ON THE DISCREPANCY OF HALTON-KRONECKER SEQUENCES

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Dedicated to Robert F. Tichy on the occasion of his 60th birthday

ABSTRACT. We study the discrepancy D_N of sequences $(z_n)_{n\geq 1} = ((\mathbf{x}_n, y_n))_{n\geq 1} \in [0,1)^{s+1}$ where $(\mathbf{x}_n)_{n\geq 0}$ is the s-dimensional Halton sequence and $(y_n)_{n\geq 1}$ is the one-dimensional Kronecker-sequence $(\{n\alpha\})_{n\geq 1}$. We show that for α algebraic we have $ND_N = \mathcal{O}(N^{\varepsilon})$ for all $\varepsilon > 0$. On the other hand, we show that for α with bounded continued fraction we have $ND_N = \mathcal{O}\left(N^{\frac{1}{2}} (\log N)^s\right)$ which is (almost) optimal since there exist α with bounded continued fraction coefficients such that $ND_N = \Omega\left(N^{\frac{1}{2}}\right)$.

1. Introduction and statement of results

Let $(\mathbf{z}_n)_{n\geq 0}$ be a sequence in the d-dimensional unit-cube $[0,1)^d$, then the discrepancy of the first N points of the sequence is defined by

$$D_N = \sup_{B\subseteq[0,1)^d} \left| \frac{A_N(B)}{N} - \lambda(B) \right|,$$

where

$$A_N(B) := \# \{n : 0 \leq n < N, \mathbf{z}_n \in B\},$$

λ is the d-dimensional volume and the supremum is taken over all axis-parallel subintervals $B \subseteq [0,1)^d$.
The sequence $(\mathbf{z}_n)_{n\geq 0}$ is called uniformly distributed if $\lim_{N\to\infty} D_N = 0$.

It is the most well-known conjecture in the theory of irregularities of distribution, that for every sequence $(\mathbf{z}_n)_{n\geq 0}$ in $[0,1)^d$ we have

$$D_N \geq c_d \cdot \frac{(\log N)^d}{N}$$

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for a constant $c_d > 0$ and for infinitely many N. Hence sequences whose discrepancy satisfies $D_N = \mathcal{O}\left(\frac{(\log N)^d}{N}\right)$ are called *low-discrepancy sequences*. Note that recent investigations of Bilyk, Lacey et al., see for example [1] or [2] have led some people to conjecture that $\frac{(\log N)^{\frac{d+1}{2}}}{N}$ instead of $\frac{(\log N)^d}{N}$ is the best possible order for the discrepancy of sequences in $[0,1)^d$.

Well-known examples of low-discrepancy sequences are the s-dimensional Halton-sequence $(\mathbf{x}_n)_{n\geq 1} \in [0,1)^s$, or the one-dimensional Kronecker sequence $(y_n)_{n\geq 1} = (\{n\alpha\})_{n\geq 1} \in [0,1)$ where α is a given irrational number with bounded continued fraction coefficients.
That is, the s-dimensional Halton-sequence satisfies

$$ND_N = \mathcal{O}\left((\log N)^s\right)$$

and the Kronecker sequence with suitable α (namely if α has bounded continued fraction expansion) satisfies

$$ND_N = \mathcal{O}\left(\log N\right).$$

If α is an algebraic number then with the help of the Thue-Siegel-Roth Theorem it can be shown that in this case for the discrepancy of the one-dimensional Kronecker-sequence we have $ND_N = \mathcal{O}\left(N^{\varepsilon}\right)$ for all $\varepsilon > 0$.

For the sake of completeness we remind the definition of the Halton sequence $\mathbf{x}_n$:
We choose a basis $b_1, b_2, \ldots, b_s$ of pairwise relatively prime integers larger than 1. To construct the i-th coordinate $x_n^{(i)} \in [0,1)$ of the n-th sequence point $\mathbf{x}_n = \left(x_n^{(i)}, \ldots, x_n^{(s)}\right) \in [0,1)^s$ we represent $n = n_0^{(i)} + n_1^{(i)} b_i + n_2^{(i)} b_i^{(2)} + n_3^{(i)} b_i^3 + \ldots.$ in base b_i and set

$$x_n^{(i)} := \frac{n_0^{(i)}}{b_i} + \frac{n_1^{(i)}}{b_i^2} + \frac{n_2^{(i)}}{b_i^3} + \ldots.$$

