Extremal Parameters in Sub-Critical Graph Classes

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Abstract

We analyze several extremal parameters like the diameter or the maximum degree in sub-critial graph classes. Sub-critical graph classes cover several well-known classes of graphs like trees, outerplanar graph or series-parallel graphs which have been intensively studied during the last few years. However, this paper is the first one, where these kind of parameters are studied from a general point of view.

1 Introduction

In this paper we present a general framework to analyze in a unified way extremal parameters in a wide variety of labelled classes of graphs that are called sub-critical. Informally a graph class is subcritical if the average size of 2-connected components is bounded so that the block-decomposition looks tree-like. (We will make this definition more precise in the Section 2.)

Sub-critical graph classes include important graph classes like trees, outerplanar graphs, or series-parallel graphs. During the last few years these kind of graphs have been studied from various points of view [4, 7, 8, 9]. In particular we mention here the paper by Drmota et al. [7], where it has been shown that several (additive) parameters like the number of edges, the number of blocks or the number of vertices of given degree satisfy a central limit theorem. The purpose of the present paper is to provide a first systematic treatment on extremal parameters of sub-critical graph classes.

In particular we concentrate ourselves to three important parameters, the diameter, the maximum block-size and the maximum degree.

Outline of the paper. The paper is organized as follows. In Section 2 we set up the combinatorial and analytic background. The main results are then collected in Section 3. Finally, in Section 4–6 we present the proofs. They are all based on generating function and methods from analytic combinatorics [13].

2 Combinatorial and Analytic Background

For $k \geq 0$, a graph is k-connected if one needs to delete at least k vertices to disconnect it. Obviously, a graph G is a set of its connected components. For the decomposition from

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connected graphs into 2-connected graphs we use the block structure of a connected graph. A block of a graph G is a maximal 2-connected induced subgraph of G. We say a vertex of G is incident to a block B of G if it belongs to B. The block structure of G yields a bipartite tree with the vertex set consisting of two types of nodes, i.e. cut-vertices and blocks of G, and the edge set describing the incidences between the cut-vertices and blocks of G. This suggests a natural decomposition of connected graphs into 2-connected graphs and this holds also for rooted graphs. The root-vertex v of a rooted graph G is incident to a set of blocks and to each non-root vertex on these blocks is attached a rooted connected graph. In other words, a rooted connected graph rooted at v is uniquely obtained as follows: take a set of derived 2-connected graphs and merge them at their pointed (distinguished but not labelled) vertices so that v is incident to these derived 2-connected graphs, then replace each non-root vertex w in these blocks by a rooted connected graph rooted at w (which is allowed to consist of a single vertex and in this case it has no effect).

Let $\mathcal{G}$ be a family of vertex labeled graphs and g_n the number of graphs in this family of size n then

$$\mathcal{G}(z) = \sum_{n\geq 0} g_n \frac{z^n}{n!}$$

denotes the corresponding exponential generating function. The derived familty $\mathcal{G}'$ is the set of graphs, where a specific vertex is chosen, for example, the vertex with largest label and this vertex is discounted, that is, the label of this vertex is deleted and is it not counted for the size. Note that the corresponding generating function of $\mathcal{G}'$ is precisely the derivative

$$\mathcal{G}'(z) = \sum_{n\geq 1} g_n \frac{z^{n-1}}{(n-1)!}.$$

Similarly we define the family $\mathcal{G}^\bullet$ of rooted graphs, where one vertex is chosen as a so-called *root*. Since we have precisely n choices for choosing the root, the number of rooted objects of size n equals ng_n and the generating function is given by

$$\mathcal{G}^\bullet(z) = z\mathcal{G}'(z) = \sum_{n\geq 0} n g_n \frac{z^n}{n!}.$$

