ON GENERALIZED FIBONACCI NUMBERS OF GRAPHS

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Abstract. The Fibonacci number of a graph has been introduced as the number of independent vertex-sets. In this paper asymptotic formulas for the average number and size of very large (2–independent, 3–independent) independent subsets of vertices are derived for simply generated families of trees.

1. Introduction

A subset I of vertices of a graph G is called independent if no two vertices of I are joined by an edge of G. It is of some interest to determine the number $g(G)$ of independent vertex-sets. For example, consider the graphs G_n with n vertices $x_1, \ldots, x_n$ such that only the pairs (x_i, x_{i+1}), $i = 1, \ldots, n-1$, are joined by an edge. It is easy to see that the numbers $g_n = g(G_n)$ satisfy the relation

$$g_{n+1} = g_n + g_{n-1}$$

with the initial conditions $g_1 = 2$, $g_2 = 3$. Therefore the numbers g_n are essentially the Fibonacci numbers.

This is the motivation to call the number $g(G)$ of a graph G the Fibonacci number of G. This notation has been introduced by H. Prodinger and R. F. Tichy [8], (see also [3], [4]). P. Kirschenhofer, H. Prodinger and R. F. Tichy derived some explicit and asymptotic formulas for the Fibonacci numbers of complete t–ary trees and for the average number of some special families of trees. Later on A. Meir and J. W. Moon [7] determined asymptotic formulas for the average number of maximal independent subsets of simply generated families of trees. The intention of this paper is to generalize this concept.

Let T denote a rooted tree with root x_{root}. A subset I is called i–independent $(i \geq 1)$, if I is independent and if every vertex $x \in T$ not contained in I can by joined to a vertex $y \in I$ by a path $x_0 = x, x_1, \ldots, x_j = y$ of length $j \leq i$ such that the sequence of distances to the root $d(x_i, x_{root})$ is monotone. Obviously a subset I is maximal independent if and only if it is 1–independent. Furthermore every i–ind. subset is j–ind. if $i \leq j$.

The last part of this definition seems to be unnatural due to the monotonicity condition. The first and obvious reason for this restriction is that is much more easy to handle with this notion in simply generated families of trees as it is done in this paper. But there is another reason. If such a tree is interpreted as a recursive data structure it is apparently easier to store the information of the path to the next vertex of an i–ind. subset with this monotonicity restriction than without it. Now let $\mathcal{F}$ denote a family of rooted trees and $\mathcal{F}_n$ the subset of $\mathcal{F}$ of trees T with

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$|T| = n$. Such a family of trees is called simply generated family of trees if the generating function

$$Y(x) = \sum_{n\ge 1} y_n x^n \tag{1}$$

of the sequence $y_n = |\mathcal{F}_n|$ satisfies a functional equation of the type

$$Y = x\varphi(Y)\ , \tag{2}$$

where

$$\varphi(t) = 1 + \varphi_1 t + \varphi_2 t^2 + \cdots \tag{3}$$

is a power series with non-negative coefficients. The coefficients of $\varphi(t)$ can be used to define weights $\omega(T)$ of rooted trees T such that $\omega(T) > 0$ if and only if $T \in \mathcal{F}$. Let $D_j(T)$ denote the number of vertices of T incident with j edges leading away from the root, then the weight $\omega(T)$ can be defined by ($\varphi_0 = 1$)

$$\omega(T) = \prod_{j\ge 0} \varphi_j^{D_j(T)}\ . \tag{4}$$

Now it is easy to see that

$$\sum_{n\ge 1} \left(\sum_{|T|=n} \omega(T) \right) x^n$$

is the solution of (2). Thus

$$y_n = \sum_{|T|=n} \omega(T)\ . \tag{5}$$

For example, we have $\varphi(t) = 1 + t^2$ for binary trees, $\varphi(t) = 1/(1-t)$ for plane trees, and $\varphi(t) = e^t$ for labelled trees.

In the next section we derive functional equations for the generating functions of the average number of i–ind. subsets. In the third section a singularity analysis will give asymptotic formulas for the average number and size, and in the last section we present some numerical results.

