THE DISTRIBUTION OF PATTERNS IN DIGITAL EXPANSIONS

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Abstract. By using a generating function approach it is shown that certain functionals of digitial expansion (related to specific finite linear recurrences), e.g. the number of specific digital patterns, satisfy a central limit theorem. We provide asymptotic expansions for mean value and variance and prove a global and a local limit law.

1. Introduction

Let $G = (G_j)_{j\ge 0}$ be a strictly increasing sequence of integers with $G_0 = 1$. Then every non-negative integer n has a (unique) proper G-ary digital expansion

$$n = \sum_{j\ge 0} \epsilon_j(n) G_j$$

with integer digits $\epsilon_j(n) \ge 0$ provided that

$$\sum_{j<k} \epsilon_j(n) G_j < G_k$$

for all $k \ge 0$.

Now assume that all digits are uniformly bounded by some (integer) bound Q (i.e. the sequence of fractions G_{j+1}/G_j is bounded) and let $F : \{0,1,\ldots,Q\}^{L+1} \to \mathbf{R}$ be any given function (for some $L \ge 0$) with $F(0,0,\ldots,0) = 0$. Furthermore, set

$$s_F(n) = \sum_{j\ge 0} F\left(\epsilon_{j+L}(n), \epsilon_{j+L-1}(n), \ldots, \epsilon_j(n)\right).$$

This means that we consider a weighted sum over all subsequent digital patterns of length $L+1$ of the digital expansion of n. For example, for $L = 0$ and $F(\epsilon) = \epsilon$ we just obtain the sum-of-digits function, or if $L = 1$ and $F(\epsilon,\eta) = 1 - \delta_{\epsilon,\eta}$ ($\delta_{x,y}$ denoting the Kronecker delta) then $s_F(n)$ is just counting the number of times that a digit is different from the preceding one etc.

The aim of this paper is to get an insight into the distribution of $s_F(n)$. It is convenient to consider a related sequence of random variables X_N, $N \ge 1$, defined by

$$\mathbf{Pr}[X_N \le x] = \frac{1}{N} |\{n < N : s_F(n) \le x\}|$$

Expected value and variance of X_N are given by

$$\mathbf{E}X_N = \frac{1}{N} \sum_{n<N} s_F(n) \quad \text{and by} \quad \mathbf{V}X_N = \frac{1}{N} \sum_{n<N} (s_F(n) - \mathbf{E}X_N)^2 \tag{1.1}$$

The main purpose of this paper is to prove asymptotic normality (of the distribution of X_N) by the use of generating functions, where it is also possible to derive a local limit law if F attains only integer values. The analytic methods are adapted from [6], [3], and [4].

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It should be noted that special cases of these problems, especially the sum-of-digits function (see [3] for a list of references), have been discussed in the literature quite frequently, e.g. by Kirschenhofer [9], by Cateland [2], by Tenenbaum [12], and by Barat, Tichy, and Tijdeman [1].

2. Results

In the present paper we will deal with basis sequences $G = (G_j)_{j\ge 0}$ which satisfy specific linear recurrences.

2.1. q-Ary Expansions. This is the classical case $G_j = q^j$ (with $q > 1$ an integer). Here the digits ϵ_j are bounded by $0 \le \epsilon_j < q$ and all digital patterns acutally occur. Note that the sequence $G_j = q^j$ satisfies the recurrence $G_j = qG_{j-1}$.

2.2. Finite Recurrences. Generalizing q-ary expansions we consider the following situation: There exist non-negative integers $a_1 \ge a_2 \ge \cdots \ge a_r > 0$ such that

$$G_j = \sum_{i=1}^{j} a_i G_{j-i} + 1$$

for $j < r$ and

$$G_j = \sum_{i=1}^{r} a_i G_{j-i}$$

for $j \ge r$.

It is well known that the characterictic polynomial

$$P(u) = u^r - \sum_{i=1}^{r} a_i u^{r-i}$$

is irreducible and that there is a unique root α of maximal modulus which is real and positive and all other roots have modulus < 1, i.e. α is a Pisot number (compare with [7]). Obviously we have

$$G_j \sim C\alpha^j \quad (j \to \infty)$$

for some constant $C > 0$. Furthermore, a sequence of non-negative integers ϵ_j, $j \ge 0$, constitutes the digital expansions of

$$n = \sum_{j\ge 0} \epsilon_j G_j,$$

i.e. $\epsilon_j(n) = \epsilon_j$, $j \ge 0$, if and only if $\epsilon_j = 0$ for all but finitely many j and if

$$(\epsilon_k, \epsilon_{k-1}, \ldots, \epsilon_{k-r+1}) < (a_1, a_2, \ldots, a_r) \qquad \text{for all } k \ge 0,$$

where "$<$" denotes the lexicographic order. ($\epsilon_j = 0$ for $j < 0$.)

As already mentioned, the usual q-ary case $G_j = q^j$ is contained in this concept with $r = 1$ and $a_1 = q$. Here $\alpha = q$.

2.3. Asymptotic Properties. First we state a theorem concerning expected value and variance of X_N (defined) in (1.1).

Theorem 2.1. *Suppose that $G = (G_j)_{j\ge 0}$ satisfies a linear recurrence of the above type and let a function F be as above. Let $\alpha(z)$ be the analytic function (in a sufficiently small (complex) neighbourhood of $z = 1$) which is defined by the maximal solution of the equation $G(t, z) = \det(t\mathbf{I} - \mathbf{A}(z)) = 0$ (see Lemma 3.3 and (4.1)). Then*

$$\mathbf{E}X_N = \frac{1}{N} \sum_{n<N} s_G(n) = \mu \frac{\log N}{\log \alpha} + \mathcal{O}(1) \tag{2.1}$$

and

$$\mathbf{V}X_N = \frac{1}{N} \sum_{n<N} (s_G(n) - \mathbf{E}X_N)^2 = \sigma^2 \frac{\log N}{\log \alpha} + \mathcal{O}(1), \tag{2.2}$$

where

$$\mu = \frac{\alpha'(1)}{\alpha} \quad \text{and} \quad \sigma^2 = \frac{\alpha''(1)}{\alpha} + \mu - \mu^2.$$

Remark . The asymptotic expansion of $\mathbf{E}X_N$ has been discussed for several special cases ([2], [12], [1]). Expecially the $\mathcal{O}(1)$-term has been analyzed in detail.

Our main result concerns the distribution properties of X_N. We prove asymptotic normality in the weak sense and provide a local limit law.