Through the entire paper, given a class of graphs $\mathcal{G}$, we denote by $\mathcal{C}$ (resp. $\mathcal{B}$) the subfamily of connected (resp. 2-connected) graphs in $\mathcal{G}$. In the language of symbolic combinatorics from [3, 13], the block-decomposition described above translates into the fundamental equations:

$$\mathcal{G} = \mathrm{Set}(\mathcal{C}), \tag{2.1}$$

$$\mathcal{C}^\bullet = \mathcal{Z} \cdot \mathrm{Set}(\mathcal{B}' \circ \mathcal{C}^\bullet), \tag{2.2}$$

where the factor $\mathcal{Z}$ in the last equation takes account of the root vertex (which is distinguished and labelled), the symbol $\cdot$ denotes the partitional product on combinatorial classes, and the symbol $\circ$ denotes substitution at an atom (see [3, 13] for definitions).

A graph class $\mathcal{G}$ is called *block-stable* if it contains the link-graph ℓ, which is a graph with one edge together with its two (labelled) end vertices, and satisfies the property that a graph G belongs to $\mathcal{G}$ iff all the blocks of G belong to $\mathcal{G}$. Block-stable classes include classes of graph specified by a finite list of forbidden minors that are all 2-connected, for instance, planar graphs (Forbid($K_5, K_{3,3}$)), series-parallel graphs (Forbid(K_4)), and outerplanar graphs

(Forbid($K_4, K_{3,2}$)). For a block-stable graph class, (2.1) and (2.2) translates into equations of EGFs in the labelled setting:

$$\mathcal{G}(z) = \exp(\mathcal{C}(z)),$$
$$\mathcal{C}^\bullet(z) = z\exp(\mathcal{B}'(\mathcal{C}^\bullet(z))).$$

In particular this means that the generating function $\mathcal{B}(z)$ determines $\mathcal{C}(z)$ and consequently $\mathcal{G}(z)$, too. In order to apply complex analytic methods we will assume that $\mathcal{B}(z)$ has a positive radius of convergence that is *sufficiently large* in the following sense.

Definition 2.1. *Let $\mathcal{G}$ be a block-stable graph class with the connected subclass $\mathcal{C}$ and the 2-connected subclass $\mathcal{B}$ such that the generating function $\mathcal{B}(z)$ has a positive radius of convergence η.*

$\mathcal{G}$ is called sub-critical*, if*

$$\mathcal{C}^\bullet(\rho) < \eta,$$

where ρ denotes the radius of convergence of $\mathcal{C}(z)$.

Note that the implicit function theorem implies that $\mathcal{C}^\bullet(z)$ is regular for $z = 0$ has consequently it has a positive radius of convergence ρ. Furthermore it is well known that the equation for determining ρ (in the sub-critical case) is

$$\rho = y\exp(-\mathcal{B}'(y)),$$

where $y = \mathcal{C}^\bullet(\rho)$ is given by the equation $1 = y\mathcal{B}''(y)$. Hence, we have subcritiallity if and only if $\eta\mathcal{B}''(\eta) > 1$, compare with [4].

In particular we are in the sub-critial case if $\lim_{z\to\eta-}\mathcal{B}''(z) = +\infty$.

It is also well known that $\mathcal{C}^\bullet(z)$ has a squareroot singularity at $z = \rho$:

$$\mathcal{C}^\bullet(z) = g(z) - h(z)\sqrt{1 - \frac{z}{\rho}},$$

where $g(z)$ and $h(z)$ represent non-zero analytic function in a neighborhood of $z = \rho$. If we also assume that we are in the aperiodic case, that is, we have $g_n > 0$ for all n (with at most finitely many exceptions) then this implies that

$$g_n \sim cn^{-3/2}\rho^{-n}n!$$

for some constant $c > 0$, see [12, 13, 6]. For simplicity we will always assume for the proofs that we are in the aperiodic case (in the periodic case we have to restrict to $n \equiv 1 \bmod d$ for some $d > 1$, which is reflected by the property that there are also squareroot singularities for $z = \rho\exp(2\pi ij/d)$, $1 \le j < d$, however, all results are of the same kind).

Several well known graph classes are sub-critial. For expample, labelled trees, cacti-graphs, outerplanar graphs and series-parallel graphs are sub-critical, see [4, 7]. However, the class of labelled planar graphs is not sub-critical.