2. Functional Equations

The aim of this section is to derive functional equations for the generating functions related to the average number

$$e^{(i)}(n) = \frac{1}{y_n} \sum_{|T|=n} \omega(T) g^{(i)}(T) = \frac{g_n}{y_n} \tag{6}$$

and the average size

$$\mu^{(i)}(n) = \frac{1}{g_n} \sum_{|T|=n} \omega(T) \sum_{k=1}^{n} k g_k^{(i)}(T) \tag{7}$$

of i–ind. subsets. ($g_k^{(i)}(T)$ and $g^{(i)}$ are defined in the sequel.)

Let T denote a rooted tree. Then let $a_k^{(i)}(T)$ denote the number of i–ind. subsets of size k of vertices of T that contain the root, $b_k^{(i)}$ the correspooding number that do not contain the root, and $g_k^{(i)}(T) = a_k^{(i)}(T) + b_k^{(i)}(T)$. Now set

$$a_{nk}^{(i)} = \sum_{|T|=n} \omega(T) a_k^{(i)}(T) \quad , \quad b_{nk}^{(i)} = \sum_{|T|=n} \omega(T) b_k^{(i)}(T) \ , \tag{8}$$

and $g_{nk}^{(i)} = a_{nk}^{(i)} + b_{nk}^{(i)}$, furthermore

$$a_n^{(i)} = \sum_{k \geq 0} a_{nk}^{(i)} \quad , \quad b_n^{(i)} = \sum_{k \geq 0} b_{nk}^{(i)} \ , \tag{9}$$

and $g_n^{(i)} = a_n^{(i)} + b_n^{(i)}$. Thus $g_n^{(i)}$ is the total number of i–ind. subsets in trees in $\mathcal{F}_n$. If we introduce the formal generating functions

$$A^{(i)}(x,z) = \sum_{n,k} a_{nk}^{(i)} z^k x^n \quad , \quad B^{(i)}(x,z) = \sum_{n,k} b_{nk}^{(i)} z^k x^n \ , \tag{10}$$

and

$$G^{(i)}(x,z) = A^{(i)}(x,z) + B^{(i)}(x,z) = \sum_{n,k} g_{nk}^{(i)} z^k x^n \tag{11}$$

we get immediately (compare with [7])

Lemma 1.

$$G^{(i)}(x,1) = \sum_{n \geq 1} g_n^{(i)} x^n = \sum_{n \geq 1} e^{(i)}(n) y_n x^n \tag{12}$$

$$G_z^{(i)}(x,1) = \sum_{n \geq 1} \mu^{(i)}(n) g_n^{(i)} x^n \tag{13}$$

Now we define two sequences of functions recursively by

$$\begin{aligned} I^{(1)}(x,z,u,v) &= xz\varphi(x\varphi(v)) \\ I^{(i+1)}(x,z,u,v) &= I^{(i)}(x,z,u,u+x\varphi(v)) \quad (i \geq 1) \end{aligned} \tag{14}$$

and

$$\begin{aligned} J^{(1)}(x,u,v,w) &= x\varphi(v) - x\varphi(w) \\ J^{(i+1)}(x,u,v,w) &= J^{(i)}(x,u,v,u - x\varphi(v) + x\varphi(w)) \quad (i \geq 1) \ . \end{aligned} \tag{15}$$

Then we can formulate

Theorem 1. *If $Y(x) = x\varphi(Y(x))$ we have*

$$A^{(i)}(x,z) = I^{(i)}(x,z,A^{(i)}(x,z),G^{(i)}(x,z)) \tag{16}$$

and

$$B^{(i)}(x,z) = J^{(i)}(x,B^{(i)}(x,z),G^{(i)}(x,z),B^{(i)}(x,z)) \ . \tag{17}$$

For $i = 1$ we have

$$A^{(1)} = xz\varphi(x\varphi(G^{(1)})) \quad \text{and} B^{(1)} = x\varphi(G^{(1)}) - x\varphi(B^{(1)})$$

or after some simple calculation

$$B^{(1)} = x\varphi(B^{(1)} + xz\varphi(B^{(1)} + x\varphi(B^{(1)}))) - x\varphi(B^{(1)})$$

and

$$G^{(1)} = xz\varphi(x\varphi(G^{(1)})) + x\varphi(G^{(1)}) - x\varphi(G^{(1)} - xz\varphi(x\varphi(G^{(1)}))) \ .$$

