ON IRREGULARITIES OF DISTRIBUTION ON THE HYPERBOLIC PLANE

Michael Drmota

Technical University of Vienna

Abstract. This paper deals with an application of J. Beck's Fourier transform method for irregularities of distribution on the hyperbolic plane by using a hyperbolic Fourier transform. The discrepancy of a point distribution in a convex set X with respect to hyperbolic circles and with respect to the convex hull of four symmetric points is estimated below. Furthermore it is indicated that these lower bounds are best possible despite of a logarithmic factor.

1. Introduction

Let H denote the hyperbolic plane, $d(\cdot,\cdot)$ the geodesic distance on H, and μ the Riemannian measure. Now consider a convex set X and a set $\mathcal{P}_N = \{p_1, \dots, p_N\}$ of N points contained in X. The aim of this paper is to discuss the distance between the two measures

$$\mu_X(A) = \mu(A \cap X)/\mu(X) \tag{1.1}$$

and

$$\zeta_{\mathcal{P}_N}(A) = \frac{1}{N}\sum_{n=1}^{N} 1_{A\cap X}(p_n) \tag{1.2}$$

on X (where 1_M denotes the characteristic function of M). Obviously there are different kinds to define the distance between two measures. The first one that will be discussed is

$$\Delta(\mathcal{P}_N, B) = \sup_{r\ge 0, y\in H} \left|\zeta_{\mathcal{P}_N}(B(y,r)) - \mu_X(B(y,r))\right|, \tag{1.3}$$

where $B(y,r) = \{x \in H : d(x,y) \le r\}$ is the closed geodesic ball with center y and radius r. It is usual to call such a distance discrepancy of the sequence $\{p_1, p_2, \dots, p_N\}$.

We will prove

Theorem 1. *Let $\mathcal{P}_N \subseteq X$ be a point distribution, then $\Delta(\mathcal{P}_N, B)$ is bounded below by*

$$\Delta(\mathcal{P}_N, B) >> N^{-3/4}. \tag{1.4}$$

On the other hand, there exsits a point distribution $\mathcal{P}_N \subseteq X$ such that

$$\Delta(\mathcal{P}_N, B) << N^{-3/4}(\log N)^{1/2}. \tag{1.5}$$

The constants implied by >> and << are independent of N and the point distribution $\mathcal{P}_N$.

It should be noticed that Theorem 1 could be proved by Euclidian methods by using a proper imbedding. But the aim of this paper is to find a method for proving Theorem 1 that uses the structure of the hyperbolic plane. Only in this way it is possible to generalize Theorem 1 as it is done in section 5. Nevertheless the method used here is an analogon to the Euclidian one that has been developed by J. Beck and is described in detail in his and W. Chen's book *"Irregularities of Distribution"* [1].

2. A Fourier Transform on the Hyperbolic Plane

In this section we list some facts about the hyperbolic plane and its Fourier transform. More details and proofs can be found in S. Helgason's book *"Topics in Harmonic Analysis on Homogeneous Spaces"* [3].

Consider the locally compact group

$$G = \mathrm{SU}(1,1) = \left\{ g = \begin{pmatrix} a & b \\ \overline{a} & \overline{b} \end{pmatrix} \,:\, |a|^2 - |b|^2 = 1 \right\}$$

that acts on $D = \{z \in \mathbf{C} : \, |z| < 1\}$ by

$$g \cdot z = \frac{az + b}{\overline{b}z + \overline{a}}.$$

Since the subgroup fixing 0 is $\mathrm{SO}(2)$ we have the identification

$$D = \mathrm{SU}(1,1)/\mathrm{SO}(2).$$

Define a Riemannian structure on D by

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{(1 - x^2 - y^2)^2}, \tag{2.1}$$

where $z = x + iy$, then it is easy show that the related Riemannian measure

$$d\mu(z) = \frac{dx\, dy}{(1 - x^2 - y^2)^2} \tag{2.2}$$

is preserved by maps of $\mathrm{SU}(1,1)$. Therefore μ is induced by a Haar measure dg on $\mathrm{SU}(1,1)$.

The distance between two points $z_1, z_2 \in D$ is given by

$$d(z_1, z_2) = \frac{1}{2} \log \frac{|1 - \overline{z}_1 z_2| + |z_2 - z_1|}{|1 - \overline{z}_1 z_2| - |z_2 - z_1|} \tag{2.3}$$

which gives $d(0, z) = \mathrm{artanh}(|z|)$. Therefore it is easy to evaluate

$$\mu(B(y, r)) = \pi (\sinh(R))^2 \tag{2.4}$$

and the arclength of the boundary

$$l(\partial B(y,r)) = \pi \sinh(2R). \tag{2.5}$$

D with this metric is Klein's model for the hyperbolic plane H.