3 Results

As mentioned in the Introduction we restrict ourselves to the maximum block size, to the diameter, and to the maximum degree.

Theorem 3.1. *Let $M_n^{(2)}$ denote the maximum block size of a random connected graph in an aperiodic sub-critical graph class. Then we have*

$$\mathbb{E}\, M_n^{(2)} = O(\log n).$$

Furthermore, if the limit $\lim b_{n+1}/(nb_n)$ *exists and is positive then we have*

$$\mathbb{P}[M_n^{(2)} \le k] \sim \exp\left(-\exp(\log n - f(k))\right)$$

uniformly for $n, k \to \infty$, where $f(k)$ is a function with $f(k) \sim ck$ for some constant $c > 0$.

We note that we cannot expect in general a more precise theorem. For example in trees the maximum block size equals 2 whereas in series-parallel graphs it is definitely of order $\log n$.

Theorem 3.2. *Let D_n denote the diameter of a random connected aperiodic sub-critical graph class. Then there exist two positive constants c_1, c_2 with*

$$c_1\sqrt{n} \le \mathbb{E}\, D_n \le c_2\sqrt{n\log n}.$$

This result is probably not best possible. We conjecture that there is a univeral limit law for $D_n/\sqrt{n}$. However, this is already an open problem for series-parallel graphs.

Theorem 3.3. *Let Δ_n denote the maximum degree of a random connected aperiodic sub-critical graph class. Then there exist two positive constants c_1, c_2 with*

$$c_1 \frac{\log n}{\log\log n} \le \mathbb{E}\,\Delta_n \le c_2 \log n.$$

This result is best possible if we do not impose further conditions. For example, for labelled trees the maximum degree is of order $\log n/\log\log n$, see [14], whereas for series-parallel graphs it is of order $\log n$, see [9].

4 Maximum Block Size

For every positive integer k let $\mathcal{B}_k$ the sub-class of 2-connected graphs of size $\le k$ and $\mathcal{C}_k$ the class of connected graphs, where all blocks have size $\le k$. Then the relation

$$\mathcal{C}_k^\bullet = \mathcal{Z} \cdot \mathrm{Set}(\mathcal{B}_k' \circ \mathcal{C}_k^\bullet),$$

translates to

$$\mathcal{C}_k^\bullet(z) = z\exp(\mathcal{B}_k'(\mathcal{C}_k^\bullet(z))),$$

Hence, if ρ_k denotes the radius of convergence of $\mathcal{C}_k^\bullet(z)$ then then we obtain (for sufficiently large k)

$$\mathcal{C}_k^\bullet(z) = g_k(z) - h_k(z)\sqrt{1 - \frac{z}{\rho_k}}$$

(for certain analytic functions g_k and h_k) and consequently

$$[z^n]\mathcal{C}_k^\bullet(z) \sim c_k \rho_k^{-n} n^{-3/2} n!$$

Thus, for fixed k we have, as $n \to \infty$,

$$\mathbb{P}[M_n^{(2)} \le k] = \frac{[z^n]\mathcal{C}_k^\bullet(z)}{[z^n]\mathcal{C}^\bullet(z)} \sim \frac{c_k}{c_0}\left(\frac{\rho}{\rho_k}\right)^n.$$

Actually it is not very difficult to prove that this relation holds uniform for $n, k \to \infty$, see [1]. The main idea is to start with a boots-trapping procedure to obtain uniform error estimates and to apply the singularity analysis methods of Flajolet and Odlyzko [12], that provide uniform error terms, too. (For the sake of shortness we omit the technical details.)

