \sum_{A\leqslant a<A'} \left| \sum_{0\leqslant b<B} \tilde{F}_{ij}(b)\, \mathrm{e}(\mathbf{x}(a,h)\cdot \mathbf{y}(b)) \right| \right)^2 \ll_d \prod_{k=1}^{3} (1+\Delta_k^{-1}) \mathcal{B}_1 \mathcal{B}_2, \tag{13}$$

where $\mathcal{B}_1$ represents the number of quadruples (h_1, h_2, a_1, a_2) with $H \leqslant h_1, h_2 \leqslant 2H$ and $A \leqslant a_1, a_2 \leqslant A'$ such that

$$\left| h_1 a_1^{\gamma-k} - h_2 a_2^{\gamma-k} \right| \leqslant (2Y_k)^{-1}, \qquad k = 1, \dots, 3,$$

and $\mathcal{B}_2$ represents the number of pairs (b_1, b_2) with $0 \leqslant b_1, b_2 < B$ such that

$$\left| \gamma_k B^{\gamma-k}(b_1+u)^k - \gamma_k B^{\gamma-k}(b_2+u)^k \right| \leqslant (2X_k)^{-1}, \qquad k = 1, \dots, 3.$$

Note, that the right-hand side of (13) is independent of i and j, since the sets $\mathcal{X}$ and $\mathcal{Y}$ as well as the numbers X_k and Y_k for $k = 1, \dots, 3$ are independent of i and j. Mauduit and Rivat have shown (see [12, Sections 3 and 4]) that

$$\prod_{k=1}^{3} (1+\Delta_k^{-1}) \mathcal{B}_1 \mathcal{B}_2 \ll_\gamma H^{9/4} M^{1+\gamma/2} (1 + H^{-1} M^{1-\gamma}) \log(3M).$$

Thus, we obtain that S is bounded by some constant times

$$H^{9/8} M^{(2+\gamma)/4} (1 + H^{-1/2} M^{(1-\gamma)/2}) \sqrt{\log(3M)} + HA + HB + H^2 B^4 M^{\gamma-3}.$$

Exactly the same way as at the end of [12, Sections 4], we obtain that

$$HB + H^2 B^4 M^{\gamma-3} \ll H^{9/8} M^{(2+\gamma)/4} (1 + H^{-1/2} M^{(1-\gamma)/2}) \sqrt{\log(3M)}.$$

Furthermore, we have

$$HA \ll HB^{-1} M \ll H^{5/4} M^{\gamma/2} \ll H^{9/8} M^{(2+\gamma)/4}.$$

The last inequality follows from the fact that (note, that $H \leqslant M$ and $\gamma < 1$)

$$H^{1/8} M^{-1/2+\gamma/4} \leqslant M^{1/8-1/2+1/4} \leqslant 1.$$

Finally, this completes the proof of Proposition 2. $\square$

3.2. **Proof of Theorem 1.** Before we start with the proof of Theorem 1, we show that Theorem 5 implies the following result.

Lemma 2. *Let $c \in (1, 7/5)$, $q \geqslant 2$, $d \geqslant 1$ and assume that F is a generalized q-multiplicative function in $\mathbb{C}^{d\times d}$ and there exists a submultiplicative norm $\|\cdot\|_s$ such that we have $\|F(n)\|_s \leqslant 1$, $\|G_k^{(1)}(n)\|_s \leqslant 1$ and $\|G_k^{(2)}(n)\|_s \leqslant 1$ for all $k \geqslant 0$ and $n \geqslant 0$. Set $\gamma = 1/c$. Then we have*

$$\left\| \sum_{1\leqslant n\leqslant x} \frac{F(\lfloor n^c \rfloor)}{n} - \sum_{1\leqslant m\leqslant x^c} \gamma \frac{F(m)}{m} \right\|_s \ll_{c,d} 1, \tag{14}$$

and

$$\left\| \sum_{1\leqslant n\leqslant x^\gamma} c n^{c-1} F(\lfloor n^c \rfloor) - \sum_{1\leqslant m\leqslant x} F(m) \right\|_s \ll_{c,\delta,d} x^{1-\delta\gamma}, \tag{15}$$

for all $\delta \in (0, (7-5c)/9)$.

Proof. In what follows, we set

$$\Xi(u) := \sum_{1\leqslant n\leqslant u} F(\lfloor n^c \rfloor) - \sum_{1\leqslant m\leqslant u^c} \gamma m^{\gamma-1} F(m).$$

We start with proving inequality (14). By partial summation we obtain

$$\sum_{1\leqslant n\leqslant x} \frac{F(\lfloor n^c \rfloor)}{n} - \sum_{1\leqslant m\leqslant x^c} \gamma \frac{F(m)}{m} = \frac{1}{x}\Xi(x) + I(x),$$

where

$$I(x) = \int_1^x \left(\sum_{1\leqslant n\leqslant t} F(\lfloor n^c \rfloor) \right) \frac{1}{t^2} \mathrm{d}t - \gamma \int_1^{x^c} \left(\sum_{1\leqslant m\leqslant t} \gamma m^{\gamma-1} F(m) \right) \frac{1}{t^{\gamma+1}} \mathrm{d}t.$$

Changing the variable in the last integral yields

$$I(x) = \int_1^x \Xi(t) \frac{1}{t^2} \mathrm{d}t.$$

Thus we obtain

$$\left\| \sum_{1\leqslant n\leqslant x} \frac{F(\lfloor n^c \rfloor)}{n} - \sum_{1\leqslant m\leqslant x^c} \gamma \frac{F(m)}{m} \right\|_s \ll \frac{1}{x} \|\Xi(x)\|_s + \int_1^x \|\Xi(t)\|_s \frac{1}{t^2} \mathrm{d}t.$$

We can use Theorem 5 with some fixed $\delta < (7-5c)/9$ and get

$$\left\| \sum_{1\leqslant n\leqslant x} \frac{F(\lfloor n^c \rfloor)}{n} - \sum_{1\leqslant m\leqslant x^c} \gamma \frac{F(m)}{m} \right\|_s \ll_{c,d} \frac{1}{x^\delta} + \int_1^x \frac{1}{t^{1+\delta}} \mathrm{d}t \ll_{c,d} 1.$$

One can show (15) using the same ideas but for brevity we do not give a proof. (Partial summation, integration by substitution and Theorem 5 yields the desired result.) ☐

Proof of Theorem 1. Recall that we have $1 < c < 7/5$ and $q \geqslant 2$. Let $a \in E$. As we have seen in Section 2.1, there exist q transition matrices $M(0), \ldots, M(q-1) \in \mathbb{C}^{d\times d}$ (for some $d \geqslant 1$) corresponding to the automatic sequence $\mathbf{u}$ and a vector $\mathfrak{z}_a \in \mathbb{C}^d$ such that

$$\mathfrak{z}_a^T S(n) e_1 = \begin{cases} 1, & \text{if } \mathbf{u}(n) = a, \\ 0, & \text{otherwise,} \end{cases}$$

where $S(n)$ is given by (4). Note, that S is a is a generalized q-multiplicative function in $\mathbb{C}^{d\times d}$. Indeed, we have for every triple $(a, b, k) \in \mathbb{N}^3$ with $a > 0$ and $b < q^k$ that

$$S(q^k a + b) = S(b) M(0)^{k-\ell(b)} S(a),$$

where $\ell(b) = \lfloor \log_q(b) \rfloor + 1$ for $b \geqslant 1$ and $\ell(0) = 0$ (the expression $\ell(b)$ is equal to the number of digits of b in the base-q representation system). Thus, we can set

$$G_k^{(1)}(n) = S(n) M(0)^{k-\ell(n)} \quad \text{and} \quad G_k^{(2)}(n) = S(n).$$