This result is due to A. Meir and J. W. Moon [7]. The difference between $i = 1$ and $i > 1$ is that the functional equations (15) and (16) cannnot be reduced to one equation for $i > 1$. For example, we get for $i = 2$

$$\begin{aligned} A^{(2)} &= xz\varphi(x\varphi(A^{(2)} + x\varphi(G^{(2)}))) \\ B^{(2)} &= x\varphi(G^{(2)}) - x\varphi(B^{(2)} - x\varphi(G^{(2)}) + x\varphi(B^{(2)})) \end{aligned} \tag{18}$$

and for $i = 3$

$$\begin{aligned} A^{(3)} &= xz\varphi(x\varphi(A^{(3)} + x\varphi(A^{(3)} + x\varphi(G^{(3)})))) \\ B^{(3)} &= \\ =&x\varphi(G^{(3)}) - x\varphi(B^{(3)} - x\varphi(G^{(3)}) + x\varphi(B^{(3)} - x\varphi(G^{(3)}) + x\varphi(B^{(3)}))) \ . \end{aligned} \tag{19}$$

Proof. We will only prove the cases $i = 2, 3$ in order to demonstrate the methods and ideas. The proof of the general case runs along the same lines.

In order to formulate the arguments more clearly we will use following additional notations. If u is any vertex in a tree T with root-vertex x_{root}, let T_u denote the subtree (rooted at u) that contains all vertices v such that the path from x_{root} to v passes through u. And if I is any subset of the vertex-set $V(T)$ of T, let $I_u = I \cap V(T_u)$ for any vertex u.

$\underline{i = 2}$: We first observe that if $C^{(2)}(x, z) = A^{(2)}(x, x) + x\varphi(G^{(2)}(x, z))$, then $C^{(2)}$ enumerates those subsets I of the vertex-sets of trees T in $\mathcal{F}$ such that

(1.1) $x_{root} \in I$ and I is a 2–ind. subset of T, or
(1.2) $x_{root} \notin I$ and I_u is a 2–ind. subset for every vertex u joined to x_{root} in T.

We now consider the generating function $A^{(2)}$ that enumerates 2–ind. subsets I of T such that $x_{root} \in I$. Since $x_{root} \in I$ no vertex joined to the root of T can belong to any such subset I. Now let a denote any vertex in T at distance two from the root x_{root} of T. Then it follows that readily from the definition of a 2–ind. set that a subset I of the vertices of T is counted in $A^{(2)}$ if and only if $x_{root} \in I$ and either

(2.1) $a \in I_a$ and I_a is a 2–ind. subset of T_a, or
(2.2) $a \notin I_a$ and I_b is a 2–ind. subset of T_b for every vertex b joined to a in T_a.

If we compare (2.1) and (2.2) with (1.1) and (1.2) and observe that

$$|I| = 1 + \sum_{d(a, x_{root})=2} |I_a| \ ,$$

it is not difficult to see that

$$\begin{aligned} A^{(2)} &= xz\varphi(x\varphi(C^{(2)})) \\ &= xz\varphi(x\varphi(A^{(2)} + x\varphi(G^{(2)}))) \\ &= I^{(2)}(x, z, A^{(2)}, G^{(2)}) \ , \end{aligned}$$

as required.

Figure 1 tries to give a graphical interpretation for the two equations

$$\begin{aligned} A^{(2)} &= xz\varphi(x\varphi(C^{(2)})) \\ C^{(2)} &= x\varphi(G^{(2)}) + A^{(2)} \end{aligned}$$

for binary trees.

FIGURE 1

Now let $D^{(2)}(x, z)$ enumerate those subsets I of the vertex-sets of trees T in $\mathcal{F}$ such that

(3.1) $x_{root} \notin I$ and I is a 2–ind. subset of T and

(3.2) $u \notin I$ for every vertex u of T joined to x_{root}.

