The Parity of the Sum-of-Digits-Function of Generalized Zeckendorf Representations

Michael Drmota and Johannes Gajdosik[1]

Department of Discrete Mathematics
Technical University of Vienna
Wiedner Hauptstraße 8–10, A-1040 Vienna

1 Introduction

Let $G = (G_n)$ be a strictly increasing sequence of positive integers with $G_1 = 1$. Then every non-negative integer n has a digital expansion

$$n = \sum_{i\ge 1} \epsilon_i G_i$$

with respect to basis G, where the digits $\epsilon_i = \epsilon_i(n) \ge 0$ are integers. This digital expansion is unique, when one assumes that the digits ϵ_i are chosen in a way that the digital sum $\sum_{i\ge 1} \epsilon_i$ is as small as possible, in this case we will call the digital expansion a proper digital expansion. It is easy to see that the following algorithm provides this expansion.

1. For $n = 0$ we have $\epsilon_i(n) = 0$ for every $i \ge 1$.
2. If $G_j \le n < G_{j+1}$ and $n' = n - G_j$ has the proper expansion $n' = \sum_{i\ge 1} \epsilon_i' G_i$, then the expansion of $n = \sum_{i\ge 1} \epsilon_i G_i$ is given by $\epsilon_i = \epsilon_i'$ for $i \ne j$ and by $\epsilon_j = \epsilon_j' + 1$.

The most prominent digital expansions are related to linear recurring sequences $G = (G_n)$, e.g. the binary (resp. the q-ary) expansion relies on $G_n = 2^{n-1}$ (resp. on $G_n = q^{n-1}$). If G_n are the Fibonacci numbers, i.e. $G_n = F_{n+1}$, then we obtain the Zeckendorf expansion.

For each digital expansion with respect to a basis G we can define a partial order in a quite natural way. We will say $a \le_G b$ if and only if $\epsilon_i(a) \le \epsilon_i(b)$ for every $i \ge 1$. It is well known that for every partial order there is a Möbius function (see [10, 13]). Let $s_G(n)$ denote the sum

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of digits of n. Then it will turn out that the Möbius function μ_G of a digital expansion to a basis G is given by $\mu_G(n) = (-1)^{s_G(n)}$ if $\max_{i\geq 1} \epsilon_i(n) \leq 1$ and $\mu_G(n) = 0$ otherwise.

If G is a proper linear recurring sequence and if the initial conditions of G are properly chosen (see section 3), then

$$M_G(N) := \sum_{n=0}^{N-1} \mu_G(n)$$

is either bounded or

$$M_G(N) = S_G(N) := \sum_{n=0}^{N-1} (-1)^{s_G(n)},$$

which we will see from calculating the Möbius function in section 2. (We always define empty sums to be zero, i.e. $M_G(N) = S_G(N) := 0$ for $N \leq 0$.)

In section 3 we will formulate conditions for G, under which we will be able to derive formulas for $S_G(N)$. We will also get a recursive formula for the generating function of $S_G(G_n)$, which we will analyse in section 4 in order to get asymptotic information about $S_G(N)$.

Our main interrest lies in the distribution of the $S_G(N)$ (resp. $M_G(N)$) when $0 \leq N < m$ for large m. This means we count the number of times $S_G(N)$ takes a certain value k when $0 \leq N < m$: let $d_m(k) := |\{0 \leq N < m : S_G(N) = k\}|$ be this number and let X_m be a random variable with probability distribution $\mathbf{P}(X_m = k) = d_m(k)/m$. Then we are interested in the asymptoctic distribution of X_m for $m \to \infty$. Depending on the nature of the recurrence relation for G we will observe significantly different behaviour of X_m. First of all we distinguish two cases:

1. Either $S_G(G_n)$ is bounded for all initial conditions of G (section 4.1),

2. or there are initial conditions of G, such that $S_G(G_n)$ is unbounded (section 4.2).

Since we can establish a linear recurrence relation for the $S_G(G_n)$, the first case is equivalent to the assumtion, that the characteristic polynomial of this recursion is a product of some z^{r-v} $(r - v \geq 0)$ and certain different cyclotomic polynomials. In this case we can derive asymptotic formulas for $\mathbf{E}X_m$ and $\mathbf{V}X_m$, provided that the sequence G satisfies a certain technical condition. Our main result (Theorem 2) says that in the case of unbounded variance X_m satisfies a central limit theorem. (Note that there are sequences G, for which $\mathbf{V}X_m$ is bounded, e.g. $G_n = 2^{n-1}$.)

2 The Möbius function of a digital expansion

Let $G = (G_n)$ be a strictly increasing sequence of integers with $G_1 = 1$. As mentioned above, every non-negative integer n has a digital expansion $n = \sum_{i\ge 1} \epsilon_i G_i$ with non-negative integral digits ϵ_i. It is called proper digital expansion for n if the digital sum $\sum_{i\ge 1} \epsilon_i$ is as small as possible.

Lemma 1 *Let $n = \sum_{i\ge 1} \epsilon_i G_i$ be a proper digital expansion for n. Then any sum of the form $\sum_{i\ge 1} \epsilon_i' G_i$ with integral digits ϵ_i', $i \ge 1$, satisfying $0 \le \epsilon_i' \le \epsilon_i$ is a proper digital representation for some $n' \le n$.*

Proof. First, note that it follows form the algorithm stated in the Introduction that any digital expansion of the form $n_j = \sum_{i=1}^{j} \epsilon_i G_i \le n$ is a proper one.

Next, we will use induction on the digital sum $s' = \sum_{i\ge 1} \epsilon_i'$, where $0 \le \epsilon_i' \le \epsilon_i$. Obviously there is nothing to show if $s' = 0$.

Now suppose that $n' = \sum_{i\ge 1} \epsilon_i' G_i$ has digital sum s'. There exists $j \ge 1$ such that $\epsilon_j' > 0$ and $\epsilon_i' = 0$ for $i > j$. Then $G_j \le n' \le n_j < G_{j+1}$. Hence $n'' = n' - G_j$ can be represented by $n'' = \sum_{i=1}^{j} \epsilon_i'' G_i$ with $\epsilon_j'' = \epsilon_j' - 1$ and $\epsilon_i'' = \epsilon_i'$ for $i \ne j$. Since $0 \le \epsilon_i'' \le \epsilon_i$ and its digital sum satisfies $\sum_{i\ge 1} \epsilon_i'' = s' - 1 < s'$ this expansion for n'' is proper. Consequently $\sum_{i\ge 1} \epsilon_i' G_i$ is a proper expansion for n'. $\square$

Now we introduce the Möbius functions $\mu(x, y)$ of a locally finite partial order $\le$ on a set X, i.e. all intervals $[x, y] = \{u \in X : x \le u \le y\}$ are finite (see [10, 13]). Any function $f : X^2 \to \mathbf{C}$ which satisfies $f(x, y) = 0$ for $x \not\le y$ will be called arithmetical function. The convolution $f * g$ of two arithmetical functions f, g is given by

$$(f * g)(x, y) = \sum_{x\le u\le y} f(x, u)g(u, y).$$

Obviously δ, defined by $\delta(x, y) = 1$ for $x = y$ and $\delta(x, y) = 0$ otherwise, is the unit element of $*$. Furthermore, if $f(x, x) \ne 0$ for every $x \in X$ then there always exists an inverse arithmetical functions f^{-1} satisfying $f^{-1} * f = \delta$. The Möbius function μ is defined as the inverse function of ζ given by $\zeta(x, y) = 1$ if $x \le y$ and $\zeta(x, y) = 0$ otherwise. Especially, if $g = \zeta * f$ then f can be recovered by $f = \mu * g$. (It is intended to use this Möbius function in future work for sieve methods in connection with specific problems of digital expansions.)

Theorem 1 *Let $\leq_G$ be the partial order on the non-negative integers induced by the digital expansion with respect to a stricly increasing sequence of integers $G = (G_n)$ and suppose that $m = \sum_{i\geq 1} \epsilon_i' G_i$ and $n = \sum_{i\geq 1} \epsilon_i'' G_i$ are proper digital expansions of non-negative integers m, n with $m \leq_G n$, i.e. $\epsilon_i' \leq \epsilon_i''$ for all i. Then*

$$\mu(m,n) = \begin{cases} 0 & \text{if there is an } i \text{ with } \epsilon_i'' - \epsilon_i' > 1 \\ (-1)^{\sum_{i\geq 1}(\epsilon_i''-\epsilon_i')} & \text{otherwise} \end{cases} .$$

Proof. Since there is a natural bijection between $[m,n] = \{d \in \mathbf{N}_0 \mid m \leq_G d \leq_G n\}$ and $[0, n-m]$, we have $\mu(m,n) = \mu(0, n-m)$ if $m \leq_G n$ then. (For $m \not\leq_G n$ we have $\mu(m,n) = 0$.)