Let $\|\cdot\|_1$ denote the submultiplicative norm induced by the 1-norm in $\mathbb{C}^d$. Alternatively, if $A = (a_{ij})_{1\leqslant i,j\leqslant d}$, then $\|A\|_1$ is also given by $\|A\|_1 = \max_j \sum_i |a_{ij}|$ (maximum absolute column sum norm). Since for each n there is exactly one entry equal to 1 in each column of $S(n)$, we have $\|S(n)\|_1 = 1$ (the same holds clearly for $G_k^{(1)}(n)$ and $G_k^{(2)}(n)$). Hence, we will be able to apply Theorem 5 and its consequences.

We start with showing that the logarithmic density of a in $\mathbf{u}_c$ (recall that $\mathbf{u}_c$ is defined by (3)) exists and that it is the same as the logarithmic density log-dens$(\mathbf{u}, a)$ of a in $\mathbf{u}$ (which exists since this sequence is q-automatic). We have

$$\text{log-dens}(\mathbf{u}, a) = \lim_{x\to\infty} \frac{1}{\log x} \sum_{\substack{1\leqslant n\leqslant x \\ \mathbf{u}(n)=a}} \frac{1}{n} = \lim_{x\to\infty} \frac{1}{\log x} \sum_{1\leqslant n\leqslant x} \frac{\mathfrak{z}_a^T S(n) e_1}{n}.$$

In what follows, we show that

$$\frac{1}{\log x} \sum_{1\leqslant n\leqslant x} \frac{\mathfrak{z}_a^T S(\lfloor n^c \rfloor) e_1}{n} - \text{log-dens}(\mathbf{u}, a) = o(1),$$

which implies the desired result. We can write

$$\frac{1}{\log x} \sum_{1\leqslant n\leqslant x} \frac{\mathfrak{z}_a^T S(\lfloor n^c \rfloor) e_1}{n} - \text{log-dens}(\mathbf{u}, a) = E_{\log}^{(1)} + E_{\log}^{(2)},$$

with

$$E_{\log}^{(1)} = \frac{1}{\log x} \mathfrak{z}_a^T \left(\sum_{1\leqslant n\leqslant x} \frac{S(\lfloor n^c \rfloor)}{n} - \sum_{1\leqslant m\leqslant x^c} \gamma \frac{S(m)}{m} \right) e_1,$$

$$E_{\log}^{(2)} = \frac{1}{\log x^c} \sum_{1\leqslant m\leqslant x^c} \frac{\mathfrak{z}_a^T S(m) e_1}{m} - \text{log-dens}(\mathbf{u}, a).$$

The Cauchy-Schwarz inequality yields

$$\begin{aligned}\left|E^{(1)}_{\log}\right| &\leqslant \frac{1}{\log x}\,\|\mathfrak{z}_a\|_2\cdot\left\|\left(\sum_{1\leqslant n\leqslant x}\frac{S(\lfloor n^c\rfloor)}{n}-\sum_{1\leqslant m\leqslant x^c}\gamma\frac{S(m)}{m}\right)e_1\right\|_2\\ &\leqslant \frac{\sqrt{d}}{\log x}\left\|\sum_{1\leqslant n\leqslant x}\frac{S(\lfloor n^c\rfloor)}{n}-\sum_{1\leqslant m\leqslant x^c}\gamma\frac{S(m)}{m}\right\|_2, \qquad (16)\end{aligned}$$

where $\|\cdot\|_2$ denotes the norm induced by the 2-norm in $\mathbb{C}^d$. Inequality (14) of Lemma 2 implies that $E^{(1)}_{\log}=o(1)$. Since $E^{(2)}_{\log}=o(1)$ holds trivially, we are done.

Next, we assume that the quantity $\mathrm{dens}(\mathbf{u},a)$ exists. We will show that in this case the quantity $\mathrm{dens}(\mathbf{u}_c,a)$ also exists and that they are equal, i.e.,

$$\sum_{1\leqslant n\leqslant x}\left(\mathfrak{z}_a^T S(\lfloor n^c\rfloor)e_1-\mathrm{dens}(\mathbf{u},a)\right)=o(x). \qquad (17)$$

Again, we split up the occurring sum in different parts. We write

$$\sum_{1\leqslant n\leqslant x}\left(\mathfrak{z}_a^T S(\lfloor n^c\rfloor)e_1-\mathrm{dens}(\mathbf{u},a)\right)=E^{(1)}+E^{(2)}+E^{(3)}, \qquad (18)$$

with

$$\begin{aligned}E^{(1)}&=\mathfrak{z}_a^T\left(\sum_{1\leqslant n\leqslant x}S(\lfloor n^c\rfloor)-\sum_{1\leqslant m\leqslant x^c}\gamma m^{\gamma-1}S(m)\right)e_1,\\ E^{(2)}&=\sum_{1\leqslant m\leqslant x^c}\gamma m^{\gamma-1}\left(\mathfrak{z}_a^T S(m)e_1-\mathrm{dens}(\mathbf{u},a)\right),\\ E^{(3)}&=\mathrm{dens}(\mathbf{u},a)\left(\sum_{1\leqslant m\leqslant x^c}\gamma m^{\gamma-1}-\sum_{1\leqslant n\leqslant x}1\right).\end{aligned}$$

Similar to (16) we get

$$\left|E^{(1)}\right|\leqslant\sqrt{d}\left\|\sum_{1\leqslant n\leqslant x}S(\lfloor n^c\rfloor)-\sum_{1\leqslant m\leqslant x^c}\gamma m^{\gamma-1}S(m)\right\|_2.$$

Theorem 5 implies that $E^{(1)}=o(x)$. In order to treat $E^{(2)}$ note that there exists a continuous, positive and monotonic function $g(t)$ with $g(t)\to 0$ for $t\to\infty$ such that

$$\left|\sum_{1\leqslant m\leqslant t}\left(\mathfrak{z}_a^T S(m)e_1-\mathrm{dens}(\mathbf{u},a)\right)\right|\leqslant t g(t).$$

(This easily follows from the fact that the density of a exists.) By partial summation we obtain

$$E^{(2)} = \gamma x^{1-c} \sum_{1 \leqslant m \leqslant x^c} \left(\mathfrak{z}_a^T S(m) e_1 - \mathrm{dens}(\mathbf{u}, a)\right) - \int_1^{x^c} \left(\sum_{1 \leqslant m \leqslant t} \left(\mathfrak{z}_a^T S(m) e_1 - \mathrm{dens}(\mathbf{u}, a)\right) \right) \gamma(\gamma-1) t^{\gamma-2} \mathrm{d}t.$$

Hence we have

$$\begin{aligned} \left|E^{(2)}\right| &\leqslant \gamma x^{1-c} x^c g(x^c) + \int_1^{x^{c/2}} g(t)\gamma(1-\gamma)t^{\gamma-1}\mathrm{d}t + \int_{x^{c/2}}^{x^c} g(t)\gamma(1-\gamma)t^{\gamma-1}\mathrm{d}t \\ &\ll x g(x^c) + x^{1/2} + g(x^{c/2})x. \end{aligned} \tag{19}$$

This implies $E^{(2)} = o(x)$. That $E^{(3)} = o(x)$ is a simply consequence of Euler-Maclaurin's summation formula. We finally obtain that (17) holds true.