Therefore it remains to get asymptotics for ρ_k and c_k. First we use the fact that $\mathcal{B}(z)$ is analytic if z varies in a sufficiently small complex neighborhoud of $y = \mathcal{C}^\bullet(\rho)$. Hence it follow that there exists $\gamma < 1$ with

$$\mathcal{B}(z) = \mathcal{B}_k(z) + O(\gamma^k)$$

uniformly in this neighborhood. Actually the same holds for $\mathcal{B}'(z)$ and $\mathcal{B}''(z)$. Now the equation for $y_k = \mathcal{C}^\bullet(\rho_k)$ is

$$1 = y_k \mathcal{B}_k'(y_k), \tag{4.1}$$

from which we get $\rho_k = y_k \exp\left(-\mathcal{B}_k'(y_k)\right)$. Recall that $y = \mathcal{C}^\bullet(\rho)$ satisfies $1 = y\mathcal{B}'(y)$. Hence, we obtain $y_k = y + O(\gamma^k)$ and consequently

$$\rho_k = \rho + O(\gamma^k).$$

Actually this also leads to $c_k = c_0 + O(\gamma^k)$ and to

$$\mathbb{P}[M_n^{(2)} \le k] \sim \left(1 + O(\gamma^k)\right)\left(1 + O(\gamma^k)\right)^n = \exp\left(O(n\gamma^k)\right).$$

In particular, if $k \ge \log n / \log(1/\gamma)$ then

$$\mathbb{P}[M_n^{(2)} > k] = O\left(n\gamma^k\right).$$

Thus, the expected value is bounded by

$$\begin{aligned}\mathbb{E}\, M_n^{(2)} &= \sum_{k\ge 0} \mathbb{P}[M_n^{(2)} > k] \\ &\le \frac{\log n}{\log(1/\gamma)} + \sum_{k \ge \log n/\log(1/\gamma)} O\left(n\gamma^k\right) \\ &= O(\log n).\end{aligned}$$

If the limit $\lim b_{n+1}/(nb_n)$ exists then we can be much more precise. Clearly, if τ denotes the radius of convergence of $\mathcal{B}(z)$ then this limit equals τ^{-1}. It also follows that

$$\mathcal{B}(z) = \mathcal{B}_k(z) + b_k(z/\tau)^k,$$

where c_k is a sequence with $\lim b_{k+1}/b_k = 1$. (Of course, similiar relations hold for $\mathcal{B}'(z)$ and $\mathcal{B}''(z)$.) With the help of these information it follows (by boots-trapping) that $y_k = y - d_k(y/\tau)^k$ for a sequence d_k with $\lim d_{k+1}/d_k = 1$ consequently $\rho_k = \rho + e_k'(y/\tau)^k$ and $c_k = c + e_k''(y/\tau)^k$ for sequence e_k', e_k'' with $\lim e_{k+1}'/e_k' = \lim e_{k+1}''/e_k'' = 1$. This leads to

$$\mathbb{P}[M_n^{(2)} \le k] \sim \exp\left(-e_k'''(y/\tau)^k n\right) = \exp(-\exp(\log n - k\log(\tau/y) + \log e_k''')),$$

where $\log e_k''' = o(k)$ as $k \to \infty$. This completes the proof of the Theorem 3.1.

5 Diameter

The lower bound is quite easy to establish. Let $\underline{D}_n$ denote maximum number of blocks in a path from the root to any other vertex, which is clearly a lower bound for the diameter. Then we are precisely in the situation of determining the height of a special class of simply generated trees, see [6].

More precisely, let $\varphi(t) = \exp(\mathcal{B}'(t))$. Then $\varphi(t)$ is a power series with non-negative coefficients with $\varphi(0) > 0$. Hence it defines a probability distribution on rooted trees. Set $y_0(x) = x = x\exp(\mathcal{B}'(0))$ and $y_{k+1}(x) = x\varphi(y_k(t)) = x\exp(\mathcal{B}'(y_k(t)))$. Then $y_k(x)$ counts those rooted graphs where all paths starting at the root pass $\le k$ blocks. On the other hand $y_k(x)$ corresponds to those simply generated trees of height $\le k$. Hence $\underline{D}_n$ has the same distribution as the height H_n of these trees. By [11] it is known that the height in simply generated trees satisfies $\mathbb{E}\,H_n \sim c_1\sqrt{n}$ for a proper positive constant c_1. Recall that $\varphi(t)$ satisfies the necessary conditions, since we are in the sub-critical case: $\varphi(t)$ is analytic for the critical point $\tau > 0$ that is defined by $\tau\varphi'(\tau) = \varphi(\tau)$. In fact we have $\tau = \mathcal{C}^\bullet(\rho) < \eta$.