In the following we will be interested in the discrepancy of the combination

$$\mathbf{z}_n = (\mathbf{x}_n, y_n)_{n\geq 1} \in [0,1)^{s+1} =: [0,1)^d$$

in the $d := s+1-$ dimension unit-cube. For this sequence (we will call it $s+1-$dimensional Halton-Kronecker sequence) it was shown (see [9] or [8]) that this sequence is uniformly distributed for all α irrational.
In [7] (see also [5] for an earlier slightly weaker result) it was shown that for almost all choices of α the Halton-Kronecker sequence is almost a low-discrepancy sequence, i.e., for almost all α we have

$$ND_N = \mathcal{O}\left((\log N)^{s+1+\varepsilon}\right) = \mathcal{O}\left((\log N)^{d+\varepsilon}\right)$$

for all $\varepsilon > 0$.
However until now no concrete explicite choice for α, such that this discrepancy bound is valid could be given. When searching for concrete explicit examples of α providing a small discrepancy for the Halton-Kronecker sequence, then two possible ideas are near at hand:

- maybe algebraic α generate a small discrepancy of order $ND_N = \mathcal{O}(N^{\varepsilon})$ as in the pure Kronecker case
- maybe α with bounded continued fraction coefficients generate a low-discrepancy Halton-Kronecker sequence, i.e., $ND_N = \mathcal{O}\left((\log N)^{s+1}\right)$ as in the pure Kronecker case.

We will show in the following that the first assertion is true (see Theorem 1) and that the second assertion in general is not true (see Theorem 2).
So, our results are:

Theorem 1. *Let α be irrational and algebraic then for the discrepancy D_N of the $(s+1)$-dimensional Halton-Kronecker sequence $(\mathbf{z}_n)_{n\geq 1} = ((\mathbf{x}_n, y_n))_{n\geq 1}$ we have*

$$ND_N = \mathcal{O}(N^{\varepsilon})$$

for all $\varepsilon > 0$.

For the proof of this result we will essentially use Ridout's p-adic version of the Thue-Siegel-Roth-Theorem. Maybe it is possible to prove an analog to Theorem 1 for t-dimensional vectors $\boldsymbol{\alpha}$, i.e., for an $s+t$-dimensional Halton-Kronecker sequence, probably based on multidimensional variants of Ridout's Theorem, as were given for example by Schlickewei in [10]. However at the moment we still are not able to give such a proof and leave this as an open problem.
Concerning α with bounded continued fraction coefficients we show:

Theorem 2. *Let α be irrational with bounded continued fraction coeffcients. Then the discrepancy D_N of the $(s+1)$-dimensional Halton-Kronecker sequence $(\mathbf{z}_n)_{n\geq 1} = ((\mathbf{x}_n, y_n))_{n\geq 1}$ satisfies*

$$ND_N = \mathcal{O}\left(N^{\frac{1}{2}}(\log N)^s\right).$$

On the other hand there exists an irrational number α with bounded continued fraction coefficients such that

$$ND_N = \Omega\left(N^{\frac{1}{2}}\right).$$

We want to mention that the logarithmic factor $(\log N)^s$ is certainly not optimal. For example, with slightly more care we can prove $ND_N = \mathcal{O}\left((N\log N)^{\frac{1}{2}}\right)$ in the case $s = 1$. We leave the determination of the precise threshold as on open problem.

2. Proofs of the results

Proof of Theorem 1. We have $(\mathbf{z}_n) = (\mathbf{x}_n, y_n)_{n\geq 1}$ where $(\mathbf{x}_n)_{n\geq 1}$ is the s-dimensional Halton sequence in bases $b_1, \ldots, b_s$ and y_n is the one-dimensional Kronecker-sequence $(\{n\alpha\})_{n\geq 1}$. Let $I = [0, \beta) \times [0, \gamma) \subseteq [0,1)^d$, with $d = s+1, \beta = (\beta_1, \ldots, \beta_s)$ and $\gamma \in [0,1)$. We will choose in the following certain disjoint subsets I_{int} and I_{bor} of $[0,1)^d$ such that $I_{\text{int}} \subseteq I \subseteq I_{\text{int}} \cup I_{\text{bor}}$. Then with

$$A_N(I) := \#\{n : 0 \leq n < N, \mathbf{z}_n \in I\},$$

we obviously have

$$|A_N(I) - N\lambda(I)| \leq |A_N(I_{\text{int}}) - N\lambda(I_{\text{int}})| + A_N(I_{\text{bor}}) + N\lambda(I_{\text{bor}}). \tag{1}$$

