FIBONACCI NUMBERS AND CONTINUED FRACTION EXPANSIONS

Michael Drmota

Technical University of Vienna

Abstract. Let $K \geq 1$ some fixed integer and a sequence G_n defined by $G_0 = 0$, $G_1 = 1$, and $G_{n+1} = KG_n + G_{n-1}$ for $n \geq 1$. Of course, for $K = 1$ we get the usual Fibonacci numbers F_n. By definition it is clear that G_n and G_{n+1} are coprime and the continued fraction expansion of the ratio is given by $G_n/G_{n+1} = [0, K, \dots, K]$. In the first part of this paper it is shown that G_n and $G_{n+1} + 1$ are coprime for infinitely many n (at least for $n \equiv \pm 1 \pmod 6$). Moreover the continued fraction expansion of $G_n/(G_{n+1}+1)$ can be obtained explicitly. It turns out that the partial quotients of these continued fraction expansions are uniformly bounded by $2K$. Such pairs of integers can be used to obtain an optimality result for linear diophantine equations in three variables, as the second part of this paper shows.

1. Introduction

It is clear by the definition of Fibonacci numbers F_n: $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ $(n \geq 1)$, that the continued fraction expansion of F_n/F_{n+1} is given by

$$\frac{F_n}{F_{n+1}} = [0, \underbrace{1, \dots, 1}_{n-2}, 2]. \tag{1.1}$$

It is interesting that it possible to obtain the continued fraction expansion of $F_n/(F_{n+1}+1)$ explicitly, too (see [1] for the coprime case $n \equiv \pm 1 \pmod 6$):

$$\frac{F_n}{F_{n+1}+1} = \begin{cases} [0, 1^{\frac{n-1}{2}}, 5, 4^{\frac{n-13}{6}}] & \text{for} \quad n \equiv 1 \pmod{12}, \\ [0, 1^{\frac{n-3}{2}}, 2, 3, 4^{\frac{n-9}{6}}] & \text{for} \quad n \equiv 3 \pmod{12}, \\ [0, 1^{\frac{n-1}{2}}, 5, 4^{\frac{n-11}{6}}, 3] & \text{for} \quad n \equiv 5 \pmod{12}, \\ [0, 1^{\frac{n-3}{2}}, 2, 3, 4^{\frac{n-13}{6}}, 5] & \text{for} \quad n \equiv 7 \pmod{12}, \\ [0, 1^{\frac{n-1}{2}}, 5, 4^{\frac{n-9}{6}}] & \text{for} \quad n \equiv 9 \pmod{12}, \\ [0, 1^{\frac{n-3}{2}}, 2, 3, 4^{\frac{n-11}{6}}, 3] & \text{for} \quad n \equiv 11 \pmod{12}, \\ [0, 1^{\frac{n}{2}-2}, 2] & \text{for} \quad n \equiv 0 \pmod 4, \\ [0, 1^{\frac{n}{2}-1}, 3] & \text{for} \quad n \equiv 2 \pmod 4, \end{cases} \tag{1.2}$$

where the notation a^k means k repetitions of the partial quotient a. Similar expansions can be obtained for $F_n/(F_{n+1}+2)$ and so on. But the effort significanly increases. Notice that the expansions for even n are of minor interest since they only represent already known expansions, e.g. for $n \equiv 0 \pmod 4$ we have

$$\frac{F_n}{F_{n+1}+1} = \frac{F_{\frac{n}{2}}}{F_{\frac{n}{2}+1}}.$$

In the fist part of this paper these expansions are generalized to sequences G_n satisfying a recurrence relation of the form

$$G_{n+1} = K\ G_n + G_{n-1},$$

where K is a positive integer.

The most interesting property of these sequences is that they provide infinitely many examples of pairs of coprime positive integers a, b such that the (finite) continued fraction expansions of a/b and $a/(b+1)$ have uniformly bounded partial quotients. The partial quotients of F_n/F_{n+1} and $F_n/(F_{n+1}+1)$ ($n \equiv \pm 1 \pmod 6$) are uniformly bounded by 5, for example.

These pairs of integers can be used to construct linear diophantine equations of the form

$$a_1x_1 + a_2x_2 + a_3x_3 = 0, \tag{1.3}$$

where all non-zero solutions are large in the following sense.