In order to finish the proof of Theorem 1 it remains to show that existence of the quantity $\mathrm{dens}(\mathbf{u}_c, a)$ implies existence of the quantity $\mathrm{dens}(\mathbf{u}, a)$. In particular, this holds true if

$$\sum_{1 \leqslant m \leqslant x} \left(\mathfrak{z}_a^T S(m) e_1 - \mathrm{dens}(\mathbf{u}_c, a)\right) = o(x).$$

Using a similar decomposition as in (18), we can use Lemma 2 (Inequality (15)) in order to show that this holds true indeed. □

3.3. **Proof of Theorem 2.** Let us fix some $c \in (1, 7/5)$, $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$ and $k > 0$. Then we have to show that

$$\lim_{x \to \infty} \frac{1}{x} \# \Big\{ 1 \leqslant n \leqslant x : (\mathbf{t}(\lfloor n^c \rfloor), \mathbf{t}(\lfloor n^c \rfloor + k)) = (\varepsilon_1, \varepsilon_2) \Big\} = \ell_k(\varepsilon_1, \varepsilon_2).$$

where $\ell_k(\varepsilon_1, \varepsilon_2) = \lim_{x \to \infty} \frac{1}{x} \#\{1 \leqslant n \leqslant x : (\mathbf{t}(n), \mathbf{t}(n+k)) = (\varepsilon_1, \varepsilon_2)\}$. Note, that

$$\left| \sum_{n \leqslant x} (-1)^{s_2(n+k)} \right| \leqslant 2 \tag{20}$$

for all $x \geqslant 1$ and $k > 0$. Thus we have

$$\begin{aligned} &\# \Big\{ 1 \leqslant n \leqslant x : (\mathbf{t}(n), \mathbf{t}(n+k)) = (\varepsilon_1, \varepsilon_2) \Big\} \\ &\quad = \sum_{n \leqslant x} \frac{1 + (-1)^{s_2(n) + \varepsilon_1}}{2} \cdot \frac{1 + (-1)^{s_2(n+k) + \varepsilon_2}}{2} \\ &\quad = \frac{x}{4} + \frac{(-1)^{\varepsilon_1 + \varepsilon_2}}{4} \sum_{n \leqslant x} (-1)^{s_2(n) + s_2(n+k)} + O(1). \end{aligned}$$

Set $\gamma_k := \lim_{x\to\infty} 1/x \sum_{n\leqslant x}(-1)^{s_2(n)+s_2(n+k)}$. (Note, that this limit really exists, see the introduction of this article.) Then we have

$$\ell_k(\varepsilon_1,\varepsilon_2) = \frac{1}{4} + \frac{(-1)^{\varepsilon_1+\varepsilon_2}}{4}\gamma_k.$$

The same calculation as above shows that we have

$$\begin{aligned}
&\#\Big\{1 \leqslant n \leqslant x : (\mathbf{t}(\lfloor n^c \rfloor), \mathbf{t}(\lfloor n^c \rfloor + k)) = (\varepsilon_1,\varepsilon_2)\Big\} \\
&\quad = \frac{x}{4} + \frac{(-1)^{\varepsilon_1}}{4}\sum_{n\leqslant x}(-1)^{s_2(\lfloor n^c\rfloor)} + \frac{(-1)^{\varepsilon_2}}{4}\sum_{n\leqslant x}(-1)^{s_2(\lfloor n^c\rfloor+k)} \\
&\qquad + \frac{(-1)^{\varepsilon_1+\varepsilon_2}}{4}\sum_{n\leqslant x}(-1)^{s_2(\lfloor n^c\rfloor)+s_2(\lfloor n^c\rfloor+k)}.
\end{aligned} \tag{21}$$

It is now easy to show (by partial summation) that Theorem 5 and (20) imply

$$\sum_{n\leqslant x}(-1)^{s_2(\lfloor n^c\rfloor+k)} = o(x),$$

for all $k > 0$. Since the function $F(n) = (-1)^{s_2(n)+s_2(n+k)}$ is a generalized 2-multiplicative function (with $L = k$), we obtain

$$\sum_{n\leqslant x}(-1)^{s_2(\lfloor n^c\rfloor)+s_2(\lfloor n^c\rfloor+k)} = \sum_{m\leqslant x^c}\gamma m^{\gamma-1}(-1)^{s_2(m)+s_2(m+k)}.$$

Similar to the calculations in the proof of Theorem 1, partial summation shows that

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x}(-1)^{s_2(\lfloor n^c\rfloor)+s_2(\lfloor n^c\rfloor+k)} = \gamma_k.$$

Together with (21) this proves Theorem 2.

4. Correlation of consecutive terms

In this section we prove Proposition 1 and Theorem 3. In order to do so, we need some exponential sum estimates which we show in the following section.

4.1. **Exponential sums.**

Proposition 3. *Let $1 < c < 2$ be a real number and let x and ν be integers with $\nu \geqslant 1$ and $2^{\nu-1} \leqslant x \leqslant 2^\nu$. Furthermore, let $\alpha, \beta \in \mathbb{R}$ such that $\|\alpha+\beta\| \geqslant 2^{\nu(1-c)(5-c)/3}$. Then we have*[3]

$$\sum_{2^{\nu-1}<n\leqslant x} \mathrm{e}(\alpha\lfloor n^c\rfloor + \beta\lfloor (n+1)^c\rfloor) \ll \nu^2 2^{\nu(7+c)/9}.$$

[3] In this section, the implied constants may depend on c.

Lemma 3. *Let $1 < c < 2$ be a real number and let x, ν and H be integers with $\nu \geqslant 1$, $1 \leqslant H \leqslant 2^{\nu(2-c)/3}$ and $2^{\nu-1} \leqslant x \leqslant 2^{\nu}$. Furthermore, let $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $2^{\nu(1-c)(5-c)/3} \leqslant |\gamma_1 + \gamma_2|$ and $|\gamma_1|, |\gamma_2| \leqslant H$. Then we have*

$$\sum_{2^{\nu-1} < n \leqslant x} \mathrm{e}(\gamma_1 n^c + \gamma_2 (n+1)^c) \ll H^{1/2} 2^{\nu c/2}.$$

Proof. Let us denote the considered sum by T and set $\gamma = \gamma_1 + \gamma_2$. For $n \in [2^{\nu-1}, 2^{\nu})$ we have

$$\mathrm{e}(\gamma_1 n^c + \gamma_2 (n+1)^c) = \mathrm{e}\left(\gamma n^c + \gamma_2 c n^{c-1} + \gamma_2 \frac{c(c-1)}{2} n^{c-2}\right) + O(H 2^{\nu(c-3)}).$$