The Fourier transform of a function $f : D \to \mathbf{C}$ is defined by

$$\hat{f}(\lambda, b) = \int_D f(z) \left(\frac{\sqrt{1-|z|^2}}{|z-b|} \right)^{-i\lambda+1} d\mu(z) \tag{2.6}$$

for all $\lambda \in \mathbf{C}$ and $b \in B = \partial D$ for which this integral exists. This mapping can be extended to an isometry of $\mathrm{L}^2(D)$ onto
$\mathrm{L}^2\left(\mathbf{R}^+ \times B, \frac{\lambda}{2\pi}\tanh\left(\frac{\lambda\pi}{2}\right)\, d\lambda\, db\right)$; db is the normalized measure on B that is invariant under $SO(2)$. Therefore we have

$$\int_D |f(z)|^2\, d\mu(z) = \int_B \int_{\mathbf{R}^+} |\hat{f}(\lambda,b)|^2 \frac{\lambda}{2\pi} \tanh\left(\frac{\lambda\pi}{2}\right) d\lambda\, db. \tag{2.7}$$

A function $f : D \to \mathbf{R}$ is called radial if $f(z\, e^{i\theta}) = f(z)$ for all $\theta \in [0, 2\pi)$. If a convolution $f_1 * f_2$ is defined by

$$f_1 * f_2(z) = \int_G f_1(h \cdot 0) f_2(h^{-1} \cdot z)\, dh \tag{2.8}$$

and f_1 or f_2 is radial then

$$\widehat{f_1 * f_2}(\lambda, b) = \hat{f}_1(\lambda, b)\, \hat{f}_2(\lambda, b). \tag{2.9}$$

Note that the Fourier transform of a radial function f is essentially an even function of λ and independent of b.

3. Plan of the Proof

Let

$$\chi_r(z) = 1_{B(0,r)}(z) \tag{3.1}$$

be the characteristic function of the closed disc with center 0 and hyperbolic radius r, that is of course radial. Then the difference

$$\zeta_{\mathcal{P}_N}(B(z,r)) - \mu_X(B(z,r)) = \left[\chi_r * (d\zeta_{\mathcal{P}_N} - d\mu_X)\right](z) \tag{3.2}$$

can be interpreted as a convolution. Therefore

$$\begin{aligned}
\Delta_2(r)^2 &= \int_D \left|\zeta_{\mathcal{P}_N}(B(z,r)) - \mu_X(B(z,r))\right|^2\, dz \\
&= \int_B \int_{\mathbf{R}^+} |\hat{\chi}_r(\lambda)|^2 \left|(\widehat{d\zeta_{\mathcal{P}_N} - \mu_X})(\lambda, b)\right|^2 \frac{\lambda}{2\pi}\tanh\left(\frac{\lambda\pi}{2}\right) d\lambda\, db.
\end{aligned}$$

Using following notations

$$\varphi_u(\lambda) = \frac{1}{u}\int_u^{2u} |\hat{\chi}_r(\lambda)|^2\, dr,$$

$$\Phi(u) = \frac{1}{u}\int_u^{2u} \Delta_2(u)^2\, du,$$

$$\Psi(\lambda) = \int_B \left|(d\widehat{\zeta_{\mathcal{P}_N} - \mu_X})(\lambda, b)\right|^2 db,$$

we have

$$\Delta(\mathcal{P}_N, B)^2 \ge \Phi(u) = \int_{\mathbf{R}^+} \varphi_u(\lambda)\, \Psi(\lambda)\, d\lambda. \tag{3.3}$$

In the sequel we will prove following two Lemmata:

Lemma 1. *If $0 < u < v \le C$ (for some constant $C > 0$), then*

$$\frac{\varphi_v(\lambda)}{\varphi_u(\lambda)} >> \frac{v}{u} \tag{3.4}$$

uniformly for all $\lambda > 0$.

Lemma 2. *If $u = \frac{1}{2}\mathrm{arsinh}\sqrt{\mu(X)/(2\pi N)}$, then*

$$\Phi(u) >> \frac{1}{N^2} \tag{3.5}$$

for all $N \ge 1$.