Fist observe that $x\varphi(G^{(2)}(x, z))$ enumerates those subsets I of the vertex-sets of trees T in $\mathcal{F}$ such that

(4.1) $x_{root} \notin I$ and

(4.2) I_u is a 2–ind. subset of T_u for every vertex u of T joined to x_{root}.

It is easy to verify that I satisfies (4.1) and (4.2) if and only if

(5.1) $x_{root} \notin I$ and I is a 2–ind. subset of T, or

(5.2) $x_{root} \notin I$ and for every vertex u joined to x_{root}

(5.2.1) $u \notin I_u$ and I_u is a 2–ind. subset of T_u, and

(5.2.2) $v \notin I_u$ for every vertex v of T_u joined to u.

Note that (5.1) and (5.2) cannot occur at once. Comparing (3.1) and (3.2) with (5.2.1) and (5.2.2) we get $x\varphi(G^{(2)}) = B^{(2)} + x\varphi(D^{(2)})$. Similarly it can be shown that $x\varphi(B^{(2)}) = D^{(2)} + x\varphi(D^{(2)})$.

Thus we get $B^{(2)} - D^{(2)} = x\varphi(G^{(2)}) - x\varphi(B^{(2)})$ and after substituting $D^{(2)}$ in the first equation

$$\begin{aligned} B^{(2)} &= x\varphi(G^{(2)}) - x\varphi(B^{(2)} - x\varphi(G^{(2)}) + x\varphi(B^{(2)})) \\ &= J^{(2)}(x, B^{(2)}, G^{(2)}, B^{(2)}) \ . \end{aligned}$$

Figure 2 gives a graphical interpretation of the two equations

$$\begin{aligned} B^{(2)} &= x\varphi(G^{(2)}) - x\varphi(D^{(2)}) \\ D^{(2)} &= x\varphi(B^{(2)}) - x\varphi(D^{(2)}) \end{aligned}$$

for binary trees.

FIGURE 2

$\underline{i=3}$: First we deal with 3–ind. subsets that contain the root. We omit the verbal description. It can be easily deduced from the graphical interpretation

FIGURE 3

This directly yields the functional equation

$$A^{(3)} = xz\varphi(x\varphi(A^{(3)} + x\varphi(A^{(3)} + x\varphi(G^{(3)})))) \ .$$

For the 3–ind. subsets that do not contain the root we have

FIGURE 4

This gives

$$\begin{aligned} B^{(3)} &= x\varphi(G^{(3)}) \\ &\quad - x\varphi(B^{(3)} - x\varphi(G^{(3)}) + x\varphi(B^{(3)} - x\varphi(G^{(3)}) + x\varphi(B^{(3)}))) \\ &= J^{(3)}(x, B^{(3)}, G^{(3)}, B^{(3)}) \ . \end{aligned}$$

3. Asymptotic Results

If we set

$$\begin{aligned} F_1^{(i)}(x,z,a,b) &= a - I^{(i)}(x,z,a,a+b) \\ F_2^{(i)}(x,z,a,b) &= b - J^{(i)}(x,b,a+b,b) \end{aligned} \tag{20}$$

we formally have

$$F_1^{(i)}(x,z,A^{(i)}(x,z),B^{(i)}(x,z)) = F_2^{(i)}(x,z,A^{(i)}(x,z),B^{(i)}(x,z)) \equiv 0 \ .$$

Now suppose that $\varphi(t)$ has a positive radius of convergence which implies that $F_1^{(i)}(x,1,a,b)$ and $F_2^{(i)}(x,1,a,b)$ are analytic around $(0,1,0,0)$. (We have $A^{(i)}(0,1) = B^{(i)}(0,1) = 0$.) If

$$\frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,b)}(0,1,0,0) = (F_{1a}^{(i)}F_{2b}^{(i)} - F_{1b}^{(i)}F_{2a}^{(i)})(0,1,0,0) \neq 0$$

the implicit function theorem says that there is a unique analytic solution $A(x) = A^{(i)}(x,1)$, $B(x) = B^{(i)}(x,1)$ around 0 of the implicit equations $F_1^{(i)}(x,1,A(x),B(x)) = F_2^{(i)}(x,1,A(x),B(x)) \equiv 0$. Since we know that the Taylor coefficients of $A^{(i)}(x,1)$ and $B^{(i)}(x,1)$ are non-negative the first singularity on the positive real axis is the radius of convergence. As long as

$$\frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,b)}(x_0,1,A^{(i)}(x_0,1),B^{(i)}(x_0,1)) \neq 0 \quad \text{with} \quad x_0 > 0$$

$A^{(i)}(x,1)$ and $B^{(i)}(x,1)$ are analytic in the circle $|x| < x_0$.