Since $H 2^{\nu(c-2)} \ll 1$, we get

$$T \ll 1 + \sum_{2^{\nu-1} < n \leqslant x} \mathrm{e}\left(\gamma n^c + \gamma_2 c n^{c-1} + \gamma_2 \frac{c(c-1)}{2} n^{c-2}\right).$$

Set $g(y) = \gamma y^c + \gamma_2 c y^{c-1} + \gamma_2 (c(c-1)/2) y^{c-2}$. Then the second derivative of g is given by $g''(y) = \gamma c(c-1) y^{c-2} + \gamma_2 c(c-1)(c-2) y^{c-3} + \gamma_2 (c(c-1)(c-2)(c-3)/2) y^{c-4}$. Since $|\gamma| \geqslant 2^{\nu(1-c)(5-c)/3}$ and $H \leqslant 2^{\nu(2-c)/3}$, we have for $y \in [2^{\nu-1}, 2^{\nu})$ the inequalities

$$\left|\gamma c(c-1) y^{c-2}\right| \gg 2^{\nu(1-c)(5-c)/3 + \nu(c-2)} = 2^{\nu(c^2-3c-1)/3},$$

and

$$\begin{aligned} &\left|\gamma_2 c(c-1)(c-2) y^{c-3} + \gamma_2 (c(c-1)(c-2)(c-3)/2) y^{c-4}\right| \\ &\qquad \ll H 2^{\nu(c-3)} \ll 2^{\nu(2-c)/3 + \nu(c-3)} = 2^{\nu(2c-7)/3}. \end{aligned}$$

We see that we can ignore the second and the third term of the derivative (note that $c^2 - 3c - 1 > 2c - 7$ if $c < 2$) and we obtain

$$|\gamma| 2^{\nu(c-2)} \ll |g''(y)| \ll |\gamma| 2^{\nu(c-2)}$$

for every $y \in [2^{\nu-1}, 2^{\nu})$. Thereom 2.2 of [6] implies that

$$T \ll |\gamma|^{1/2} 2^{\nu c/2} + \frac{1}{|\gamma|^{1/2}} 2^{\nu(1-c/2)}.$$

Since we have

$$(c-1)(5-c)/6 + 1 - c/2 = (-c^2 + 6c - 5)/6 + 1 - c/2 < (6c-6)/6 + 1 - c/2 = c/2,$$

the constraints on γ and H imply the desired result. □

Proof of Proposition 3. Let us denote the considered sum by S. Without loss of generality, we can assume that $0 \leqslant \alpha, \beta < 1$. Let k be a positive integer (which we choose later on) and set

$$I_\ell := \left[\frac{\ell}{k}, \frac{\ell+1}{k}\right) \qquad \ell = 0, \ldots, k-1.$$

We start with the following correlation:

$$S = \sum_{0\leqslant \ell_1,\ell_2<k}\ \sum_{n\in\mathcal{I}_{\ell_1,\ell_2}} \mathrm{e}(\alpha\lfloor n^c\rfloor + \beta\lfloor (n+1)^c\rfloor),$$

where $\mathcal{I}_{\ell_1,\ell_2} := \{2^{\nu-1} < n \leqslant x : \{n^c\} \in I_{\ell_1}, \{(n+1)^c\} \in I_{\ell_2}\}$. If $n \in \mathcal{I}_{\ell_1,\ell_2}$, then there exist real numbers $0 \leqslant \theta_1, \theta_2 < 1$, such that

$$\begin{aligned}\mathrm{e}(\alpha\lfloor n^c\rfloor + \beta\lfloor (n+1)^c\rfloor) &= \mathrm{e}\left(\alpha n^c + \beta(n+1)^c - \alpha\frac{\ell_1}{k} - \beta\frac{\ell_2}{k} - \alpha\frac{\theta_1}{k} - \beta\frac{\theta_2}{k}\right)\\ &= \mathrm{e}\left(\alpha n^c + \beta(n+1)^c - \alpha\frac{\ell_1}{k} - \beta\frac{\ell_2}{k}\right) + O\left(\frac{1}{k}\right).\end{aligned}$$

Thus, we obtain

$$|S| \ll \sum_{0\leqslant \ell_1,\ell_2<k}\left|\sum_{n\in\mathcal{I}_{\ell_1,\ell_2}} \mathrm{e}(\alpha n^c + \beta(n+1)^c)\right| + \frac{2^\nu}{k}. \tag{22}$$

If we set $f_\ell(x) := \mathbf{1}_{I_\ell}(\{x\})$, where $\mathbf{1}_A$ denotes the characteristic function of a set A, then inequality (22) reads as follows:

$$|S| \ll \sum_{0\leqslant \ell_1,\ell_2<k}\left|\sum_{2^{\nu-1}<n\leqslant x} \mathrm{e}(\alpha n^c + \beta(n+1)^c) f_{\ell_1}(n^c) f_{\ell_2}((n+1)^c)\right| + \frac{2^\nu}{k}. \tag{23}$$

Next, we approximate the function f_ℓ by trigonometric polynomials (similar to Section 3.1). Let $H \geqslant 1$ be an integer. Then there exist coefficients $a_{\ell,H}(h)$ with $|a_{\ell,H}(h)| \leqslant 2$, such that the function

$$f^*_{\ell,H}(t) = (\gamma_2 - \gamma_1) + \frac{1}{2\pi i}\sum_{1\leqslant |h|\leqslant H} \frac{a_{\ell,H}(h)}{h}\,\mathrm{e}(ht)$$

verifies

$$|f_\ell(t) - f^*_{\ell,H}(t)| \leqslant \frac{1}{2H+2}\left(\kappa_H\left(t - \frac{\ell}{k}\right) + \kappa_H\left(t - \frac{\ell+1}{k}\right)\right),$$

where $\kappa_H(t)$ is defined by (7). This follows from [19, Theorem 19] and a simple continuity argument (even though f_ℓ does not satisfy Vaaler's normalizing condition). We obtain (the integer H is chosen in the last step of the proof)

$$\left|\sum_{2^{\nu-1}<n\leqslant x} \mathrm{e}(\alpha n^c + \beta(n+1)^c) f_{\ell_1}(n^c) f_{\ell_2}((n+1)^c)\right| \leqslant S^*_{\ell_1,\ell_2} + R(H), \tag{24}$$

where

$$S^*_{\ell_1,\ell_2} := \left|\sum_{2^{\nu-1}<n\leqslant x} \mathrm{e}(\alpha n^c + \beta(n+1)^c) f^*_{\ell_1,H}(n^c) f^*_{\ell_2,H}((n+1)^c)\right|, \tag{25}$$

and

$$R(H) := \sum_{2^{\nu-1}<n\leqslant x} \left| f_{\ell_1}(n^c) f_{\ell_2}((n+1)^c) - f^*_{\ell_1,H}(n^c) f^*_{\ell_2,H}((n+1)^c) \right|.$$