The proof of the upper bound is much more involved, however, we use again the tree like structure of a connected graph G that is induced by the block decomposition of a vertex rooted graph. Actually, for every block B we have a unique vertex that acts as a *local root*: it is a cut vertex of G and has miminum distance to the root of G of all vertices in B. We define the *block height of* B as the maximum distance from the local root in B. Furthermore for any vertex v in G we denote by $\overline{d}(v)$ the sum of block-heights of those blocks that are passed on a path from the root of G to v. Finally we define $\overline{D}_n$ as the maximum $\overline{d}(v)$ over all vertices v of G of size n.

Let $\mathcal{B}'_{=k}$ denote the set of rooted blocks of block-height k and $L_h(z,u)$ the generating function of rooted graphs, where z corresponds to the size and u to those vertices with $\overline{d}(v) = h$, that is, is describes the random variable $Y_{n,h}$ of the number vertices v with $\overline{d}(v) = h$ in a graph of size n. The tree-like structure translates to the following recurrence relation:

$$L_h(z,u) = z\exp\left(\sum_{1\le k\le h} \mathcal{B}'_{=k}\left(L_{h-k}(z,u)\right)\right) \qquad (h \ge 1)$$

with $L_0(z,u) = zu$. We simplify this equation by considering the expectations $\mathbb{E}\,Y_{n,h}$. Formally this means that we take the derivative with respect to u and set $u = 1$. In particular by setting $M_h(z) = \frac{\partial}{\partial u} L_h(z,u)|_{u=1}$ we get

$$M_h(z) = z\exp\left(\sum_{1\le k\le h} \mathcal{B}'_{=k}\left(\mathcal{C}^\bullet(z)\right)\right) \sum_{1\le k\le h} \mathcal{B}'_{=k}\left(\mathcal{C}^\bullet(z)\right) M_{h-k}(z).$$

Since we will be interested in upper bounds for $\mathbb{E}\,Y_{n,h}$ we replace this recurrence by

$$\begin{aligned}\overline{M}_h(z) &= z\exp\left(\sum_{k\ge 1} \mathcal{B}'_{=k}\left(\mathcal{C}^\bullet(z)\right)\right) \sum_{1\le k\le h} \mathcal{B}''_{=k}\left(\mathcal{C}^\bullet(z)\right) \overline{M}_{h-k}(z) \\ &= \mathcal{C}^\bullet(z) \sum_{1\le k\le h} \mathcal{B}''_{=k}\left(\mathcal{C}^\bullet(z)\right) \overline{M}_{h-k}(z). \qquad (5.1)\end{aligned}$$

Inductively it follows that all coefficients of $\overline{M}_h(z)$ are larger or equal than the corresponding ones of $M_h(z)$ if we assume that $\overline{M}_0(z) = M_0(z) = z$.

Next we set $\overline{M}(z,v) = \sum_{h\ge 0} \overline{M}_h(z) v^k$. Then (5.1) translates to

$$\overline{M}(z,v) = \frac{z}{1 - \mathcal{C}^\bullet(z) \sum_{k\ge 1} \mathcal{B}''_{=k}\left(\mathcal{C}^\bullet(z)\right) v^k}.$$

If $z = \rho$ then we have the singular equation $1 = \mathcal{C}^\bullet(\rho)\mathcal{B}''(\mathcal{C}^\bullet(\rho))$. Consequently the denominator of $\overline{M}(\rho, v)$ vanishes for $v = 1$. The next step is to show that we have a polar singularity of the function $v \mapsto \overline{M}(z,v)$ if z is close to ρ.

