The Sum-of-Digits-Function and Uniform Distribution Modulo 1

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Abstract

The aim of this paper is to provide detailed estimates for the discrepancy of the sequences $(\{\alpha \cdot s_q(n)\})$, ($\{x\}$ denotes the fractional part of x), and results concerning the uniform distribution and the discrepancy of the sequences $(\{\alpha_1 \cdot s_{q_1}(n)\}, \ldots, \{\alpha_d \cdot s_{q_d}(n)\})$, where $\alpha, \alpha_1, \ldots, \alpha_d$ are fixed irrational numbers and $s_q(n)$ denotes the q-ary sum-of-digits-function.

1 Introduction

The discrepancy $D_N = D_N(x_n)$ of a sequence (x_n) ; $n = 0, 1, \ldots, N-1$ in $[0,1)^d$ is defined by

$$D_N = \sup_{[\mathbf{a},\mathbf{b}]\subseteq[0,1)^d} \left| \frac{1}{N} \sum_{n=0}^{N-1} \chi_{[\mathbf{a},\mathbf{b}]}(x_n) - \lambda_d([\mathbf{a},\mathbf{b}]) \right|,$$

where χ_E denotes the characteristic function (of the set E), $[\mathbf{a},\mathbf{b}]$ denotes a d-dimensional interval $[\mathbf{a},\mathbf{b}] = [a_1, b_1] \times \cdots \times [a_d, b_d]$ with $a_j \le b_j$ $(1 \le j \le n)$, and λ_d denotes the d-dimensional Lebesgue measure.

A sequence (x_n) in $[0,1)^d$ is called *uniformly distributed modulo* 1 (u.d.) if

$$\lim_{N\to\infty} D_N(x_n) = 0.$$

Equivalently the sequence of (weighted) counting measures

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{x_n}$$

(with $\delta_x(E) = \chi_E(x)$) converges weakly to the Lebesgue measure. For more details on this subject we refer to the monographs [5, 8].

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Let $q > 1$ be an integer. Then any non-negative integer n has a unique q-ary digital expansion

$$n = \sum_{k \geq 0} d_k(n) q^k$$

with integer digits $0 \leq d_k(n) < q$. The q-ary sum-of-digits-function $s_q(n)$ is then given by

$$s_q(n) = \sum_{k \geq 0} d_k(n).$$

We will consider in the following distribution properties of the sequences $(\{\alpha \cdot s_q(n)\})$ and of the sequences $(\{\alpha_1 \cdot s_{q_1}(n)\}, \ldots, \{\alpha_d \cdot s_{q_d}(n)\})$ where $\alpha, \alpha_1, \ldots, \alpha_d$ are fixed irrational numbers and $q > 1$ is a fixed integer resp. $q_1, q_2, \ldots, q_d > 1$ are fixed pairwise coprime integers.

First we will give detailed discrepancy estimates for the one-dimensional sequences, thereby considerably refining previously known results. We will state these estimates in Section 2. The proofs are given in Sections 4 and 5.

In Sections 3 and 6 we will state and prove a first result concerning the uniform distribution of d-dimensional sequences $(\{\alpha_1 \cdot s_{q_1}(n)\}, \ldots, \{\alpha_d \cdot s_{q_d}(n)\})$.

2 Results in Dimension 1

It was shown by Mendès France [10] and later by Coquet [1] that $(\{\alpha \cdot s_q(n)\})$ is uniformly distributed modulo 1 for irrational α. (There are also very interesting related results, see [11, 12, 2].) Furthermore, Tichy und Turnwald in [15] derived estimates for the discrepancy D_N of $(\{\alpha \cdot s_q(n)\})$ based on estimates for exponential sums and the inequalities of Erdős-Turán and Koksma. They proved essentially the following properties. (Note that the statements in [15] are a little bit weaker than the following ones.)

> *Suppose that there exist constants $c_1 > 0$, $\eta \geq 1$ such that $\|h\alpha\| \geq c_1 h^{-\eta}$ for all integers $h \geq 0$. Then the discrepancy D_N of the sequence $(\{\alpha \cdot s_q(n)\})$ is bounded by*
>
> $$D_N \leq c(c_1, \eta, q) \left(\frac{\log \log N}{\log N} \right)^{\frac{1}{2\eta}}$$
>
> *with some constant $c(c_1, \eta, q)$ depending on c_1, η, and q. Conversely, if for some $\eta \geq 1$ and some constant $c_2 > 0$ we have $\|h\alpha\| < c_2 h^{-\eta}$ for infinitely many integers $h > 0$ then*
>
> $$D_N \geq c'(c_2, \eta, q) \frac{1}{(\log N)^{\frac{1}{2\eta}}} \geq \frac{1}{(\log N)^{\frac{1}{2}}}$$
>
> *for infinitely many $N \geq 0$, where the constant $c'(c_2, \eta, q) > 0$ depends on c_2, η, and q.*

From these results we easily obtain a metrical bound.

for almost all α in the sense of Lebesgue measure we have

$$D_N = O\left(\frac{1}{(\log N)^{\frac{1}{2}-\epsilon}}\right) \text{ and } D_N = \Omega\left(\frac{1}{(\log N)^{\frac{1}{2}}}\right)$$

for every $\epsilon > 0$.

Note that for α with bounded continued fraction expansion we have $\|h\alpha\| \geq c/h$ for all integers $h \neq 0$ which leads to

$$D_N \leq c(\alpha, q)\frac{\sqrt{\log \log N}}{\sqrt{\log N}} \tag{1}$$

(for some constant $c(\alpha, q)$ depending on α and q) which was until now the best known upper bound for sequences of the form $(\{\alpha \cdot s_q(n)\})$.

We also note that the above results partly were generalized to sequences of the form $(\{\alpha \cdot f(n)\})$ where $f(n)$ is a (non-zero) strongly q-additive function (see [5]).

It is the intention of the first part of this paper to refine the above results, i.e., to give estimates for D_N which take into account the essential approximation properties of α in more detail.

By doing this we frequently will have to stress the continued fraction expansion of α.

Let α (w.l.o.g. $0 < \alpha < 1$) have continued fraction expansion

$$\alpha = [0; a_1, a_2, \ldots],$$

and denote by

$$q_{-1} = 0 \text{ and } 1 = q_0 \leq q_1 < q_2 < \ldots$$

the best approximation denominators of α. They satisfy the recurrent relations $q_{i+1} = a_{i+1}q_i + q_{i-1}$ for $i \geq 0$.

We already mentioned that the previously known results were all obtained via the Erdős-Turán inequality, respectively the Koksma inequality, and hence via the estimation of certain exponential sums. For our refined estimates we have to use a direct approach, which gives more insight into the structure of the sequence.

We will show first the following lower bound

Theorem 1 *There are positive constants $c(q)$ and $d(q)$ with the following property: For given N let i be such that the approximation denominator q_i satisfies $q_i \leq d(q)\sqrt{\log N} < q_{i+1}$. Then the discrepancy D_N of $(\{\alpha \cdot s_q(n)\})$ satisfies*

$$D_N > \frac{c(q)}{q_i}.$$

We state some immediate consequences.

Corollary 1 *For all N and all irrational α we have*

$$D_N > c'(q) \cdot \frac{1}{\sqrt{\log N}},$$

with a constant $c'(q) > 0$ depending only on q.

Remark. The result of Corollary 1 even holds for sequences $(\{\alpha \cdot f(n)\})$ with f strictly q-additive, as easily can be seen by following the first part of the proof of Theorem 1.

Corollary 2 *Let N be such that for some approximation denominator q_j of α we have $q_j = \lceil d(q)\sqrt{\log N}\rceil$ then*

$$D_N > c''(q) \cdot \frac{a_j}{\sqrt{\log N}}$$

with a constant $c''(q) > 0$ only depending on q.

Remark. Corollary 2 will lead to an optimal metric lower bound for D_N (see Corollary 4).

It is more difficult to obtain a satisfying upper bound for D_N. So, in a first step, it is no problem to obtain a result of (roughly) the following form (this indeed is an easy consequence of the subsequent Theorem 2):

There are positive constants $c(q)$ and $d(q)$ such that

$$D_N \quad \leq \quad \frac{c(q)}{\sqrt{\log N}} \sum_{1 \leq l \leq i_0+1} a_l \quad ,$$

where i_0 is defined by

$$q_{i_0} \leq d(q)\sqrt{\log N \cdot \log\log N} < q_{i_0+1}.$$

This result is motivated by, in some sense, equivalent results for the sequence $(\{\alpha \cdot n\})$. However, whereas in that case those results are sharp, the above result is not sharp for the sequence $(\{\alpha \cdot s_q(n)\})$.

So for example for α with bounded continued fraction coefficients we obtain as a consequence

$$D_N \leq c(q, \alpha)\frac{\log\log N}{\sqrt{\log N}},$$

which is a weaker bound than the previously known bound stated in (1).

The reason for the "weakness" of this result is, that indeed not all continued fraction coefficients are of the same importance for the distribution properties of $(\{\alpha \cdot s_q(n)\})$. As already indicated by the result of Theorem 1, of highest importance are continued fractions a_i with i such that q_i is in the "neighbourhood" of $\sqrt{\log N}$. The a_i with q_i "much larger" than $\sqrt{\log N}$ are of no importance, and a_i with q_i "much smaller" than $\sqrt{\log N}$ are of "little" importance. This fact is reflected in the following upper estimate.