Theorem 1. *Let a, b be coprime positive integers satisfying $b^2 \geq a$. Assume that the partial quotients of the continued fraction expansions of*

$$\frac{a}{b} = [a_0, a_1, \dots, a_N] \tag{1.4}$$

and

$$\frac{a}{b+1} = [a_0', a_1', \dots, a_M'] \tag{1.5}$$

are bounded as follows:

$$\max_{1\leq i\leq N} a_i \leq L \quad \textit{and} \quad \max_{1\leq j\leq M} a_j' \leq L'. \tag{1.6}$$

Then all non-zero integral solution x, y, z of $ax + by + (b+1)z = 0$ satisfy

$$\begin{aligned} \overline{x}\,\overline{y} &> \frac{b+1}{2(L'+1)}, \\ \overline{x}\,\overline{z} &> \frac{b}{2(L+1)}, \\ \overline{y}\,\overline{z} &> \frac{a}{(L+1)^2(L'+1)^2((L+2)(L'+2))^{1/2}}, \end{aligned} \tag{1.7}$$

where $\overline{x} = \max\{|x|, 1\}$.

This theorem is the best possible for linear diophantine equations with three variables as the following theorem shows.

Theorem 2. *Let $a_1x_1 + \cdots + a_kx_k = 0$ a linear diophantine equation with non-zero integral coefficients a_i. Then there exists a non-zero integral solution $x_1, \dots, x_k$ with*

$$\prod_{\substack{1\leq i\leq k \\ i\neq j}} \overline{x_i} \leq \begin{cases} (k-1)\,|a_j| & \textit{for} \quad j = 1, \dots, k-1, \\ |a_j| & \textit{for} \quad j = k. \end{cases} \tag{1.8}$$

Special cases of Theorems 1 and 2 are included in [1]. Both theorems will be proved in section 3.

2. Continued Fraction Expansions

For a given positive integer K, let G_n be defined by

$$G_0 = 0, \quad G_1 = 1, \quad G_{n+1} = K\,G_n + G_{n-1} \quad \text{for } n \geq 1 \tag{2.1}$$

or equivalently by

$$G_n = \frac{1}{\sqrt{K^2+4}}\left(\left(\frac{K+\sqrt{K^2+4}}{2}\right)^n - \left(\frac{K-\sqrt{K^2+4}}{2}\right)^n\right). \tag{2.2}$$

Trivially for $K = 1$ we get the Fibonacci numbers.

In this section we will discuss the relation between G_n and $G_{n+1}+1$. It should be noted that similar results can be obtained for $G_{n+1}-1$ and for $G_{n-1}\pm 1$ by exactly the same methods.

Lemma 1. *If K is even, then the greatest common divisor $(G_n, G_{n+1}+1)$ of G_n and $G_{n+1}+1$ is given by*

$$(G_n, G_{n+1}+1) = \begin{cases} 1 & \textit{for } n \equiv 1 \pmod 2, \\ G_{\frac{n}{2}-1} + G_{\frac{n}{2}+1} & \textit{for } n \equiv 0 \pmod 4, \\ 2\,G_{\frac{n}{2}} & \textit{for } n \equiv 2 \pmod 4, \end{cases} \tag{2.3}$$

and if K is odd, then we have

$$(G_n, G_{n+1}+1) = \begin{cases} 1 & \textit{for } n \equiv \pm 1 \pmod 6, \\ 2 & \textit{for } n \equiv 3 \pmod 6, \\ G_{\frac{n}{2}-1} + G_{\frac{n}{2}+1} & \textit{for } n \equiv 0 \pmod 4, \\ G_{\frac{n}{2}} & \textit{for } n \equiv 2 \pmod 4. \end{cases} \tag{2.4}$$

Proof. First it is easily follows by induction that

$$(G_n, G_{n+1}+1) = (G_{n-l+1} + (-1)^l G_{l-1}, G_{n-l} + (-1)^{l+1} G_l) \tag{2.5}$$

for $l \geq 1$. Namely $(G_n, G_{n+1}+1) = (G_n, G_{n+1}+1-KG_n) = (G_n, G_{n-1}+1)$ and so for $l \geq 1$, we get

$$\begin{aligned} &(G_{n-l+1} + (-1)^l G_{l-1}, G_{n-l} + (-1)^{l+1} G_l) \\ &\quad = (G_{n-l+1} + (-1)^l G_{l-1} - K(G_{n-l} + (-1)^{l+1} G_l), G_{n-l} + (-1)^{l+1} G_l) \\ &\quad = (G_{n-l-1} + (-1)^{l+2} G_{l+1}, G_{n-l} + (-1)^{l+1} G_l). \end{aligned}$$