Therefore we will only calculate $\mu(0,n)$. From the definition of $\mu(x,y)$ it is clear that $\mu(0,0) = 1$ and that

$$\sum_{0 \leq_G d \leq_G n} \mu(0,d) = 0 \qquad \text{for } n > 0 .$$

Assume for a moment that $\epsilon_i'' \leq 1$ for all i. We show that $\mu(0, \sum_{j=0}^{k-1} G_{i_j}) = (-1)^k$ by induction on the digital sum $s = k$. Clearly, for $s = 0$ we have $\mu(0,0) = 1 = (-1)^0$. Now assume that $s \geq 1$ and that $\mu(0, \sum_{j=0}^{k-1} G_{i_j}) = (-1)^k$ for all $k < s$. Then

$$\begin{aligned} 0 &= \sum_{0 \leq_G d \leq_G \sum_{j=0}^{s-1} G_{i_j}} \mu(0,d) \\ &= \Big(\mu(0,0)\Big) + \Big(\mu(0, G_{i_0}) + \mu(0, G_{i_1}) + \ldots + \mu(0, G_{i_{s-1}})\Big) \\ &\quad + \Big(\mu(0, G_{i_0} + G_{i_1}) + \mu(0, G_{i_0} + G_{i_2}) + \ldots + \mu(0, G_{i_{s-2}} + G_{i_{s-1}})\Big) + \ldots + \left(\mu\left(0, \sum_{j=0}^{s-1} G_{i_j}\right)\right) \\ &= 1 + \binom{s}{1}(-1)^1 + \binom{s}{2}(-1)^2 + \ldots + \binom{s}{s-1}(-1)^{s-1} + \mu\left(0, \sum_{j=0}^{s-1} G_{i_j}\right) . \end{aligned}$$

Because of $\sum_{j=0}^{s} \binom{s}{j}(-1)^j = 0$ it follows that $\mu(0, \sum_{j=0}^{s-1} G_{i_j}) = (-1)^s$ which proves the theorem in this special case.

Now suppose kG_i with $i \geq 1$ and $k > 1$ is a proper digital expansion. Then $0 = \mu(0,0) + \mu(0, G_i) + \ldots + \mu(0, kG_i)$. Notice that $\mu(0,0) + \mu(0, G_i) = 0$. Hence it follows that $\mu(0, 2G_i) = \mu(0, 3G_i) = \ldots = \mu(0, kG_i) = 0$.

Next we show by induction on the digitial sum $s(n) = \sum_{i\geq 1} \epsilon_i''$ that $\mu(0,n) = 0$ whenever there is an i with $\epsilon_i'' > 1$. We have to start with $s(n) = 2$ because $\epsilon_i'' > 1$ cannot be satisfied when $s(n) < 2$. Suppose that $s(n) = 2$ and there is some i with $\epsilon_i'' > 1$. Then $m = 2G_i$ and

$\mu(0,m)=0$. Now assume the assertion holds for all natural numbers l with $s(l)<s(n)$ and assume that there is a j with $\epsilon_j''>1$. Then

$$\begin{aligned}-\mu(0,n) &= \sum_{0\leq_G d<_G n}\mu(0,d)=\sum_{0\leq_G d<_G n,\forall i:\epsilon_i(d)\leq 1}\mu(0,d)+\sum_{0\leq_G d<_G n,\exists i:\epsilon_i(d)>1}\mu(0,d)\\ &= \sum_{0\leq_G d<_G n,\forall i:\epsilon_i(d)\leq 1}\mu(0,d).\end{aligned}$$

Define $n_1:=\sum_{i\geq 1}\min(\epsilon_i'',1)G_i$. Because of the existence of j with $\epsilon_j''>1$ we have $0<n_1<n$ and

$$\sum_{0\leq_G d<_G n,\forall i:\epsilon_i(d)\leq 1}\mu(0,d)=\sum_{0\leq_G d\leq_G n_1}\mu(0,d).$$

The righthand side is of course zero due to (2), which completes our proof. □

Since $\mu(m,n)=\mu(0,n-m)$ (if $m\leq_G n$) it is sufficient to consider the restricted Möbius function $\mu_G(n)=\mu_G(0,n)$. As mentioned above the main topic of this paper is to discuss the partial sums

$$M_G(N)=\sum_{n=0}^{N-1}\mu_G(n).$$

Nevertheless we will rather discuss the partial sums $S_G(N)$ (see (1)), which will be motivated by the following

Proposition 1 *Suppose that $G_n\geq 2G_{n-1}$ for all $n>1$. Then $M_G(N)$ is bounded by 1.*

On the other hand, if $G_n\leq 2G_{n-1}$ for all $n>1$, then

$$M_G(N)=S_G(N):=\sum_{n=0}^{N-1}(-1)^{s_G(n)},$$

where $s_G(n)$ denotes the digital sum $s_G(n)=\sum_{i\geq 1}\epsilon_i$ of the proper digital representation $n=\sum_{i\geq 1}\epsilon_i G_i$.

Proof. Due to Theorem 1 only those n with expansion coefficients 0 or 1 enter the sum. If $G_n\geq 2G_{n-1}$ for all $n>1$ then all digital expansions $\sum_{i\geq 1}\epsilon_i G_i$ with $\epsilon_i\in\{0,1\}$ are proper ones. Hence $M_G(N)$ attains only the same values as in the binary case where the corresponding sum is 0 or ± 1.

If $G_n\leq 2G_{n-1}$ for all $n>1$ then in all proper digital expansions only the digits 0 and 1 can occur and the assertion follows from Theorem 1 with $m=0$. □

Remark 1. We will see later, that for all G that we consider, $(a_1+1)G_{n-1}\geq G_n\geq a_1 G_{n-1}$ holds for $n>r$, and therefore $G_n\leq 2G_{n-1}$ for all $n>1$ is equivalent to $a_1=2$ and $r=1$

or $a_1 = 1$, when the initial conditions of G are properly chosen. But if $a_1 > 2$ or $a_1 = 2$ and $r > 1$, and if $G_n \geq 2G_{n-1}$ holds for the initial values, then Proposition 1 applies and $M_G(N)$ is bounded. Because of this we will investigate the function $S_G(N)$ rather than $M_G(N)$, keeping in mind, that in most cases, when $M_G(N)$ is of interest, both are the same.

Remark 2. If $G_n = 2^{n-1}$ then $t_n = (-1)^{s_G(n)}$ is the Thue-Morse sequence [11]. Since $t_{2n} + t_{2n+1} = 0$ we have $S_G(2n+1) = t_{2n} = t_n$, and we also have $S_G(2n) = 0$. Thus it is not really interesting to study $S_G(N)$ in this case.

3 Digital expansions and generating functions

From now on we will just consider integral linear recurring sequences $G = (G_n)_{n \geq 1}$ which satisfy assumptions 1.–5. (in section 4.1 we will also need assumption 6.):

1. $G_1 = 1$ and $G_{n+1} > G_n$ for $n \geq 1$.

2. $G_n = \sum_{i=1}^{r} a_i G_{n-i}$ for $n > r$ with some integers $a_i \geq 0$.

3. $G_{n-j} \geq \sum_{i=j+1}^{r} a_i G_{n-i}$ for $n > r$ and $1 \leq j < r$.

4. G satisfies no linear recursion with constant integer coefficients with a smaller degree.

5. The characteristic polynomial $z^r - \displaystyle\sum_{i=1}^{r} a_i z^{r-i} = \prod_{i=1}^{r}(z - \alpha_i)$ (of the above recursion) has only one real, positive, and simple root α_1 of maximal modulus.

6. Let $b_i = (a_i \bmod 2)(-1)^{a_1 + \cdots + a_{i-1}}$ ($a_i \bmod 2 = 0$ if a_i is even and $a_i \bmod 2 = 1$ otherwise). Then

$$z^r - \sum_{i=1}^{r} b_i z^{r-i} \quad = \quad z^{r-v} \prod_{h=1}^{k'} \Phi_{k_h}(z) \tag{1}$$

   is a product of z^{r-v} ($r - v \geq 0$) and different cyclotomic polynomials $\Phi_{k_h}(z)$ ($k_1 < k_2 < \ldots < k_{k'}$), all of them dividing $z^p - 1$ with some fixed $p > r$. Furthermore none of the α_i and no quotient α_i/α_j ($i \neq j$) is a p-th root of unity.

Assumptions 1., 2., and 4. are natural. Therefore only the conditions 3., 5., and 6. need to be motivated.

Assumption 3. is necessary to show that $S(G_n)$ satisfies a linear recurrence, especially it impies (6) in Proposition 2.