Theorem 2.2. *Suppose that $G = (G_j)_{j\geq 0}$ satisfies a linear recurrence of the above type and $F : \{0, \dots, Q\}^{L+1} \to \mathbf{R}$ be a given function (with $Q = q - 1$ in the q-ary case and $Q = a_1$ in the more general case). If $\sigma^2 \neq 0$ then for every $\varepsilon > 0$*

$$\frac{1}{N}|\{n < N : s_F(n) < \mathbf{E}X_N + x\mathbf{V}X_N\}| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt + \mathcal{O}((\log N)^{-1/2+\varepsilon}) \tag{2.3}$$

uniformly for all real x as $N \to \infty$.
Furthermore, if F just attains integer values and if

$$d = \gcd\{\kappa(\epsilon, B) : (\epsilon, B) \in \mathcal{B}_{L+1}\} = 1$$

(where $\mathcal{B}_{L+1}$ and $\kappa(\epsilon, B)$ are defined in the sequel) then

$$|\{n < N : s_F(n) = k\}| = \frac{N}{\sqrt{2\pi\mathbf{V}X_N}} \left(\exp\left(-\frac{(k - \mathbf{E}X_N)^2}{2\mathbf{V}X_N} \right) + \mathcal{O}((\log N)^{-1/2+\varepsilon}) \right) \tag{2.4}$$

uniformly for all non-negative integers k as $N \to \infty$.

Remark 1. The case $d > 1$ can be treated in a similar way as we will see in the proof of Theorem 2.2. However, for the sake of shortness and simplicity we just formulated Theorem 2.2 for $d = 1$.

Remark 2. It is an open problem if corresponding theorems hold for a number system G where G_j satisfies an infinite linear recurrence related to the Parry expansion of a real number $\alpha > 1$ (see [11], [8]). The case $L = 0$ is discussed in [3].

In section 3 show some preliminary facts about the q-ary case in every detail. In section 4 we indicate which changes have to be made for finite recurrences of the above type. In section 5 we prove Theorem 2.1, i.e. the asymptotic expansion for mean value and variance. Finally, sections 6 and 7 are devoted to the global and local limit laws of Theorem 2.2.

3. q-Ary Expansions

Let

$$\mathcal{B}_L = \{(\epsilon_{L-1}(n), \epsilon_{L-2}(n), \dots, \epsilon_0(n)) : n < G_L = q^L\} \tag{3.1}$$

be the set of blocks $B \subseteq \{0, 1, \dots, q-1\}^L$ of length L which actually occur in digital expansions. In the q-ary case we trivially have

$$\mathcal{B}_L = \{0, 1, \dots, q - 1\}^L.$$

Now, let $F : \mathcal{B}_{L+1} \to \mathbf{R}$ be any given function (with $F(0, 0, \dots, 0) = 0$). For any block $B = (\eta_1, \dots, \eta_L) \in \mathcal{B}_L$ we consider the functions

$$a_j^B(z) := \sum_{n<q^j, (\epsilon_{j-1}(n), \dots, \epsilon_{j-L}(n))=B} z^{s_F(n)}.$$

In order to obtain recurrent relations for these functions we need the following notation.

For $B = (\eta_1, \ldots, \eta_L) \in \mathcal{B}_L$ let $B' = (\eta_2, \ldots, \eta_L)$ denote the block consisting of the last $L-1$ element of B and η_B the first element η_1, i.e. $B = (\eta_B, B')$. (Similarly $'B = (\eta_1, \ldots, \eta_{L-1})$.) Furthermore, for $(\epsilon, B) = (\epsilon, \eta_1, \ldots, \eta_L) \in \mathcal{B}_{L+1}$ set

$$\kappa(\epsilon, B) = \sum_{i=0}^{L-1} (F(0, \ldots, 0, \epsilon, \eta_1, \ldots, \eta_{L-i}) - F(0, 0, \ldots, 0, \eta_1, \ldots, \eta_{L-i})) + F(0, 0, \ldots, 0, \epsilon). \tag{3.2}$$

Note that $\kappa(0, B) = 0$.

Lemma 3.1. *The functions $a_j^B(z)$, $j > L$, are recursively given by*

$$a_j^B(z) = \sum_{\epsilon=0}^{q-1} a_{j-1}^{(B',\epsilon)}(z)\, z^{\kappa(\eta_B,(B',\epsilon))}.$$

Proof. Obviously the set $\{n < q^j : (\epsilon_{j-1}(n), \ldots, \epsilon_{j-L}(n)) = B\}$ is divided into q subsets of the form

$$\begin{aligned} &\{n < q^j : \epsilon_{j-1}(n) = \eta_B,\ (\epsilon_{j-2}(n), \epsilon_{j-3}(n), \ldots, \epsilon_{j-L-1}(n)) = (B', \epsilon)\} \\ &\quad = \{n < q^{j-1} : (\epsilon_{j-2}(n), \epsilon_{j-3}(n), \ldots, \epsilon_{j-L-1}(n)) = (B', \epsilon)\} + \eta_B q^{j-1} \end{aligned}$$

with $0 \le \epsilon < q$. Furthermore, for $n < q^{j-1}$ with $(\epsilon_{j-2}(n), \epsilon_{j-3}(n), \ldots, \epsilon_{j-L-1}(n)) = (B', \epsilon)$ we have

$$s_F(n + \eta_B q^{j-1}) = s_F(n) + \kappa(\eta_B, (B', \epsilon)).$$

This proves the lemma. □

Corollary . *The vector $\mathbf{a}_j(z) = (a_j^B(z))_{B\in\mathcal{B}_L}$ satisfies the matrix recursion*

$$\mathbf{a}_j(z) = \mathbf{A}(z)\mathbf{a}_{j-1}(z) \qquad (j > L)$$

where the $q^L \times q^L$-matrix $\mathbf{A}(z) = (a_{B,C}(z))_{B,C\in\mathcal{B}_L}$ is given by

$$a_{B,C}(z) = \begin{cases} z^{\kappa(\epsilon,C)} & \text{if } \eta_B = \epsilon \text{ and } 'C = B'. \\ 0 & \text{otherwise.} \end{cases}$$

Remark . For example, for $q = L = 2$, the matrix $\mathbf{A}(1)$ has the form

$$\mathbf{A}(1) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

In what follows we will need the eigenvalues of $\mathbf{A}(1)$.