Let $(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)})$ be the first solution of

$$\begin{aligned} F_1^{(i)}(x,1,a,b) &= 0 \\ F_2^{(i)}(x,1,a,b) &= 0 \\ \frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,b)}(x,1,a,b) &= 0 \end{aligned} \tag{21}$$

with $r^{(i)} > 0$ and suppose that

$$\frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,x)}(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)}) \neq 0 \tag{22'}$$

and

$$\frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(x,b)}(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)}) \neq 0\ . \tag{22"}$$

Then $r^{(i)}$ is the common radius of convergence of $A^{(i)}(x,1)$, $B^{(i)}(x,1)$. This can be seen by following observation. If $A^{(i)}(x,1)$ is regular at $x = r^{(i)}$ then $A_x^{(i)}(x,1)$ is regular, too. But this is a contradiction to the fact that the limit $x \to r^{(i)}-$ of

$$A_x^{(i)}(x,1) = -\frac{\frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(x,b)}(x,1,A^{(i)}(x,1),B^{(i)}(x,1))}{\frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,b)}(x,1,A^{(i)}(x,1),B^{(i)}(x,1))}$$

does not exist. The same is true for $B^{(i)}(x,1)$.

Similarly to A. Meir and J. W. Moon [7] it can be shown that $A^{(i)}(x,1)$ and $B^{(i)}(x,1)$ converge at $x = r^{(i)}$ if there is a $k \geq 2$ with $\varphi_k > 0$. Thus

$$\lim_{x\to r^{(i)}} A^{(i)}(x,1) = \alpha_0^{(i)} \quad \text{and} \quad \lim_{x\to r^{(i)}} B^{(i)}(x,1) = \beta_0^{(i)}\ .$$

If

$$\left[\left(F^{(i)}_{1bb}\frac{(F^{(i)}_{1a})^2}{(F^{(i)}_{1b})^2}-2F^{(i)}_{1ab}\frac{F^{(i)}_{1a}}{F^{(i)}_{1b}}+F^{(i)}_{1aa}\right)F^{(i)}_{2b}-\right. \tag{23}$$

$$\left.-\left(F^{(i)}_{2bb}\frac{(F^{(i)}_{2a})^2}{(F^{(i)}_{2b})^2}-2F^{(i)}_{2ab}\frac{F^{(i)}_{2a}}{F^{(i)}_{2b}}+F^{(i)}_{2aa}\right)F^{(i)}_{1b}\right](r^{(i)},1,\alpha^{(i)}_0,\beta^{(i)}_0)\neq 0$$

and

$$\left[\left(F^{(i)}_{1aa}\frac{(F^{(i)}_{1b})^2}{(F^{(i)}_{1a})^2}-2F^{(i)}_{1ab}\frac{F^{(i)}_{1b}}{F^{(i)}_{1a}}+F^{(i)}_{1bb}\right)F^{(i)}_{2a}-\right. \tag{24}$$

$$\left.-\left(F^{(i)}_{2aa}\frac{(F^{(i)}_{2b})^2}{(F^{(i)}_{2a})^2}-2F^{(i)}_{2ab}\frac{F^{(i)}_{2b}}{F^{(i)}_{2a}}+F^{(i)}_{2bb}\right)F^{(i)}_{1a}\right](r^{(i)},1,\alpha^{(i)}_0,\beta^{(i)}_0)\neq 0$$

the inverse functions $X_1(a)$ of $A^{(i)}(x,1)$ and $X_2(b)$ of $B^{(i)}(x,1)$ satisfy $X_1'(\alpha_0^{(i)}) = X_2'(\beta_0^{(i)}) = 0$ and $X_1''(\alpha_0^{(i)}) \neq 0$, $X_2''(\beta_0^{(i)}) \neq 0$. Hence we have a local expansion of the form