From 5. we obtain $G_n = \beta\alpha_1^{n-1} + O((\alpha_1\gamma)^n)$ with some $\beta > 0$ and $0 \le \gamma < 1$. Note that 2. and 3. imply $(a_1+1)G_{n-1} \ge G_n \ge a_1 G_{n-1}$ for $n > r$, which gives $a_1 \le \alpha_1 \le a_1 + 1$. Similarily we get $a_1 \ge a_i$ for all i.

The first part of assumption 6. (concerning the cyclotomic factors) ensures that $S(G_n)$ is bounded. The assumption that α_i and α_i/α_j are no p-th roots of unity is frequently used in problems concerning digital expansions with respect to linear recurring sequences and avoids technical difficulties (see Lemma 2).

Usually assumptions 3. and 5. are replaced by the stronger condition $a_1 \ge a_2 \ge \ldots \ge a_r$ and certain assumptions on the initial values of G. (see e.g. [8], in this case the second part of 6. is also satisfied). However, there are other interesting examples, e.g. $a_1 = a_r = 1$, $a_2 = \ldots = a_{r-1} = 0$, which satisfy the above assumptions and are not of the form $a_1 \ge a_2 \ge \ldots \ge a_r$.

From now on let $G = (G_n)$ be a fixed linear recurring sequence with assumptions 1.–5. For notational convenience we will omit the index G in the sequel.

Proposition 2 *Let $b_i = (a_i \bmod 2)(-1)^{a_1+\cdots+a_{i-1}}$ ($a_i \bmod 2 = 0$ if a_i is even and $a_i \bmod 2 = 1$ otherwise). Then $S(G_n) = S_G(G_n)$ satisfies the linear recurrence*

$$S(G_n) = \sum_{i=1}^{r} b_i S(G_{n-i}) \qquad \textit{for } n > r. \tag{2}$$

Furthermore, if n has the proper digital expansion $n = \sum_{j=1}^{l} \epsilon_j G_j$ then

$$S\left(\sum_{j=1}^{l} \epsilon_j G_j\right) = \sum_{j=1}^{l} (\epsilon_j \bmod 2)(-1)^{\epsilon_{j+1}+\cdots\epsilon_l} S(G_j). \tag{3}$$

Proof. We will first establish a set identity that holds for all non-negative integers ϵ_j, regardless whether $\sum_{j=1}^{l} \epsilon_j G_j$ is a proper digital expansion or not:

$$\begin{aligned}
\left\{ a \,\middle|\, 0 \le a < \sum_{j=1}^{l} \epsilon_j G_j \right\} &= \bigcup_{j=1}^{l} \left\{ a \,\middle|\, \sum_{h=j+1}^{l} \epsilon_h G_h \le a < \sum_{h=j}^{l} \epsilon_h G_h \right\} \\
= \bigcup_{j=1}^{l} \left\{ \sum_{h=j+1}^{l} \epsilon_h G_h + a \,\middle|\, 0 \le a < \epsilon_j G_j \right\} &= \bigcup_{j=1}^{l} \bigcup_{i=0}^{\epsilon_j - 1} \left\{ \sum_{h=j+1}^{l} \epsilon_h G_h + a \,\middle|\, iG_j \le a < (i+1)G_j \right\} \\
= \bigcup_{j=1}^{l} \bigcup_{i=0}^{\epsilon_j - 1} \left\{ \left(\sum_{h=j+1}^{l} \epsilon_h G_h \right) + iG_j + a \,\middle|\, 0 \le a < G_j \right\} & \qquad (4)
\end{aligned}$$

where each union is disjoint. (Again empty sums are set zero.)

Now set $l = n-1$, $\epsilon_j = a_{n-j}$ for $n-r \le j < n$ and $\epsilon_j = 0$ otherwise. Then one obtains for $n > r$ after interchanging i and j and shifting $i \to n-i$, $h \to n-h$

$$\left\{ a \,\middle|\, 0 \le a < \sum_{i=n-r}^{n-1} a_{n-i} G_i \right\} = \bigcup_{i=1}^{r} \bigcup_{j=0}^{a_i-1} \left\{ \left(\sum_{h=1}^{i-1} a_h G_{n-h} \right) + jG_{n-i} + a \,\middle|\, 0 \le a < G_{n-i} \right\}. \quad (5)$$

From this we see that for $n > r$

$$\begin{aligned} S(G_n) &= \sum_{a=0}^{G_n-1} (-1)^{s(a)} = \sum_{i=1}^{r} \sum_{j=0}^{a_i-1} \sum_{a=0}^{G_{n-i}-1} (-1)^{s\left(\sum_{h=1}^{i-1} a_h G_{n-h} + jG_{n-i} + a\right)} \\ &= \sum_{i=1}^{r} \sum_{j=0}^{a_i-1} \sum_{a=0}^{G_{n-i}-1} (-1)^{\left(\sum_{h=1}^{i-1} a_h + j + s(a)\right)} = \sum_{i=1}^{r} (-1)^{\left(\sum_{h=1}^{i-1} a_h\right)} S(G_{n-i}) \sum_{j=0}^{a_i-1} (-1)^j \\ &= \sum_{i=1}^{r} (a_i \bmod 2)(-1)^{\left(\sum_{h=1}^{i-1} a_h\right)} S(G_{n-i}) = \sum_{i=1}^{r} b_i S(G_{n-i}) \end{aligned}$$

with $b_i := (a_i \bmod 2)(-1)^{a_1 + \dots + a_{i-1}}$. Note that assumption 3. from above ensures that

$$s\left(\sum_{h=1}^{i-1} a_h G_{n-h} + jG_{n-i} + a \right) = \sum_{h=1}^{i-1} a_h + j + s(a). \quad (6)$$

You only have to start with $m = \sum_{h=1}^{i-1} a_h G_{n-h} + jG_{n-i} + a$ and to apply the algorithms stated in the Introduction to deduce that $\epsilon_{n-h}(m) = a_h$, $1 \le h < i$ and $\epsilon_{n-i}(m) = j$. (Of course, this procedure is standard in the study of such digital sequences, compare with [8, 9]). This proves (2).

The proof of (3) is quite similar. If we set $\sum_{j=1}^{l} \epsilon_j G_j =: m + \epsilon_l G_l$ in (4) we get

$$\{a \mid 0 \le a < m + \epsilon_l G_l\} = \bigcup_{i=0}^{\epsilon_l - 1} \{iG_l + a \mid 0 \le a < G_l\} \cup \{\epsilon_l G_l + a \mid 0 \le a < m\}$$

Let $\epsilon_l G_l + m = \sum_{j=1}^{l} \epsilon_j G_j$ be a proper digital expansion. Then it follows that

$$\begin{aligned} S(\epsilon_l G_l + m) &= \sum_{a=0}^{\epsilon_l G_l + m - 1} (-1)^{s(a)} = \sum_{i=0}^{\epsilon_l - 1} \sum_{a=0}^{G_l - 1} (-1)^{s(iG_l + a)} + \sum_{a=0}^{m-1} (-1)^{s(\epsilon_l G_l + a)} \\ &= \sum_{i=0}^{\epsilon_l - 1} (-1)^i \sum_{a=0}^{G_l - 1} (-1)^{s(a)} + (-1)^{\epsilon_l} \sum_{a=0}^{m-1} (-1)^{s(a)} = (\epsilon_l \bmod 2) S(G_l) + (-1)^{\epsilon_l} S(m) \quad (7) \end{aligned}$$

Iterated use of this equation gives (3). $\square$

Corollary *Let* $d_m(k) := |\{0 \le a < m \mid S(a) = k\}|$ *and* $D_m(z)$ *the corresponding generating function*

$$D_m(z) = \sum_{k \in \mathbf{Z}} d_m(k) z^k = \sum_{a=0}^{m-1} z^{S(a)}. \quad (8)$$

Then $D_{G_n}(z)$ (and $D_{G_n}(z^{-1})$) satisfy for $n > r$ the relation

$$D_{G_n}(z) = \sum_{i=1}^{r} \sum_{j=0}^{a_i-1} z^{\left(\sum_{h=1}^{i-1} b_h S(G_{n-h}) + (-1)^{a_1+\dots+a_{i-1}} (j \bmod 2) S(G_{n-i})\right)} D_{G_{n-i}}\left(z^{(-1)^{a_1+\dots+a_{i-1}+j}}\right). \quad (9)$$

Proof. Suppose $n > r$. An iterated use of (7) gives for $1 \le i \le r$, $j < a_i$ and $m < G_{n-i}$

$$\begin{aligned} &S(a_1 G_{n-1} + \ldots + a_{i-1} G_{n-i+1} + j G_{n-i} + m) \\ &= (a_1 \bmod 2) S(G_{n-1}) + (-1)^{a_1} (a_2 \bmod 2) S(G_{n-2}) + \ldots \\ &\quad + (-1)^{a_1+\dots+a_{i-2}} (a_{i-1} \bmod 2) S(G_{n-i+1}) + (-1)^{a_1+\dots+a_{i-1}} (j \bmod 2) S(G_{n-i}) \\ &\quad + (-1)^{a_1+\dots+a_{i-1}+j} S(m) \\ &= \sum_{h=1}^{i-1} b_h S(G_{n-h}) + (-1)^{a_1+\dots+a_{i-1}} (j \bmod 2) S(G_{n-i}) + (-1)^{a_1+\dots+a_{i-1}+j} S(m). \end{aligned}$$