Lemma 3.2. *The eigenvalues of $\mathbf{A}(1)$ are q (which is simple) and 0.*

Proof. Since $a_j^B(1) = q^j$ it is clear that q is an eigenvalue of $\mathbf{A}(1)$ with eigenvector $(1, 1, \ldots, 1)^t$. Next, it is an easy exercise to show that $\mathbf{A}(1)^L = (1)$ is the matrix where all elements are 1. Since this matrix has rank $q^L - 1$ it follows that 0 is the only other eigenvalue of $\mathbf{A}(1)$ with multiplicity $q^L - 1$. Hence, the eigenvalue q is simple. □

Lemma 3.3. *Let $G(t, z) = \det(t\mathbf{I} - \mathbf{A}(z))$ be the characteristic polynomial of the matrix $\mathbf{A}(z)$. Then there exists a (complex) neighbourhood of $z = 1$ such that $G(t, z) = 0$ has a unique solution $t = \alpha(z)$ of maximal modulus. Furthermore, the function $\alpha(z)$ is analytic in this neighbourhood.*

Proof. Since the eigenvalues of $\mathbf{A}(1)$ are q (which is simple) and 0 there exists a neigbourhood of $z = 1$ such that $q^L - 1$ eigenvalues of $\mathbf{A}(z)$ are bounded by $1/2$ and one eigenvalue is contained in the circle $|t - q| < 1$. Hence, the equation $G(t, z) = 0$ has a unique solution $t = \alpha(z)$ of maximal modulus. Since q is a simple root of $G(t, 1) = 0$ we also have $\frac{\partial}{\partial t} G(t, 1)|_{t=q} \ne 0$. Hence, by the implicit function theorem, there exists a neighbourhood of $z = 1$ such that $\alpha(z)$ is analytic, too. □

Corollary . *There exists a neighbourhood of $z=1$ such that for every block $B \in \mathcal{B}_L$*

$$a_j^B(z) = a^B(z)\alpha(z)^j + \mathcal{O}(\gamma^j)$$

as $j \to \infty$, where $0<\gamma<1$ and $a^B(z)$ is a properly chosen analytic function.

In order to get an insight into the distribution behaviour of $s_F(n)$ we now introduce the function

$$c_N(z) = \sum_{n<N} z^{s_F(n)}. \tag{3.3}$$

By the preceding Corollary we know the following.

Lemma 3.4. *The asymptotic behaviour of $c_{\eta q^j}(z)$, $j \ge 0$, $1 \le \eta < q$, locally around $z=1$ is given by*

$$c_{\eta q^j}(z) = C_\eta(z)\alpha(z)^j + \mathcal{O}(\gamma^j)$$

as $j \to \infty$, where $0<\gamma<1$ and $C_\eta(z)$ is a properly chosen analytic function with $C_\eta(1) = \eta$.

Proof. We just have to add up over all $a_{j+1}^B(z)$ with blocks $B \in \mathcal{B}_L$ with $\eta_B = \eta$ and observe that $c_{\eta q^j}(1) = \eta q^j$. □

The next recurrence will help to extend this property to general N.

Lemma 3.5. *Suppose that $N = \eta q^k + N'$, $k \ge L$, with $1 \le \eta < q$ and $N' < q^k$ and let $B_N = (\epsilon_{k-1}(N), \ldots, \epsilon_{k-L}(N))$ be the block of digits of N preceding $\eta = \epsilon_k(N)$. Then*

$$c_N(z) = c_{\eta q^k}(z) + c_{N'}(z)\, z^{\kappa(\eta, B_N)}. \tag{3.4}$$

Proof. For $n < N'$ we have

$$s_F(n + \eta q^k) = s_F(n) + \kappa(\eta, B_N).$$

Hence, (3.4) follows immediately. □

4. Finite Recurrences

Now we assume that the sequence G_j satisfies a linear recurrence

$$G_j = \sum_{i=1}^{r} a_i G_{j-i}$$

with integers $a_1 \ge a_2, \ge \cdots \ge a_r > 0$ and $r > 1$. As above let $\mathcal{B}_L$ denote the set

$$\mathcal{B}_L = \{(\epsilon_{L-1}(n), \epsilon_{L-2}(n), \ldots, \epsilon_0(n)) : n < G_L\}$$

which is exactly the set of digital pattern of lenght L (in G-ary digital expansions) that actually occur. Of course we have $|\mathcal{B}_L| = G_L$. We now consider a function $F : \mathcal{B}_{L+1} \to \mathbf{R}$ (with $F(0,0,\ldots,0) = 0$). Without loss of generality we may assume that $L \ge r$. (If we are only interested in $L+1$ subsequent digits with $L < r$ then we consider a new function $\tilde{F} : \mathcal{B}_{r+1} \to \mathbf{R}$ that does not depend on the first $(r-L)$ digits.)

For all $B \in \mathcal{B}_L$ we consider the functions

$$a_j^B(z) := \sum_{n<G_j, (\epsilon_{j-1}(n), \ldots, \epsilon_{j-L}(n)) = B} z^{s_F(n)}.$$

As above these functions satisfy a recurrent relation.

Lemma 4.1. *Suppose that $L \ge r$. Then the functions $a_j^B(z)$, $j > L$, $B \in \mathcal{B}_L$ are recursively given by*

$$a_j^B(z) = \sum_{C \in \mathcal{B}_L, 'C = B'} a_{j-1}^C(z)\, z^{\kappa(\eta_B, C)}.$$

Proof. The proof is exactly the same as that of Lemma 3.1. We only have to observe that $L \ge r$ ensures that $\{(\eta_B, C) : C \in \mathcal{B}_L, 'C = B'\} = \mathcal{B}_{L+1}$, i.e. we cover all possible cases. □

Corollary . *The vector* $\mathbf{a}_j(z) = (a_j^B(z))_{B\in\mathcal{B}_L}$ *satisfies the matrix recursion*

$$\mathbf{a}_j(z) = \mathbf{A}(z)\mathbf{a}_{j-1}(z) \qquad (j > L)$$

where the $G_L \times G_L$*-matrix* $\mathbf{A}(z) = (a_{B,C}(z))_{B,C\in\mathcal{B}_L}$ *is given by*

$$a_{B,C}(z) = \begin{cases} z^{\kappa(\eta_B,C)} & \text{if } 'C = B'. \\ 0 & \text{otherwise.} \end{cases}$$

Remark . For example, for $r = L = 2$ and $a_1 = a_2 = 1$ we have $\mathcal{B}_2 = \{00, 01, 10\}$ and

$$\mathbf{A}(1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Again we will need the maximal eigenvalue of $\mathbf{A}(1)$.