We choose I_{int} as follows:
Let

$$\beta_i = \frac{\beta_1^{(i)}}{b_i^1} + \frac{\beta_2^{(i)}}{b_i^2} + \ldots$$

with $\beta_j^{(i)} \in \{0, 1, \ldots, b_i - 1\}$. Then let

$$I(j_1, \ldots, j_s, k_1, \ldots, k_s, \gamma) := \prod_{i=1}^{s} \left[\sum_{l=1}^{j_i - 1} \frac{\beta_l^{(i)}}{b_i^l} + \frac{k_i}{b_i^{j_i}}, \sum_{l=1}^{j_i - 1} \frac{\beta_l^{(i)}}{b_i^l} + \frac{k_i + 1}{b_i^{j_i}} \right) \times [0, \gamma)$$

for positive integers $j_1, \ldots, j_s$ and $k_i \in \{0, 1, \ldots, b_i - 1\}$ for $i = 1, \ldots, s$. By the construction of the Halton sequence there is a unique

$$r = r(j_1, \ldots, j_s, k_1, \ldots, k_s) \in \left\{0, 1, \ldots, b_1^{j_1} b_2^{j_2} \ldots b_s^{j_s} - 1\right\}$$

such that $\mathbf{z}_n \in I(j_1, \ldots, j_s, k_1, \ldots, k_s, \gamma)$ if and only if

$$n \equiv r \mod \left(b_1^{j_1} b_2^{j_2} \ldots b_s^{j_s}\right) \text{ and } y_n \in [0, \gamma). \tag{2}$$

For $x \in \mathbb{R}$ let $[x]$ denote the largest integer less or equal x. Then let $L_i := \left[\log_{b_i} N\right] + 1$ and define I_{int} as union of disjoint intervals by

$$I_{\text{int}} := \bigcup_{j_1=1}^{L_1} \ldots \bigcup_{j_s=1}^{L_s} \bigcup_{k_1=0}^{\beta_{j_1}^{(1)}} \ldots \bigcup_{k_s=0}^{\beta_{j_s}^{(s)}} I(j_1, \ldots, j_s, k_1, \ldots, k_s, \gamma).$$

Further let

$$I_{\text{bor}} := \bigcup_{i=1}^{s} \left(\prod_{j=1}^{i-1} [0,1) \times \left[\sum_{l=1}^{L_i} \frac{\beta_l^{(i)}}{b_i^l}, \sum_{l=1}^{L_i} \frac{\beta_l^{(i)}}{b_i^l} + \frac{1}{b_i^{L_i}} \right) \times \prod_{j=i+1}^{s} [0,1) \right) \times [0, \gamma).$$

Then indeed we have

$$I_{\text{int}} \subseteq I \subseteq I_{\text{int}} \cup I_{\text{bor}}$$

and by (1) and (2) (where we use the notation $\mathbf{j} := (j_1, \ldots, j_s)$, $\mathbf{k} := (k_1, \ldots, k_s)$, $\theta(\mathbf{j}, \mathbf{k}) := r(\mathbf{j}, \mathbf{k}) \cdot \alpha$, $b(\mathbf{j}) := b_1^{j_1} b_2^{j_2} \ldots b_s^{j_s}$, $N(\mathbf{j}) := \lfloor N / b(\mathbf{j}) \rfloor$)

$$|A_N(I) - N\lambda(I)| \leq \sum_{\mathbf{j},\mathbf{k}} \# \left\{ m \middle| 0 \leq n = r(\mathbf{j}, \mathbf{k}) + mb(\mathbf{j}) < N \text{ and } \{n\alpha\} \in [0, \gamma) \right\} - N \frac{1}{b(\mathbf{j})} \lambda([0, \gamma)) \Bigg|$$