Note that for $i = 1$ we just obtain $S(jG_{n-1} + m) = (j \bmod 2) S(G_{n-1}) + (-1)^j S(m)$. Hence by using (5) and (8) we get

$$\begin{aligned} D_{G_n}(z) &= \sum_{m=0}^{G_n-1} z^{S(m)} = \sum_{i=1}^{r} \sum_{j=0}^{a_i-1} \sum_{m=0}^{G_{n-i}-1} z^{S(a_1 G_{n-1} + \dots + a_{i-1} G_{n-i+1} + j G_{n-i} + m)} \\ &= \sum_{i=1}^{r} \sum_{j=0}^{a_i-1} z^{\left(\sum_{h=1}^{i-1} b_h S(G_{n-h}) + (-1)^{a_1+\dots+a_{i-1}} (j \bmod 2) S(G_{n-i})\right)} \sum_{m=0}^{G_{n-i}-1} z^{(-1)^{a_1+\dots+a_{i-1}+j} S(m)} \\ &= \sum_{i=1}^{r} \sum_{j=0}^{a_i-1} z^{\left(\sum_{h=1}^{i-1} b_h S(G_{n-h}) + (-1)^{a_1+\dots+a_{i-1}} (j \bmod 2) S(G_{n-i})\right)} D_{G_{n-i}}\left(z^{(-1)^{a_1+\dots+a_{i-1}+j}}\right). \quad \square \end{aligned}$$

4 Asymptotic analysis

We distinguish two cases: either $S(G_n)$ is bounded for all suitable initial conditions of G or not. The first case will be of special interest. It turns out that in this case the distribution of the values of $S(N)$ approximates a normal distribution for all suitable initial conditions of G (see Theorem 2).

4.1 Bounded $S(G_n)$

Proposition 3 *Suppose that $S(G_n)$ is bounded. Then $S(G_n)$ satisfies a linear recursion for $n > N$ with some N, whose characteristic polynomial is a product of different cyclotomic polynomials.*

Remark. This motivates the first part of assumption 6. in section 3.

Proof. We know that every $S(m)$ is a natural number and therefore can only attain a finite number of distinct values. So we see from (2) that $S(G_n)$ must be periodic (in n) for $n > N$. Let $p > r$ be some period of $S(G_n)$ and assume $n > N$. Then $S(G_{n+p}) - S(G_n) = 0$ which implies that $S(G_n)$ is a linear combination of powers of p-th roots of unity. Let $m(z)$ be the product of all cyclotomic polynomials corresponding to those roots of unity which appear in the representation of $S(G_n)$. Then $S(G_n)$ satisfies the linear recurrence related to $m(z)$. □

Proposition 4 *Suppose that $S(G_n)$ is bounded. Then $D_{G_n}(z)$ (defined in (8)) and $D_{G_n}(z^{-1})$ satisfy for $n > N$ a homogenous linear recurrence with (in n) constant coefficients $a_i(z)$ which are analytic around $z = 1$ and satisfy $a_i(z) = a_i(z^{-1})$.*

Proof. Let $p > r$ be a period of $S(G_n)$. Then by splitting up (9) into four parts we get

$$\begin{aligned} D_{G_{k+sp}}(z) &= \sum_{i=\max(0,k-r)}^{k-1} \gamma_{k,i}(z) D_{G_{i+sp}}(z) + \sum_{i=k+p-r}^{p-1} \zeta_{k,i}(z) D_{G_{i+(s-1)p}}(z) \\ &\quad + \sum_{i=\max(0,k-r)}^{k-1} \gamma_{k,p+i}(z) D_{G_{i+sp}}\left(z^{-1}\right) + \sum_{i=k+p-r}^{p-1} \zeta_{k,p+i}(z) D_{G_{i+(s-1)p}}\left(z^{-1}\right) \end{aligned} \tag{10}$$

with

$$\begin{aligned} \gamma_{k,i}(z) &= h_{k-i} z^{\left(\sum_{h=1}^{k-i-1} b_h m_{k-h} - (a_1+\ldots+a_{k-i-1} \bmod 2) m_i\right)} \\ \gamma_{k,p+i}(z) &= \bar{h}_{k-i} z^{\left(\sum_{h=1}^{k-i-1} b_h m_{k-h} + (a_1+\ldots+a_{k-i-1}+1 \bmod 2) m_i\right)} \\ \zeta_{k,i}(z) &= h_{k+p-i} z^{\left(\sum_{h=1}^{k+p-i-1} b_h m_{k-h} - (a_1+\ldots+a_{k+p-i-1} \bmod 2) m_i\right)} \\ \zeta_{k,p+i}(z) &= \bar{h}_{k+p-i} z^{\left(\sum_{h=1}^{k+p-i-1} b_h m_{k-h} + (a_1+\ldots+a_{k+p-i-1}+1 \bmod 2) m_i\right)} \end{aligned}$$

where $m_i := S(G_i)$, $0 \le k < p$ and $0 \le i < p$ and

$$\begin{aligned} h_i &= \begin{cases} |\{0 \le j < a_i \mid j \equiv a_1 + \ldots + a_{i-1} (2)\}| & \text{for } 1 \le i \le r \\ 0 & \text{otherwise} \end{cases}, \\ \bar{h}_i &= \begin{cases} |\{0 \le j < a_i \mid j \equiv a_1 + \ldots + a_{i-1} + 1 (2)\}| & \text{for } 1 \le i \le r \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

In the case $1 \le i \le r$ we can calculate

$$\begin{aligned} h_i &= \left\lfloor \frac{a_i + 1}{2} \right\rfloor - \begin{cases} 1 & \text{for } a_i \equiv a_1 + \ldots + a_{i-1}\,(2) \equiv 1\,(2) \\ 0 & \text{otherwise} \end{cases} \\ h_i + \bar{h}_i &= a_i \\ h_i - \bar{h}_i &= b_i \quad . \end{aligned} \tag{11}$$

Furthermore we define $\gamma_{p+k,p+i}\left(z^{-1}\right) = \gamma_{k,i}(z)$, $\gamma_{p+k,i}\left(z^{-1}\right) = \gamma_{k,p+i}(z)$, $\zeta_{p+k,p+i}\left(z^{-1}\right) = \zeta_{k,i}(z)$, $\zeta_{p+k,i}\left(z^{-1}\right) = \zeta_{k,p+i}(z)$ and

$$
\begin{aligned}
\mathbf{d}_s(z) &= \begin{pmatrix} \mathbf{d}_{1,s}(z) \\ \mathbf{d}_{2,s}(z) \end{pmatrix} = \begin{pmatrix} \left(D_{G_{0+sp}}(z), D_{G_{1+sp}}(z), \ldots, D_{G_{p-1+sp}}(z)\right)^T \\ \left(D_{G_{0+sp}}\left(z^{-1}\right), D_{G_{1+sp}}\left(z^{-1}\right), \ldots, D_{G_{p-1+sp}}\left(z^{-1}\right)\right)^T \end{pmatrix} \\
\mathbf{\Gamma}(z) &= \begin{pmatrix} \mathbf{\Gamma}_{1,1}(z) & \mathbf{\Gamma}_{1,2}(z) \\ \mathbf{\Gamma}_{2,1}(z) & \mathbf{\Gamma}_{2,2}(z) \end{pmatrix} = \begin{pmatrix} \left(\gamma_{k,i}(z)\right)_{0\le k,i<p} & \left(\gamma_{k,p+i}(z)\right)_{0\le k,i<p} \\ \left(\gamma_{p+k,i}(z)\right)_{0\le k,i<p} & \left(\gamma_{p+k,p+i}(z)\right)_{0\le k,i<p} \end{pmatrix} \\
\mathbf{Z}(z) &= \begin{pmatrix} \mathbf{Z}_{1,1}(z) & \mathbf{Z}_{1,2}(z) \\ \mathbf{Z}_{2,1}(z) & \mathbf{Z}_{2,2}(z) \end{pmatrix} = \begin{pmatrix} \left(\zeta_{k,i}(z)\right)_{0\le k,i<p} & \left(\zeta_{k,p+i}(z)\right)_{0\le k,i<p} \\ \left(\zeta_{p+k,i}(z)\right)_{0\le k,i<p} & \left(\zeta_{p+k,p+i}(z)\right)_{0\le k,i<p} \end{pmatrix} .
\end{aligned}
$$