Lemma 4.2. *Suppose that* $L \ge r$. *Then the matrix* $\mathbf{A}(1)$ *has a unique eigenvalue* α *of maximal modulus with is real, positive, and simple. It conincides with the maximal root of the characteristic polynomial* $P(u) = u^r - a_1u^{r-1} - \cdots - a_r$ *of the sequence* G_j.

Proof. First of all the matrix $\mathbf{A}(1)$ is a primitive irreducible nonnegative matrix. This follows from the fact that for every pair of blocks $B, C \in \mathcal{B}_L$ there exists an integer n such that both, B and C, occur in the digital expansion of n. (We just have to consider n with digits $(B, 0, C)$.) Hence, all elements of $\mathbf{A}(1)^{L+1}$ are positive and consequently by [10, Theorem 2.1, p. 49] $\mathbf{A}(1)$ is primitive and irreducible.

Hence, by Perron-Frobenius theory (see [10]) there exists a unique simple eigenvalue $\tilde\alpha$ of $\mathbf{A}(1)$ of maximal modul which is positive. Furthermore, there is an eigenvector $\mathbf{v} = (v_B)_{B\in\mathcal{B}_L}$ (corresponding to $\tilde\alpha$) with positive coordinates $v_B > 0$.

Since $\mathbf{a}_j(1)$ satisfy the recurrence $\mathbf{a}_j(1) = \mathbf{A}(1)\mathbf{a}_{j-1}(1)$ we surely have

$$a_j^B(1) = \tilde a_B \tilde\alpha^j + \mathcal{O}(\tilde\alpha^{j(1-\delta)}) \quad (B \in \mathcal{B}_L)$$

for some constants $\tilde a_B \ge 0$ and some $\delta > 0$. Thus

$$G_j = \sum_{b\in\mathcal{B}_L} a_j^B(1) = \tilde C \tilde\alpha^j + \mathcal{O}(\tilde\alpha^{j(1-\delta)})$$

with $\tilde C \ge 0$ implies that $\tilde\alpha \ge \alpha$ (where α denotes the maximal root of $P(u) = u^r - a_1u^{r-1} - \cdots - a_r$).

Conversely, fix $j_0 > 0$ such that $a_{j_0}^B(1) \ge v_B$ for all $B \in \mathcal{B}_L$. (This is surely possible since $a_j^B(1)$ is unbounded for $j \to \infty$ and $B \in \mathcal{B}_L$.) Now it follows by induction that $a_j^B(1) \ge v_B\tilde\alpha^{j-j_0}$ for $j \ge j_0$. Of course this gives $\tilde\alpha \le \alpha$ and consequently $\tilde\alpha = \alpha$. ☐

Remark . It seems that the eigenvalues of $\mathbf{A}(1)$ are exactly the roots of the characteristic polynomial and 0. However, we do not need this precise information.

Now as above it follows that there exists a (complex) neighbourhood of $z = 1$ such that the equation

$$G(t, z) = \det(t\mathbf{I} - \mathbf{A}(z)) = 0 \tag{4.1}$$

has a unique solution $t = \alpha(z)$ of maximal modulus. The function $\alpha(z)$ is analytic in this neighbourhood. For every block $B \in \mathcal{B}_L$

$$a_j^B(z) = a^B(z)\alpha(z)^j + \mathcal{O}(\alpha^{(1-\delta)j})$$

as $j \to \infty$, where $\delta > 0$ and $a^B(z)$ is a properly chosen analytic function.

In order to get an insight into the distribution behaviour of $s_F(n)$ we again introduce the function

$$c_N(z) = \sum_{n<N} z^{s_F(n)}$$

and obtain that the asymptotic behaviour of $c_{\eta G_j}(z)$, $j \geq 0$, $1 \leq \eta \leq a_1$, locally around $z = 1$ is given by

$$c_{\eta G_j}(z) = C_\eta(z)\alpha(z)^j + \mathcal{O}(\alpha^{(1-\delta)j})$$

as $j \to \infty$, where $C_\eta(z)$ is a properly chosen analytic function with $C_\eta(1) = \eta C$ (where $C = \lim_{j\to\infty} G_j \alpha^{-j}$). Furthermore, if $N = \eta G_k + N'$, $k \geq L$, with $1 \leq \eta \leq a_1$ and $N' < G_k$ and if $B_N = (\epsilon_{k-1}(N), \ldots, \epsilon_{k-L}(N))$ is the block of digits of N preceding $\eta = \epsilon_k(N)$. Then

$$c_N(z) = c_{\eta G_k}(z) + c_{N'}(z)\, z^{\kappa(\eta, B_N)}. \tag{4.2}$$

Hence, we are in the same situation as in the q-ary case.

5. Expected Value and Variance

Obviously, we have

$$\mathbf{E}X_N = \frac{c_N'(1)}{N}$$

and

$$\mathbf{V}X_N = \frac{c_N''(1)}{N} + \frac{c_N'(1)}{N} - \left(\frac{c_N'(1)}{N}\right)^2.$$

Therefore we need expressions for $c_N'(1)$ and $c_N''(1)$.

Lemma 5.1. *Suppose that the digital representation of N with respect to G_j is given by*

$$N = \sum_{m=1}^{M} \eta_m G_{j_m} \tag{5.1}$$

with $G_{j_1} > G_{j_2} > \cdots > G_{j_M}$ and $\eta_m > 0$. Then there exist κ_m depending on N such that

$$c_N(z) = \sum_{m=1}^{M} z^{\kappa_m} c_{\eta_m G_{j_m}}(z) \tag{5.2}$$

and $|\kappa_m| \leq Dm$ for some constant $D > 0$.

Proof. We just have to apply Lemma 3.5 (resp. (4.2)) recursively. Furthermore $|\kappa_m| \leq Dm$ with

$$D = \max_{(\eta,B)\in\mathcal{B}_{L+1}} |\kappa(\eta, B)|.$$

□

Corollary . *With the same notation as in Lemma 5.1 we have*

$$c_N'(1) = \sum_{m=1}^{M} \left(\kappa_m \eta_m + C_{\eta_m}'(1) + j_m \frac{\alpha'(1)}{\alpha(1)}\eta_m\right) G_{j_m} + \mathcal{O}(N^{1-\delta})$$

and

$$\begin{aligned} c_N''(1) = &\sum_{m=1}^{M} \left(\kappa_m(\kappa_m - 1)\eta_m + 2\kappa_m C_{\eta_m}'(1) + C_{\eta_m}''(1)\right) G_{j_m} \\ &+ \sum_{m=1}^{M} \left(2\kappa_m\eta_m \frac{\alpha'(1)}{\alpha(1)} + 2C_{\eta_m}'(1)\frac{\alpha'(1)}{\alpha(1)} + \frac{\alpha''(1)}{\alpha(1)}\eta_m\right) j_m G_{j_m} \\ &+ \sum_{m=1}^{M} \left(\eta_m j_m(j_m - 1)\frac{\alpha'(1)^2}{\alpha(1)^2}\right) G_{j_m} + \mathcal{O}(N^{1-\delta}) \end{aligned}$$

for some $\delta > 0$.