Since $f_{\ell_1} f_{\ell_2} - f^*_{\ell_1,H} f^*_{\ell_2,H}$ is equal to

$$(f_{\ell_1} - f^*_{\ell_1,H})(f_{\ell_2} + f^*_{\ell_2,H}) + f_{\ell_1}(f_{\ell_2} - f^*_{\ell_2,H}) + f_{\ell_2}(f^*_{\ell_1,H} - f_{\ell_1}),$$

and $|f_\ell(x)| \leqslant 1$, $|f^*_{\ell,H}(x)| \leqslant |f^*_{\ell,H}(x) - f_\ell(x)| + 1 \leqslant 2$, we have

$$R(H) \ll \frac{1}{2H+2} \max_{\ell \in \{\ell_1,\ell_2\}} \sum_{2^{\nu-1}\leqslant n\leqslant x} \left(\kappa_H\left(n^c - \frac{\ell}{k}\right) + \kappa_H\left(n^c - \frac{\ell+1}{k}\right)\right).$$

Using the definition of κ_H, we obtain

$$R(H) \ll \frac{1}{2H+2} \sum_{0\leqslant|h|\leqslant H} \left| \sum_{2^{\nu-1}\leqslant n\leqslant x} \mathrm{e}\left(hn^c\right)\right|.$$

We separate the case $h = 0$ from $h \neq 0$ and apply Lemma 3 (with $\gamma_1 = h$ and $\gamma_2 = 0$). This is admissible as long as $H \leqslant 2^{\nu(2-c)/3}$. We obtain

$$R(H) \ll \frac{2^\nu}{H} + H^{1/2} 2^{\nu c/2}. \tag{26}$$

Next, we use the definition of $f^*_{\ell,H}$ to deal with $S^*_{\ell_1,\ell_2}$. We get

$$\begin{aligned} S^*_{\ell_1,\ell_2} &= \left| \sum_{2^{\nu-1}<n\leqslant x} \mathrm{e}(\alpha n^c + \beta(n+1)^c) \left(\frac{1}{k} + \frac{1}{2\pi i} \sum_{1\leqslant|h_1|\leqslant H} \frac{a_{\ell_1,H}(h_1)}{h_1} \mathrm{e}(h_1 n^c)\right) \right. \\ &\qquad \left. \cdot \left(\frac{1}{k} + \frac{1}{2\pi i} \sum_{1\leqslant|h_2|\leqslant H} \frac{a_{\ell_2,H}(h_2)}{h_2} \mathrm{e}(h_2 (n+1)^c)\right) \right| \\ &\leqslant \frac{T(\alpha,\beta)}{k^2} + \frac{1}{k} \sum_{1\leqslant|h_1|\leqslant H} \frac{T(\alpha+h_1,\beta)}{|h_1|} + \frac{1}{k} \sum_{1\leqslant|h_2|\leqslant H} \frac{T(\alpha,\beta+h_2)}{|h_2|} \\ &\qquad + \sum_{1\leqslant|h_1|,|h_2|\leqslant H} \frac{T(\alpha+h_1,\beta+h_2)}{|h_1|\cdot|h_2|}, \end{aligned}$$

where

$$T(\gamma_1,\gamma_2) = \left| \sum_{2^{\nu-1}<n\leqslant x} \mathrm{e}(\gamma_1 n^c + \gamma_2 (n+1)^c)\right|. \tag{27}$$

Since $\|\alpha+\beta\| \geqslant 2^{\nu(1-c)(5-c)/3}$ we have $|\alpha+\beta+h_1+h_2| > 2^{\nu(1-c)(5-c)/3}$ for all integers h_1 and h_2. Assuming that $H \leqslant 2^{\nu(2-c)/3}$, Lemma 3 implies that $S^*_{\ell_1,\ell_2}$ is bounded by some constant times

$$\frac{1}{k^2} H^{1/2} 2^{\nu c/2} + \frac{1}{k}(\log H) H^{1/2} 2^{\nu c/2} + \frac{1}{k}(\log H) H^{1/2} 2^{\nu c/2} + (\log H)^2 H^{1/2} 2^{\nu c/2}$$

for all $0 \leqslant \ell_1, \ell_2 < k$. Hence we obtain (see (23), (24) and (26))

$$S \ll k^2 (\log H)^2 H^{1/2} 2^{\nu c/2} + k^2 2^\nu / H + 2^\nu / k.$$

If we set $H = \lfloor 2^{\nu(2-c)/3} \rfloor$ and $k = \lfloor 2^{\nu(2-c)/9} \rfloor$, then we obtain

$$S \ll \nu^2 2^{\nu(2(2-c)/9+(2-c)/6+c/2)} + 2^{\nu(2(2-c)/9+1-(2-c)/3)} + 2^{\nu(1-(2-c)/9)} \ll \nu^2 2^{\nu(7+c)/9}.$$

This proves the desired result. □

Proposition 4. *Let $1 < c < 2$ be a real number and let x and ν, ρ be integers with $\nu \geqslant 1$, $\rho < \nu(2-c)/6$ and $2^{\nu-1} \leqslant x \leqslant 2^\nu$. Furthermore, let $\alpha, \beta \in \mathbb{R}$ such that $\|\beta\| \geqslant 2^{\nu(1-c)+2\rho}$ and such that $\alpha + \beta \in \mathbb{Z}$. Then we have*

$$\sum_{2^{\nu-1} < n \leqslant x} \mathrm{e}(\alpha \lfloor n^c \rfloor + \beta \lfloor (n+1)^c \rfloor) \ll \nu^2 2^{\nu - 2\rho/3}.$$

Lemma 4. *Let $1 < c < 2$ be a real number and let x, ν, ρ and H be integers with $\nu \geqslant 1$, $\rho \geqslant 0$, $1 \leqslant H \leqslant 2^{\nu(2-c)/3}$ and $2^{\nu-1} \leqslant x \leqslant 2^\nu$. Furthermore, let $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $2^{\nu(1-c)+2\rho} < |\gamma_2| \leqslant H$ and $\gamma_1 + \gamma_2 = 0$. Then we have*

$$\sum_{2^{\nu-1} < n \leqslant x} \mathrm{e}(\gamma_1 n^c + \gamma_2 (n+1)^c) \ll 2^{\nu - 2\rho}.$$

Proof. Let us denote the considered sum by T. For $n \in [2^{\nu-1}, 2^\nu)$ we have (note, that $\gamma_1 + \gamma_2 = 0$)

$$\mathrm{e}(\gamma_1 n^c + \gamma_2 (n+1)^c) = \mathrm{e}\left(\gamma_2 c n^{c-1} + \gamma_2 \frac{c(c-1)}{2} n^{c-2}\right) + O(H 2^{\nu(c-3)}).$$

Since $H 2^{\nu(c-2)} \ll 1$, we get

$$T \ll 1 + \sum_{2^{\nu-1} < n \leqslant x} \mathrm{e}\left(\gamma_2 c n^{c-1} + \gamma_2 \frac{c(c-1)}{2} n^{c-2}\right).$$