Then the identities $\mathbf{d}_{2,s}(z) = \mathbf{d}_{1,s}\left(z^{-1}\right)$, $\mathbf{\Gamma}_{2,2}(z) = \mathbf{\Gamma}_{1,1}\left(z^{-1}\right)$, $\mathbf{\Gamma}_{2,1}(z) = \mathbf{\Gamma}_{1,2}\left(z^{-1}\right)$, $\mathbf{Z}_{2,2}(z) = \mathbf{Z}_{1,1}\left(z^{-1}\right)$ and $\mathbf{Z}_{2,1}(z) = \mathbf{Z}_{1,2}\left(z^{-1}\right)$ hold and (10) becomes

$$
\begin{aligned}
\mathbf{d}_s(z) &= \mathbf{\Gamma}(z)\mathbf{d}_s(z) + \mathbf{Z}(z)\mathbf{d}_{s-1}(z) \ , \quad \text{or formally} \\
\mathbf{d}_s(z) &= \left((\mathbf{I} - \mathbf{\Gamma}(z))^{-1}\mathbf{Z}(z)\right)\mathbf{d}_{s-1}(z) \ .
\end{aligned}
$$

Since the quadratic matrix $\mathbf{\Gamma}(1)$ consists of four quadratic $p \times p$-blocks which are lower triangle matrices with zero diagonal it is an easy exercise to show that $\mathbf{I} - \mathbf{\Gamma}(1)$ is invertible. Hence $(\mathbf{I} - \mathbf{\Gamma}(z))$ is invertible in a neighbourhood of $z = 1$.

Call $\mathbf{P}_{(z)}(l) := \det(l\mathbf{I} - \mathbf{\Theta}(z))$ the characteristic polynomial of the matrix $\mathbf{\Theta}(z) := (\mathbf{I} - \mathbf{\Gamma}(z))^{-1}\mathbf{Z}(z)$. Then by the theorem of Cayley-Hamilton $\mathbf{P}_{(z)}(\mathbf{\Theta}(z)) = \mathbf{0}$. From this we see that the sequence $\left(D_{G_{i+sp}}(z)\right)_{s\ge 0}$ satisfies a linear homogenous recursion.

Finally it follows from the definition of $\mathbf{\Gamma}$ and $\mathbf{Z}$ that $\mathbf{P}_{(z)}(l) = \mathbf{P}_{(z^{-1})}(l)$ from which we see that $a_i(z) = a_i\left(z^{-1}\right)$. □

Let $A_i(z)$, $1 \le i \le 2p$, denote the roots of the polynomial $\mathbf{P}_{(z)}(l)$, where z varies in a sufficiently small neighbourhood of $z = 1$. Since $a_i\left(z^{-1}\right) = a_i(z)$ they satisfy $A_i\left(z^{-1}\right) = A_i(z)$. Furthermore there exist functions $B_{k,i}(z,s)$ which are polynomials in s such that

$$
D_{G_{k+sp}}(z) = \sum_i B_{k,i}(z,s)A_i(z)^s. \tag{12}
$$

Since $D_{G_{k+sp}}(1) = G_{k+sp} \sim \beta_1\alpha_1^{k-1}(\alpha_1^p)^s$ it might be expected that (locally around $z = 1$) there exists a unique root $A_1(z)$ (satisfying $A_1(1) = \alpha_1^p$) of maximal modulus which is simple. The following lemma shows that this is true if condition 6. in section 3 holds.

Lemma 2 *Suppose that conditions 1.–6. in section 3 hold and let* $v := \max\{1 \le i \le r | b_i \ne 0\}$. *Then with the above notation the* $2p$ *roots of* $\mathbf{P}_{(1)}(l)$ *are* α_i^p, $1 \le i \le r$, *where* α_i, $1 \le i \le r$, *denote the roots of* $z^r - \sum_{j=1}^r a_j z^{r-j}$, 0 *with multiplicity* $2p - r - v$, *and* 1 *with multiplicity* v.

Proof. From $D_{G_{k+sp}}(1) = G_{k+sp} = \sum_i \beta_i (k+sp)\alpha_i^{k+sp-1} \sim \beta_1 \alpha_1^{k-1}(\alpha_1^p)^s$ we see that α_i^p are surely roots of $\mathbf{P}_{(1)}(l)$.

Since $\mathbf{I} - \mathbf{\Gamma}(1)$ is invertible the multiplicity of 0 is $2p$ minus the rank of $\mathbf{Z}(1)$. $\mathbf{Z}(1)$ has a simple block structure. It is an easy exercise to show that its rank equals $r + v$. (Recall that $h_i + \overline{h}_i = a_i$ and $h_i - \overline{h}_i = b_i$.)

Similarly the multiplicity of 1 is $2p$ minus the rank of $\mathbf{K} = \begin{pmatrix} \mathbf{K}_{1,1} & \mathbf{K}_{1,2} \\ \mathbf{K}_{1,2} & \mathbf{K}_{1,1} \end{pmatrix} = \mathbf{I} - \mathbf{\Gamma}(1) - \mathbf{Z}(1)$. Observe that

$$\mathrm{rk} \begin{pmatrix} \mathbf{K}_{1,1} & \mathbf{K}_{1,2} \\ \mathbf{K}_{1,2} & \mathbf{K}_{1,1} \end{pmatrix} = \mathrm{rk} \begin{pmatrix} \mathbf{K}_{1,1} + \mathbf{K}_{1,2} & \mathbf{0} \\ \mathbf{K}_{1,2} & \mathbf{K}_{1,1} - \mathbf{K}_{1,2} \end{pmatrix}$$

and that $\mathbf{K}_{1,1} + \mathbf{K}_{1,2}$ resp. $\mathbf{K}_{1,1} - \mathbf{K}_{1,2}$ are cyclic matrices with entries $1, -a_1, \ldots, -a_r, 0, \ldots, 0$ resp. $1, -b_1, \ldots, -b_r, 0, \ldots, 0$. By [3, Lemma 3] the rank of $\mathbf{K}_{1,1} + \mathbf{K}_{1,2}$ is p resp. the rank of $\mathbf{K}_{1,1} - \mathbf{K}_{1,2}$ is $p - v$, v being equal to the number of different p-th roots of unity which are roots of $z^r - \sum_{j=1}^r b_j z^{r-j}$. Thus $\mathrm{rk}\mathbf{K} = 2p - v$ which completes the proof of the lemma. □

Let us define discrete random variables X_m by

$$\mathbf{P}(X_m = k) = \frac{d_m(k)}{m}. \tag{13}$$

(Recall that $d_m(k) := |\{0 \le a < m \mid S(a) = k\}|$.) It is well known that one can calculate mean and variance using the generating function:

$$\begin{aligned} \mu_m = \mathbf{E}X_m &= \frac{1}{m} D_m'(1) \\ \sigma_m^2 = \mathbf{V}X_m &= \frac{1}{m}\left(D_m''(1) + D_m'(1) - \frac{1}{m} D_m'(1)^2 \right) . \end{aligned}$$

From now on we will assume 1.–6. in section 3.

Lemma 3 *Let* $A_1(z)$ *be the unique root of maximal modulus of* $\mathbf{P}_{(l)}(z)$. *Then we have* $A_1''(1) \ge 0$,

$$\mu_{G_{k+sp}} := \mathbf{E}X_{G_{k+sp}} = O(1) \quad \text{and} \quad \sigma^2_{G_{k+sp}} := \mathbf{V}X_{G_{k+sp}} = s\frac{A_1''(1)}{A_1(1)} + O(1)$$

as $s \to \infty$. *Furthermore, if* $A_1''(1) \ne 0$ *then*

$$\mathbf{E}\exp\left(it \frac{X_{G_{k+sp}} - \mu_{G_{k+sp}}}{\sigma_{G_{k+sp}}} \right) = \exp\left(-\frac{t^2}{2}\right)\left(1 + O\left(\frac{1}{\sqrt{s}}\right)\right)$$

as $s \to \infty$. This means that X_{G_m} is asymptotically gaussian with mean μ_{G_m} and variance $\sigma^2_{G_m}$.