Now (2.5) immediately gives

$$\begin{aligned} (G_n, G_{n+1}+1) &= (G_{\frac{n}{2}+1} + G_{\frac{n}{2}-1}, 0) \\ &= G_{\frac{n}{2}+1} + G_{\frac{n}{2}-1} \end{aligned}$$

for $n \equiv 0 \pmod 4$ and

$$\begin{aligned} (G_n, G_{n+1}+1) &= (G_{\frac{n}{2}+1} - G_{\frac{n}{2}-1}, 2G_{\frac{n}{2}}) \\ &= (KG_{\frac{n}{2}}, 2G_{\frac{n}{2}}) \\ &= (K, 2)G_{\frac{n}{2}} \end{aligned}$$

for $n \equiv 2 \pmod 4$. For odd n we first get

$$\begin{aligned}(G_n, G_{n+1}+1) &= (G_{\frac{n+1}{2}} + G_{\frac{n-1}{2}}, G_{\frac{n+1}{2}} - G_{\frac{n-1}{2}}) \\ &= (G_{\frac{n+1}{2}} + G_{\frac{n-1}{2}}, 2G_{\frac{n-1}{2}}).\end{aligned}$$

Note that $G_m + G_{m+1}$ is even if and only if K is odd and $m \equiv 1 \pmod 3$ and that G_m and $G_m + G_{m+1}$ are always coprime. Hence we get $(G_n, G_{n+1}+1) = 2$ for odd K and $n \equiv 3 \pmod 6$ and $(G_n, G_{n+1}+1) = 1$ otherwise. $\square$

Theorem 3. *If $K = 2$, then the continued fraction expansion of $G_n/(G_{n+1}+1)$ is given by*

$$\begin{aligned}&[0, 2^{\frac{n-1}{2}}, 4, 2^{\frac{n-5}{2}}, 3] \quad &&\text{for} \quad n \equiv 1 \pmod 4, \\ &[0, 2^{\frac{n-1}{2}}, 1, 1, 1, 2^{\frac{n-5}{2}}, 3] \quad &&\text{for} \quad n \equiv 3 \pmod 4,\end{aligned} \tag{2.6}$$

if $K > 2$ is even, then we get

$$\begin{aligned}&[0, K^{\frac{n-1}{2}}, K+1, \frac{K-2}{2}, (1, 1, \frac{K-2}{2})^{\frac{n-3}{2}}, 2] \quad &&\text{for} \quad n \equiv 1 \pmod 4, \\ &[0, K^{\frac{n-1}{2}}, K-1, \frac{K}{2}, (1, 1, \frac{K-2}{2})^{\frac{n-3}{2}}, 2] \quad &&\text{for} \quad n \equiv 3 \pmod 4,\end{aligned} \tag{2.7}$$

and if $K > 2$ is odd, then we have

$$\begin{aligned}&[0, K^{\frac{n-1}{2}}, K+1, \frac{K-1}{2}, (2K, \frac{K-1}{2}, 1, 1, \frac{K-1}{2})^{\frac{n-7}{6}}, 2K, \frac{K-1}{2}, 2] \\ &\qquad\qquad\qquad\qquad \text{for } n \equiv 1 \pmod{12}, \\ &[0, K^{\frac{n-1}{2}}, K-1, \frac{K+1}{2}, (2K, \frac{K-1}{2}, 1, 1, \frac{K-1}{2})^{\frac{n-3}{6}}] \\ &\qquad\qquad\qquad\qquad \text{for } n \equiv 3 \pmod{12}, \\ &[0, K^{\frac{n-1}{2}}, K+1, \frac{K-1}{2}, (2K, \frac{K-1}{2}, 1, 1, \frac{K-1}{2})^{\frac{n-5}{6}}, 2K] \\ &\qquad\qquad\qquad\qquad \text{for } n \equiv 5 \pmod{12}, \\ &[0, K^{\frac{n-1}{2}}, K-1, \frac{K+1}{2}, (2K, \frac{K-1}{2}, 1, 1, \frac{K-1}{2})^{\frac{n-7}{6}}, 2K, \frac{K-1}{2}, 2] \\ &\qquad\qquad\qquad\qquad \text{for } n \equiv 7 \pmod{12}, \\ &[0, K^{\frac{n-1}{2}}, K+1, \frac{K-1}{2}, (2K, \frac{K-1}{2}, 1, 1, \frac{K-1}{2})^{\frac{n-3}{6}}] \\ &\qquad\qquad\qquad\qquad \text{for } n \equiv 9 \pmod{12}, \\ &[0, K^{\frac{n-1}{2}}, K-1, \frac{K+1}{2}, (2K, \frac{K-1}{2}, 1, 1, \frac{K-1}{2})^{\frac{n-5}{6}}, 2K] \\ &\qquad\qquad\qquad\qquad \text{for } n \equiv 11 \pmod{12},\end{aligned} \tag{2.8}$$