$$\begin{aligned} A^{(i)}(x,1) &= \alpha_0^{(i)} - \alpha_1^{(i)}(r^{(i)}-x)^{1/2} - \alpha_2^{(i)}(r^{(i)}-x) + \cdots \\ B^{(i)}(x,1) &= \beta_0^{(i)} - \beta_1^{(i)}(r^{(i)}-x)^{1/2} - \beta_2^{(i)}(r^{(i)}-x) + \cdots\ , \end{aligned} \tag{25}$$

where

$$(\alpha_1^{(i)})^2 =$$

$$\frac{(F^{(i)}_{1x}F^{(i)}_{2b}-F^{(i)}_{2x}F^{(i)}_{1b})\,(r^{(i)},1,\alpha^{(i)}_0,\beta^{(i)}_0)}{\left[\left(F^{(i)}_{1bb}\frac{(F^{(i)}_{1a})^2}{(F^{(i)}_{1b})^2}-2F^{(i)}_{1ab}\frac{F^{(i)}_{1a}}{F^{(i)}_{1b}}+F^{(i)}_{1aa}\right)F^{(i)}_{2b}-\left(F^{(i)}_{2bb}\frac{(F^{(i)}_{2a})^2}{(F^{(i)}_{2b})^2}-2F^{(i)}_{2ab}\frac{F^{(i)}_{2a}}{F^{(i)}_{2b}}+F^{(i)}_{2aa}\right)F^{(i)}_{1b}\right]}$$

$$(\beta_1^{(i)})^2 = \tag{26}$$

$$\frac{(F^{(i)}_{1x}F^{(i)}_{2a}-F^{(i)}_{2x}F^{(i)}_{1a})\ (r^{(i)},1,\alpha^{(i)}_0,\beta^{(i)}_0)}{\left[\left(F^{(i)}_{1bb}\frac{(F^{(i)}_{1a})^2}{(F^{(i)}_{1b})^2}-2F^{(i)}_{1ab}\frac{F^{(i)}_{1a}}{F^{(i)}_{1b}}+F^{(i)}_{1aa}\right)F^{(i)}_{2a}-\left(F^{(i)}_{2bb}\frac{(F^{(i)}_{2a})^2}{(F^{(i)}_{2b})^2}-2F^{(i)}_{2ab}\frac{F^{(i)}_{2a}}{F^{(i)}_{2b}}+F^{(i)}_{2aa}\right)F^{(i)}_{1a}\right]}$$

If there are in addition two distinct j,k with $\gcd(j,k)=1$, $\varphi_j>0$, and $\varphi_k>0$ then it can be shown similarly to A. Meir and J. W. Moon [7] but in a more complicated way that $F^{(i)}_{1a}F^{(i)}_{2b} - F^{(i)}_{1b}F^{(i)}_{2a} \neq 0$ on the circle of convergence except at $x = r^{(i)}$. Thus $r^{(i)}$ is the only singularity on the circle of convergence. (Compare also with A. Bender [1,Theorem 4].) Therefore we get by Darboux's theorem [2] that

$$g_n^{(i)} \sim \frac{\alpha_1^{(i)}+\beta_1^{(i)}}{2\sqrt{\pi}}\,(r^{(i)})^{-n+1/2}\,n^{-3/2}\ . \tag{27}$$

If τ is the smallest positive solution of $\tau\varphi'(\tau)=\varphi(\tau)$ then (see [6])

$$y_n \sim \sqrt{\frac{\varphi(\tau)}{2\pi\varphi''(\tau)}}\left(\frac{\tau}{\varphi(\tau)}\right)^{-n} n^{-3/2} \tag{28}$$

and therefore

$$e^{(i)}(n) = \frac{g_n^{(i)}}{y_n} \sim (\alpha_1^{(i)} + \beta_1^{(i)}) \sqrt{\frac{r^{(i)}\varphi''(\tau)}{2\varphi(\tau)}} \left(\frac{\tau/\varphi(\tau)}{r^{(i)}}\right)^n \tag{29'}$$

or

$$\log e^{(i)}(n) \sim E^{(i)} n \tag{29''}$$

with $E^{(i)} = \log(\tau/(r^{(i)}\varphi(\tau)))$.