$$+ \sum_{i=1}^{s} \left(\left[N \frac{1}{b_i^{L_i}} \right] + 1 \right) + N \sum_{i=1}^{s} \frac{1}{b_i^{L_i}} \leq$$

$$\left| \sum_{\mathbf{j}} \sum_{\mathbf{k}} \# \{0 \leq m < N(\mathbf{j}) \mid \{mb(\mathbf{j})\alpha\} \in [\theta(\mathbf{j}, \mathbf{k}), \gamma + \theta(\mathbf{j}, \mathbf{k}))\} - N(\mathbf{j}) \lambda([\theta(\mathbf{j}, \mathbf{k}), \gamma + \theta(\mathbf{j}, \mathbf{k})))\right|$$

$$+\sum_{\mathbf{j}}\sum_{\mathbf{k}}1+\sum_{i=1}^{s}3\leq c\left(\alpha,s,b_1,\ldots,b_s\right)\cdot\left(\sum_{\mathbf{j}}N\left(\mathbf{j}\right)D_{N(\mathbf{j})}\left(b\left(\mathbf{j}\right)\alpha\right)+\left(\log N\right)^s\right).$$

and the sums are always interpreted as

$$\sum_{\mathbf{j}}:=\sum_{j_1=0}^{L_1}\cdots\sum_{j_s=0}^{L_s},\qquad \sum_{\mathbf{k}}:=\sum_{k_1=1}^{\beta_{j_1}^{(1)}-1}\cdots\sum_{k_s=1}^{\beta_{j_s}^{(s)}-1}.$$

Hence, to prove our Theorem 1 it suffices to show that

$$\sum_{\mathbf{j}}N\left(\mathbf{j}\right)D_{N(\mathbf{j})}\left(b(\mathbf{j})\alpha\right)=\mathcal{O}\left(N^{\varepsilon}\right). \tag{3}$$

To provide this estimate we use the well-known Koksma-Erdös-Turan inequality (see [6] or [4]) together with Ridout's p-adic version of the Thue-Siegel-Roth-Theorem. The discrepancy D_M of a point set $x_0,\ldots,x_{M-1}$ in $[0,1)$ can be estimated with the Koksma-Erdös-Turan inequality by

$$D_M\leq c_1\cdot\left(\frac{1}{H}+\sum_{h=1}^{H}\frac{1}{h}\left|\frac{1}{M}\sum_{n=0}^{M-1}e^{2\pi i h x_n}\right|\right),$$

for arbitrary $H\geq 1$.
If $x_n=\{n\alpha\}$, then

$$\left|\sum_{n=0}^{M-1}e^{2\pi i h x_n}\right|\leq c_2\cdot\frac{1}{\|h\alpha\|}.$$

Here, and in the following $c_1,c_2,\ldots$ are absolute constants, and $\|x\|$ denotes the distance of x to the nearest integer. Hence we have (choosing $H=N(\mathbf{j})$)

$$N\left(\mathbf{j}\right)D_{N(\mathbf{j})}\left(b(\mathbf{j})\alpha\right)\leq c_3\cdot\sum_{h=1}^{N(\mathbf{j})}\frac{1}{h}\cdot\frac{1}{\|hb(\mathbf{j})\alpha\|}, \tag{4}$$

and it suffices to show that

$$\sum_{\mathbf{j}}\sum_{h=1}^{N(\mathbf{j})}\frac{1}{h}\cdot\frac{1}{\|hb(\mathbf{j})\alpha\|}=\mathcal{O}\left(N^{\varepsilon}\right). \tag{5}$$

Now we use a result which was shown in [3] with the help of Ridout's theorem:
suppose that ϕ is algebraic and that $q_1,q_2,\ldots,q_s\geq 2$ are pairwise coprime integers. Then for every $\varepsilon>0$ there exists a constant $C=C\left(\phi,\varepsilon,q_1,\ldots,q_s\right)$ such that for all integers $j_1,\ldots,j_s\geq 0$ and $H\geq 1$

$$\sum_{h=1}^{H}\frac{1}{h}\frac{1}{\left\|q_1^{j_1}\cdots q_s^{j_s}h\phi\right\|}\leq c\cdot\left(q_1^{j_1}\cdots q_s^{j_s}H\right)^{\varepsilon}.$$