where $(a, \ldots, b)^k$ denotes k repetitions of the partial quotients $a, \ldots, b$. (Of course, the cases $n \equiv 3 \pmod{12}$ and $n \equiv 9 \pmod{12}$ must be interpreted as $(G_n/2)/((G_{n+1}+1)/2)$.)

Proof. The method of proving continued fraction expansions for rationals is, of course, the Euclidean algorithm. For brevity we will only prove two typical cases. The other ones can be handled along the same lines.

First, let $K > 2$ be even and $n \equiv 1 \pmod 4$, say $n = 4m+1$. Here we have

$$G_{n+1} + 1 = KG_n + (G_{n-1} + 1),$$

where $0 < G_{n-1} + 1 < G_n$ and then

$$G_{n-l} + (-1)^{l+1} G_l = K(G_{n-l-1} + (-1)^{l+2} G_{l+1}) + (G_{n-l-2} + (-1)^{l+3} G_{l+2}),$$

for $0 \leq l \leq 2m-2$, where $0 < G_{n-l-2} + (-1)^{l+3} G_{l+2} < G_{n-l-1} + (-1)^{l+2} G_{l+1}$. Now we get

$$G_{2m+2} + G_{2m-1} = (K+1)(G_{2m+1} - G_{2m}) + 2G_{2m},$$

where $0 < 2G_{2m} < G_{2m+1} - G_{2m}$ and

$$G_{2m+1} - G_{2m} = \frac{K-2}{2} 2G_{2m} + (G_{2m} + G_{2m-1}),$$

where $0 < G_{2m} + G_{2m-1} < 2G_{2m}$. Similarly, we have for $0 \leq l \leq 2m-2$

$$2G_{2m-l} = 1(G_{2m-l} + G_{2m-l-1}) + (G_{2m-l} - G_{2m-l-1}),$$

where $0 < G_{2m-l} - G_{2m-l-1} < G_{2m-l} + G_{2m-l-1}$,

$$G_{2m-l} + G_{2m-l-1} = 1(G_{2m-l} - G_{2m-l-1}) + 2G_{2m-l-1},$$

where $0 < 2G_{2m-l-1} < G_{2m-l} - G_{2m-l-1}$, and

$$G_{2m-l} - G_{2m-l-1} = \frac{K-2}{2} 2G_{2m-l-1} + (G_{2m-l-1} + G_{2m-l-2}),$$

where $0 < G_{2m-(l+1)} + G_{2m-(l+1)-1} < 2G_{2m-(l+1)}$. Finally we have

$$2G_1 = 2(G_1 + G_0) + 0,$$

leading to the continued fraction expansion of $G_n/(G_{n+1}+1)$ in this case.

Now suppose that $K > 2$ is odd, and that $n \equiv 7 \pmod{12}$, say $n = 12m+7$. The Euclidean algorithm starts with

$$G_{n+1} + 1 = KG_n + (G_{n-1} + 1),$$

where $0 < G_{n-1} + 1 < G_n$ and with

$$G_{n-l} + (-1)^{l+1} G_l = K(G_{n-l-1} + (-1)^{l+2} G_{l+1}) + (G_{n-l-2} + (-1)^{l+3} G_{l+2}),$$

for $0 \leq l \leq 6m+1$, where $0 < G_{n-l-2} + (-1)^{l+3} G_{l+2} < G_{n-l-1} + (-1)^{l+2} G_{l+1}$. Now we have