If $\varphi(t) = 1 + t^2$ (binary trees) $y_{2k} = a_{2k}^{(i)} = b_{2k}^{(i)} = 0$. Thus $A^{(i)}(-x,1) = -A^{(i)}(x,1)$ and $B^{(i)}(-x,1) = -B^{(i)}(x,1)$. Therefore there are at least two singularities $x = \pm r^{(i)}$. Since $f(x)$ with $f(x^2) = \frac{1}{x}A^{(i)}(x^2,1)$ has only one singularity $A^{(i)}(x,1)$ has only these two singularities. (The same is true for $B^{(i)}(x,1)$.) Therefore the above considerations can easily be adapted for this case and we get an analoguous asymptotic formula

$$\log e^{(i)}(2n+1) \sim (2n+1)\log\frac{\tau/\varphi(\tau)}{r^{(i)}} \ . \tag{30}$$

We have proved

Theorem 2. *Let $\mathcal{F}$ be a simply generated family of trees where $\varphi(t)$ has a radius of convergence large enough that $(r^{(i)}, 1, \alpha_0^{(i)}, \beta_0^{(i)})$, the solution of (21) where $r^{(i)} > 0$ is minimal, is a regular point of $F_1^{(i)}(x,z,a,b)$ and $F_2^{(i)}(x,z,a,b)$. Suppose that (22), (23), and (24) are satisfied and that there are two distinct positive integers j,k with $\gcd(j,k) = 1$, $\varphi_j > 0$, and $\varphi_k > 0$. Then the average number $e^{(i)}(n)$ of i–independent subsets in trees $T \in \mathcal{F}_n$ can be determined asymptotically by*

$$e^{(i)}(n) \sim C^{(i)}(D^{(i)})^n$$

where

$$C^{(i)} = (\alpha_1^{(i)} + \beta_1^{(i)}) \sqrt{\frac{r^{(i)}\varphi''(\tau)}{2\varphi(\tau)}} \quad \textit{and} \quad D^{(i)} = \frac{\tau/\varphi(\tau)}{r^{(i)}} \ .$$

($\alpha_1^{(i)}$ and $\beta_1^{(i)}$ are defined in (25) and (26) and τ is the smallest positive solution of $t\varphi'(t) = \varphi(t)$.)

Since $A^{(i)}(x,1)$ and $B^{(i)}(x,1)$ are absolutely convergent for $|x| \le r^{(i)}$, $A^{(i)}(x,z)$ and $B^{(i)}(x,z)$ are analytic functions for $|x| < r^{(i)}$ and $|z| < 1$ and satisfy

$$F_1^{(i)}(x,z,A^{(i)}(x,z),B^{(i)}(x,z)) = F_2^{(i)}(x,z,A^{(i)}(x,z),B^{(i)}(x,z)) \equiv 0 \ .$$

Thus we get

$$\begin{aligned} G_x^{(i)}(x,z) &= A_x^{(i)}(x,z) + B_x^{(i)}(x,z) \\ &= -\frac{\frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(x,b)} + \frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,x)}}{\frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,b)}}(x,z,A^{(i)}(x,z),B^{(i)}(x,z)) \end{aligned}$$

$$\begin{aligned} G_z^{(i)}(x,z) &= A_z^{(i)}(x,z) + B_z^{(i)}(x,z) \\ &= -\frac{\frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(z,b)} + \frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,z)}}{\frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,b)}}\,(x,z,A^{(i)}(x,z),B^{(i)}(x,z)) \end{aligned}$$

and therefore

$$\begin{aligned} &G_z^{(i)}(x,1) = \\ &= \frac{\frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(z,b)} + \frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,z)}}{\frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(x,b)} + \frac{\partial(F_1^{(i)},F_2^{(i)})}{\partial(a,x)}}\,(x,1,A^{(i)}(x,1),B^{(i)}(x,1))\,G_x^{(i)}(x,1) \\ &= H^{(i)}(x,1,A^{(i)}(x,1),B^{(i)}(x,1))\,G_x^{(i)}(x,1)\ . \end{aligned} \tag{31}$$