Assuming that these two Lemmata are true it is easy to prove the first part of Theorem 1 by combining (3.3), (3.4), and (3.5). You only have to set $v = C$ and $u = \frac{1}{2}\mathrm{arsinh}\sqrt{\mu(X)/(2\pi N)}$ to get

$$\Delta(\mathcal{P}_N, B)^2 \ge \Phi(C) >> \Phi(u)\,\frac{1}{u} >> \frac{1}{N^{3/2}} \tag{3.6}$$

for a proper chosen constant C.

The upper bound (1.5) of Theorem 1 can be proved by verbally the same arguments as in [1,pp. 229]. You only have to subdivide X in a proper way.

4. An Asymptotic Expansion

As an essential tool for the proof of Lemma 1 we will need the asymptotic behaviour of the hyperbolic Fourier transform of the characteristic function of a disc of hyperbolic radius r and center 0

$$\hat{\chi}_r(\lambda) = \int_{|z|\le\tanh(r)} \left(\frac{\sqrt{1-|z|^2}}{|z-1|}\right)^{-i\lambda+1} d\mu(z) \tag{4.1}$$

that is of course independent of b. Using the substitution

$$\begin{aligned} x &= \frac{1+t}{2} - \frac{1-t}{2}\cos\varphi \\ y &= \frac{1-t}{2}\sin\varphi \end{aligned} \qquad 0 \le \varphi \le \arccos\frac{1+\tanh(t)^2 - 2\tanh(r)^2}{1-\tanh(t)^2}$$

we get

$$\begin{aligned} \hat{\chi}_r(\lambda) &= 4\int_0^r \cos(\lambda t)\sqrt{\cosh(r)^2 - \cosh(t)^2}\,dt \\ &= 2\int_{-r}^r e^{i\lambda t}\sqrt{\cosh(r)^2 - \cosh(t)^2}\,dt. \end{aligned} \tag{4.2}$$

This representation motivates the use of following complex path of integration:

Figure 1

We have

$$\begin{aligned} 0 &= \int_\gamma 2\,e^{i\lambda t}\sqrt{\cosh(r)^2 - \cosh(t)^2}\,dt \\ &= \hat{\chi}_r(\lambda) - e^{-\lambda\pi}\,\hat{\chi}_r(\lambda) \\ &\quad + 2i\int_0^\pi e^{i\lambda(r+is)}\sqrt{\cosh(r)^2 - \cosh(r+is)^2}\,ds \\ &\quad - 2i\int_0^\pi e^{i\lambda(-r+is)}\sqrt{\cosh(r)^2 - \cosh(-r+is)^2}\,ds \end{aligned}$$

or after some simple calculations

$$\hat{\chi}_r(\lambda) = \frac{4}{1-e^{-\lambda\pi}}\left(\sin(\lambda r)\,I_1(\lambda, r) - \cos(\lambda r)\,I_2(\lambda, r)\right), \tag{4.3}$$

where

$$I_1(\lambda, r) = \frac{1}{\sqrt{2}}\int_0^\pi e^{-\lambda s}\sqrt{\sin(s)\sqrt{\sin(s)^2 + \sinh(2r)^2} + \sin(s)^2\cosh(2r)}\,ds$$

and

$$\begin{aligned} I_2(\lambda, r) = \frac{1}{\sqrt{2}}\int_0^{\pi/2} &\left(e^{-\lambda s} - e^{-\lambda(\pi - s)}\right) \times \\ &\times \sqrt{\sin(s)\sqrt{\sin(s)^2 + \sinh(2r)^2} - \sin(s)^2\cosh(2r)}\,ds. \end{aligned}$$

With help of these two representations (4.2) and (4.3) we can prove

Proposition 1. *For $r \to 0$ we have uniformly in $\lambda > 0$*

$$\hat{\chi}_r(\lambda) = \frac{2r\pi}{\lambda}\, J_1(\lambda r) + O\left(\min\left(\frac{r^2}{\lambda}, r^3\right)\right), \tag{4.4}$$

where $J_1(x)$ is the first Bessel function,
for $\lambda \to \infty$ we have uniformly in $r > 0$

$$\hat{\chi}_r(\lambda) = \frac{2\sqrt{\pi\, sinh(2r)}}{\lambda^{3/2}} \left(\sin\left(\lambda r - \frac{\pi}{4}\right) + O\left(\frac{1}{\lambda^2}\right)\right) + O\left(\frac{1}{\lambda^2}\right), \tag{4.5}$$