Using this result we obtain

$$\sum_{\mathbf{j}} \sum_{h=1}^{N(j)} \frac{1}{h} \cdot \frac{1}{\|hb(\mathbf{j})\alpha\|} \le C \cdot N^{\varepsilon} \cdot \sum_{j_1=1}^{L_1} \dots \sum_{j_s=1}^{L_s} q_1^{\varepsilon j_1} \dots q_s^{\varepsilon j_s} \le C \cdot N^{(s+1)\varepsilon},$$

and the result follows. □

Proof of Theorem 2. In order to prove the upper bound we proceed similarly to the proof of Theorem 1. However, instead of (4) we use the trivial estimate

$$N(\mathbf{j})\, D_{N(\mathbf{j})}(b(\mathbf{j})\alpha) \le \sqrt{N}$$

if $N(\mathbf{j}) \le \sqrt{N}$ and

$$N(\mathbf{j})\, D_{N(\mathbf{j})}(b(\mathbf{j})\alpha) \le c_1 \frac{N(\mathbf{j})}{H(\mathbf{j})} + c_1 c_3 \sum_{h=1}^{H(\mathbf{j})} \frac{1}{h} \cdot \frac{1}{\|hb(\mathbf{j})\alpha\|},$$

if $N(\mathbf{j}) > \sqrt{N}$, where we set $H(\mathbf{j}) = \lfloor N(\mathbf{j})/\sqrt{N}) \rfloor$.
If α has bounded continued fraction coefficients then we have $\|\alpha h\| \ge C/h$ for all positive integers. Hence it follows (in the case $N(\mathbf{j}) > \sqrt{N}$) that

$$\sum_{h=1}^{H(\mathbf{j})} \frac{1}{h} \cdot \frac{1}{\|hb(\mathbf{j})\alpha\|} \le H(\mathbf{j})\, b(\mathbf{j}) \le \sqrt{N}$$

and consequently

$$N(\mathbf{j})\, D_{N(\mathbf{j})}(b(\mathbf{j})\alpha) \le c_4 \sqrt{N}.$$

Thus we certainly have

$$\sum_{\mathbf{j}} N(\mathbf{j})\, D_{N(\mathbf{j})}(b(\mathbf{j})\alpha) = \mathcal{O}\left(\sqrt{N}\,(\log N)^s\right)$$

which proves the upper bound.
In order to otbtain the lower bound we use the real number $\alpha = \sum_{m=1}^{\infty} \frac{1}{b_1^{2^m}}$, where we first suppose that $b_1 = \max\{b_1, \dots, b_s\} \ge 3$. By [11] it is known that α has bounded continued fraction coefficients. Let $k \ge 1$ be fixed and $N = b_1^{2^{k+1}-1}$. We consider the interval $B := \left[0, \frac{1}{b_1^{2^k}}\right) \times [0,1)^{s-1} \times \left[0, \frac{1}{2}\right)$. By definition it is clear that $x_n^{(1)} \in [0, b_1^{-2^k})$ if and only if $n = \ell b_1^{2^k}$ for some $\ell \le \lfloor N/b_1^{2^k} \rfloor = b_1^{2^k-1}$. However, for all these n we have

$$n\alpha - \lfloor n\alpha \rfloor = \ell \sum_{m>k} b_1^{2^k-2^m} \le b_1^{2^k-1} \frac{b_1^{-2^k}}{1-b_1^{-2^{-k}}} \le \frac{1}{2}$$

provided that k is sufficiently large. Hence, for this interval B we have

$$A_N(B) - N \cdot \lambda(B) \ge b_1^{2^k-1} - b_1^{2^{k+1}-1} b_1^{-2^k} \frac{1}{2} = \frac{\sqrt{N}}{2\sqrt{b_1}}.$$

This proves the result in the case $b_1 \ge 3$.

If $b_1 = 2$ then we can proceed in precisely the same way by using $\alpha = \sum_{m=1}^{\infty} \frac{1}{4^{2^m}}$. □

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