Set $\tilde{g}(y) = \gamma_2 c y^{c-1} + \gamma_2 (c(c-1)/2) y^{c-2}$. Then the first derivative of $\tilde{g}$ is given by $\tilde{g}'(y) = \gamma_2 c(c-1) y^{c-2} + \gamma_2 (c(c-1)(c-2)/2) y^{c-3}$. Thus we have

$$|\gamma_2| 2^{\nu(c-2)} \ll |\tilde{g}'(y)| \ll |\gamma_2| 2^{\nu(c-2)}$$

for all $y \in [2^{\nu-1}, 2^\nu)$. Since $|\gamma_2| 2^{\nu(c-2)} \leqslant H 2^{\nu(c-2)} \leqslant 2^{2\nu(c-2)/3}$ and $2(c-2)/3 < 0$, we see that we can use Theorem 2.1 of [6] (at least for ν sufficiently large) in order to obtain

$$T \ll \frac{1}{|\gamma_2| 2^{\nu(c-2)}} \ll 2^{\nu - 2\rho}.$$

This proves Lemma 4. □

Proof of Proposition 4. Let us denote the considered sum again by S. The first steps of the proof are as in the proof of Propostion 3. Without loss of generality, we can assume that $0 \leqslant \alpha, \beta < 1$ (that is, $\alpha + \beta = 1$). Then we obtain (see (23), (24) and (26))

$$|S| \ll \left(\sum_{0 \leqslant \ell_1, \ell_2 < k} S^*_{\ell_1,\ell_2} \right) + k^2 \frac{2^\nu}{H} + k^2 H^{1/2} 2^{\nu c/2} + \frac{2^\nu}{k},$$

where $H \leqslant 2^{\nu(2-c)/3}$ and k are positive integers (chosen in the last step of the proof) and $S^*_{\ell_1,\ell_2}$ is defined by (25). We get (as in the proof of Propostion 3)

$$S^*_{\ell_1,\ell_2} \leqslant \frac{T(\alpha,\beta)}{k^2} + \frac{1}{k} \sum_{1 \leqslant |h_1| \leqslant H} \frac{T(\alpha + h_1, \beta)}{|h_1|} + \frac{1}{k} \sum_{1 \leqslant |h_2| \leqslant H} \frac{T(\alpha, \beta + h_2)}{|h_2|} + \sum_{1 \leqslant |h_1|, |h_2| \leqslant H} \frac{T(\alpha + h_1, \beta + h_2)}{|h_1| \cdot |h_2|},$$

where $T(\gamma_1, \gamma_2)$ is defined by (27). If h_1 and h_2 are two integers, then

$$\begin{cases} |\alpha + \beta + h_1 + h_2| \geqslant 1 & \text{if } h_1 + h_2 \neq -1, \\ |\alpha + \beta + h_1 + h_2| = 0 & \text{otherwise.} \end{cases}$$

Thus, if $h_1 + h_2 \neq -1$, Lemma 3 implies $T(\alpha + h_1, \beta + h_2) \ll H^{1/2} 2^{\nu c/2}$. Thus we obtain

$$S^*_{\ell_1,\ell_2} \ll \frac{1}{k} T(\alpha - 1, \beta) + \frac{1}{k} T(\alpha, \beta - 1) + \sum_{1 \leqslant h_1 < H} \frac{T(\alpha + h_1, \beta - h_1 - 1)}{|h_1| \cdot |h_1 + 1|} + \sum_{1 \leqslant h_2 < H} \frac{T(\alpha - h_2 - 1, \beta + h_2)}{|h_2 + 1| \cdot |h_2|} + H^{1/2} (\log H)^2 2^{\nu c/2}.$$

If $h_1 + h_2 = -1$, then we can use Lemma 4 and we obtain $T(\alpha + h_1, \beta + h_2) \ll 2^{\nu - 2\rho}$. Since the sums in the last expression converge for H to infinity, we get

$$S \ll k^2 2^{\nu - 2\rho} + k^2 H^{1/2} (\log H)^2 2^{\nu c/2} + k^2 \frac{2^\nu}{H} + \frac{2^\nu}{k}.$$

We set $H = \lfloor 2^{2\rho} \rfloor$ and $k = \lfloor 2^{2\rho/3} \rfloor$. This is admissible, since $\rho < \nu(2-c)/6$ implies $H \leqslant 2^{\nu(2-c)/3}$. Furthermore, the assumption on ρ also implies

$$k^2 H^{1/2} (\log H)^2 2^{\nu c/2} \ll \nu^2 \, 2^{\nu c/2 + 7\rho/3} \ll \nu^2 \, 2^{\nu - 2\rho/3}.$$

Thus, we finally have $S \ll \nu^2 \, 2^{\nu - 2\rho/3}$. This proves the desired result. □

4.2. **Proof of Proposition 1 and Theorem 3.** At first we show that we can replace the sum-of-digits function by a truncated version of it. In the following lemma, we use the fact that the higher placed digit of $\lfloor n^c \rfloor$ and $\lfloor (n+1)^c \rfloor$ do not differ in most of the cases. A similar idea has been used in [13] and [14]. (In [13, Lemma 16], Mauduit and Rivat showed a slightly more general result for $c = 2$.) We set

$$\lambda = \lfloor \nu(c-1) \rfloor + 2\rho,$$

where ρ is an integer satisfying

$$\rho < \nu(2-c)(c-1)/6, \tag{28}$$

and

$$s_\lambda(m) = \varepsilon_{\lambda-1}(m) + \varepsilon_{\lambda-2}(m) + \cdots + \varepsilon_0(m),$$

where $\varepsilon_j(m)$, $j \geqslant 0$ are the binary digits of m.

Lemma 5. *For all integers $\nu > 0$ and x with $2^{\nu-1} \leqslant x \leqslant 2^\nu$ we denote by $E(\nu, x)$ the set of integers n such that $2^{\nu-1} < n \leqslant x$ and*

$$s_2(\lfloor n^c \rfloor) - s_2(\lfloor (n+1)^c \rfloor \neq s_\lambda(\lfloor n^c \rfloor) - s_\lambda(\lfloor (n+1)^c \rfloor.$$

Then we have

$$\# E(\nu, x) \ll 2^{\nu-\rho}$$

Proof. This lemma can be shown in the same way (with minor modifications) as in the case $c = 2$. Thus, we omit the proof (see the proof of [13, Lemma 16] for details). $\square$

Proof of Proposition 1. Lemma 5 implies

$$\sum_{2^{\nu-1} < n \leqslant x} (-1)^{s_2(\lfloor n^c \rfloor) + s_2(\lfloor (n+1)^c \rfloor)} \ll S_\lambda + 2^{\nu-\rho}, \tag{29}$$

where

$$S_\lambda := \sum_{2^{\nu-1} < n \leqslant x} (-1)^{s_\lambda(\lfloor n^c \rfloor) + s_\lambda(\lfloor (n+1)^c \rfloor)}.$$

We get

$$S_\lambda = \sum_{0 \leqslant u,v < 2^\lambda} \sum_{2^{\nu-1} < n \leqslant x} (-1)^{s_\lambda(u) + s_\lambda(v)} \cdot \left(\frac{1}{2^\lambda} \sum_{0 \leqslant h < 2^\lambda} \mathrm{e}\left(\frac{h\left(\lfloor n^c \rfloor - u\right)}{2^\lambda} \right) \right) \cdot \left(\frac{1}{2^\lambda} \sum_{0 \leqslant k < 2^\lambda} \mathrm{e}\left(\frac{k\left(\lfloor (n+1)^c \rfloor - v\right)}{2^\lambda} \right) \right).$$