$$G_{6m+5} - G_{6m+2} = (K-1)(G_{6m+4} + G_{6m+3}) + 2G_{6m+3},$$

where $0 < G_{6m+3} < G_{6m+4} + G_{6m+3}$ and

$$G_{6m+4} + G_{6m+3} = \frac{K+1}{2}\, 2G_{6m+3} + G_{6m+2},$$

where $0 < G_{6m+2} < 2G_{6m+3}$. Then we get for $0 \le l \le 2m-1$,

$$2G_{6m-3l+3} = 2KG_{6m-3l+2} + 2G_{6m-3l+1},$$

where $0 < 2G_{6m-3l+1} < G_{6m-3l+2}$,

$$G_{6m-3l+2} = \frac{K-1}{2}\, 2G_{6m-3l+1} + (G_{6m-3l+1} + G_{6m-3l}),$$

where $0 < G_{6m-3l+1} + G_{6m-3l} < 2G_{6m-3l+1}$,

$$2G_{6m-3l+1} = 1(G_{6m-3l+1} + G_{6m-3l}) + (G_{6m-3l+1} - G_{6m-3l}),$$

where $0 < G_{6m-3l+1} - +G_{6m-3l} < G_{6m-3l+1} + G_{6m-3l}$,

$$G_{6m-3l+1} + G_{6m-3l} = 1(G_{6m-3l+1} - G_{6m-3l}) + 2G_{6m-3l},$$

where $0 < 2G_{6m-3l} < G_{6m-3l+1} - G_{6m-3l}$, and

$$G_{6m-3l+1} - G_{6m-3l} = \frac{K-1}{2}\, 2G_{6m-3l} + G_{6m-3l-1},$$

where $0 < G_{6m-3(l+1)+2} < 2G_{6m-3(l+1)+3}$. Finally we have

$$\begin{aligned} 2G_3 &= 2KG_2 + 2G_1, && \text{where } 2G_1 < G_2, \\ G_2 &= \frac{K-1}{2}\, 2G_1 + G_1 && \text{where } 1 = G_1 < 2G_1, \end{aligned}$$

and

$$2G_1 = 2\,G_1 + 0,$$

which completes the derivation of the continued fraction expansion of $G_n/(G_{n+1} + 1)$. □

3. Linear Diophantine Equations

In order to prove Theorem 1 we will use the following Lemmas.

Lemma 2. *Let p, q be positive coprime integers and*

$$\frac{p}{q} = [a_0, a_1, a_2, \dots, a_N] \qquad (a_N > 1), \tag{3.1}$$

the continued fraction expansion of p/q, and

$$\frac{p_n}{q_n} = [a_0, a_1, a_2, \dots, a_n] \qquad (n \le N) \tag{3.2}$$

the principal convergents of p/q. Then we have for $n < N$

$$\frac{q}{2q_{n+1}} < |pq_n - qp_n| = \min_{q_n \le |x| < q_{n+1}, y \in \mathbf{Z}} |px + qy| < \frac{q}{q_{n+1}}. \tag{3.3}$$

Proof. (3.3) is a simple combination of Theorems 5 and 6 in Chapter I of [3]. □

Lemma 3. *Let p, q, p_n, q_n be defined as in Lemma 2. Then we have*

$$\frac{q_{N-1}}{q} = [0, a_N, a_{N-1}, \dots, a_1] \tag{3.4}$$

and

$$\frac{p_{N-1}}{p} = \begin{cases} [0, a_N, a_{N-1}, \dots, a_1, a_0] & \text{for} \quad a_0 \neq 0, \\ [0, a_N, a_{N-1}, \dots, a_2] & \text{for} \quad a_0 = 0. \end{cases} \tag{3.5}$$

Furthermore if we define P_n, Q_n by

$$\frac{P_n}{Q_n} = [0, a_N, a_{N-1}, \dots, a_{N-n+1}] \tag{3.6}$$

or recursively by $P_{-1} = 1$, $P_0 = 0$, $P_{n+1} = a_{N-n}P_n + P_{n-1}$, $Q_{-1} = 0$, $Q_0 = 1$, $Q_{n+1} = a_{N-n}Q_n + Q_{n-1}$ then we have

$$q = Q_N, \quad p = \begin{cases} Q_{N+1} & \text{for} \quad a_0 \neq 0, \\ Q_{N-1} & \text{for} \quad a_0 = 0 \end{cases} \tag{3.7}$$

and

$$p_{N-k}Q_k + p_{N-k-1}Q_{k-1} = p \tag{3.8}$$

for $k \geq 0$.

Proof. The continued fraction expansions (3.4) and (3.5) are classical. They can be found in [2, p 27] or in [3] as a consequences of Theorem 4 in Chapter I. Hence, (3.7) follows.