Theorem 2 *Let $q > 1$ be an integer and let α be an irrational number with continued fraction expansion $\alpha = [a_0; a_1, \ldots]$ and best approximation denominators $1 = q_0 \leq q_1 < q_2 \ldots$.*

Then there is a positive constant $d(q)$ and for every $\gamma \geq 0$ there is a constant $c(q, \gamma) > 0$ such that the discrepancy D_N of $(\{\alpha \cdot s_q(n)\})$ can be estimated by

$$
\begin{aligned}
D_N \quad \leq \quad \frac{c(q,\gamma)}{\sqrt{\log N}} \Bigg(& \sum_{i_0 - \gamma \log\log\log N \leq l \leq i_0} a_l + \frac{a_{i_0+1}}{(\log\log N)^{\gamma \frac{\log 2}{2} - \frac{1}{2}}} \\
& + \frac{(\log\log\log N)^{3/2}}{(\log\log N)^{\gamma \frac{\log 2}{2}}} \cdot \max_{l \leq i_0 - \gamma \log\log\log N} a_l \Bigg),
\end{aligned}
$$

where i_0 is defined by

$$q_{i_0} \leq d(q) \sqrt{\log N \cdot \log\log N} < q_{i_0+1}.$$

Remark The following proof shows that we can do a little bit better. a_{i_0+1} may be replace by the smaller value

$$\bar{a}_{i_0+1} := \frac{d(q)\sqrt{\log N \cdot \log\log N}}{q_{i_0}} \leq a_{i_0+1}.$$

As a first corollary we obtain a sharpening of (1).

Corollary 3 *If α has bounded continued fraction coefficients, then with a constant $c(q, \alpha) > 0$ we have for the discrepancy D_N of the sequence $(\{\alpha \cdot s_q(n)\})$:*

$$D_N \leq c(q, \alpha) \frac{\log\log\log N}{\sqrt{\log N}}.$$

Remark. This result can be obtained by choosing $\gamma = \frac{1}{\log 2}$, for example. The previously known bound (1) corresponds to the choice $\gamma = 0$.

We also obtain the following optimal metrical bound

Corollary 4 *For almost all α the discrepancy D_N of the sequence $(\{\alpha \cdot s_q(n)\})$ satisfies for all $\epsilon > 0$*

$$D_N = O\left(\frac{(\log\log N)^{1+\epsilon}}{\sqrt{\log N}}\right) \text{ and } D_N = \Omega\left(\frac{\log\log N}{\sqrt{\log N}}\right).$$

Proof. Corollary 4 follows from Theorem 2 and from the facts that for almost all α we have

$$\limsup_{i\to\infty} \frac{a_i}{i} = \infty$$

whereas for every $\epsilon > 0$

$$\lim_{i\to\infty} \frac{a_i}{i^{1+\epsilon}} = 0,$$

and that

$$\left(\frac{\log q_i}{i}\right)_{i=1,2,\dots}$$

is bounded. $\square$

Finally, we slightly sharpen the upper bound for D_N if α has approximation type $\eta < 2$. The lower bound is the same as above (but obtained by a completely different proof without exponential sums and Koksma's inequality).

Corollary 5 *Suppose that there exist constants $c_1 > 0$, $\eta > 1$ such that $||h\alpha|| \geq c_1\, h^{-\eta}$ for all integers $h \geq 0$. Then the discrepancy D_N of the sequence $(\{\alpha \cdot s_q(n)\})$ is bounded by*

$$D_N \leq c(c_1, \eta, q) \frac{(\log\log N)^{\frac{1}{2}-\frac{1}{2\eta}}}{(\log N)^{\frac{1}{2\eta}}}$$

with a constant $c(c_1, \eta, q) > 0$ depending on c_1, η, and q.
Conversely, if for some $\eta \geq 1$ and some constant $c_2 > 0$ we have $||h\alpha|| < c_2\, h^{-\eta}$ for infinitely many integers $h > 0$ then

$$D_N \geq c'(c_2, \eta, q) \frac{1}{(\log N)^{\frac{1}{2\eta}}}$$

for infinitely many $N \geq 0$, where the constant $c'(c_2, \eta, q)$ depends on c_2, η, and q.

Proof. We have $||q_i\alpha|| \geq \frac{c}{q_i^\eta}$ for all i. Hence, we obtain from

$$\frac{1}{2q_i a_{i+1}} \leq ||q_i\alpha|| \leq \frac{1}{q_i a_{i+1}},$$

that

$$a_{i+1} \leq \frac{1}{c} q_i^{\eta-1} \leq \frac{1}{c} \left(\frac{q_{i+1}}{a_{i+1}}\right)^{\eta-1},$$

and therefore (with an absolute constant $\tilde{c}$)

$$a_{i+1} \leq \tilde{c} q_{i+1}^{1-\frac{1}{\eta}} \text{ for all i.}$$

Now by Theorem 2 (note that for $\eta > 1$ we have $\sum_{j\leq k} q_j^{1-\frac{1}{\eta}} \leq \bar{c} \cdot q_k^{1-\frac{1}{\eta}}$) the upper bound follows.

For the lower bound we similarly observe that for infinitely many i

$$a_{i+1} \geq \frac{1}{c'} q_i^{\eta-1} \geq \frac{1}{c'} \left(\frac{q_{i+1}}{2a_{i+1}} \right)^{\eta-1},$$

which gives (with an absolute constant $\tilde{c}'$)

$$a_{i+1} \geq \tilde{c}' q_{i+1}^{1-\frac{1}{\eta}} \text{ for infinitely many i.}$$

Now we just have to apply Corollary 2 and the result follows. $\square$

So as main open problem in the one-dimensional case there remains

Problem. *Are there irrational numbers α, such that for the discrepancy D_N of $(\{\alpha \cdot s_q(n)\})$; we have*

$$D_N \leq \frac{c}{\sqrt{\log N}}$$

as $N \to \infty$?

3 Results in higher dimensions for coprime bases

First of all we want to state a direct generalization of the well-known one-dimensional result saying that the sequence $(\{\alpha s_q(n)\})$ is uniformly distributed modulo 1 if and only if α is irrational.

Theorem 3 *Let $q_1, q_2, \ldots, q_d$ be pairwisely comprime integers > 1. Then the d-dimensional sequence $(\{\alpha_1 s_{q_1}(n)\}, \{\alpha_2 s_{q_2}(n)\}, \ldots, \{\alpha_d s_{q_d}(n)\})$ $(n \geq 0)$ is uniformly distributed modulo 1 if and only if $\alpha_1, \alpha_2, \ldots, \alpha_d$ are irrational.*

Remark It is possible to prove a little more, namely that the d-dimensional sequence $(\{\alpha_1 s_{q_1}(n)\}, \{\alpha_2 s_{q_2}(n)\}, \ldots, \{\alpha_d s_{q_d}(n)\})$ is well distributed modulo 1, too (see [5]). This fact was also observed by Grabner, Liardet and Tichy [6] via ergodic means, even for Cantor representations. However, our method works with exponential sums. So we can be much more precise, especially if we know approximation properties of α_j $(1 \leq j \leq d)$.

Theorem 4 *Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be irrational numbers. Suppose that there exists $\eta \geq 1$ and a constant $c_1 > 0$ such that for all integers $h > 0$ we have $\|h\alpha_j\| \geq c_1 h^{-\eta}$ for all $1 \leq j \leq d$. Then the discrepancy D_N of the sequence $(\{\alpha_1 s_{q_1}(n)\}, \{\alpha_2 s_{q_2}(n)\}, \ldots, \{\alpha_d s_{q_d}(n)\})$ $(n \geq 0)$ is bounded by*

$$D_N \leq c(c_1, \eta, q) \left(\frac{\log \log N}{\log N} \right)^{\frac{1}{2\eta}}$$

with some constant $c(c_1, \eta, q) > 0$ depending on c_1, η, and $q = (q_1, q_2, \dots, q_d)$. Conversely, if for some $\eta \ge 1$ and some constant $c_2 > 0$ there exists j such that $\|h\alpha_j\| \le c_2\, h^{-\eta}$ for infinitely many integers $h > 0$ then

$$D_N \ge c'(c_2, \eta, q) \frac{1}{(\log N)^{\frac{1}{2\eta}}}$$

for infinitely many $N \ge 0$, where the constant $c(c_1, \eta, q) > 0$ depends on c_2, η, and $q = (q_1, q_2, \dots, q_d)$.

4 Auxiliary results

In this section we collect and, if necessary prove, some technical results which will be used in the proofs of Theorems 1 and 2.

For integers $q \ge 2$, $n \ge 1$, and k with $0 \le k \le n(q-1)$ let

$$T_q(n,k) := \left|\{(i_0, \dots, i_{n-1}) \in \{0, \dots, q-1\}^n \ :\ i_0 + \dots + i_{n-1} = k\}\right|.$$

This is of course the number of $j < q^n$ with $s_q(j) = k$. Obviously, $T_q(n,k)$ is the coefficient of the power series

$$\sum_{k \ge 0} T_q(n,k) x^k = (1 + x + x^2 + \dots + x^{q-1})^n.$$

So we can apply Theorem 1 of [13] and observe that there exists $n(q)$ such that for $n \ge n(q)$ the function $T_q(n,k)$ is unimodular in k. The fact will sometimes be used implicitely.

We further have

Lemma 1

$$T_q(n,k) = \frac{q^n}{\sqrt{2\pi n \sigma_q^2}} \cdot e^{-\frac{1}{2}x_{n,k}^2} \left(1 + \frac{A_1(x_{n,k})}{\sqrt{n}} + \frac{A_2(x_{n,k})}{n}\right) + O\left(\frac{q^n}{n^2}\right),$$

for certain polynomials $A_1(x)$, $A_2(x)$, where $A_1(x)$ is odd and where $x_{n,k}$ denotes

$$x_{n,k} = \frac{k - \frac{n(q-1)}{2}}{\sigma_q \sqrt{n}}$$

and $\sigma_q := \sqrt{\frac{q^2-1}{12}}$. The O-constant is uniform for all k and only depends on q.