Clearly if $\overline{d}(v) = k$ then the block B has at least k vertices. Hence it follows that $\mathcal{B}''_{=k}(y) = O(\gamma^k)$ for some $\gamma < 1$ uniformly for z in a sufficiently small complex neighborhood N of $y = \mathcal{C}^\bullet(\rho)$. This implies that the function

$$v \mapsto \sum_{k\ge 1} \mathcal{B}''_{=k}(y)\, v^k$$

is well defined for $|v-1| < \varepsilon$ and $y \in N$ (for a suitably chosen $\varepsilon > 0$). Hence by the implicit function theorem the equation

$$y \sum_{k\ge 1} \mathcal{B}''_{=k}(y)\, v^k = 1 \tag{5.2}$$

has a unique analytic solution $v = \beta(y)$ in a complex neighborhood of $y = \mathcal{C}^\bullet(\rho)$ with the properties $\beta(\mathcal{C}^\bullet(\rho)) = 1$ and $\beta'(\mathcal{C}^\bullet(\rho)) \ne 0$. Finally if we set $\alpha(z) = 1/\beta(\mathcal{C}^\bullet(z))$ then $1/\alpha(z)$ is the unique polar singulariy of $v \mapsto \overline{M}(z,v)$ if z is close to ρ. Since $\mathcal{C}^\bullet(z)$ has a squareroot singularity and $\beta'(\mathcal{C}^\bullet(\rho)) \ne 0$ it follows that $\alpha(z)$ has a local expansion of the form

$$\alpha(z) = 1 - c'\sqrt{1 - z/\rho} + O(|\rho - z|)$$

for some positive constant c'. It is also easy to check that $1/\alpha(z)$ is the only singularity in the dics $|v| \le 1/|\alpha(z)| + \delta$ for some sufficiently small $\delta > 0$. This follows from the fact that the left hand side of (5.2) is a power series with non-negative coefficients and the aperiodicity assumption.

Hence by a simple Cauchy integration it follows that

$$\overline{M}_h(z) = C(z)\alpha(z)^h (1 + O(\gamma^k))$$

for some $\gamma < 1$ and a suitable function $C(z)$ (that has a squareroot singularity, too).

Hence we are in a situation, where we can apply a slight generalization of [5]. It follows that

$$[z^n]\overline{M}_h(z) \sim c_1 k n^{-3/2} \rho^{-n} \exp\left(-c_2 \frac{k^2}{n}\right)$$

uniformly for $k \le C\sqrt{n}\log n$ (for an arbitrary constant $C > 0$), where c_1 and c_2 are positive constants. Furthermore, by setting $z = \rho(1 - 1/n)$ we get the upper bound

$$[z^n]\overline{M}_h(z) \le z^{-n}\overline{M}_h(z) = O\left(\rho^{-n} \exp\left(-c_3 \frac{k}{\sqrt{n}}\right)\right).$$

for some positive constant $c_3 > 0$. Since $[z^n]\mathcal{C}^\bullet(z) \sim c n^{-3/2}\rho^{-n}$ it follows that

$$\mathbb{E}\, Y_{n,h} \le \begin{cases} c_4 k \exp\left(-c_2 \frac{k^2}{n}\right) & \text{for } k \le C\sqrt{n}\log n, \\ c_4 n^{3/2} \exp\left(-c_3 \frac{k}{\sqrt{n}}\right) & \text{for } k > C\sqrt{n}\log n \end{cases}$$

and consequently

$$\sum_{\ell \ge h} \mathbb{E}\, Y_{n,\ell} \le \begin{cases} c_5 n \exp\left(-c_2 \frac{k^2}{n}\right) & \text{for } k \le C\sqrt{n}\log n, \\ c_5 n^{3/2} \exp\left(-c_3 \frac{k}{\sqrt{n}}\right) & \text{for } k > C\sqrt{n}\log n, \end{cases}$$

where C has to be chosen sufficiently large. Recall that $Z_{n,h} = \sum_{\ell \ge h} Y_{n,\ell}$ denotes the number of vertices v with $\overline{d}(v) \ge h$.