Equation (3.8) is surely true for $k = 0$ by definition. Now assume that (3.8) is true for some $k \geq 0$. Hence we get

$$\begin{aligned} &p_{N-(k+1)}Q_{k+1} + p_{N-(k+1)-1}Q_{(k+1)-1} \\ &\quad = p_{N-k-1}(a_{N-k}Q_k + Q_{k-1}) + p_{N-k-2}Q_k \\ &\quad = (a_{N-k}p_{N-k-1} + p_{N-k-2})Q_k + p_{N-k-1}Q_{k-1} \\ &\quad = p_{N-k}Q_k + p_{N-k-1}Q_{k-1}, \end{aligned}$$

which proves (3.8). $\square$

Proof of Theorem 1. First, assume that one component of a solution x, y, z is zero. Of course, a non-zero solution must contain at least two non-zero components. So, if $x = 0$ then y, z must be of the form $y = m(b+1)$, $y = -mb$, where m is any non-zero integer. Thus $\overline{x}\,\overline{y} \geq b+1$, $\overline{x}\,\overline{z} \geq b$, and $\overline{y}\,\overline{z} \geq b(b+1) > b^2 \geq a$. The cases $y = 0$ and $z = 0$ can be treated similarly.

Since $a + ab - a(b+1) = 0$ we have a special solution $x_0 = 1$, $y_0 = a$, $z_0 = -a$, and all solutions can be described explicitly by

$$\begin{aligned} x &= k, \\ y &= ka + m(b+1), \\ z &= -ka - mb, \end{aligned} \tag{3.9}$$

where $k, m \in \mathbf{Z}$ are arbitrary integers. First we will discuss the product $\overline{x}\,\overline{z}$. Recall that a/b has the continued fraction expansion (1.4) which partial quotients a_i are bounded by L. Furthermore let p_n/q_n denote the principal convergents

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n].$$

The case $x = 0$ is already settled and if $|x| \ge b$, then we get the trivial bound $\overline{x}\,\overline{z} \ge b$. Now if $1 \le |x| < b$, there is some n with $0 \le n < N$ such that $q_n \le |x| < q_{n+1}$. Hence we get by (3.3)

$$|xz| \ge q_n \frac{b}{2q_{n+1}} > \frac{b}{2(L+1)}$$

since $q_{n+1} = a_{n+1}q_n + q_{n-1} < (L+1)q_n$. Therefore we have in all possible cases

$$\overline{x}\,\overline{z} > \frac{b}{2(L+1)}.$$

Similarly we get

$$\overline{x}\,\overline{y} > \frac{b+1}{2(L'+1)}.$$

The most difficult case is the product $\overline{y}\,\overline{z}$. Here we have to use two different representations of all solutions. Since $aq_{N-1} - bp_{N-1} = (-1)^{N-1}$ a special solution is given by $x_0 = q_{N-1}$, $y_0 = -(p_{N-1} + (-1)^N)$, $z_0 = (-1)^N$, and so all solutions can be stated as

$$\begin{aligned} x &= kq_{N-1} + mb, \\ y &= -k(p_{N-1} + (-1)^N) - ma, \\ z &= (-1)^N k, \end{aligned} \tag{3.10}$$

where $k, m \in \mathbf{Z}$ are arbitrary integers. On the other hand let

$$\frac{p'_n}{q'_n} = [a'_0, a'_1, \ldots, a'_n]$$

denote the principal convergents of $a/(b+1)$. Hence we get a similar representation of the kind

$$\begin{aligned} x &= kq'_{M-1} + m(b+1), \\ y &= (-1)^{M+1} k, \\ z &= -k(p'_{M-1} + (-1)^M) - ma. \end{aligned} \tag{3.11}$$

If

$$|y| > \frac{a^{1/2}}{(L'+1)^2(L'+2)^{1/2}} \quad \text{and} \quad |z| > \frac{a^{1/2}}{(L+1)^2(L+2)^{1/2}} \tag{3.12}$$

then we are done. So we only have to discuss where one of the inequalities in (3.12) is not satisfied. First, assume that

$$|z| \le \frac{a^{1/2}}{(L+1)^2(L+2)^{1/2}}.$$

In order to get a lower bound for $\overline{y}\,\overline{z}$ in this case we need some information about the partial quotients b_i of $(p_{N-1}+(-1)^N)/a = [0, b_1, \ldots, b_{N_1}]$. It is well known that the principal convergents of a real number are the best rational approximation in the class of all rationals with bounded denominator. Therefore the first principal convergents and consequently the first partial quotients of $(p_{N-1}+(-1)^N)/a$ and p_{N-1}/a are the same. This means that $b_i = a_N - i + 1$ for $1 \le i \le N_2$. The first problem will be to specify N_2.