Proof. This is a simple application of local limit theorems for sums of independent random variables (see for example [14] or [3]). □

Lemma 2 *Let $q > 1$ be an integer. Then there exists a constant $c(q) > 0$ such that*

$$\sum_{j \geq x\sqrt{n}} T_q\left(n, \frac{n(q-1)}{2} - j\right) \leq q^n e^{-c(q)x^2}$$

for all $x \geq 0$.

Proof. This again is a well known "tail estimate" for sums of independent random variables (see for example [4], Proposition 4.1 and 4.2). $\Box$

Remark. From Lemma 2 easily the following extension of the above result can be obtained:

> *Let $q > 1$ be an integer. Then there exists a constant $c'(q) > 0$ such that*
>
> $$\left| \{n < N : |s_q(n) - \frac{q-1}{2} \cdot \frac{\log N}{\log q}| \geq x\sqrt{\log N}\} \right| \leq N e^{-c'(q)x^2},$$
>
> *for all N and all $x \geq 0$.*

Lemma 3 *Let $q, n > 1$ be integers, such that $n(q-1)$ is even. Let S and σ_k, $k \geq 0$, be positive reals with $1 \leq S < \frac{n(q-1)}{2}$ and $\sigma_k \leq c \cdot S$ for all k, where c is a given positive constant such that $kS + \sigma_k$ is an integer for all k. Then with a constant $c(q,c)$ depending only on c and q*

$$\sum_{k \geq 0} T_q\left(n, \frac{n(q-1)}{2} - kS - \sigma_k\right) \leq c(q,c) \cdot T_q\left(n, \frac{n(q-1)}{2}\right) + \frac{q^n}{2S}.$$

Proof. Let n be so large that $T_q(n,k)$ is unimodular (for smaller n (if they exist at all) the result is trivial). Then

$$\begin{aligned} \sum_{k \geq 0} T_q\left(n, \frac{n(q-1)}{2} - kS - \sigma_k\right) &\leq (c+2) \cdot T_q\left(n, \frac{n(q-1)}{2}\right) \\ &+ \sum_{k \geq 1} T_q\left(n, \frac{n(q-1)}{2} - [kS] - 1\right) \\ &\leq (c+2) \cdot T_q\left(n, \frac{n(q-1)}{2}\right) + \frac{q^n}{2S}. \quad \Box \end{aligned}$$

Lemma 4 *Let $q > 1$ be an integer. There is a constant $c'(q) > 0$ such that for all integers n, k, and a with $0 \leq k < k + a \leq n(q-1)$ we have*

$$|T_q(n, k+a) - T_q(n,k)| \leq c'(q) \cdot \frac{q^n a}{n}.$$

Proof. This immediately follows from Lemma 1. □

Lemma 5 *Let $1 = G_0 \leq G_1 < G_2 < \ldots$ be the best approximation denominators of a real number δ. Let l_0 be a positive integer. Then*

$$\sum_{l \geq l_0} \frac{1}{G_{l+1}} \leq \frac{2}{\sqrt{2}-1} \cdot \frac{1}{2^{\frac{l_0}{2}}}.$$

Proof. This follows easily from the fact that $G_l \geq 2^{\frac{l-1}{2}}$ for all l. □

Lemma 6 *Let p, q be positive integers with $\gcd(p,q) = 1$ and let $\frac{p}{q} := [g_0; g_1, \ldots, g_r]$ be the continued fraction expansion of $\frac{p}{q}$.*

Then for all integers v, w with $0 \leq v < v + w \leq q$ we have

$$\left| \sum_{k=v}^{v+w-1} \{k \cdot \frac{p}{q}\} - \frac{w}{2} \right| \leq 4 \cdot (g_1 + \cdots + g_r).$$

Proof. This follows immediately by applying Koksma's inequality (Theorem 5.1 in [8], Chapter 2) for $f(x) = x$ and by formula (3.18) in [8], Chapter 2, concerning the discrepancy of the point set $\{k \cdot \frac{p}{q}\}; k = v, v+1, \ldots, v+w-1$. □

Lemma 7 *Let $q, n > 1$ be integers. Let $S \geq 1$ and σ and τ be positive reals with $0 \leq \sigma, \tau < S$.*

Let $y_0, y_1, \ldots$ be a sequence in $[0,1)$ with the property that for all integers $k \geq 0$ we have

$$\left| \sum_{l=0}^{k} y_l - \frac{k+1}{2} \right| < \Delta \quad \text{for some } \Delta \geq 1, \tag{2}$$

and that $kS + \sigma + \tau \cdot y_k$ are integers. Then with a constant $c(q) > 0$ only depending on q we have

$$\sum_{k \geq 0} T(n, kS + \sigma + \tau \cdot y_k) \leq \frac{q^n}{S} + c(q) \frac{q^n}{n} S \Delta.$$

Remark. One could try to estimate the above sum with the help of the two-dimensional version of the Koksma-Hlawka inequality. Indeed, one obtains non-trivial results with the help of this inequality. However, these results are much too weak.

Proof. First observe that we only have to prove Lemma 7 for $S \leq \sqrt{n}$. If $S > \sqrt{n}$ then Lemma 3 provides better bounds.

We now replace the values $T_q(n,k)$ for $0 \leq k \leq (q-1)n$ by

$$\bar{T}_q(n,k) := \frac{q^n}{\sqrt{2\pi n \sigma_q}} \cdot e^{-\frac{1}{2} x_{n,k}^2} \left(1 + \frac{A_1(x_{n,k})}{\sqrt{n}} + \frac{A_2(x_{n,k})}{n} \right)$$

(with the notation of Lemma 1). In view of Lemma 1 and 2 it is sufficient to prove the result for $\bar{T}_q(n,k)$. We obviously have

$$\sum_{k\ge 0} \left| T(n,kS+\sigma+\tau\cdot y_k) - \bar{T}(n,kS+\sigma+\tau\cdot y_k) \right| = O\left(\frac{q^n}{n}\right).$$

The advantage of the use of $\bar{T}_q(n,k)$ is that it provides a natural analytic continuation to non-integral k.

For the rest of the proof we slightly change the range of summation. We also set $z_k = y_{k+[\frac{n(q-1)}{2}]} - \frac{1}{2}$ for $-\left[\frac{n(q-1)}{2}\right] \le k \le \left[\frac{n(q-1)}{2}\right]$ and $z_k = 0$ for $k < -\left[\frac{n(q-1)}{2}\right]$ and $k > \left[\frac{n(q-1)}{2}\right]$. In fact, we will compare

$$S_1 := \sum_{k\in\mathbf{Z}} \bar{T}\left(n, kS + \frac{n(q-1)}{2} + \sigma + \tau\cdot z_k\right)$$

and

$$S_2 := \sum_{k\in\mathbf{Z}} \bar{T}\left(n, kS + \frac{n(q-1)}{2} + \sigma\right).$$

Note that this change only effects σ by a finite shift $< S$ and that we now have

$$\left| \sum_{l=-\infty}^{k} z_k \right| < \Delta$$

for all $k \in \mathbf{Z}$.

First of all we have

$$\begin{aligned}
\bar{T}\left(n, kS + \frac{n(q-1)}{2} + \sigma + \tau\cdot z_k\right) &= \bar{T}\left(n, kS + \frac{n(q-1)}{2} + \sigma\right) \\
&+ \bar{T}'\left(n, kS + \frac{n(q-1)}{2} + \sigma\right)\tau z_k \\
&+ \frac{1}{2}\bar{T}''\left(n, kS + \frac{n(q-1)}{2}\sigma + \eta_k\right)\tau^2 z_k^2
\end{aligned}$$

with some $|\eta_k| \le \tau$. By the same method as in the proof of Lemma 3 it follows that

$$\sum_{k\in\mathbf{Z}} \bar{T}''\left(n, kS + \frac{n(q-1)}{2} + \sigma + \eta_k\right) = O\left(q^n\left(\frac{1}{Sn} + \frac{1}{n^{3/2}}\right)\right) = O\left(\frac{q^n}{Sn}\right).$$

Hence,

$$S_1 - S_2 = \tau \sum_{k\in\mathbf{Z}} \bar{T}'\left(n, kS + \frac{n(q-1)}{2} + \sigma\right) z_k + O\left(\frac{q^n S}{n}\right).$$

Now, by partial summation

$$\begin{aligned}
&\left|\sum_{k\in\mathbf{Z}} \bar T'\left(n, kS+\frac{n(q-1)}{2}+\sigma\right) z_k\right| \\
&\le \sum_{k\in\mathbf{Z}} \left|\bar T'\left(n,(k+1)S+\frac{n(q-1)}{2}+\sigma\right) - \bar T'\left(n,kS+\frac{n(q-1)}{2}+\sigma\right)\right| \cdot \left|\sum_{l=-\infty}^{k} z_l\right| \\
&\le \Delta \sum_{k\in\mathbf{Z}} \left|\bar T'\left(n,(k+1)S+\frac{n(q-1)}{2}+\sigma\right) - \bar T'\left(n,kS+\frac{n(q-1)}{2}+\sigma\right)\right|.
\end{aligned}$$

Since the number of turning points of $\bar T(n,x)$ (i.e., of points x_0 with $\bar T(n,x_0)'' = 0$) is bounded by a number depending only on q it follows that

$$\begin{aligned}
&\sum_{k\in\mathbf{Z}} \left|\bar T'\left(n,(k+1)S+\frac{n(q-1)}{2}+\sigma\right) - \bar T'\left(n,kS+\frac{n(q-1)}{2}+\sigma\right)\right| \\
&\quad = O\left(\max_{x\ge 0} \bar T'(n,x)\right) = O\left(\frac{q^n}{n}\right).
\end{aligned}$$