Finally we use the property that $\overline{D}_n > h$ if and only if $Z_{n,h} > 0$. Hence, by applying the first moment method we obtain for every $\lambda > 0$

$$\begin{aligned} \mathbb{E}\, \overline{D}_n &= \sum_{h \ge 0} \mathbb{P}[\overline{D}_n > h] \\ &= \sum_{h \ge 0} \mathbb{P}[Z_{n,h} > 0] \\ &\le \lambda \sqrt{n \log n} + \sum_{h \ge \lambda\sqrt{n \log n}} \mathbb{E}\, Z_{n,h} \\ &\le \lambda \sqrt{n \log n} + c_6 n \exp\left(-c_2 \lambda^2 \log n\right). \end{aligned}$$

Consequently, by setting $\lambda = c_2^{-1/2}$ this leads to

$$\mathbb{E}\, \overline{D}_n = O\left(\sqrt{n \log n}\right).$$

and completes the proof since the diameter is upper bounded by $2\overline{D}_n$.

6 Maximum Degree

In order to prove the lower bound we use the concept of Boltzmann sampling. In particular we fix $z = \rho$ and *sample* rooted connected graphs G with probability distribution $\mathbb{P}[G] = \rho^{|G|}/(|G|!\mathcal{C}^\bullet(\rho))$. An important feature of this sampling is that conditioning on the size leads to uniform distribution (that is, the distribution we are interested in). Furthermore, the Boltzmann distribution is *compatible* with the combinatorial (block) decomposition. In particular one can sample a connected graph by a recursive procedure and by using independent samples of 2-connected components (again according to the corresponding Boltzmann distribution), for details see [10].

Instead of the maximum degree we will consider the maximum number of blocks that are attached to a cut vertex (when we are looking at the tree-like block decomposition of rooted graphs). This random variable will be denoted by $\underline{\Delta}$ and is obviously a lower bound for the maximum degree.

Let (p_k) be the distribution of the number of blocks that are attached to the root in Boltzmann sampled connected graphs. Then we have

$$p_k = \frac{\rho \mathcal{B}'(\mathcal{C}^\bullet(\rho))^k}{k!\mathcal{C}^\bullet(\rho)}$$

Next let E_1 denote the event that the number of vertices of G equals n and E_2 the event that the number of blocks is larger than κn (where $\kappa > 0$ will be chosen sufficiently small).

It is well known that $\mathbb{P}[E_1] \sim c_1 n^{-3/2}$ for a proper constant $c_1 > 0$. Furthermore, since we know that (conditioned on E_1) the number of blocks satisfies a central limit theorem with asymptotically linear mean and variance it follows that we can choose $\kappa > 0$ in a way that $\mathbb{P}[E_2|E_1] \ge 1 - n^{-2}$. Note that this also implies $\mathbb{P}[E_1 \cap E_2] = \mathbb{P}[E_2|E_1]\mathbb{P}[E_1] \ge c_2 n^{-3/2}$. Furthermore, for every cut vertex v let X_v the indicator function that there are at least $c \log n / \log\log n$ blocks attached to v (for some positive constant $c < 1$), and set $X = \sum X_v$, where the sum it taken over all cut vertices v of G. Obviously, if $X > 0$ then $\underline{\Delta}(G) \ge c \log n$. Hence we obtain

$$\begin{aligned}
\mathbb{P}[\underline{\Delta}(G) < c \log n / \log\log n | E_1] &\le \mathbb{P}[X = 0 | E_1] \\
&= \mathbb{P}[(X = 0) \cap E_2^c | E_1] + \mathbb{P}[(X = 0) \cap E_2 | E_1] \\
&\le n^{-2} + \mathbb{P}[(X = 0) | E_1 \cap E_2] \\
&\le n^{-2} + \frac{\mathbb{P}[(X = 0) | E_2]}{\mathbb{P}[E_1 \cap E_2]} \\
&\le n^{-2} + c_3 n^{3/2} \mathbb{P}[(X = 0) | E_2].
\end{aligned}$$