Proof. We just apply that $c_{\eta G_j}(z)$ and its derivatives are given by

$$\begin{aligned}
c_{\eta G_j}(z) &= C_\eta(z)\alpha(z)^j + \mathcal{O}(\alpha^{(1-\delta)j}),\\
c'_{\eta G_j}(z) &= C'_\eta(z)\alpha(z)^j + C_\eta(z)j\alpha'(z)\alpha(z)^{j-1} + \mathcal{O}(\alpha^{(1-\delta)j}),\\
c'_{\eta G_j}(z) &= C''_\eta(z)\alpha(z)^j + 2C'_\eta(z)j\alpha'(z)\alpha(z)^{j-1}\\
&\quad + C_\eta(z)j\alpha''(z)\alpha(z)^{j-1} + C_\eta(z)j(j-1)\alpha'(z)^2\alpha(z)^{j-2} + \mathcal{O}(\alpha^{(1-\delta)j})
\end{aligned}$$

in a sufficiently small neighbourhood of $z = 1$ and for some $\delta > 0$. Note that by Cauchy's formula, the absolute value of the derivative of an analytic function f in some circle $|z-1| \le R$ can be essentially bounded by $\max|f|$ in a slightly larger circle $|z-1| \le R+\varepsilon$. $\square$

Now we can prove Theorem 2.1.

Proof. Let us start with $\mathbf{E}X_N$. Firstly, we have

$$\begin{aligned}
\mathbf{E}X_N &= \frac{c'_N(1)}{N}\\
&= \frac{1}{N}\sum_{m=1}^{M}\left(\kappa_m\eta_m + C'_{\eta_m}(1) + j_m\frac{\alpha'(1)}{\alpha(1)}\eta_m\right)G_{j_m} + \mathcal{O}(N^{-\delta})\\
&= j_1\frac{\alpha'(1)}{\alpha(1)}\frac{1}{N}\sum_{m=1}^{M}\eta_m G_{j_m} - \frac{1}{N}\sum_{m=1}^{M}\frac{\alpha'(1)}{\alpha(1)}(j_1-j_m)\eta_m G_{j_m}\\
&\quad + \frac{1}{N}\sum_{m=1}^{M}(\kappa_m\eta_m + C'_{\eta_m}(1))G_{j_m} + \mathcal{O}(N^{-\delta}).
\end{aligned}$$

Now note that

$$\sum_{m=1}^{M}(j_1-j_m)\eta_m G_{j_m} \le \sum_{j=0}^{j_1} j a_1 G_{j_1-1} = \mathcal{O}(N)$$

and that

$$\sum_{m=1}^{M}|\kappa_m|\eta_m G_{j_m} \le \sum_{m=1}^{M} Dma_1 G_{j_m} = \mathcal{O}(N).$$

Hence,

$$\mathbf{E}X_N = j_1\frac{\alpha'(1)}{\alpha(1)} + \mathcal{O}(1).$$

Finally, since $j_1 = \log N/\log\alpha + \mathcal{O}(1)$ the representation (2.1) for $\mathbf{E}X_N$ follows.

For the variance $\mathbf{V}X_N$ we have to be a little bit more careful. Let us start with the full expansion of

$$
\begin{aligned}
N(c_N''(1)+c_N'(1))-c_N'(1)^2 = N\sum_{m=1}^{M}&\left(\kappa_m^2\eta_m+2\kappa_m C_{\eta_m}'(1)+C_{\eta_m}'(1)+C_{\eta_m}''(1)\right)G_{j_m}\\
&+2N\frac{\alpha'(1)}{\alpha(1)}\sum_{m=1}^{M}(\kappa_m\eta_m+C_{\eta_m}'(1))j_mG_{j_m}\\
&-2\frac{\alpha'(1)}{\alpha(1)}\sum_{m=1}^{M}(\kappa_m\eta_m+C_{\eta_m}'(1))G_{j_m}\sum_{k=1}^{M}j_k\eta_kG_{j_k}\\
&+N\sum_{m=1}^{M}\left(\frac{\alpha''(1)}{\alpha(1)}+\frac{\alpha'(1)}{\alpha(1)}-\frac{\alpha'(1)^2}{\alpha(1)^2}\right)j_m\eta_mG_{j_m}\\
&+N\frac{\alpha'(1)^2}{\alpha(1)^2}\sum_{m=1}^{M}j_m^2\eta_mG_{j_m}-\frac{\alpha'(1)^2}{\alpha(1)^2}\left(\sum_{m=1}^{M}j_m\eta_mG_{j_m}\right)^2\\
&-\left(\sum_{m=1}^{M}(\kappa_m\eta_m+C_{\eta_m}'(1))G_{j_m}\right)^2+\mathcal{O}(N^{2-\delta}).
\end{aligned}
$$

Now we apply the estimates

$$
\begin{aligned}
\sum_{m=1}^{M}m^2\eta_mG_{j_m}&=\mathcal{O}(N),\\
\sum_{m=1}^{M}|\kappa_m\eta_m+C_{\eta_m}'(1)|(j_1-j_m)G_{j_m}&=\mathcal{O}(N),\\
\sum_{m=1}^{M}(j_1-j_m)\eta_mG_{j_m}&=\mathcal{O}(N),
\end{aligned}
$$

and

$$
\begin{aligned}
N\sum_{m=1}^{M}j_m^2\eta_mG_{j_m}-\left(\sum_{m=1}^{M}j_m\eta_mG_{j_m}\right)^2&=\sum_{k,m=1}^{M}j_m(j_m-j_k)\eta_m\eta_kG_{j_m}G_{j_k}\\
&=\sum_{k,m=1}^{M}(j_m-j_1)((j_m-j_1)-(j_k-j_1))\eta_m\eta_kG_{j_m}G_{j_k}\\
&=\mathcal{O}(N^2)
\end{aligned}
$$

and directly obtain

$$
\mathbf{V}X_N=j_1\left(\frac{\alpha''(1)}{\alpha(1)}+\frac{\alpha'(1)}{\alpha(1)}-\frac{\alpha'(1)^2}{\alpha(1)^2}\right)+\mathcal{O}(1).
$$

As above, $j_1=\log N/\log\alpha+\mathcal{O}(1)$ and so the proof of Theorem 2.1 is complete. □

6. Global Limit Law

With help of Lemma 3.4 and Lemma 3.5 we can prove asymptotic normality of X_N. Observe that

$$
\frac{1}{N}c_N(e^{it})=\mathbf{E}e^{itX_N}
$$

is the characteristic function of X_N.