This proves that

$$S_1 - S_2 = O\left(\frac{q^n \Delta S}{n}\right) + O\left(\frac{q^n S}{n}\right) = O\left(\frac{q^n \Delta S}{n}\right).$$

Finally we have to consider S_2. Similarly to the above considerations, it follows for $|\sigma| \le S$ that

$$\begin{aligned}
&\sum_{k\in\mathbf{Z}} \left(\bar T\left(n,kS+\frac{n(q-1)}{2}+\sigma\right) - \bar T\left(n,kS+\frac{n(q-1)}{2}\right)\right) \\
&\quad = \sigma \sum_{k\in\mathbf{Z}} \bar T'\left(n,kS+\frac{n(q-1)}{2}\right) + O\left(\frac{q^n S}{n}\right)
\end{aligned}$$

It is an easy exercise by using the symmetry of $\bar T\left(n, \frac{n(q-1)}{2}+x\right)$ and the fact that the polynomial $A_1(x)$ is odd that

$$\sum_{k\in\mathbf{Z}} \bar T\left(n,kS+\frac{n(q-1)}{2}\right) = \frac{q^n}{S} + O\left(\frac{q^n}{n}\right)$$

and

$$\sum_{k\in\mathbf{Z}} \bar T'\left(n,kS+\frac{n(q-1)}{2}\right) = O\left(\frac{q^n}{Sn}\right).$$

Thus,

$$S_2 = \frac{q^n}{S} + O\left(\frac{q^n S}{n}\right)$$

and the proof is complete. □

Lemma 8 *Let $\delta = [0; g_1, g_2, \ldots]$ be the continued fraction expansion of the irrational number δ.*

Let $1 = G_0 \le G_1 < G_2 < \ldots$ be the best approximation denominators of δ. Let l_0 be a positive integer, then

$$\sum_{l \ge l_0} \frac{g_1 + \cdots + g_l}{G_l} \le c \frac{l_0}{2^{\frac{l_0}{2}}}$$

with an absolute constant c.

Proof. Since $G_l \ge \max_{j \le l} g_j \cdot 2^{\frac{l-2}{2}}$, we obtain

$$\sum_{l \ge l_0} \frac{g_1 + \cdots + g_l}{G_l} \le \sum_{l \ge l_0} \frac{l}{2^{\frac{l-2}{2}}} \le c \cdot \frac{l_0}{2^{\frac{l_0}{2}}}. \quad \square$$

Lemma 9 *Let D_m denote the discrepancy of $(\{\alpha \cdot s_q(n)\}); n = 0, 1, \ldots, m-1$.*

Let $N = b_r q^r + \cdots + b_1 q + b_0$ be the base q representation of the positive integer N. Then with a constant $c(q) > 0$ we have

$$D_N \le c(q) \frac{1}{N} \sum_{i=0}^{r} q^i \cdot D_{q^i}.$$

Proof. This follows from basic principles in the theory of uniform distribution, and from the fact that for all positive integers i and v the point set $(\{\alpha \cdot s_q(n)\}); n = v \cdot q^i, v \cdot q^i + 1, \ldots, v \cdot q^i + q^i - 1$ just is a shifted version of $(\{\alpha \cdot s_q(n)\}); n = 0, 1, \ldots, q^i - 1$. $\square$

5 The proofs for Theorems 1 and 2

Proof of Theorem 1. Let n be the unique even integer satisfying $q^n \le N < q^{n+2}$. Then we obviously have

$$\begin{aligned} \left| \left\{ k < N : \{s_q(k)\alpha\} = \left\{ \frac{(q-1)n}{2} \alpha \right\} \right\} \right| &= \left| \left\{ k < N : s_q(k) = \frac{(q-1)n}{2} \right\} \right| \\ &\ge T_q\left(n, \frac{(q-1)n}{2}\right). \end{aligned}$$

Thus, by Lemma 1

$$D_N \ge \frac{1}{N} T_q\left(n, \frac{(q-1)n}{2}\right) \ge c_1(q) \frac{1}{N} \frac{q^n}{\sqrt{n}} \ge c(q) \frac{1}{\sqrt{\log N}}. \tag{3}$$

We now choose

$$d(q) = \frac{q^3}{2\sqrt{\log q}}.$$

If $q_i > \sqrt{n}$, then the assertion of the theorem follows from (3).

If $q_i < \sqrt{n}$ then let $a := \left[\frac{\sqrt{n}}{q_i}\right]$, and consider the set of points

$$P := \left\{ \left\{ \frac{(q-1)n}{2}\alpha \right\}, \left\{ \left(\frac{(q-1)n}{2} + q_i \right) \alpha \right\}, \dots, \left\{ \left(\frac{(q-1)n}{2} + aq_i \right) \alpha \right\} \right\}.$$

It is $\alpha = \frac{p_i}{q_i} + \frac{\delta_i}{q_i q_{i+1}}$ for some integer p_i and some δ_i with $|\delta_i| < 1$. We have $\delta_i > 0$ if and only if i is even and we may w.r.o.g. assume that this is the case. Hence, the points in P are contained in the interval

$$\left[\left\{ \frac{(q-1)n}{2}\alpha \right\}, \left\{ \frac{(q-1)n}{2}\alpha \right\} + \frac{a}{q_{i+1}} \right) \mod 1 .$$

Furthermore, we have

$$\begin{aligned} &|\{k < N : \{s_q(k)\alpha\} \in P\}| \\ &\quad \geq T_q\left(n, \frac{(q-1)n}{2}\right) + T_q\left(n, \frac{(q-1)n}{2} + q_i\right) + \dots + T_q\left(n, \frac{(q-1)n}{2} + aq_i\right) \\ &\quad \geq aT_q\left(n, n + [\sqrt{n}]\right) . \end{aligned}$$

Thus, the discrepancy D_N of the sequence $\{s_q(k)\alpha\}$ is bounded below by

$$D_N \geq \frac{1}{N}\left(aT_q\left(n, n + [\sqrt{n}]\right) - q^{n+2}\frac{a}{q_{i+1}} \right) .$$

By Lemma 1 there exists $n_0 = n_0(q)$ such that for $n > n_0$

$$T_q(n, n + [\sqrt{n}]) > \frac{q^n}{\sqrt{n}}\sqrt{\frac{6}{\pi(q^2-1)}}$$

If $n < n_0$ then the result of the theorem is trivially true. However, if $n \geq n_0$ then

$$\begin{aligned} D_N &\geq \frac{1}{N}a\frac{q^n}{\sqrt{n}}\left(\sqrt{\frac{6}{\pi(q^2-1)}} - q^2\frac{2\sqrt{\log q}}{q^3\frac{\sqrt{\log N}}{\sqrt{n}}} \right) \\ &\geq \frac{aq^{-1}}{\sqrt{n}}\left(\sqrt{\frac{6}{\pi(q^2-1)}} - \frac{2}{q} \right), \end{aligned}$$

and the result follows. □

Proof of Theorem 2. We consider the sequence $(v_k) = (\{\alpha s_q(k)\})$, $0 \leq k < N$.

We remind that $\alpha := [0; a_1, a_2, \ldots]$ is the continued fraction expansion of α, $1 = q_0 \le q_1 < q_2 < \ldots$ are the best approximation denominators of α, which are recursively given by $q_{i+1} = a_{i+1} q_i + q_{i-1}$. Furthermore, we have $\alpha = \frac{p_i}{q_i} + \frac{\delta_i}{q_i q_{i+1}}$, with $\frac{1}{2} < |\delta_i| < 1$ for all i, and $\delta_i > 0$ if and only if i is even.

Let first $N = q^n$ for some $n \ge 1$. Then $s_q(k)$ attains the values j with $0 \le j \le (q-1)n$ with frequency $T_q(n, j)$.

We assume in the following that $(q-1)n$ is even. (The other case is treated quite analogously.)

Then with $\xi := \left\{\frac{(q-1)n}{2}\alpha\right\}$, the sequence $(\{\alpha s_q(k)\})$, $0 \le k < q^n$ consists of the points $\{\xi \pm j\alpha\}$, $0 \le j \le \frac{(q-1)n}{2}$, with frequency $T_q(n, \frac{(q-1)n}{2} \pm j) = T_q(n, \frac{(q-1)n}{2} - j)$. Firstly, we shift the sequence (v_k) by ξ, i.e., we consider $w_k := \{v_k - \xi\}$, $0 \le k < q^n$. Secondly we append $T_q(n, \frac{(q-1)n}{2})$ times the point 1, i.e. we set $w_k := 1$ for $q^n \le k < q^n + T_q(n, \frac{(q-1)n}{2})$. This means that we can subdivide the sequence (w_k), $0 \le k \le q^n + T_q(n, \frac{(q-1)n}{2})$, into two subsequences W_1, W_2 such that W_1 consists of the points $\{j\alpha\}$, $0 \le j \le \frac{(q-1)n}{2}$, with frequency $T_q(n, \frac{(q-1)n}{2} - j)$ and W_2 consists of the points $1 - \{j\alpha\}$, $0 \le j \le \frac{(q-1)n}{2}$, also with frequency $T_q(n, \frac{(q-1)n}{2} - j)$.