Suppose that $(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)})$ is a regular point of $H^{(i)}(x,1,a,b)$ then $f(x) = H^{(i)}(x,1,A^{(i)}(x,1),B^{(i)}(x,1))$ has an expansion of the form

$$f(x) = \delta_0^{(i)} - \delta_1^{(i)}(r^{(i)}-x)^{1/2} - \delta_2^{(i)}(r^{(i)}-x) - \cdots$$

with $\delta_0^{(i)} = f(r^{(i)})$. Since

$$G_x^{(i)}(x,1) = \frac{\alpha_1^{(i)}+\beta_1^{(i)}}{2}(r^{(i)}-x)^{-1/2} + (\alpha_2^{(i)}+\beta_2^{(i)}) + \cdots$$

we have

$$G_z^{(i)}(x,1) = \frac{\delta_0^{(i)}(\alpha_1^{(i)}+\beta_1^{(i)})}{2}(r^{(i)}-x)^{-1/2} + \cdots\ . \tag{32}$$

Using again Darboux's theorem [2] and comparing this with Lemma 1 and Theorem 2 we conclude

Theorem 3. *If we assume in addition to the assumtions of Theorem 2 that* $(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)})$ *is a regular point of* $H^{(i)}(x,z,a,b)$ *(defined in (31)), the average size* $\mu^{(i)}(n)$ *of* i*–independent subsets of trees* $T \in \mathcal{F}_n$ *can evaluated asymptotically by*

$$\mu^{(i)}(n) \sim S^{(i)} n \tag{33}$$

where

$$S^{(i)} = \frac{\delta_0^{(i)}}{r^{(i)}} = \frac{H^{(i)}(r^{(i)},1,\alpha_0^{(i)},\beta_0^{(i)})}{r^{(i)}}\ .$$

Again we have the same asymtotic formula

$$\mu\ (2n+1) \sim S^{(i)}(2n+1)$$

for binary trees

4. Numerical Results

The author has approximately evaluated the numerical constants $E^{(i)}$ and $S^{(i)}$ (see (29) and (33)) for binary trees ($\varphi(t) = 1 + t^2$), plane trees ($\varphi(t) = 1/(1-t)$), and for labelled trees ($\varphi(t) = e^t$) for $i = 2, 3$. $r^{(i)}$, $\alpha_0^{(i)}$, and $\beta_0^{(i)}$ can be determined by solving the equations (21) approximately by a Newton iteration method. Then it is easy to examine the conditions (22), (23), (24) and to determine $D^{(i)}$ and $S^{(i)}$. In the following two tables these constants are compared with $D^{(1)}$, $S^{(1)}$ [7], with the corresponding numbers $D^{(\infty)}$, $S^{(\infty)}$ for all ind. subsets [3], and with $S^{(max)}$, the average size of the largest ind. subset of a tree $T \in \mathcal{F}_n$ [5]. (All constants are rounded.)

	$D^{(\infty)}$	$D^{(3)}$	$D^{(2)}$	$D^{(1)}$
binary trees	1.657	1.531	1.433	1.250
plane trees	1.687	1.564	1.455	1.240
labelled trees	1.655	1.562	1.474	1.274

Table 1

	$S^{(\infty)}$	$S^{(3)}$	$S^{(2)}$	$S^{(1)}$	$S^{(max)}$
binary trees	0.309	0.401	0.438	0.482	0.585
plane trees	0.333	0.410	0.416	0.5	0.618
labelled trees	0.307	0.380	0.393	0.463	0.567

Table 2

It would be interesting to determine the asymptotoic expansion of $D^{(i)}$ and $S^{(i)}$ if i tends to infinity. The author conjectures that $\lim_{i\to\infty} D^{(i)} = D^\infty$ and $\lim_{i\to\infty} S^{(i)} = S^\infty$ and that the order of convergence depends on $\varphi(t)$ but he could not find a way to prove this.

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