Let P_n, Q_n be defined by

$$\frac{P_n}{Q_n} = [0, a_N, a_{N-1}, \ldots, a_{N-n-1}],$$

this means that Q_n satisfies the recurrence relation $Q_{n+1} = a_{N-n}Q_n + Q_{n-1}$ and the initial conditions $Q_{-1} = 0$ and $Q_0 = 1$. Furthermore we have $Q_N = b$ and $Q_{N-1} = a$ for $a_0 = 0$ or $Q_{N+1} = a$ for $a_0 > 0$. Now set

$$r_n = p_n + (-1)^{n+1} Q_{N-n-1}.$$

Then r_n satisfies the recurrence relation $r_{n+1} = a_{n+1}r_n + r_{n-1}$ with the initial conditions $r_{-1} = b+1$ and $r_0 = -a$ for $a_0 = 0$ or $r_{-2} = -a$ for $a_0 > 0$. Furthermore we have $r_N = a$ and $r_{N-1} = p_{N-1} + (-1)^N$. Since $(r_{m-1}, r_m) = (r_m, r_{m+1})$ it follows that

$$(a, p_{N-1} + (-1)^N) = (a, b+1) = 1.$$

So a and $p_{N-1} + (-1)^N$ are coprime. Now assume that $r_{N-n-2} < r_{N-n-1}$ for $1 \le n \le N_2$. Hence by the Euclidean algorithm the recurrence relation $r_{n+1} = a_{n+1}r_n + r_{n-1}$, where $r_N = a$, and $r_{N-1} = p_{N-1} + (-1)^N$, implies that the partial quotients b_n of $a/(p_{N-1} + (-1)^N)$ are given by $b_n = a_{N-n+1}$ for $1 \le n \le N_2$. In the following we will prove that

$$N_2 = \max\left\{ n \,:\, Q_{n+2} \le \left(\frac{a}{K+2}\right)^{1/2} \right\}$$

is a proper choice such that $r_{N-n-2} < r_{N-n-1}$ for $1 \le n \le N_2$ by showing that $p_{N-N_2-n} + Q_{N_2+1} < p_{N-N_2-1} - Q_{N_2}$. Since $Q_{N_2+1} + Q_{N_2} \le Q_{N_2+2}$ and $p_{N-N_2-1} - p_{N-N_2-2} \ge p_{N-N_2-3}$ it suffices to check $Q_{N_2+2} \le p_{N-N_2-3}$. By (3.8) we have

$$\begin{aligned} a &= p_{N-k}Q_k + p_{N-k-1}Q_{k-1} \\ &< p_{N-k}(Q_k + Q_{k-1}) \\ &< p_{N-k}(K+2)Q_{k-1} \end{aligned}$$

and therefore

$$p_{N-N_2-3} > \frac{a}{K+2}\, Q_{N_2+2}^{-1} \ge \left(\frac{a}{K+2}\right)^{1/2} \ge Q_{N_2+2}.$$

As above, we get

$$|yz| > Q_n \frac{a}{2Q_{n+1}}$$

for $Q_n \le |z| < Q_{n+1}$ and $n \le N_2$. Thus

$$\min_{1\le|z|<Q_{N_2+1}} |zy| > \frac{a}{2(L+1)}.$$

From

$$Q_{N_2+2} \le \left(\frac{a}{K+2}\right)^{1/2} < Q_{N_2+3} < (L+1)^2 Q_{N_2+1}$$

we get

$$Q_{N_2+1} > \frac{a^{1/2}}{(L+1)^2(L+2)^{1/2}}.$$

Hence, if

$$1 \le |z| \le \frac{a^{1/2}}{(L+1)^2(L+2)^{1/2}}$$

then we have $1 \le |z| < Q_{N_2+1}$ and therefore

$$\overline{y}\,\overline{z} > \frac{a}{2(L+1)} \ge \frac{a}{(L+1)^2(L'+1)^2((L+2)(L'+2))^{1/2}},$$

as proposed. Of course, the case

$$1 \le |y| \le \frac{a^{1/2}}{(L'+1)^2(L'+2)^{1/2}}$$

can be treated similarly. $\square$

Theorem 2 is a generalization of [1, Theorem 1]. So we can proceed along similar lines.