For x with $0 < x \le 1$ let $A([0,x))$ denote the number of points of w_k, $0 \le k < q^n + T_q(n, \frac{(q-1)n}{2})$, in $[0,x)$:

$$A([0,x)) := \left|\left\{k < q^n + T_q\left(n, \frac{(q-1)n}{2}\right) : w_k \in [0,x)\right\}\right|.$$

Due to the symmtry of the sequence w_k the function

$$\Delta(x) := A([0,x)) - x\left(q^n + T_q\left(n, \frac{(q-1)n}{2}\right)\right)$$

satisfies $\Delta(x) = -\Delta(1-x)$ (if x is not of the for $x = \{j\alpha\}$ or $x = 1 - \{j\alpha\}$, $0 \le j \le \frac{(q-1)n}{2}$) and consequently

$$\sup_{0 \le x \le 1} |\Delta(x)| = \sup_{0 \le x \le 1} \Delta(x).$$

Let $\Delta_{q^n} := q^n D_{q^n}(v_k)$ denote the discrepancy of the sequence of interest (multiplied by q^n.) Then we trivially have

$$\begin{aligned}
\Delta_{q^n} &\le 2q^n D_{q^n}(w_k) \\
&\le 2T_q\left(n, \frac{(q-1)n}{2}\right) + 2\sup_{0 \le x \le 1} \Delta(x) \\
&\le 4T_q\left(n, \frac{(q-1)n}{2}\right) + 2\sup_{0 \le x \le 1} (A([0,x)) - q^n x).
\end{aligned}$$

So it remains to estimate $A([0,x))$ from above.

Now let i be be defined by $q_i \leq \sqrt{\frac{2}{c(q)}}\sqrt{n\log n} < q_{i+1}$, where $c(q)$ is the constant in Lemma 2.

Let $x := \frac{b}{q_i}$ for some $0 < b < q_i$. We have

$$\sup_y (A([0,y)) - q^n y) \leq \max_b \left(A\left(\left[0, \frac{b}{q_i}\right)\right) - q^n \frac{b}{q_i}\right) + \frac{q^n}{q_i},$$

hence

$$\Delta_{q^n} \leq 4T_q\left(n, \frac{(q-1)n}{2}\right) + 2\frac{q^n}{q_i} + 2\max_b \left(A\left(\left[0, \frac{b}{q_i}\right)\right) - q^n \frac{b}{q_i}\right).$$

We assume in the following that i is even (the case "i odd" is treated quite analogously).

Instead of $A([0,\frac{b}{q_i}))$ we will consider $B([0,\frac{b}{q_i}))$ which denotes the number of points of w_n in $[0,\frac{b}{q_i})$ which are of the form

$$\{j\alpha\} \text{ or } 1 - \{j\alpha\} \text{ for } j = 0, \ldots, \bar{a}q_i - 1,$$

where

$$\bar{a} := \left[\sqrt{\frac{2}{c(q)}}\frac{\sqrt{n\log n}}{q_i}\right] \leq a_{i+1}.$$

By Lemma 2 we have

$$\begin{aligned} A\left(\left[0, \frac{b}{q_i}\right)\right) &\leq B\left(\left[0, \frac{b}{q_i}\right)\right) + 2\sum_{j \geq \sqrt{\frac{n\log n}{2c(q)}}} T_q\left(n, \frac{(q-1)n}{2} - j\right) \\ &\leq B\left(\left[0, \frac{b}{q_i}\right)\right) + c_1(q)\frac{q^n}{\sqrt{n}}, \end{aligned}$$

and hence

$$\Delta_{q^n} \leq 4T_q(n, \frac{(q-1)n}{2}) + 2\frac{q^n}{q_i} + 2c_1(q)\frac{q^n}{\sqrt{n}} + 2\max_b \left(B\left(\left[0, \frac{b}{q_i}\right)\right) - q^n \frac{b}{q_i}\right).$$

We recall that the rational number $\frac{q_i - q_{i-1}}{q_i}$ has continued fraction expansion

$$\begin{aligned} \frac{q_i - q_{i-1}}{q_i} &= [0; e_1, e_2, \ldots, e_{i'}] \\ &= \begin{cases} [0; 1 + a_{i-1}, a_{i-2}, \ldots, a_1] \text{ if } a_i = 1 \\ [0; 1, a_i - 1, a_{i-1}, \ldots, a_1] \text{ if } a_i > 1 \end{cases}. \end{aligned}$$

Let $1 = Q_0 \leq Q_1 < Q_2 < \ldots < Q_{i'} = q_i$ be the best approximation denominators of $\frac{q_i - q_{i-1}}{q_i}$ and let b be represented by

$$b = c_{i'-1}Q_{i'-1} + \cdots + c_l Q_l + \cdots + c_0 Q_0$$

with $0 \le c_l < e_{l+1}$. We further have

$$\frac{q_i - q_{i-1}}{q_i} = \frac{P_j}{Q_j} + \frac{\gamma_j}{Q_j Q_{j+1}},$$

with $\frac{1}{2} \le |\gamma_j| \le 1$ for all j.

We now consider j of the form $j = aq_i + h$ with $aq_i \le j < (a+1)q_i$, i.e., $0 \le h < q_i$. Then we have

$$\{j\alpha\} = \frac{\{hp_i\}_{q_i}}{q_i} + \frac{a\delta_i}{q_{i+1}} + \frac{h\delta_i}{q_i q_{i+1}} \in \left[0, \frac{b}{q_j}\right)$$

(where "$\{y\}_p$" means "y modulo p") if and only if

$$\{hp_i\}_{q_i} \in \{0, 1, \ldots, b-1\}$$

(note that i even and therefore $0 \le \frac{a\delta_i}{q_{i+1}} + \frac{h\delta_i}{q_i q_{i+1}} < \frac{1}{q_i}$).

Now for i even we have $p_i(q_i - q_{i-1}) \equiv 1 (\mathrm{mod}\ q_i)$, so that h must be of the form $h = t(q_i - q_{i-1})$ modulo q_i for some $t = 0, \ldots, b-1$.

We subdivide $0, \ldots, b-1$ into the following parts:

$$\Omega_{l,c} := \{c_{i'-1}Q_{i'-1} + \cdots + c_{l+1}Q_{l+1} + cQ_l + m : 0 \le m < Q_l\}$$

for $c = 0, 1, \ldots, c_l - 1$ and $l = 0, \ldots, i'-1$. Take some $t = c_{i'-1}Q_{i'-1} + \cdots + c_{l+1}Q_{l+1} + cQ_l + m \in \Omega_{l,c}$ then

$$\begin{aligned}
h &= t(q_i - q_{i-1}) \bmod q_i \\
&= q_i \cdot \Bigg[c_{i'-1}Q_{i'-1} \cdot \left(\frac{P_{i'-1}}{Q_{i'-1}} + \frac{\gamma_{i'-1}}{Q_{i'-1}Q_{i'}}\right) + \ldots \\
&\qquad + c_{l+1}Q_{l+1} \cdot \left(\frac{P_{l+1}}{Q_{l+1}} + \frac{\gamma_{l+1}}{Q_{l+1}Q_{l+2}}\right) \\
&\qquad + (cQ_l + m) \cdot \left(\frac{P_l}{Q_l} + \frac{\gamma_l}{Q_l Q_{l+1}}\right)\Bigg] (\bmod q_i) \\
&= q_i \cdot \left(\frac{c_{i'-1}\gamma_{i'-1}}{Q_{i'}} + \cdots + \frac{c_{l+1}\gamma_{l+1}}{Q_{l+2}} + \frac{c\gamma_l}{Q_{l+1}}\right) \\
&\quad + q_i \cdot \left(\frac{mP_l}{Q_l} + \frac{m\gamma_l}{Q_l Q_{l+1}}\right) (\bmod q_i) \\
&=: q_i \phi(l,c) + \{mP_l\}_{Q_l} \cdot \frac{q_i}{Q_l} + m \cdot \frac{\gamma_l q_i}{Q_l Q_{l+1}} (\mathrm{mod}\ q_i).
\end{aligned}$$

If m runs through $0, \ldots, Q_l - 1$, then $k := \{mP_l\}_{Q_l}$ runs through $0, \ldots, Q_l - 1$. Conversely, if k is given, then $m = m_k$ with $k := \{mP_l\}_{Q_l}$ is given by

$$m_k := \begin{cases} \{k(Q_l - Q_{l-1})\}_{Q_l} & \text{if } l \text{ is even} \\ \{kQ_{l-1}\}_{Q_l} & \text{if } l \text{ is odd.} \end{cases}$$

We further represent $\phi(l,c)$ by $\phi(l,c) = \frac{\psi}{Q_l} + \kappa$ with some integer ψ and $0 \leq \kappa < \frac{1}{Q_l}$.

So we get the representation

$$h = t(q_i - q_{i-1}) \bmod q_i = q_i\kappa + (k+\psi)\frac{q_i}{Q_l} + m_k \frac{\gamma_l q_i}{Q_l Q_{l+1}} \pmod{q_i},$$

with $k = 0, \dots, Q_l - 1$, or after a shift

$$\begin{aligned} h &= k\frac{q_i}{Q_l} + q_i\kappa + m_{k-\psi}\frac{\gamma_l q_i}{Q_l Q_{l+1}} \pmod{q_i} \\ &= k\frac{q_i}{Q_l} + q_i\kappa + x_k \pmod{q_i}, \end{aligned} \tag{4}$$

with $k = 0, \dots, Q_l - 1$.