Proposition 6.1. *Suppose that $\sigma^2 \neq 0$ and set $\mu_N = \mathbf{E}X_N$ and $\sigma_N^2 = \mathbf{V}X_N$. Then for every $\varepsilon > 0$ we have uniformly for $|t| \le (\log N)^{1/2-\varepsilon}$*

$$e^{-it\mu_N/\sigma_N}\frac{1}{N}c_N(e^{it/\sigma_N}) = e^{-t^2/2} + \mathcal{O}((\log N)^{-1/2+\varepsilon}). \tag{6.1}$$

Proof. Set $f(z) = \log\alpha(e^z)$ in an open neighbourhood of $z = 0$. Then we have

$$\alpha(e^{it}) = \alpha e^{i\mu t - \sigma^2 t^2/2 + \mathcal{O}(|t|^3)},$$

with $\mu = f'(0) = \alpha'(1)/\alpha$ and $\sigma^2 = f''(0) = \alpha''(1)/\alpha + \mu - \mu^2$ (see Theorem 2.2). Hence, by using Lemma 3.4 resp. (4.2)

$$c_{\eta G_j}(e^{it}) = \eta G_j e^{j(i\mu t - \sigma^2 t^2/2)} e^{\mathcal{O}(|t| + j|t|^3)} + \mathcal{O}\left(\alpha^{(1-\delta)j}\right)$$

in an open neighbourhood of $t = 0$ in $\mathbf{R}$. Now suppose that $N = \sum_{m=1}^{M}\eta_m G_{j_m}$ with $j_1 > j_2 > \cdots > j_M$ and $\eta_m > 0$ is the G-ary expansion of N. Then by Lemma 5.1

$$\begin{aligned}
c_N(e^{it}) &= \sum_{m=1}^{M} c_{\eta_m G_{j_m}}(e^{it}) e^{it\kappa_m} \\
&= \sum_{m=1}^{M} \eta_m G_{j_m} e^{ij_m\mu t - j_m\sigma^2 t^2/2} e^{\mathcal{O}(m|t| + j_m|t|^3)} + \mathcal{O}(N^{(1-\delta)}).
\end{aligned}$$

Now observe that

$$\frac{it}{\sigma_N} = \frac{it}{\sigma\, j_1^{1/2}}\left(1 + \mathcal{O}(j_1^{-1})\right),$$

and that

$$e^{-it\mu_N/\sigma_N} = e^{-it(\mu/\sigma)j_1^{1/2}(1+\mathcal{O}(j_1^{-1}))}.$$

Hence

$$\begin{aligned}
\mathbf{E}e^{it(X_N-\mu_N)/\sigma_N} &= e^{-it\mu_N/\sigma_N}\frac{1}{N}c_N(e^{it/\sigma_N}) \\
&= e^{-t^2/2}\frac{1}{N}\sum_{m=1}^{M}\eta_m G_{j_m} e^{it(j_m-j_1)/(\sigma j_1^{1/2}) - (j_m-j_0)t^2/(2j_1)} \\
&\qquad \times e^{\mathcal{O}\left(m|t|/j_1^{1/2} + t^2/j_1 + |t^3|/j_1^{1/2}\right)} + \mathcal{O}(N^{-\delta}).
\end{aligned}$$

Let $\varepsilon > 0$ be a (small) real number and let τ be defined by $j_{\tau-1} > j_0 - j_0^\varepsilon \ge j_\tau$. Then $\tau \le j_0 - j_\tau < j_0^\varepsilon$ and consequently

$$\begin{aligned}
\mathbf{E}e^{it(X_N-\mu_N)/\sigma_N} &= e^{-t^2/2}\sum_{m=1}^{\tau}\frac{\eta_m G_{j_m}}{N} e^{\mathcal{O}\left(|t|\sigma j_1^{\varepsilon-1/2} + t^2 j_1^{\varepsilon-1} + |t|^3 j_1^{-1/2}\right)} \\
&\quad + \mathcal{O}\left(\sum_{m=\tau+1}^{M}\frac{\eta_m G_{j_m}}{N}\right) + \mathcal{O}(N^{-\delta}) \\
&= e^{-t^2/2} e^{\mathcal{O}\left(|t|\sigma j_1^{\varepsilon-1/2} + t^2 j_1^{\varepsilon-1} + |t|^3 j_1^{-1/2}\right)} + \mathcal{O}(\alpha^{-j_1^\varepsilon}).
\end{aligned}$$

Since $j_1 = (\log N)/(\log\alpha) + \mathcal{O}(1)$ this implies (6.1) directly for $|t| \le (\log N)^{\varepsilon/3}$. Furthermore, since

$$e^{-t^2/2 + \mathcal{O}(|t^3|j_1^{-1/2})} \le e^{-cj_1^{2\varepsilon/3}} = \mathcal{O}(j_1^{-1})$$

for $(\log N)^{\varepsilon/3} \le |t| \le (\log N)^{1/2-\varepsilon}$ and a sufficiently small $c > 0$ we finally obtain the full version of (6.1). $\square$

We can now use Proposition 6.1 to prove the first part of Theorem 2.2.