Using the discrete Fourier transform of s_2, we can write

$$S_\lambda = S_\lambda^{(1)} + S_\lambda^{(2)} + S_\lambda^{(3)}, \tag{30}$$

where we set

$$S_\lambda^{(i)} = \sum_{(h,k) \in I_i} F_\lambda(h, 1/2) F_\lambda(k, 1/2) \sum_{2^{\nu-1} < n \leqslant x} \mathrm{e}\left(\frac{h}{2^\lambda} \lfloor n^c \rfloor + \frac{k}{2^\lambda} \lfloor (n+1)^c \rfloor \right).$$

for $1 \leqslant i \leqslant 3$, and

$$I_1 := \{(h,k) : 0 \leqslant h, k < 2^\lambda, h + k \not\equiv 0 \bmod 2^\lambda\},$$
$$I_2 := \{(h,k) : 0 \leqslant h, k < 2^\lambda, h + k \equiv 0 \bmod 2^\lambda, \|k/2^\lambda\| \geqslant 2^{\nu(1-c)+2\rho}\},$$
$$I_3 := \{(h,k) : 0 \leqslant h, k < 2^\lambda, h + k \equiv 0 \bmod 2^\lambda, \|k/2^\lambda\| < 2^{\nu(1-c)+2\rho}\}.$$

Since $\rho < \nu(2-c)(c-1)/6$ we have $\nu(1-c) - 2\rho > \nu(1-c)(5-c)/3$ and we can employ Proposition 3 in order to obtain

$$\begin{aligned} S_\lambda^{(1)} &\ll v^2 2^{\nu(7+c)/9} \sum_{(h,k) \in I_1} |F_\lambda(h,1/2) F_\lambda(k,1/2)| \\ &\ll v^2 2^{\nu(7+c)/9} \sum_{0 \leqslant h,k < 2^\lambda} |F_\lambda(h,1/2) F_\lambda(k,1/2)|. \end{aligned}$$

Part 2 of Lemma 1 implies

$$S_\lambda^{(1)} \ll v^2 2^{\nu(7+c)/9 + 2\eta_2 \lambda},$$

where $\eta_2 = \log(2+\sqrt{2})/\log 16$. Next, we apply Proposition 4 (note, that (28) implies $\rho < \nu(2-c)/6$) and get

$$\begin{aligned} S_\lambda^{(2)} &\ll v^2 2^{\nu - 2\rho/3} \sum_{\substack{0 \leqslant k < 2^\lambda \\ \|k/2^\lambda\| \geqslant 2^{\nu(1-c)+2\rho}}} |F_\lambda(k,1/2) F_\lambda(-k,1/2)| \\ &\ll v^2 2^{\nu - 2\rho/3} \sum_{0 \leqslant k < 2^\lambda} |F_\lambda(k,1/2) F_\lambda(-k,1/2)|. \end{aligned}$$

Part 3 of Lemma 1 implies

$$S_\lambda^{(2)} \ll v^2 2^{\nu - 2\rho/3}.$$

If $(h,k) \in I_3$ we bound the exponential sum trivially by 2^ν and apply Lemma 1 (part 1) to obtain

$$S_\lambda^{(3)} \ll 2^\nu \sum_{(h,k) \in I_3} |F_\lambda(h,1/2) F_\lambda(k,1/2)| \ll 2^{\nu + \lambda + \nu(1-c) + 2\rho - c_2\lambda/2},$$

where $c_2 > 0$ is defined in Lemma 1. Hence we obtain (see (29), (30), and the estimates for $S_\lambda^{(1)}$, $S_\lambda^{(2)}$ and $S_\lambda^{(3)}$)

$$\begin{aligned} &\sum_{2^{\nu-1} < n \leqslant x} (-1)^{s_2(\lfloor n^c \rfloor) + s_2(\lfloor (n+1)^c \rfloor)} \\ &\qquad \leqslant v^2 2^{\nu(7+c)/9 + 2\eta_2\lambda} + v^2 2^{\nu - 2\rho/3} + 2^{\nu + \lambda + \nu(1-c) + 2\rho - c_2\lambda/2} + 2^{\nu - \rho} \\ &\qquad \leqslant v^2 2^{\nu((7+c)/9 + 2\eta_2(c-1)) + 4\eta_2\rho} + v^2 2^{\nu - 2\rho/3} + 2^{\nu(1 - c_2(c-1)/2) + \rho(4 - c_2)}. \end{aligned}$$

Finally, if

$$c < \frac{18\eta_2 + 2}{18\eta_2 + 1} = \frac{4\log 2 + 9\log(2+\sqrt{2})}{2\log 2 + 9\log(2+\sqrt{2})},$$

then we have $(7+c)/9 + 2\eta_2(c-1) < 1$ and we can choose ρ in an appropriate way (also satisfying (28)), such that

$$\sum_{2^{\nu-1} < n \leqslant x} (-1)^{s_2(\lfloor n^c \rfloor) + s_2(\lfloor (n+1)^c \rfloor)} \ll 2^{\nu(1-\sigma_c)}$$

for a constant $\sigma_c > 0$. A standard argument using geometric series finally implies the desired result. $\square$

Proof of Theorem 3. Using standard Fourier analysis (cf. the proof of Theorem 2), Proposition 1 implies the desired result. $\square$

5. The $\lfloor n \log n \rfloor$-case

Suppose that a sequence a_n has the property that the limit

$$\lim_{x\to\infty} \frac{1}{x} \sum_{1\leqslant n\leqslant x} a_n = a \tag{31}$$

exists. Then it also follows that for every $\gamma < 1$ we have

$$\lim_{x\to\infty} \frac{1}{x} \sum_{\gamma x\leqslant n\leqslant x} a_n = (1-\gamma)a. \tag{32}$$

The main idea of the proof of Theorem 4 is to show that (32) cannot hold for

$$a_n = (-1)^{s_2(\lfloor n\log n\rfloor)+s_2(\lfloor (n+1)\log(n+1)\rfloor)} \tag{33}$$

and for a properly chosen constant $\gamma < 1$. This will be done in several steps.

The first one is to show a result similar to the one stated in Theorem 5. In particular, we are interested in the sum

$$\sum_{1\leqslant n\leqslant x} (-1)^{s_2(\lfloor n\log n\rfloor)+s_2(\lfloor n\log n\rfloor+\ell)},$$

for ℓ not to large. In order to analyze this sum one has to study the inverse of the function $x\log x$ (which we denote by $\gamma(x)$). Note, that in Section 3.1 (where we are interested in the function x^c), we have to deal with the inverse function x^γ, $\gamma = 1/c$. The considerations in Section 3.1 are relatively easy since the inverse of x^c can be written in an explicit form. Contrary to this situation, the function $\gamma(x)$ cannot be expressed in terms of elementary functions. However, it can be written as

$$\gamma(x) = \frac{x}{W(x)},$$

where $W(x)$ is the (principal branch of the) Lambert W function (see [4]). The function $W(x)$ satisfies the functional equation

$$W(x)e^{W(x)} = x,$$

and we have

$$\log\left(\frac{x}{\log x}\right) < W(x) < \log(x)$$

for $x > e$. Thus it follows that

$$\frac{x}{\log x} < \gamma(x) < \frac{x}{\log x - \log\log x}$$

for $x > e$. Differentiating in the functional equation of the Lambert W function (for example, one has $W'(x) = W(x)/(x + xW(x))$), it is also possible to give lower and upper bounds on the derivatives of $\gamma(x)$. As in Section 3.1, these calculations (which get quite cumbersome) lead to the study of exponential sums. For the sake of brevity we do not give a proof of the following result.