In a similar way we can consider the condition

$$1 - \{j\alpha\} \in \left[0, \frac{b}{q_i}\right)$$

Let again $aq_i \leq j < (a+1)q_i$, $j = aq_i + h$, then

$$1 - \{j\alpha\} \in \left[0, \frac{b}{q_i}\right) \Longleftrightarrow \{j\alpha\} \in \left(1 - \frac{b}{q_i}, 1\right]$$

which is equivalent to

$$\frac{\{hp_i\}}{q_i} + \frac{a\delta_i}{q_{i+1}} + \frac{h\delta_i}{q_i q_{i+1}} \in \left(1 - \frac{b}{q_i}, 1\right]$$

or to

$$\{hp_i\}_{q_i} \in \{q_i - b, \dots, q_i - 1\}.$$

So h must be of the form

$$h = t(q_i - q_{i-1}) \bmod q_i \text{ for some } t = q_i - b, \dots, q_i - 1.$$

We subdivide $q_i - b, \dots, q_i - 1$ into the following parts:

$$\Lambda_{l,c} := \{q_i - (c_{i'-1}Q_{i'-1} + \cdots + c_{l+1}Q_{l+1} + cQ_l + m) : 0 \leq m < Q_l\}$$

for $c = 0, 1, \dots, c_l - 1$ and $l = 0, \dots, i' - 1$.

Indeed in this subdivision also $t = q_i$ but not $t = q_i - b$ is contained. This fact does change $B([0, \frac{b}{q_i}))$ at most by $T_q(n, \frac{(q-1)n}{2})$. We denote the corresponding quantity by $\tilde{B}([0, \frac{b}{q_i}))$, and get

$$\Delta_{q^n} \leq 6T_q\left(n, \frac{(q-1)n}{2}\right) + 2\frac{q^n}{q_i} + 2c_1(q)\frac{q^n}{\sqrt{n}} + 2\max_b\left(\tilde{B}\left(\left[0, \frac{b}{q_i}\right)\right) - q^n\frac{b}{q_i}\right).$$

If $t = q_i - (c_{i'-1}Q_{i'-1} + \cdots + c_{l+1}Q_{l+1} + cQ_l + m) \in \Lambda_{l,c}$ then as above it can be seen that $h = t(q_i - q_{i-1})$ mod q_i runs through

$$h = (Q_l - k)\frac{q_i}{Q_l} - q_i\kappa - x_k \pmod{q_i}, \tag{5}$$

with $k = 0, \ldots, Q_l - 1$.

Note that the sequences in (4) and (5) do not depend on a, i.e., for all a the "admissible" j with $aq_i \le j < (a+1)q_i$, $j = aq_i + h$ have the same h.

Now denote by $\tilde{B}(c,l)$ the number of points of the form $\{j\alpha\}$ respectively $1 - \{j\alpha\}$ in $[0, \frac{b}{q_i})$ with $j = aq_i + h$ for some $0 \le a < \bar{a}$ and $0 \le h < q_i$ with $h \equiv t(q_i - q_{i-1})\text{mod } q_i$ and $t \in \Omega_{l,c}$ respectively $\Lambda_{l,c}$. We have to estimate

$$\tilde{B}\left(\left[0, \frac{b}{q_i}\right)\right) = \sum_{l=0}^{i'-1}\sum_{c=0}^{c_l-1} \tilde{B}(c,l).$$

Let $l_0 = \gamma \log\log \frac{n}{2}$ with a non-negative absolute real constant γ.

For $l \le l_0$ we use the following trivial estimate for $\tilde{B}(c,l)$:

$$\tilde{B}(c,l) \le \frac{q^n}{q_i} Q_l + c(q)T_q\left(n, \frac{(q-1)n}{2}\right)$$

which is obtained from Lemma 3 with the parameters $S = \frac{q_i}{Q_l}$ and $\sigma_k = q_i\kappa + x_k$, respectively $\sigma_k = -q_i\kappa - x_k$, and from the fact that $|q_i\kappa + x_k| \le 2\frac{q_i}{Q_l}$ and hence $c = 2$ in Lemma 3. Therefore

$$\sum_{l\le l_0}\sum_{c=0}^{c_l-1} \tilde{B}(c,l) \le \frac{q^n}{q_i}\sum_{l=0}^{l_0} c_l Q_l + c(q)T_q\left(n, \frac{(q-1)n}{2}\right)\sum_{l\le l_0} c_l.$$

In the following we consider $\sum_{l>l_0}\sum_{c=0}^{c_l-1} \tilde{B}(c,l)$. First we once more replace $\tilde{B}(c,l)$ by a similar quantity. Now denote by $\bar{B}(c,l)$ the number of points of the form $\{j\alpha\}$ respectively $1 - \{j\alpha\}$ in $[0, \frac{b}{q_i})$ of the form

$$j = k\frac{q_i}{Q_l} + q_i\kappa + x_k$$

with $0 \le k < \bar{a}Q_l$ resp. of the form

$$j = k\frac{q_i}{Q_l} - q_i\kappa - x_{Q_l - k}$$

with $0 < k \le \bar{a}Q_l$. This means that we removed the expression "mod q_i" in (4) and (5) . This effects (4) and (5) if $\gamma_l > 0$ eventually for $k = Q_l - 1$, and if $\gamma_l < 0$ it eventually effects (4) and (5) for $k = 0$.

Assume $\gamma_l > 0$ ($\gamma_l < 0$ is treated in absolutely the same way): (4) and (5) then are effected both if and only if

$$q_i\kappa + x_{Q_l-1} \geq \frac{q_i}{Q_l}.$$

In (4)

$$-\frac{q_i}{Q_l} + q_i\kappa + x_{Q_l-1} \quad \text{is replaced by} \quad q_i - \frac{q_i}{Q_l} + q_i\kappa + x_{Q_l-1},$$

and in (5)

$$q_i + \frac{q_i}{Q_l} - q_i\kappa - x_{Q_l-1} \quad \text{is replaced by} \quad \frac{q_i}{Q_l} - q_i\kappa - x_{Q_l-1}.$$

Hence (by using Lemma 4)

$$\begin{aligned}
\tilde{B}(c,l) \quad \leq \quad & \bar{B}(c,l) \\
- \quad & \sum_{a=0}^{\bar{a}-1} \Bigg(T_q(n, \frac{(q-1)n}{2} - q_i + \frac{q_i}{Q_l} - q_i\kappa - x_{Q_l-1} - aq_i) \\
& \quad + T_q\left(n, \frac{(q-1)n}{2} - q_i - \frac{q_i}{Q_l} + q_i\kappa + x_{Q_l-1} - aq_i\right) \\
& \quad + T_q\left(n, \frac{(q-1)n}{2} + \frac{q_i}{Q_l} - q_i\kappa - x_{Q_l-1} - aq_i\right) \\
& \quad - T_q\left(n, \frac{(q-1)n}{2} - \frac{q_i}{Q_l} + q_i\kappa + x_{Q_l-1} - aq_i\right)\Bigg) \\
\leq \quad & \bar{B}(c,l) + 4\bar{a}c'(q)\frac{q^n}{n}\frac{q_i}{Q_{l+1}}.
\end{aligned}$$

Therefore (by using Lemma 5)

$$\begin{aligned}
\sum_{l>l_0}\sum_{c=0}^{c_l-1} \tilde{B}(c,l) \quad & \leq \quad 4c'(q)\bar{a}\frac{q^n}{n}q_i \sum_{l>l_0}\frac{1}{Q_{l+1}} + \sum_{l>l_0}\sum_{c=0}^{c_l-1}\bar{B}(c,l) \\
& \leq \quad c''(q)\bar{a}q_i\frac{q^n}{n}\frac{1}{Q_{l_0}} + \sum_{l>l_0}\sum_{c=0}^{c_l-1}\bar{B}(c,l) \\
& \leq \quad c_1(q)\frac{\bar{a}}{(\log n)^{\gamma\frac{\log 2}{2}-\frac{1}{2}}}\frac{q^n}{\sqrt{n}} + \sum_{l>l_0}\sum_{c=0}^{c_l-1}\bar{B}(c,l).
\end{aligned}$$

(Note that $Q_{l_0} \leq 2^{\frac{l_0-1}{2}}$.)

So let us consider finally

$$\begin{aligned}\bar{B}(c,l) &= \sum_{k=0}^{\bar{a}Q_l-1} T_q\left(n,\frac{(q-1)n}{2}+k\frac{q_i}{Q_l}+q_i\kappa+x_k\right)\\ &+ \sum_{k=1}^{\bar{a}Q_l} T_q\left(n,\frac{(q-1)n}{2}-k\frac{q_i}{Q_l}+q_i\kappa+x_{Q_l-k}\right).\end{aligned}$$

Here $x_u := x_{u \pmod{Q_l}}$.

We have to investigate first the sequences $x_k;\ k=0,\dots,\bar{a}Q_l-1$, respectively $x_{Q_l-k};\ k=1,\dots,\bar{a}Q_l$. These sequences consist of $\bar{a}$-times the part $x_0,\dots,x_{Q_l-1}$ (resp. $x_{Q_l},\dots,x_1$).