Proof. Set
$$\Delta_N(t) = e^{-t^2/2} - \mathbf{E}e^{it(X_N-\mu_N)/\sigma_N}.$$
Then by Esseen's inequality [5, p. 32] we have
$$\frac{1}{N}|\{n < N : s_F(n) < \mathbf{E}X_N + x\mathbf{V}X_N\}| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}dt + \mathcal{O}\left(\frac{1}{T} + \int_{-T}^{T}\left|\frac{\Delta_N(t)}{t}\right| dt\right).$$
We choose $T = (\log N)^{1/2-\varepsilon}$ and use the estimate
$$e^{-it\mu_N/\sigma_N}\frac{1}{N}c_N(e^{it/\sigma_N}) = 1 + \mathcal{O}(t^2)$$
for $|t| \le (\log N)^{-2}$. Combining this with Proposition 6.1 we directly get
$$\begin{aligned}\int_{-T}^{T}\left|\frac{\Delta_N(t)}{t}\right| dt &= \int_{|t|\le(\log N)^{-2}}\left|\frac{\Delta_N(t)}{t}\right| dt + \int_{(\log N)^{-2}<|t|\le T}\left|\frac{\Delta_N(t)}{t}\right| dt\\ &= \mathcal{O}\left((\log N)^{-1/2+\varepsilon}(\log\log N)\right).\end{aligned}$$
Hence, (2.3) follows. $\square$

7. Local Limit Law

In order to prove a local limit law for X_N, i.e. the second part or Theorem 2.2, we need more precise information on the behaviour of $c_{\eta G_j}(z)$.

Proposition 7.1. *Suppose that $G = (G_j)_{j\ge0}$ satisfies a finite recurrence of the above type suppose that*
$$d = \gcd\{\kappa(\eta, B) :\in \mathcal{B}_{L+1}\} = 1.$$
Then there exist $\varepsilon > 0$ and $\delta > 0$ such that
$$c_{\eta G_j}(e^{it}) = C_\eta(e^{it})\alpha(e^{it})^j + \mathcal{O}\left(\alpha^{(1-\delta)j}\right) \tag{7.1}$$
uniformly for $|t| \le \varepsilon$, where $C_\eta(z)$ and $\alpha(z)$ are as in Lemma 3.4 resp. in (4.2), and
$$c_{\eta G_j}(e^{it}) = \mathcal{O}\left(\alpha^{(1-\delta)j}\right) \tag{7.2}$$
uniformly for $\varepsilon \le |t| \le \pi$.

Proof. Obviously, (7.1) follows from Lemma 3.4 resp. in (4.2) for some $\varepsilon > 0$.

For the proof of (7.2) we just have to observe that $d = 1$ implies that all solutions t of $G(t,z) = \det(t\mathbf{I} - \mathbf{A}(z)) = 0$ for $z = e^{i\varphi}$, $0 < \varphi < 2\pi$, are strictly bounded by α.

If $|z| = 1$ then all entries of $\mathbf{A}(z)$ are complex numbers whose abolute values are bounded by those of $\mathbf{A}(1)$. Hence by [10, Theorem 2.1, p. 36] all eigenvalues β of $\mathbf{A}(z)$ are bounded by $|\beta| \le \alpha$. Furthermore, $|\beta| = \alpha$ if and only if there exists a complex numbers λ with $|\lambda| = 1$ and a diagonal matrix $D = \mathrm{diag}(\lambda_B)_{B\in\mathcal{B}_L}$ with complex numbers λ_B with $|\lambda_B| = 1$ such that
$$\mathbf{A}(z) = \lambda\, D\, \mathbf{A}(1)\, D^{-1}.$$
Without loss of generality we may assume that $\lambda_{00\cdots0} = 1$.

We now show that in this case we have $\lambda = 1$ and $\lambda_B = 1$ for all $B \in \mathcal{B}_L$, i.e. $\mathbf{A}(z) = \mathbf{A}(1)$. First observe that $a_{00\cdots0,00\cdots0}(z) = 1$ for all z. Thus, $\lambda = 1$. More generally, if $a_{B,C}(z) = a_{B,C}(1) = 1$ then $\lambda_B = \lambda_C$. Obviously, we have $a_{B,C}(z) = a_{B,C}(1) = 1$ if $B' = 'C$ and $\eta_B = 0$. Thus, if $B = (\eta_1, \ldots, \eta_L)$ is any block in $\mathcal{B}_L$ then we can consider the sequence
$$B_0 = (0,0,\ldots,0),\ B_1 = (0,\ldots,0,\eta_1),\ B_2 = (0,\ldots,0,\eta_1,\eta_2),\ \ldots,\ B_{L-1} = B$$
and can conclude that
$$1 = \lambda_{B_0} = \lambda_{B_1} = \cdots = \lambda_B.$$
However, if $d = 1$ then for every φ, $0 < \varphi < 2\pi$, there exists $(\epsilon, B) \in \mathcal{B}_{L+1}$ with $e^{it\kappa(\epsilon,B)} \ne 1$. Consequently, if $d = 1$ then all eigenvalues β of $\mathbf{A}(e^{it})$ are strictly bounded by $|\beta| < \alpha$. $\square$

Remark . If $d > 1$ then we have $\mathbf{A}(z) = \mathbf{A}(1)$ if and only if z is a d-th root of unity, and therefore we get a periodic structure, i.e. a local limit law for every residue class of d.

With help of Proposition 7.1 it is possible to derive asymptotic expansions for the coefficients

$$c_{N,k} = |\{n < N : s_F(n) = k\}|$$

of

$$c_N(z) = \sum_{k\ge 0} c_{N,k} z^k$$

for $N = \eta G_j$, $1 \le \eta \le Q$, via saddle point approximations.

Proposition 7.2. *We have*

$$c_{\eta G_j,k} = \frac{\eta G_j}{\sqrt{2\pi j\sigma^2}} \left(\exp\left(-\frac{(k-j\mu)^2}{2j\sigma^2} \right) + \mathcal{O}(j^{-1/2}) \right)$$

uniformly for all $j, k \ge 0$.

Proof. We again use Cauchy's formula

$$c_{\eta G_j,k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} c_{\eta G_j}(e^{it}) e^{-ikt}\, dt.$$

Since

$$\int_{\varepsilon \le |t| \le \pi} |c_{\eta G_j}(e^{it})|\, dt = \mathcal{O}\left(\alpha^{(1-\delta)j}\right) = \mathcal{O}(G_j/j)$$

we just have to evaluate

$$I = \frac{1}{2\pi} \int_{|t|\le j^{-\nu}} c_{\eta G_j}(e^{it}) e^{-ikt}\, dt + \frac{1}{2\pi} \int_{j^{-\nu} \le |t| \le \varepsilon} b_j(e^{it}) e^{-ikt}\, dt = I_1 + I_2,$$

where $0 < \nu < \frac{1}{6}$. From $\alpha(e^{it}) = \alpha e^{i\mu t - \sigma^2 t^2/2 + \mathcal{O}(|t|^3)}$ it follows that there exists a constant $c > 0$ such that $|\alpha(e^{it})| \le e^{-ct^2}$ for $|t| \le \varepsilon$. Hence,

$$I_2 \le \frac{1}{\pi} \int_{j^{-\nu}}^{\infty} e^{-cjt^2}\, dt + \mathcal{O}\left(\alpha^{(1-\delta)j}\right) = \mathcal{O}\left(e^{-cj^{1-2\varepsilon}}\right) + \mathcal{O}\left(\alpha^{(1-\delta)j}\right) = \mathcal{O}(G_j/j)$$