Lemma 6. *We have*

$$\sum_{1\leqslant n\leqslant x} (-1)^{s_2(\lfloor n\log n\rfloor)+s_2(\lfloor n\log n\rfloor+\ell)} = \sum_{1\leqslant m\leqslant x\log x} \frac{1}{\log m}(-1)^{s_2(m)+s_2(m+\ell)} + o(x)$$

uniformly for $\ell \leqslant C\log x$ (for any given constant $C > 0$).

The second lemma follows from partial summation

Lemma 7. *Suppose that b_m is a bounded sequence and $y \leqslant x$. Then*

$$\sum_{y\leqslant m\leqslant x} \frac{b_m}{\log m} = \frac{1}{\log x}\sum_{y\leqslant m\leqslant x} b_m + O\left(\frac{x}{(\log x)^2}\right).$$

Finally we need the following limit relations.

Lemma 8. *We have*

$$\frac{1}{x}\sum_{n\leqslant x}(-1)^{s_2(n)+s_2(n+2^k)} = -\frac{1}{3} + o(1),$$

and

$$\frac{1}{x}\sum_{n\leqslant x}(-1)^{s_2(n)+s_2(n+2^k+2^{k+1})} = \frac{1}{3} + o(1)$$

uniformly for all k with $2^k \leqslant C\log x$ (for any given constant $C > 0$).

Proof. It is well known (see for example [10]) that the result holds true for $k = 0$. Furthermore we have

$$\sum_{n<2^L}(-1)^{s_2(n)+s_2(n+2^k)} = 2^k \sum_{n<2^{L-k}}(-1)^{s_2(n)+s_2(n+1)},$$

and

$$\sum_{n<2^L}(-1)^{s_2(n)+s_2(n+2^k+2^{k+1})} = 2^k \sum_{n<2^{L-k}}(-1)^{s_2(n)+s_2(n+3)}.$$

By splitting up x into subintervals of powers of 2 (according to the binary expansion) and by combining the two mentioned properties the result follows easily. □

Proof of Theorem 4. We first consider real numbers x such that $\log x + 1$ is close to a power of 2. In particular, we suppose that there exists a positive integer k such that for all $n \in [\gamma x, x)$ we have $\log n \in [2^k - 1, 2^k - 1 + \eta)$, where $\eta > 0$ will be chosen to be sufficiently small and $\gamma < 1$ satisfies $-\log\gamma < \eta$.

Since the sequence $n\log n$ is uniformly distributed modulo 1 it follows that

$$\#\{n \in [\gamma x, x) : \{n\log n\} \in [\eta/2, 1-\eta)\} \sim (1-\gamma)x(1-3\eta/2)$$

as $x \to \infty$. Now we use the relation

$$(n+1)\log(n+1) = n\log n + 1 + \log n - \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

and the above construction of x to derive

$$\#\{n \in [\gamma x, x) : \lfloor (n+1)\log(n+1)\rfloor = \lfloor n\log n\rfloor + 2^k\} \geqslant (1-\gamma)x(1-\eta) + o(x).$$

(Note that $\{\log n\} \leqslant \eta$ and $\lfloor \log n \rfloor = 2^k - 1$). This implies (with a_n from (33))

$$\sum_{\gamma x \leqslant n \leqslant x} a_n = \sum_{\gamma x \leqslant n \leqslant x} (-1)^{s_2(\lfloor n\log n\rfloor) + s_2(\lfloor n \log n\rfloor + 2^k)} + R,$$

where $|R| \leqslant 2(1-\gamma)x\eta$ (for x large enough). By applying Lemmas 6–8 it follows that (with $f_k(y) = (-1)^{s_2(y)+s_2(y+2^k)}$)

$$\begin{aligned}\sum_{\gamma x \leqslant n \leqslant x} f_k(\lfloor n\log n\rfloor) &= \sum_{\gamma x\log(\gamma x) \leqslant m \leqslant x\log x} \frac{f_k(m)}{\log m} + o(x) \\ &= \frac{1}{\log x} \sum_{\gamma x\log(\gamma x) \leqslant m \leqslant x\log x} f_k(m) + o(x) = -\frac{1-\gamma}{3}\, x + o(x).\end{aligned}$$

Hence, by choosing $\eta = \frac{1}{12}$ (and $\gamma < 1$ accordingly, for example $\gamma = \frac{12}{13}$) it follows that

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{\gamma x \leqslant n \leqslant x} a_n \leqslant -\frac{1-\gamma}{6}.$$

Similarly we can proceed by choosing x in a way that $\log x + 1$ is close to $2^k + 2^{k+1}$ for some integer k, and we obtain

$$\limsup_{x\to\infty} \frac{1}{x} \sum_{\gamma x \leqslant n \leqslant x} a_n \geqslant \frac{1-\gamma}{6}.$$

Of course this makes it impossible that the limit

$$\lim_{x\to\infty} \frac{1}{x} \sum_{1 \leqslant n \leqslant x} (-1)^{s_2(\lfloor n\log n\rfloor) + s_2(\lfloor (n+1)\log(n+1)\rfloor)}$$

exists. It remains to show that for $(\varepsilon_0, \varepsilon_1) \in \{0,1\}^2$ the asymptotic density

$$\lim_{x\to\infty} \frac{1}{x} \#\Big\{1 \leqslant n \leqslant x : (\mathbf{t}(\lfloor n\log n\rfloor), \mathbf{t}(\lfloor (n+1)\log(n+1)\rfloor)) = (\varepsilon_1, \varepsilon_2)\Big\}$$

does not exist. For $(\alpha_0, \alpha_1) \in \{0,1\}^2$, let

$$F_{\alpha_0,\alpha_1}(x) = \frac{1}{x} \sum_{1 \leqslant n \leqslant x} (-1)^{\alpha_0 s_2(\lfloor n\log n\rfloor) + \alpha_1 s_2(\lfloor (n+1)\log(n+1)\rfloor)},$$

and $G_{\alpha_0,\alpha_1}(x)$ be defined by

$$\frac{1}{x} \#\Big\{1 \leqslant n \leqslant x : (\mathbf{t}(\lfloor n\log n\rfloor), \mathbf{t}(\lfloor (n+1)\log(n+1)\rfloor)) = (\alpha_1, \alpha_2)\}.$$

As in Section 3.3 (cf. (21)) we see that

$$G_{\varepsilon_0,\varepsilon_1}(x) = \frac{1}{4}\big(F_{0,0}(x) + (-1)^{\varepsilon_0} F_{1,0}(x) + (-1)^{\varepsilon_1} F_{0,1}(x) + (-1)^{\varepsilon_0+\varepsilon_1} F_{1,1}(x)\big).$$

Note that $F_{0,0}(x) = 1$ for all positive integers x. Moreover, we have $F_{1,0} = o(1)$ and $F_{0,1} = o(1)$ (this can be proven in the same way as Lemma 6). Since $F_{0,0}$, $F_{1,0}$ and $F_{0,1}$ have a limit when x tends to infinity but not $F_{1,1}$, the expression $G_{\varepsilon_0,\varepsilon_1}$ has no limit either. This finally proves Theorem 4. □

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