Set $\zeta=\frac{\gamma_l q_i}{Q_l Q_{l+1}}$ and define $y_k\in[0,1)$ by $y_k=(Q_l-Q_{l-1})k/Q_l \bmod 1$ if l is even and by $y_k=Q_{l-1}k/Q_l \bmod 1$ if l is odd. Then we have

$$x_k=\frac{\gamma_l q_i}{Q_l Q_{l+1}}\cdot((k-\psi)\cdot(Q_l-Q_{l-1}) \pmod{Q_l})=\zeta Q_l y_{k-\psi}.$$

Now we use that $\frac{Q_l-Q_{l-1}}{Q_l}$ has continued fraction expansion

$$\frac{Q_l-Q_{l-1}}{Q_l}=\begin{cases}[0;1+e_{l-1},e_{l-2},\dots,e_1] & \text{if } e_l=1\\ [0;1,e_l-1,e_{l-1},e_{l-2},\dots,e_1] & \text{if } e_l>1\end{cases}.$$

So by Lemma 6 for all v,w with $0\le v<v+w\le Q_l-1$ we have

$$|y_v+y_{v+1}+\cdots+y_{v+w-1}-\frac{w}{2}|\le 4(e_1+\cdots+e_l).$$

Further we have

$$y_0+\cdots+y_{Q_l-1}-\frac{Q_l}{2}=-\frac{1}{2},$$

hence for all $0\le v<v+w\le\bar{a}Q_l-1$ we get

$$|y_v+y_{v+1}+\cdots+y_{v+w-1}-\frac{w}{2}|\le 4(e_1+\cdots+e_l)+\frac{\bar{a}}{2}.$$

So we can apply Lemma 7 with $\Delta=4(e_1+\cdots+e_l)+\bar{a}$, $S=\frac{q_i}{Q_l}$, $\sigma=\frac{n}{2}+q_i\kappa \pmod{\frac{q_i}{Q_l}}$, $\tau=\frac{q_i\gamma_l}{Q_{l+1}}$, and obtain

$$\bar{B}(c,l)\le\frac{q^n}{q_i}Q_l+C_1\left(\frac{q^n}{n}\frac{q_i}{Q_l}(e_1+\cdots+e_l)+\frac{q^n}{n}\frac{q_i}{Q_l}\bar{a}\right).$$

Consequently

$$\begin{aligned}\sum_{l>l_0}\sum_{c=0}^{c_l-1}\bar{B}(c,l) &\le \frac{q^n}{q_i}\sum_{l>l_0}c_lQ_l+C_2\frac{q^n}{n}q_i\cdot\sum_{l>l_0}e_{l+1}\frac{e_1+\cdots+e_l}{Q_l}\\ &+ C_3\frac{q^n}{n}q_i\bar{a}\sum_{l>l_0}\frac{1}{Q_l}.\end{aligned}$$

By Lemma 5 and by the fact that $q_i \le C_4\sqrt{n\log n}$, the last summand is bounded above by

$$C_3\frac{q^n}{n}q_i\bar{a}\sum_{l>l_0}\frac{1}{Q_l} \le C_5\frac{q^n}{\sqrt{n}}\frac{\bar{a}}{(\log n)^{\gamma\frac{\log 2}{2}-\frac{1}{2}}}.$$

The second sum above (by Lemma 8) is bounded by

$$C_6\frac{q^n}{n}q_i\cdot\frac{\gamma\log\log n}{(\log n)^{\gamma\frac{\log 2}{2}}}\cdot\max_{l>l_0}e_{l+1} \le C_7(q,\gamma)\frac{q^n}{\sqrt{n}}\cdot\frac{(\log\log n)^{\frac{3}{2}}}{(\log n)^{\gamma\frac{\log 2}{2}}}\max_{l\le i-l_0}a_l.$$

Altogether we have with a constant $c'(q,\gamma)$:

$$\begin{aligned}\tilde{B}\left(\left[0,\frac{b}{q_i}\right)\right)-q^n\frac{b}{q_i} \quad &\le \quad c'(q,\gamma)\frac{q^n}{\sqrt{n}}\cdot\left(\sum_{l=i-\gamma\log\log\frac{n}{2}}^{i}a_l\right.\\ &\qquad\left.+\frac{(\log\log n)^{\frac{3}{2}}}{(\log n)^{\gamma\frac{\log 2}{2}}}\cdot\max_{l\le i-\gamma\log\log\frac{n}{2}}a_l+\frac{1}{(\log n)^{\gamma\frac{\log 2}{2}-\frac{1}{2}}}\cdot\bar{a}\right),\end{aligned}$$

which finally gives

$$\begin{aligned}&\Delta_{q^n} \le\\ &c(q,\gamma)\frac{q^n}{\sqrt{n}}\cdot\left(\sum_{l=i-\gamma\log\log\frac{n}{2}}^{i}a_l+\frac{(\log\log n)^{\frac{3}{2}}}{(\log n)^{\gamma\frac{\log 2}{2}}}\cdot\max_{l\le i-\gamma\log\log\frac{n}{2}}a_l+\frac{\bar{a}}{(\log n)^{\gamma\frac{\log 2}{2}-\frac{1}{2}}}\right).\end{aligned}$$

This implies the desired estimate for $N = q^n$.

For arbitrary N the result now easily follows by using the principle stated in Lemma 9. By the trivial estimate $D_{q^n} \le 1$ we have

$$\begin{aligned}N\,D_N \quad &\le \quad c(q)\sum_{n\le\log_q N}q^n\,D_{q^n}\\ &\le \quad c(q)\sum_{\log_q N-\frac{1}{2}\log_q\log_q N\le n\le\log_q N}q^n\,D_{q^n}+O\left(\frac{N}{\sqrt{\log N}}\right).\end{aligned}$$

Hence, we only have to consider n with $\log_q N-\frac{1}{2}\log_q\log_q N \le n \le \log_q N$. However, in this range

$$\sqrt{n\log n}=\sqrt{[\log_q N]\log[\log_q N]}-O\left(\frac{(\log n)^{3/2}}{\sqrt{n}}\right),$$

which means that i_0 related to n attains (at most) two consecutive values (for sufficiently large N. However, the above estimates are not affected if we replace i_0 by i_0-1 or by i_0+1. This completes the proof of the theorem $\square$

6 The proofs of Theorem 3 and 4

The proofs of Theorems 3 and 4 are based on the following bound on exponential sums.

Proposition 1 *Let $q_1, q_2, \ldots, q_d$ be pairwise comprime integers > 1. Then there exists a constant $c > 0$ such that for all real numbers $\alpha_1, \alpha_2, \ldots, \alpha_d$ and all integers $h_1, h_2, \ldots, h_d$ we have*

$$\left| \frac{1}{N} \sum_{n<N} e(h_1\alpha_1 s_{q_1}(n) + h_2\alpha_2 s_{q_2}(n) + \cdots + h_d\alpha_d s_{q_d}(n)) \right| \ll N^{-c\sum\limits_{j=1}^{d} \|(q_j-1)h_j\alpha_j\|^2},$$

where $e(x) = e^{2\pi i x}$ and $\|x\| = \min\limits_{k\in\mathbf{Z}} |x-k|$ denotes the distance to the integers.

Before giving a proof of Proposition 1 we demonstrate how this Proposition implies Theorems 3 and 4.

Proof of Theorem 3. If (for some j) α_j is rational then the sequence $\alpha_j s_{q_j}(n)$ mod 1 attains only finitely many values. Hence the sequence $\alpha_j s_{q_j}(n)$ is surely not uniformly distributed modulo one and consequently the d-dimensional one $(\alpha_1 s_{q_1}(n), \alpha_2 s_{q_2}(n), \ldots, \alpha_d s_{q_d}(n))$ is not uniformly distributed modulo one, either.

Conversely, if α_j $(1 \le j \le d)$ are irrational then Proposition 1 shows that the exponential sums satisfy

$$\left| \frac{1}{N} \sum_{n<N} e(h_1\alpha_1 s_{q_1}(n) + h_2\alpha_2 s_{q_2}(n) + \cdots + h_d\alpha_d s_{q_d}(n)) \right| = o(1)$$

as $N \to \infty$ if $(h_1, h_2, \ldots, h_d) \ne (0, 0, \ldots, 0)$. Thus, by Weyl's criterion the sequence one $(\alpha_1 s_{q_1}(n), \alpha_2 s_{q_2}(n), \ldots, \alpha_d s_{q_d}(n))$ is uniformly distributed modulo 1. □

Proof of Theorem 4. We again use Proposition 1 and the inequality of Erdős-Turan-Koksma (see [5]) to estimate the discrepancy. The Erdős-Turan-Koksma inequality says that the discrepancy D_N of the sequence $x_n = (x_{n,1}, \ldots, x_{n,d}) \in \mathbf{R}^d$ can be bounded by

$$D_N \ll \frac{1}{H} + \sum_{0<\max\limits_{1\le j\le d}|h_j|\le H} \frac{1}{R(h_1,\ldots,h_d)} \left| \frac{1}{N} \sum_{n=0}^{N-1} e\left(\sum_{j=1}^{d} h_j x_{n,j} \right) \right|.$$

for every integer $H > 0$. Here $R(h_1, \ldots, h_d) = \max(|h_1|, 1) \cdots \max(|h_d|, 1)$.

Now, suppose that there exists $\eta \geq 1$ and a constant $c > 0$ such that for all integers $h > 0$ we have $\|h\alpha_j\| \geq c\, h^{-\eta}$ for all $1 \leq j \leq d$. Then with

$$H = c' \left(\frac{\log N}{\log\log N} \right)^{\frac{1}{2\eta}}$$

(where $c' > 0$ is a sufficiently small constant) it is an easy exercise to prove that

$$\begin{aligned} D_N &\ll \frac{1}{H} + \sum_{0 < \max\limits_{1 \leq j \leq d} |h_j| \leq H} \frac{1}{R(h_1, \ldots, h_d)} N^{-c'' \max(|h_1|, \ldots, |h_d|)^{-2\eta}} \\ &\ll \frac{1}{H} + (\log H)^d e^{-c'' H^{-2\eta} \log N} \\ &\ll \left(\frac{\log\log N}{\log N} \right)^{\frac{1}{2\eta}}. \end{aligned}$$

The lower bound is almost trivial since we have for any j

$$D_N(\alpha_1 s_{q_1}(n), \alpha_2 s_{q_2}(n), \ldots, \alpha_d s_{q_d}(n)) \geq D_N(\alpha_j s_{q_j}(n)).$$

Hence, we just have to apply Corollary 5 . $\Box$

For the proof of Proposition 1 we mimic the method of Kim [7].