Proof of Theorem 2. Define linear forms $L_i(x)$ $(1 \le i \le k)$ by

$$\begin{aligned} L_i(x) &= x_i \quad \text{for} \quad 1 \le i \le k-1, \\ L_k(x) &= a \cdot x = a_1x_1 + \cdots + a_kx_k \end{aligned} \tag{3.13}$$

with determinant $\Delta = a_k \ne 0$. Let $r(a)$ be an abbreviation for

$$r(a) = \prod_{i=1}^{k} |a_i|.$$

Then by Minkowski's theorem for linear forms there is a non-zero intergral point $x = (x_1, \dots, x_k)$ with

$$\begin{aligned} &|L_i(x)| \le \frac{r(a)^{\frac{1}{k-1}}}{|a_i|} \quad \text{for} \quad 1 \le i \le k-1, \\ &|L_k(x)| < 1, \end{aligned} \tag{3.14}$$

because

$$\prod_{i=1}^{k-1} \frac{r(a)^{\frac{1}{k-1}}}{|a_i|} = |a_k| = |\Delta|.$$

But $L_k(x)$ is an integer. So x solves the linear diophantine equation $a \cdot x = 0$ and we can estimate x_k by

$$|x_k| \leq \frac{1}{|a_k|} \sum_{i=1}^{k-1} |a_i x_i| \leq (k-1) \cdot \frac{r(a)^{\frac{1}{k-1}}}{|a_k|}. \tag{3.15}$$

Assume for a moment that $r(a)^{\frac{1}{k-1}}/|a_i| \geq 1$ for $1 \leq i \leq k-1$ and that $(k-1) \cdot r(a)^{\frac{1}{k-1}}/|a_k| \geq 1$. Then we trivially get

$$\prod_{\substack{1 \leq i \leq k \\ i \neq j}} \overline{x_i} \leq (k-1) \prod_{i \neq j} \frac{r(a)^{\frac{1}{k-1}}}{|a_i|} = (k-1)\, |a_j|$$

for $j = 1, \dots, k-1$ and similarly

$$\prod_{i \neq k} \overline{x_i} \leq |a_k|.$$

In general, let $i \in I$ if $r(a)^{\frac{1}{k-1}} < |a_i|$ for $1 \leq i \leq k-1$ and $k \in I$ if $(k-1) \cdot r(a)^{\frac{1}{k-1}} < |a_k|$.

For $k = 2$ we always have $I = \emptyset$ and so (1.8) holds. Now assume that $k > 2$ and that (1.8) is true for all diophantine equations with $< k$ variables. The case $I = \emptyset$ has already been discussed. Therefore we only have to consider the case $I \neq \emptyset$. First, let $j \notin K$. Note that we always have $k - |I| \geq 2$. Hence, we obtain by induction that there is a non-zero solution with $x_i = 0$ for $i \in I$ and

$$\prod_{i \neq j} \overline{x_i} = \prod_{\substack{i \notin I \\ i \neq j}} \overline{x_i} \leq \begin{cases} (k - |I| - 1)\, |a_j| & \text{for} \quad j = 1, \dots, k-1, \\ |a_j| & \text{for} \quad j = k, \end{cases}$$

which proves (1.8) in this case. Finally, let $j \in I$. As above we get by induction a non-zero solution with $x_i = 0$ for $i \in I$ and

$$\prod_{i \neq j} \overline{x_i} = \prod_{i \notin I} \overline{x_i} \leq (k - |I| - 1) \prod_{i \notin I} |a_i|^{\frac{1}{k - |I| - 1}}.$$

Since $a_i > r(a)^{\frac{1}{k-1}}$ for $i \in I$ we get

$$\begin{aligned} \prod_{i \neq j} \overline{x_i} &\leq (k-1) \prod_{i=1}^{k} |a_i|^{\frac{1}{k-|I|-1}} r(a)^{-\frac{|I|}{(k-1)(k-|I|-1)}} \\ &= (k-1) r(a)^{\frac{1}{k-1}}. \end{aligned}$$

For $1 \leq j < k$ we can use the bound $r(a)^{\frac{1}{k-1}} < |a_j|$ and for $j = k$ the bound $r(a)^{\frac{1}{k-1}} < |a_k|/(k-1)$ to obtain (1.8) in this concluding case. $\square$

References

[1] Drmota, M., *"On Linear Diophantine Equations and Fibonacci Numbers"*, J. Number Th., to appear.
[2] Perron, 0., **Die Lehre von den Kettenbrüchen**, Bd. I: Elementare Kettenbrüche, Teubner, Stuttgart, 1954.
[3] Lang, S., **Introduction to Diophantine Approximations**, Addison-Wesley, Reading, 1966.

Michael Drmota
Department of Discrete Mathematics
Technical University of Vienna
Wiedner Hauptstrasse 8–10/118
A-1040 Vienna, AUSTRIA