Finally,

$$\begin{aligned}
I_1 &= \frac{1}{2\pi} \int_{|t|\le j^{-\nu}} C_\eta(1) \alpha^j e^{it(j\mu-k) - j\sigma^2 t^2/2} \left(1 + \mathcal{O}(j|t|^3 + |t|)\right) dt + \mathcal{O}\left(\alpha^{(1-\delta)j}\right) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} C\alpha^j e^{it(j\mu-k) - j\sigma^2 t^2/2}\, dt + \mathcal{O}\left(\int_{|t| > j^{-\nu}} C_\eta(1) \alpha^j e^{-j\sigma^2 t^2/2}\, dt \right) \\
&\quad + \mathcal{O}\left(\int_{|t|\le j^{-\nu}} C_\eta(1) \alpha^j e^{-j\sigma^2 t^2/2} (j|t|^3 + |t|)\, dt \right) + \mathcal{O}\left(\alpha^{(1-\delta)j}\right) \\
&= \frac{C_\eta(1)\alpha^j}{\sqrt{2\pi j\sigma^2}} \exp\left(-\frac{(k-j\mu)^2}{2j\sigma^2} \right) + \mathcal{O}(\alpha^j/j) \\
&= \frac{\eta G_j}{\sqrt{2\pi j\sigma^2}} \exp\left(-\frac{(k-j\mu)^2}{2j\sigma^2} \right) + \mathcal{O}(G_j/j).
\end{aligned}$$

This completes the proof of Proposition 7.2. □

Finally Proposition 7.2 and Lemma 5.1 can be used to complete the proof of Theorem 2.2.

Proof. As in the proof of Proposition 6.1 we suppose that $N = \sum_{m=1}^{M} \eta_m G_{j_m}$ (with $j_1 > j_2 > \cdots > j_M$ and $\eta_m > 0$) is the G-ary expansion of N. Furthermore, let $\varepsilon > 0$ be a (small) real number and let τ be defined by $j_{\tau-1} > j_1 - j_1^\varepsilon \ge j_\tau$. Then by (5.2)

$$
\begin{aligned}
c_{N,k} &= \sum_{m=1}^{M} c_{\eta_m G_{j_m}, k-\kappa_m} \\
&= \sum_{m=1}^{\tau} c_{\eta_m G_{j_m}, k-\kappa_m} + \mathcal{O}\left(\sum_{l=\tau+1}^{M} \frac{\eta_m G_{j_m}}{j_l^{1/2}} \right) \\
&= \sum_{m=1}^{\tau} \frac{\eta_m G_{j_m}}{\sqrt{2\pi j_m \sigma^2}} \exp\left(-\frac{(k-\kappa_m - j_m\mu)^2}{2 j_m \sigma^2} \right) + \mathcal{O}(G_{j_1}/j_1).
\end{aligned}
$$

If $m < \tau$ and $|k-\mu_N| = \mathcal{O}(j_1^{1/2}\log j_1)$ then

$$
\begin{aligned}
&\frac{(k-\mu_N)^2}{2\sigma_N^2} - \frac{(k-\kappa_m - j_m\mu)^2}{2j_m\sigma^2} \\
&\qquad = \frac{(k-\mu_N)^2 - (k-\kappa_m - j_m\mu)^2}{2\sigma_N^2} + (k-\kappa_m - j_m\mu)^2 \left(\frac{1}{2\sigma_N^2} - \frac{1}{2j_m\sigma^2} \right) \\
&\qquad = \mathcal{O}\left(j_1^{\varepsilon-1/2} \log j_1 \right) + \mathcal{O}\left(j_1^{-1} (\log j_1)^2 \right),
\end{aligned}
$$

where we have used $\mu_N = j_1\mu + \mathcal{O}(1)$, $\sigma_N^2 = j_1\sigma^2 + \mathcal{O}(1)$, and $\kappa_m = \mathcal{O}(m)$. Hence, from

$$
\sqrt{\frac{j_1 + \mathcal{O}(1)}{j_m}} \exp\left(\frac{(k-\mu_N)^2}{2\sigma_N^2} - \frac{(k-\kappa_m - j_m\mu)^2}{2j_m\sigma^2} \right) = 1 + \mathcal{O}\left(j_1^{\varepsilon-1/2} \log j_1 \right)
$$

we obtain

$$
\begin{aligned}
c_{N,k} &= \frac{N}{\sqrt{2\pi\sigma_N^2}} \exp\left(-\frac{(k-\mu_N)^2}{2\sigma_N^2} \right) \sum_{m=0}^{\tau} \frac{\eta_m G_{j_m}}{N} \left(1 + \mathcal{O}\left(j_1^{\varepsilon-1/2} \log j_1 \right) \right) + \mathcal{O}(G_{j_1}/j_1) \\
&= \frac{N}{\sqrt{2\pi\sigma_N^2}} \left(\exp\left(-\frac{(k-\mu_N)^2}{2\sigma_N^2} \right) + \mathcal{O}\left(j_1^{\varepsilon-1/2} \log j_1 \right) \right).
\end{aligned}
$$

If $|k-\mu_N| \ge j_1^{1/2}\log j_1$ then we have for $m < \kappa$

$$
\begin{aligned}
c_{\eta_m G_{j_m}, k-\kappa_m} &= \mathcal{O}\left(\alpha^{j_m} j_1^{-1/2} \exp\left(-\frac{(\log j_1)^2}{4\sigma^2} \right) \right) \\
&= \mathcal{O}\left(\alpha^{j_m} j_1^{-1} \right)
\end{aligned}
$$

which finally gives

$$
\begin{aligned}
c_{N,k} &= \mathcal{O}((\alpha^{j_1} j_1^{-1}) + \mathcal{O}\left(\sum_{m=\tau+1}^{M} \frac{G_{j_m}}{j_m^{1/2}} \right) \\
&= \mathcal{O}(G_{j_1}/j_1).
\end{aligned}
$$

This completes the proof of Theorem 2.2. □

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