Firstly, let $q > 1$, $\alpha \in \mathbf{R}$, and $h \in \mathbf{Z}$ be fixed and consider the *correlation functions*

$$\Phi_N(k) = \frac{1}{N} \sum_{n=0}^{N-1} e(h\alpha(s_q(n+k) - s_q(n)))$$

and for $r = 0, 1$

$$\Phi_{K,N}(r) = \frac{1}{K} \sum_{k=0}^{K-1} \overline{\Phi_N(k)} \Phi_N(k+r).$$

Our first aim is to show the following bound.

Lemma 10 *We have uniformly for $\sqrt{N} \leq K \leq N$*

$$\Phi_{K,N}(0) \ll N^{-\frac{\|(q-1)h\alpha\|^2}{40 \log q}} \tag{6}$$

as $N \to \infty$, where the constant implied by $\ll$ only depends on q.

Proof. For convenience we set $g(n) = e(h\alpha s_q(n))$ and $\beta = \alpha h$. Then by [7] we have

$$\Phi_{qN}(qk + r) = \alpha_r \Phi_N(k) + \beta_r \Phi_N(k+1)$$

with

$$\alpha_r = \frac{1}{q} \sum_{i=0}^{q-r-1} \overline{g(i)} g(i+r) = \frac{q-r}{q} e(\beta r)$$

and

$$\beta_r = \frac{1}{q}\sum_{i=q-r}^{q-1}\overline{g(i)}g(i+r-q) = \frac{r}{q}e(\beta r),$$

and for $r=0,1$

$$\Phi_{qK,qN}(r) = \lambda_r\Phi_{K,N}(0) + \mu_r\Phi_{K,N}(1) + \nu_r\overline{\Phi_{K,N}(1)} + E_{K,N}(r),$$

where $|E_{K,N}(r)| \le 2/K$ and

$$\begin{aligned}
\lambda_r &= \frac{1}{q}\sum_{i=0}^{q-1}(\overline{\alpha_i}\alpha_{i+r} + \overline{\beta_i}\beta_{i+r}) = \frac{2q^2-3r+1}{3q^2}e(\beta r),\\
\mu_r &= \frac{1}{q}\sum_{i=0}^{q-1}\overline{\alpha_i}\beta_{i+r} = \frac{q^2+3qr+3r-1}{6q^2}e(\beta(r-q)),\\
\nu_r &= \frac{1}{q}\sum_{i=0}^{q-1}\overline{\beta_i}\alpha_{i+r} = \frac{q^2-3qr+3r-1}{6q^2}e(\beta(r+q)).
\end{aligned}$$

As in [7] this implies that (for $r=0,1$)

$$|\Phi_{q^2K,q^2N}(r)| \le \rho_r|\Phi_{K,N}(0)| + \sigma_r|\Phi_{K,N}(1)| + \frac{7}{K},$$

where

$$\begin{aligned}
\rho_r &= |\lambda_r\lambda_0 + \mu_r\lambda_1 + \nu_r\overline{\lambda_1}|,\\
\sigma_r &= |\lambda_r\mu_0 + \mu_r\mu_1 + \nu_r\overline{\nu_1}| + |\lambda_r\nu_0 + \mu_r\nu_1 + \nu_r\overline{\mu_1}|.
\end{aligned}$$

The crucial step is to realize that

$$\rho_r + \sigma_r \le 1 - \frac{\|(q-1)\beta\|^2}{4}. \tag{7}$$

Namely, if we assume that (7) holds then we can again proceed as in [7] and obtain for $t = \left[\frac{\log N}{5\log q}\right]$

$$\Phi_{K,N}(0) \ll e^{-\tau t} + \frac{q^{2t}}{\sqrt{N}},$$

where $\tau = \frac{1}{4}\|(q-1)\beta\|^2$. Now trivial estimates (compare with [7]) yield (6).

So it remains to check (7). (We again follow [7].) For $r=0,1$ we have

$$\begin{aligned}
\rho_r &= |\lambda_r\lambda_0 + \mu_r\lambda_1 + \nu_r\overline{\lambda_1}|\\
&= ||\lambda_r||\lambda_0|e(\beta r) + |\mu_r||\lambda_1|e(\beta(r+1-q)) + |\nu_r||\lambda_1|e(\beta(r+q-1))|\\
&= ||\lambda_r||\lambda_0| + |\mu_r||\lambda_1|e(\beta(1-q)) + |\nu_r||\lambda_1|e(\beta(q-1))|\\
&\le ||\lambda_r||\lambda_0| + |\mu_r||\lambda_1|e(\beta(1-q))| + |\nu_r||\lambda_1|.
\end{aligned}$$

Now we observe that $|\lambda_r||\lambda_0| \geq |\mu_r||\lambda_1|$ and use the inequality $|a + b\,e(\theta)| \leq a + b - 4b\|\theta\|^2$ (for $a \geq b > 0$) to derive

$$\begin{aligned} ||\lambda_r||\lambda_0| + |\mu_r||\lambda_1|e(\beta(1-q))| &\leq |\lambda_r||\lambda_0| + |\mu_r||\lambda_1| - 4|\mu_r||\lambda_1|\|\beta(1-q)\|^2 \\ &\leq |\lambda_r||\lambda_0| + |\mu_r||\lambda_1| - \frac{\|\beta(1-q)\|^2}{4}, \end{aligned}$$

where we also used that $|\mu_r| \geq |\mu_0| \geq \frac{1}{8}$ and $|\lambda_1| \geq \frac{1}{2}$. Hence, we finally obtain

$$\begin{aligned} \rho_r + \sigma_r &\leq |\lambda_r|(|\lambda_0| + |\mu_0| + |\nu_0|) + (|\mu_r| + |\nu_r|)((|\lambda_1| + |\mu_1| + |\nu_1|) - \frac{\|\beta(1-q)\|^2}{4} \\ &\leq 1 - \frac{\|\beta(1-q)\|^2}{4}. \end{aligned}$$

This completes the proof of the lemma. □

We finally indicate how Proposition 1 can be derived from Lemma 10. We again can proceed as in [7]. We only mention the key ideas.

We set $g_j(n) = e(h_j\alpha_j s_{q_j}(n))$ and $g(n) = \prod_{j=1}^{d} g_j(n)$ and use Weyl's inequality

$$\left|\sum_{n=0}^{N-1} g(n)\right|^2 \leq \frac{2N^2}{K} + \frac{4N}{K}\sum_{k=1}^{K}\left|\sum_{n=0}^{N-k-1}\overline{g(n)}g(n+k)\right|$$

for $K = [N^{1/(3s)}]$. The essential point is that $\sum \overline{g(n)}g(n+k)$ factors in the following way.

Set $Q_j = q_j^{t_j}$ with $t_j = [2\log K/\log q_j]$ and $Q = Q_1Q_2\cdots Q_d$. Furthermore, for every d-tuple $\mathbf{r} = (r_1, r_2 \ldots, r_d)$ of integers let $P_{\mathbf{r}}$ be the set of non-negative integers n with $n \equiv r_j \bmod Q_j$. Now, if $\mathbf{r} = (r_1, r_2 \ldots, r_d)$ is of the kind that $0 \leq r_j \leq Q_j - K - 1$ for each j then we have for every k with $1 \leq k \leq K$

$$\begin{aligned} \overline{g(n)}g(n+k) &= \prod_{j=1}^{d}\overline{g_j(n)}g_j(n+k) \\ &= \prod_{j=1}^{d}\overline{g_j(r_j)}g_j(r_j+k). \end{aligned}$$

Hence, the sum

$$\sum_{n=0}^{N-k-1}\overline{g(n)}g(n+k)$$

can be approximated by

$$\sum_{\mathbf{r}}\prod_{j=1}^{d}\overline{g_j(r_j)}g_j(r_j+k)\sum_{0\leq n\leq N-k-1, n\in P_{\mathbf{r}}} 1$$

$$= \prod_{j=1}^{d} \sum_{r_j=0}^{Q_j-1} \overline{g_j(r_j)} g_j(r_j+k) \left(\frac{N}{\prod_{j=1}^d Q_j} + O(1) \right)$$
$$= N \prod_{j=1}^{d} \frac{1}{Q_j} \sum_{r_j=0}^{Q_j-1} \overline{g_j(r_j)} g_j(r_j+k) + O\left(\prod_{j=1}^{d} Q_j \right).$$

Note that there are only few $\mathbf{r} = (r_1, r_2 \dots, r_d)$ with $r_j > Q_j - K - 1$ for some j.

This factoring property (and some trivial estimates) lead to

$$\begin{aligned}
\left| \sum_{n=0}^{N-1} g(n) \right|^2 &\le \frac{4N}{K} \sum_{k=1}^{K} \left| \prod_{j=1}^{d} \frac{1}{Q_j} \sum_{r_j=0}^{Q_j-1} \overline{g_j(r_j)} g_j(r_j+k) \right| + O\left(\frac{2N^2}{K} \right) \\
&\le \frac{4N}{K} K^{\frac{d}{d+1}} \prod_{j=1}^{d} \left(\sum_{k=1}^{K} \left| \frac{1}{Q_j} \sum_{r_j=0}^{Q_j-1} \overline{g_j(r_j)} g_j(r_j+k) \right|^{d+1} \right)^{\frac{1}{d+1}} + O\left(\frac{2N^2}{K} \right) \\
&\le 4N \prod_{j=1}^{d} \left(\frac{1}{K} \sum_{k=1}^{K} \left| \frac{1}{Q_j} \sum_{r_j=0}^{Q_j-1} \overline{g_j(r_j)} g_j(r_j+k) \right|^2 \right)^{\frac{1}{d+1}} + O\left(\frac{2N^2}{K} \right).
\end{aligned}$$

At this stage we can apply Lemma 10 and directly obtain the proposed estimate of Proposition 1.

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