Proof. Let $A(z) = A_1(z)$ and $B_k(z) = B_{k,1}(z,s)$ in (12) (the s-degree of the polynomial $B_{k,1}(z,s)$ is zero). Since $A'(1) = 0$, we obtain from (12) by differentiation

$$\begin{aligned}
D_{G_{k+sp}}(1) &= B_k(1)A(1)^s + O\left((A(1)\gamma)^s\right) \\
D'_{G_{k+sp}}(1) &= B_k(1)A(1)^s \frac{B'_k(1)}{B_k(1)} + O\left((A(1)\gamma)^s\right) \\
D''_{G_{k+sp}}(1) &= B_k(1)A(1)^s \left(s\frac{A''(1)}{A(1)} + \frac{B''_k(1)}{B_k(1)}\right) + O\left((A(1)\gamma)^s\right)
\end{aligned}$$

with some $0 \le \gamma < 1$ properly chosen. From $D_{G_{k+sp}}(1) = G_{k+sp}$ we get

$$\begin{aligned}
D'_{G_{k+sp}}(1) &= G_{k+sp}\frac{B'_k(1)}{B_k(1)}\left(1 + O\left(\gamma^s\right)\right) \\
D''_{G_{k+sp}}(1) &= G_{k+sp}\left(s\frac{A''(1)}{A(1)} + \frac{B''_k(1)}{B_k(1)}\right)\left(1 + O\left(\gamma^s\right)\right).
\end{aligned}$$

Both $D'_{G_{k+sp}}(1)$ and $D''_{G_{k+sp}}(1)$ are real and because of $B_k(1) = \beta_1\alpha_1^{k-1} \in \mathbf{R}^+$, $B'_k(1)$ is real. Furthermore, $A''(1)$ and $B''_k(1)$ are real, too. From this we obtain that

$$\begin{aligned}
\mathbf{E}X_{G_{k+sp}} &= \frac{B'_k(1)}{B_k(1)}\left(1 + O\left(\gamma^s\right)\right) \;=\; O(1) \\
\mathbf{V}X_{G_{k+sp}} &= \left(s\frac{A''(1)}{A(1)} + \frac{B''_k(1)}{B_k(1)} - \left(\frac{B'_k(1)}{B_k(1)}\right)^2\right)\left(1 + O\left(\gamma^s\right)\right) \;=\; s\frac{A''(1)}{A(1)} + O(1)
\end{aligned}$$

from which it is clear that $A''(1) \ge 0$. Using $A'(1) = A'''(1) = 0$ we get

$$A(e^t)^s = A(1)^s \exp\left(\frac{st^2}{2}\frac{A''(1)}{A(1)}\right)\left(1 + O(st^4)\right).$$

Now suppose $A''(1) > 0$ then we have

$$D_{G_{k+sp}}\left(e^{\frac{it}{\sigma_{G_{k+sp}}}}\right) = G_{k+sp}\exp\left(-\frac{t^2}{2}\right)\left(1 + O\left(\frac{t}{\sqrt{s}}\right) + O\left(\frac{t^4+1}{s}\right)\right)$$

where the O-constants are independent of k. For any fixed t we get

$$\begin{aligned}
\mathbf{E}\exp\left(it\frac{X_{G_{k+sp}} - \mu_{G_{k+sp}}}{\sigma_{G_{k+sp}}}\right) &= \frac{D_{G_{k+sp}}\left(e^{\frac{it}{\sigma_{G_{k+sp}}}}\right)}{G_{k+sp}}\exp\left(-it\frac{\mu_{G_{k+sp}}}{\sigma_{G_{k+sp}}}\right) \\
&= \exp\left(-\frac{t^2}{2}\right)\left(1 + O\left(\frac{1}{\sqrt{s}}\right)\right).
\end{aligned}$$

Thus, by Levi's theorem (see [7]) the normalized random variables $(X_{G_m} - \mu_{G_m})/\sigma_{G_m}$ converge weakly to normal distribution.

Remark. The use generating functions for the proof of asymptotic normality has probably started with Benders paper [2]. Further references can be found in [5].

Now we will turn our attention to X_m where m need not be an element of the basis G:

Theorem 2 *Suppose that $G = (G_n)$ satisfies a linear recursion with restriction 1.–6. Then with the above notation we have*

$$\mathbf{E}X_m = O(1) \quad \text{and} \quad \mathbf{V}X_m = \frac{l}{p}\frac{A''(1)}{A(1)} + O(1),$$

X_m being defined in (13) and l being the length of the digital expansion of m. If $A_1''(1) > 0$ then X_m is asymptotically Gaussian with mean value $\mathbf{E}X_m$ and variance $\mathbf{V}X_m \sim c \log m$ for some constant $c > 0$, i.e.

$$\lim_{\to\infty} \frac{1}{m}\left|\left\{N < m : S(N) \le \mathbf{E}X_m + x\sqrt{\mathbf{V}X_m}\right\}\right| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt.$$

Remark. The special case of $G_n = F_{n+1}$ (which leads to the original Zeckendorf representation) is already discussed in [4]. There are also recent contributions to simliar questions, e.g. Dumont and Thomas [6] prove asymptotic normality for substitution sequences by a different method and Barat and Grabner [1] show the existence of a limiting distribution of G-additive functions.

Proof. Let $m = \sum_{i=1}^{l} \epsilon_i G_i$ be the digital expansion of m. Iterated use of (7) yields for $1 \le j \le l$, $i < \epsilon_j$ and $a < G_j$

$$\begin{aligned}
S\left(\sum_{h=j+1}^{l} \epsilon_h G_h + iG_j + a\right) &= (\epsilon_l \bmod 2)S(G_l) + (-1)^{\epsilon_l} S\left(\sum_{h=j+1}^{l-1} \epsilon_h G_h + iG_j + a\right) \\
&= (\epsilon_l \bmod 2)S(G_l) + (-1)^{\epsilon_l}(\epsilon_{l-1} \bmod 2)S(G_{l-1}) + \dots \\
&\quad +(-1)^{\epsilon_l+\dots+\epsilon_{j+2}}(\epsilon_{j+1} \bmod 2)S(G_{j+1}) + (-1)^{\epsilon_l+\dots+\epsilon_{j+1}}(i \bmod 2)S(G_j) \\
&\quad +(-1)^{\epsilon_l+\dots+\epsilon_{j+1}+i}S(a) \\
&= \sum_{p=j+1}^{l} (-1)^{\epsilon_l+\dots+\epsilon_{p+1}}(\epsilon_p \bmod 2)S(G_p) + (-1)^{\epsilon_l+\dots+\epsilon_{j+1}}(i \bmod 2)S(G_j) \\
&\quad +(-1)^{\epsilon_l+\dots+\epsilon_{j+1}+i}S(a)
\end{aligned}$$

and from (4) we see that

$$d_m(k) = \left|\{0 \le a < \sum_{i=1}^{l} \epsilon_i G_i \mid S(a) = k\}\right|$$

$$
\begin{aligned}
&= \sum_{j=1}^{l}\sum_{i=0}^{\epsilon_j-1}\left|\left\{0 \le a < G_j \,\middle|\, S\left(\sum_{h=j+1}^{l}\epsilon_h G_h + iG_j + a\right) = k\right\}\right| \\
&= \sum_{j=1}^{l}\sum_{i=0}^{\epsilon_j-1}\left|\left\{0 \le a < G_j \,\middle|\right.\right. \\
&\qquad \left.\left. S(a) = (-1)^{\epsilon_l+\dots+\epsilon_{j+1}+i}\left(k - \sum_{\substack{p=j+1\\ \epsilon_p\equiv 1(2)}}^{l}(-1)^{\epsilon_l+\dots+\epsilon_{p+1}}S(G_p) - (-1)^{\epsilon_l+\dots+\epsilon_{j+1}}(i \bmod 2)S(G_j)\right)\right\}\right| \\
&= \sum_{j=1}^{l}\sum_{i=0}^{\epsilon_j-1} d_{G_j}\left((-1)^{\epsilon_l+\dots+\epsilon_{j+1}+i}\left(k - \sum_{\substack{p=j+1\\ \epsilon_p\equiv 1(2)}}^{l}(-1)^{\epsilon_l+\dots+\epsilon_{p+1}}m_p - (-1)^{\epsilon_l+\dots+\epsilon_{j+1}}(i \bmod 2)m_j\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
D_m(z) = \sum_{k\in\mathbf{Z}} d_m(k)z^k &= \sum_{j=1}^{l}\sum_{i=0}^{\epsilon_j-1}\sum_{k\in\mathbf{Z}} z^k \\
&\quad d_{G_j}\left((-1)^{\epsilon_l+\dots+\epsilon_{j+1}+i}\left(k - \sum_{\substack{p=j+1\\ \epsilon_p\equiv 1(2)}}^{l}(-1)^{\epsilon_l+\dots+\epsilon_{p+1}}S(G_p) - (-1)^{\epsilon_l+\dots+\epsilon_{j+1}}(i \bmod 2)S(G_j)\right)\right) \\
&= \sum_{j=1}^{l}\sum_{i=0}^{\epsilon_j-1}\sum_{k\in\mathbf{Z}} d_{G_j}(k) z^{\left((-1)^{\epsilon_l+\dots+\epsilon_{j+1}+i}k + \sum_{\substack{p=j+1\\ \epsilon_p\equiv 1(2)}}^{l}(-1)^{\epsilon_l+\dots+\epsilon_{p+1}}m_p + (-1)^{\epsilon_l+\dots+\epsilon_{j+1}}(i \bmod 2)m_j\right)} \\
&= \sum_{j=1}^{l} z^{\left(\sum_{\substack{p=j+1\\ \epsilon_p\equiv 1(2)}}^{l}(-1)^{\epsilon_l+\dots+\epsilon_{p+1}}m_p\right)} \sum_{i=0}^{\epsilon_j-1} z^{\left((-1)^{\epsilon_l+\dots+\epsilon_{j+1}}(i \bmod 2)m_j\right)} D_{G_j}\left(z^{\left((-1)^{\epsilon_l+\dots+\epsilon_{j+1}+i}\right)}\right) \\
&= \sum_{j=1}^{l}\sum_{i=0}^{\epsilon_j-1} z^{b(j,i)} D_{G_j}\left(z^{c(j,i)}\right) \qquad (14)
\end{aligned}
$$

in which

$$
\begin{aligned}
b(j,i) &= \sum_{\substack{p=j+1\\ \epsilon_p\equiv 1(2)}}^{l}(-1)^{\epsilon_l+\dots+\epsilon_{p+1}}m_p + (-1)^{\epsilon_l+\dots+\epsilon_{j+1}}(i \bmod 2)m_j \\
c(j,i) &= (-1)^{\epsilon_l+\dots+\epsilon_{j+1}+i}.
\end{aligned}
$$