Finally we note that X is binomially distributed with $\ge \kappa n$ terms and probability

$$p = \sum_{k > c \log n} p_k \sim c_4 n^{-c + O(1/\log\log n)}$$

Hence the probability $\mathbb{P}[(X = 0)|E_2]$ is bounded by

$$\mathbb{P}[(X = 0)|E_2] \le \left(1 - c_4 n^{-c+O(1/\log\log n)}\right)^{\kappa n} \le \exp\left(-c_4 \kappa\, n^{1-c+O(1/\log\log n)}\right)$$

This proves that with high probability ($\ge 1 - 2n^{-2}$) there is a cut vertex that is attached to at least $c \log n / \log\log n$ blocks. Of course this also implies that

$$\mathbb{E}\Delta_n \ge c \frac{\log n}{\log\log n}(1 - 2n^{-2})$$

and completes the proof of the lower bound.

The proof for the upper bound for the maximum degree is again an application of the first moment method. For this purpose let $Y_{n,k}$ denote (now) the number of vertices of degree k in a (connected) graph of size n. Then $\mathbb{E}\, Y_{n,k} = n p_{n,k}$, where $p_{n,k}$ denotes the probability that a given vertex has degree k (here we use the assumption that we are dealing with labelled graphs). Without loss of generality we can assume that $p_{n,k}$ denote that probability that the root of rooted graph of size n has degree k.

Now let $\mathcal{B}'(z,u)$ denote the double generating function, where the exponent of u takes care of the root degree. Then the corresponding function $\mathcal{C}^\bullet(z,u)$ is given by

$$\mathcal{C}^\bullet(z,u) = z \exp\left(\mathcal{B}'(\mathcal{C}^\bullet(z), u)\right)$$

and $p_{n,k}$ is given by

$$p_{n,k} = \frac{[z^n u^k]\, \mathcal{C}^\bullet(z,u)}{[z^n]\, \mathcal{C}^\bullet(z)}.$$

Since the degree of the root vertex is bounded by the total number of vertices we have $\mathcal{B}'(z,u) \le \mathcal{B}'(zu)$ for positive real z and u. Hence, if z is close to $\mathcal{C}^\bullet(\rho)$ then $\mathcal{B}'(z,u)$ converges for some $u_0 > 1$. This property can be used to obtain an upper bound for the coefficient

$$[z^n u^k]\, \mathcal{C}^\bullet(z,u) \le [z^{n-1}]\, u_0^{-k} \exp\left(\mathcal{B}'(\mathcal{C}^\bullet(z), u_0)\right).$$

Since $\mathcal{C}^\bullet(z)$ has a squareroot singularity the same holds fo the function $z \mapsto \exp\left(\mathcal{B}'(\mathcal{C}^\bullet(z), u_0)\right)$ with implies that

$$[z^n] \exp\left(\mathcal{B}'(\mathcal{C}^\bullet(z), u_0)\right) \sim c(u_0) n^{-3/2} \rho^{-n}$$

for some constant $c(u_0) > 0$. Putting these estimates together it follows that

$$p_{n,k} = O(u_0^{-k})$$

uniformly in n and k.

Finally we set $Z_{n,k} = \sum_{\ell > k} Y_{n,\ell}$ and observe first that the $\Delta_n > k$ if and only if $Z_{n,k} > 0$ and second that the expected value of $Z_{n,k}$ is bounded by

$$\mathbb{E}\, Z_{n,k} = O(n u_0^{-k}).$$

Consequently we obtain for $k_0 = \lfloor \log n / \log u_0 \rfloor$

$$\begin{aligned} \mathbb{E}\, \Delta_n &= \sum_{k \ge 0} \mathbb{P}[\Delta_n > k] = \sum_{k \ge 0} \mathbb{P}[Z_{n,k} > 0] \\ &\le k_0 + \sum_{k \ge k_0} \mathbb{E}\, Z_{n,k} \le k_0 + O(1) \\ &= O(\log n). \end{aligned}$$

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