Differentiation of (14) yields

$$
\begin{aligned}
zD_m'(z) &= \sum_{j=1}^{l}\sum_{i=0}^{\epsilon_j-1}\left(b(j,i)z^{b(j,i)}D_{G_j}\left(z^{c(j,i)}\right) + z^{b(j,i)}D_{G_j}'\left(z^{c(j,i)}\right)c(j,i)z^{c(j,i)}\right) \\
z\frac{\partial}{\partial z}(zD_m'(z)) &= \sum_{j=1}^{l}\sum_{i=0}^{\epsilon_j-1}\left(b(j,i)^2 z^{b(j,i)}D_{G_j}\left(z^{c(j,i)}\right)\right.
\end{aligned}
$$

$$+2b(j,i)z^{b(j,i)}D'_{G_j}\left(z^{c(j,i)}\right)c(j,i)z^{c(j,i)}$$

$$\left.+z^{b(j,i)}\left(z^{c(j,i)}D'_{G_j}\left(z^{c(j,i)}\right)+z^{2c(j,i)}D''_{G_j}\left(z^{c(j,i)}\right)\right)\right)$$

It is an easy exercise to show $\sum_{j=1}^{l}(l-j+1)^k G_j \le C_k G_l$. Because the m_j are bounded we get $b(j,i) = O(l-j+1)$ (uniformly in i) and

$$\begin{aligned} D'_m(1) &= \sum_{j=1}^{l}\sum_{i=0}^{\epsilon_j-1}\left(b(j,i)D_{G_j}(1)+c(j,i)D'_{G_j}(1)\right) \\ &= O\left(\sum_{j=1}^{l}(l-j+1)G_j\right) = O(G_l) = O(m) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial z}(zD'_m(z))\Big|_{z=1} &= \sum_{j=1}^{l}\sum_{i=0}^{\epsilon_j-1}\left(b(j,i)^2D_{G_j}(1)+2b(j,i)c(j,i)D'_{G_j}(1)+D'_{G_j}(1)+D''_{G_j}(1)\right) \\ &= \sum_{j=1}^{l}\epsilon_j D''_{G_j}(1)+O\left(\sum_{j=1}^{l}(l-j+1)^2G_j\right)+O\left(\sum_{j=1}^{l}(l-j+1)G_j\right)+O\left(\sum_{j=1}^{l}G_j\right) \\ &= \sum_{j=1}^{l}\epsilon_j G_j\frac{j}{p}\frac{A''(1)}{A(1)}\left(1+O\left(\frac{1}{j}\right)\right)+O(m) \\ &= \frac{1}{p}\frac{A''(1)}{A(1)}\left(l\sum_{j=1}^{l}\epsilon_j G_j-\sum_{j=1}^{l}\epsilon_j G_j(l-j)\right)+O\left(\sum_{j=1}^{l}\epsilon_j G_j\frac{1}{p}\frac{A''(1)}{A(1)}\right)+O(m) \\ &= m\frac{l}{p}\frac{A''(1)}{A(1)}+O(m) \end{aligned}$$

Thus we have

$$\mathbf{E}X_m = O(1) \quad \text{and} \quad \mathbf{V}X_m = \frac{l}{p}\frac{A''(1)}{A(1)}+O(1). \tag{15}$$

Furthermore, by using (14), we obtain

$$\begin{aligned} D_m\left(e^{\frac{it}{\sigma_m}}\right) &= \sum_{j=1}^{l}\exp\left(\frac{it}{\sigma_m}\sum_{\substack{p=j+1\\ \epsilon_p\equiv 1(2)}}^{l}(-1)^{\epsilon_l+\ldots+\epsilon_{p+1}}m_p\right)\sum_{i=0}^{\epsilon_j-1}\exp\left(\frac{it}{\sigma_m}(-1)^{\epsilon_l+\ldots+\epsilon_{j+1}}(i \bmod 2)m_j\right) \\ &\quad D_{G_j}\left(\exp\left(\frac{it}{\sigma_m}(-1)^{\epsilon_l+\ldots+\epsilon_{j+1}+i}\right)\right) \end{aligned}$$

and for any fixed t

$$D_{G_j}\left(\exp\left(\frac{it}{\sigma_m}(-1)^{\epsilon_l+\ldots+\epsilon_{j+1}+i}\right)\right) = D_{G_j}\left(\exp\left(\frac{it\frac{\sigma_{G_j}}{\sigma_m}}{\sigma_{G_j}}(-1)^{\epsilon_l+\ldots+\epsilon_{j+1}+i}\right)\right)$$

$$= \quad G_j \exp\left(-\frac{t^2\frac{j}{l}\left(1+O\left(\frac{1}{j}\right)\right)}{2}\right)\left(1+O\left(\frac{1}{\sqrt{j}}\right)\right) = G_j e^{-\frac{t^2}{2}} \exp\left(\frac{t^2}{2}\frac{l-j}{l} + O\left(\frac{1}{\sqrt{j}}\right)\right)$$

and

$$\sum_{i=0}^{\epsilon_j-1} \exp\left(\frac{it}{\sigma_m}(-1)^{\epsilon_l+\dots+\epsilon_{j+1}}(i \bmod 2)m_j\right)$$
$$= \sum_{i=0}^{\epsilon_j-1}\left(1+O\left(\frac{1}{\sqrt{l}}\right)\right) = \epsilon_j\left(1+O\left(\frac{1}{\sqrt{l}}\right)\right) = \epsilon_j \exp\left(O\left(\frac{1}{\sqrt{l}}\right)\right)$$

where the O-constants do not depend on l or j. Thus we get for $0 < \vartheta < \frac{1}{2}$

$$D_m\left(e^{\frac{it}{\sigma_m}}\right)e^{\frac{t^2}{2}} = \sum_{j=1}^{l}\epsilon_j G_j \exp\left(\frac{it}{\sigma_m}\sum_{\substack{p=j+1\\ \epsilon_p\equiv 1(2)}}^{l}(-1)^{\epsilon_l+\dots+\epsilon_{p+1}}m_p + O\left(\frac{l-j}{l}\right)+O\left(\frac{1}{\sqrt{j}}\right)\right)$$
$$= \sum_{l-l^\vartheta\le j\le l}\epsilon_j G_j \exp\left(O\left(\frac{l-j}{\sqrt{l}}\right)+O\left(\frac{1}{\sqrt{j}}\right)\right) + \sum_{1\le j<l-l^\vartheta}\epsilon_j G_j O(1)$$
$$= \sum_{l-l^\vartheta\le j\le l}\epsilon_j G_j \exp\left(O\left(l^{\vartheta-\frac{1}{2}}\right)+O\left(\frac{1}{\sqrt{\frac{l}{2}}}\right)\right) + O\left(G_{\lfloor l-l^\vartheta\rfloor}\right)$$
$$= \sum_{l-l^\vartheta\le j\le l}\epsilon_j G_j\left(1+O\left(l^{\vartheta-\frac{1}{2}}\right)\right)+O\left(\alpha_1^{l-l^\vartheta}\right) = \sum_{l-l^\vartheta\le j\le l}\epsilon_j G_j + O\left(ml^{\vartheta-\frac{1}{2}}\right)+O\left(\alpha_1^{l-l^\vartheta}\right)$$
$$= m + O\left(ml^{\vartheta-\frac{1}{2}}\right)+O\left(\alpha_1^{l-l^\vartheta}\right) = m+O\left(ml^{\vartheta-\frac{1}{2}}\right)$$

and finally (for any fixed t)

$$\mathbf{E}\exp\left(it\frac{X_m-\mu_m}{\sigma_m}\right) = \frac{D_m\left(e^{\frac{it}{\sigma_m}}\right)}{m}\exp\left(-it\frac{\mu_m}{\sigma_m}\right)$$
$$= \exp\left(-\frac{t^2}{2}\right)\left(1+O\left(l^{\vartheta-\frac{1}{2}}\right)\right)\exp\left(O\left(\frac{1}{\sqrt{l}}\right)\right) = \exp\left(-\frac{t^2}{2}\right)\left(1+O\left(l^{\vartheta-\frac{1}{2}}\right)\right)$$

and X_m ist asymptotically gaussian with mean μ_m und variance σ_m^2. □

The condition that $z^r - \sum_{i=1}^r b_i z^{r-i}$ (where $v = \max\{1 \le i \le r | b_i \ne 0\}$) is a product of z^{r-v} and different cyclotomic polynomials is rather restrictive in the case that $G_n \le 2G_{n-1}$ for $n > 1$:

Proposition 5 *Suppose that $G = (G_n)$ satisfies a linear recursion with restrictions 1.–5. such that $G_n \le 2G_{n-1}$ for $n > 1$. Then $z^r - \sum_{i=1}^r b_i z^{r-i}$ is a product of z^{r-v} and different cyclotomic polynomials, where $v = \max\{1 \le i \le r | b_i \ne 0\}$, if and only if one of the following conditions hold:*

1. $r=1$ and $a_1=2$: the binary system, or

2. $a_1=a_2=\ldots=a_r=1$: a generalized Zeckendorf representation.

Proof. First let $B(z)=z^r-\sum_{i=1}^r b_i z^{r-i}$ be of the above type, then if $a_1>1$ we are in the first case. So lets assume $a_1=1$, then it follows $a_i\in\{0,1\}$, $a_r=1$, and therefore $v=r$. From this we see that $z^r-\sum_{i=1}^r b_i z^{r-i}$ must be a symmetric polynomial which yields $a_i=a_{r-i}$ for all $1\le i<r$. Now suppose that $a_1=\ldots=a_{i-1}=1=a_r=\ldots=a_{r-i+1}$ and $a_i=0=a_{r-i}$ for some $1<i\le r-i$. By 3. we have $G_{n-r+i}\ge\sum_{j=r-i+1}^r a_j G_{n-j}=\sum_{j=r-i+1}^r G_{n-j}$ for $n>r$, or, equivalently, $G_n\ge\sum_{j=1}^i G_{n-j}$ for $n>i$. Because of $G_n=\sum_{j=1}^r a_j G_{n-j}$ for $n>r$ it follows that $\sum_{j=i+1}^r a_j G_{n-j}\ge G_{n-i}$ for $n>r$. On the other hand we have again by 3. $G_{n-i}\ge\sum_{j=i+1}^r a_j G_{n-j}$ for $n>r$ from which we see that $G_n=\sum_{j=1}^{r-i} a_{i+j}G_{n-j}$ for $n>r-i$, a contradiction to 4.

Now let $r=1$ and $a_1=2$, then $v=0$ and $B(z)=z$. Finally suppose $a_1=a_2=\ldots=a_r=1$. Then $b_i=(-1)^{i+1}$ and $B(z)=\sum_{i=0}^r(-1)^i z^{r-i}=\frac{z^{r+1}+(-1)^r}{z+1}$ is of the desired type. □

4.2 Unbounded $S(G_n)$

Proposition 6 *If $S(G_n)$ is unbounded then there exists some α with $1<\alpha<\alpha_1$ (α_1 being defined in section 3), $k\ge 1$, real numbers $\varphi_1,\ldots,\varphi_k$, and polynomials $\beta_1(n),\ldots,\beta_k(n),\bar\beta_1(n),\ldots,\bar\beta_k(n)$ not all of them zero such that*

$$S(G_n)=\alpha^n\sum_{i=1}^k\left(\beta_i(n)\cos(n\varphi_i)+\bar\beta_i(n)\sin(n\varphi_i)\right)+O((\gamma\alpha)^n)$$

for some $\gamma\in(0,1)$.

Proof. Since $S(G_n)$ satisfies the linear recurrence of Proposition 2 this representation follows immediately. □.

Theorem 3 *Suppose that $G=(G_n)$ satisfies a linear recurrence as above such that $S(G_n)$ is unbounded. Then*

$$\limsup_{m\to\infty}\frac{\log(|S(m)|)}{\log m}=\frac{\log\alpha}{\log\alpha_1}.$$

Proof. First it follows from Proposition 6 that

$$\limsup_{m\to\infty}\frac{\log(|S(m)|)}{\log m}\ge\limsup_{n\to\infty}\frac{\log(|S(G_n)|)}{\log G_n}=\frac{\log\alpha}{\log\alpha_1}.$$

The upper bound follows from the second part of Proposition 2 and again by an application of Proposition 6: Let $m = \sum_{j=1}^{l} \epsilon_j G_j$ be the proper digital expansion of m and let $C, K > 0$ be that large, that $|\beta_i(n) + \bar{\beta}_i(n)| < Cn^D$ for all n, i. Then we have for $l \to \infty$

$$
\begin{aligned}
\frac{\log(|S(m)|)}{\log m} &\leq \frac{\log\left(\sum_{j=1}^{l} |S(G_j)|\right)}{\log(\epsilon_l G_l)} \leq \frac{\log\left(l\alpha^l(Cl^D + C'\gamma^l)\right)}{\log \epsilon_l + \log G_l} \\
&\leq \frac{l \log \alpha + (D+1)\log l + C''}{l \log \alpha_1 + C'''} \to \frac{\log \alpha}{\log \alpha_1}
\end{aligned}
$$

which completes our proof. □

Remark. It is also possible to discuss the function $F(m) = S(m)m^{-(\log \alpha)/(\log \alpha_1)}$ in more detail. It turns out that $F(m)$ is an almost periodic function, i.e. $S(m)$ has an almost fractal structure. You just have to adapt the methods used in [8, 9].

5 Conclusions

Our starting point was the Möbius function $\mu_G(n)$ of the partial order which is induced by proper digital expansions with respect to a basis $G = (G_n)$. It turned out that $\mu_G(n) \in \{-1, 0, 1\}$ and so it is a natural question to determine the distribution of these three values $-1, 0, 1$. If $G_{n+1} \geq 2G_n$ for all $n > 1$ then the answer is very easy (see Proposition 1). So we restricted ourselves to the case $G_{n+1} \leq 2G_n$ for all $n > 1$. Here $\mu_G(n) = (-1)^{s_G(n)}$. Thus $\mu_G(n) \neq 0$ for all $n \geq 0$ and $M_G(N) = S_G(N)$ is exactly the difference between the number of $n < N$ with $\mu_G(n) = 1$ and the number of $n < N$ with $\mu_G(n) = -1$. In the case of linear recurring sequences $G = (G_n)$ (satisfying certain natural conditions) we proved that in any case $M_G(N) = o(N)$, i.e. $-1, +1$ are asymptotically equidistributed.

More precisely we discussed the distribution of values of $S_G(N)$ (which can also be considered in the case $G_{n+1} \geq 2G_n$). It turnes out that there are two essentially different cases, the case of bounded $S_G(G_n)$ and the case of unbounded $S_G(G_n)$. If $S_G(G_n)$ is unbounded then $S_G(N)$ has an almost fractal structure (see Theorem 3 and the remark following it). However, if $S_G(G_n)$ is bounded for all suitable initial conditions of G then the values $S_G(N)$ admit a gaussian limit law in the following sense. If X_N is a random variable defined by $\mathbf{P}(X_N = k) = \frac{1}{N}|\{n < N \mid S_G(n) = k\}|$ then X_N is asymptically gaussian with bounded mean value and variance $\mathbf{V}X_N \sim c \log N$, provided that $c \neq 0$ (Theorem 2).

Since $S_G(G_n)$ satisfies a linear recurrence (2) it follows that $S_G(G_n)$ is periodic (for sufficiently large n) if it is bounded. This can only occur for all suitable initial conditions of G if and only if the roots of characteristic polynomial $B(z) = z^r - \sum_{j=1}^r b_j z^{r-j}$ of (2) are 0 or roots of unity. Therefore the assumption on $B(z)$ in Theorem 2, this is assumtion 6. in section 3, is quite natural.

Finally we want to recall that the only recurring sequences $G = (G_n)$ satisfying 1.–5. such that $a_1 = 1$ (i.e. $G_{n+1} < 2G_n$) and that $B(z)$ is the product of z^{r-v} and cyclotomic polynomials are generalized Fibonacci numbers (Proposition 5). They satisfy a recursion of the form $G_n = G_{n-1} + \cdots + G_{n-r}$. Here Theorem 2 applies. Hence the values of $M_G(N)$ with respect to generalized Zeckendorf representations satisfy a central limit law.

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