for $r \to \infty$ and $\lambda \to \infty$ we have

$$\hat{\chi}_r(\lambda) = \frac{2\sqrt{\pi\, sinh(2r)}}{\lambda^{3/2}} \left(\sin\left(\lambda r - \frac{\pi}{4}\right) + O\left(\frac{1}{\lambda}\right)\right), \tag{4.6}$$

and for $r \to \infty$ and $\lambda \to 0$ we have

$$\begin{aligned} \hat{\chi}_r(\lambda) &= \left(\sqrt{sinh(2r)} + O(e^{-2r})\right)\left(2\sqrt{2}\frac{\sin(\lambda r)}{\lambda} - \sqrt{2}\pi \sin(\lambda r) + O(\lambda)\right) \\ &\quad + \left(\sqrt{sinh(2r)} + O(e^{-2r})\right)\left(\frac{4\sqrt{2}}{\pi} B \cos(\lambda r) + O(\lambda)\right), \end{aligned} \tag{4.7}$$

where

$$B = \int_0^{\pi/2} s\sqrt{\cos(s) - \cos(s)^2}\, ds.$$

Remark. Since the Bessel function $J_1(x)$ has the asymptotic expansion

$$J_1(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4}\right) + O\left(\frac{1}{x^{3/2}}\right),$$

Proposition 1 indicates a relation between and $\hat{\chi}_r(\lambda)$ and $\frac{2r\pi}{\lambda} J_1(\lambda r)$. This is not very astonishing because the ordinary Fourier transform of the characteristic function of an Euclidian disk is exasctly $\frac{2r\pi}{\lambda} J_1(\lambda r)$ and because the hyperbolic plane is locally asymptotically Euclidian.

For the proof of Theorem 1 we will only need (4.4) and (4.5). But for completeness we will also prove (4.6) and (4.7) since they can be used to get (6.17).

Proof. Using the representation (4.2) and

$$\sqrt{\cosh(r)^2 - \cosh(t)^2} = \sqrt{r^2 - t^2}\,(1 + f_r(t)),$$

where $f_r(t)$ is monotonic in t and of order $O(r^2)$, $r \to 0$, we get for $r \to 0$

$$\begin{aligned} \hat{\chi}_r(\lambda) &= 4\int_0^r \cos(\lambda t)\sqrt{r^2 - t^2}\, dt + \int_0^r \cos(\lambda t) f_r(t)\, dt \\ &= 2r^2 \int_{-1}^1 \cos(\lambda r u)\sqrt{1 - u^2}\, du + O\left(\min\left(\frac{r^2}{\lambda}, r^3\right)\right) \\ &= \frac{2r\pi}{\lambda}\, J_1(\lambda r) + O\left(\min\left(\frac{r^2}{\lambda}, r^3\right)\right), \end{aligned}$$

since the Bessel function $J_1(x)$ has the integral representation

$$J_1(x) = \frac{x}{\pi} \int_{-1}^{1} \cos(xu)\sqrt{1-u^2}\, du. \tag{4.8}$$

This gives (4.4).

It is easy to see that for $s \to 0+$

$$\begin{aligned}&\sqrt{\sin(s)\sqrt{\sin(s)^2+\sinh(2r)^2}+\sin(s)^2\cosh(2r)^2}\\ &\qquad = \sqrt{s}(1+O(s^2))\sqrt{\sinh(2r)}+O(s)\end{aligned}$$

uniformly in $r > 0$ and that for $\alpha > 0$

$$\int_0^{\pi} e^{-\lambda s} s^{\alpha}\, ds = \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} + O\left(e^{-\sqrt{\lambda}}\right).$$

The asysmptotic expansion of the integral can be proved by splitting up the integral from 0 to $\lambda^{-1/2}$ and from $\lambda^{-1/2}$ to π. This yields

$$I_1(\lambda, r) = \sqrt{\frac{\pi}{8}}\frac{\sqrt{\sinh(2r)}}{\lambda^{3/2}}\left(1+O\left(\frac{1}{\lambda^2}\right)\right)+O\left(\frac{1}{\lambda^2}\right) \quad (\lambda \to \infty).$$

$I_2(\lambda, r)$ can be handled similarly and has the same asymptotic expansion. Thus (4.5) is proved by using the representation (4.3).

The proof of (4.6) is quite similar. You only have to use the property

$$\begin{aligned}&\sqrt{\sin(s)\sqrt{\sin(s)^2+\sinh(2r)^2}+\sin(s)^2\cosh(2r)^2}\\ &\qquad = \sqrt{s}\sqrt{\sinh(2r)}(1+O(s)) \quad (s \to 0, r \geq r_0).\end{aligned}$$

From $e^{-\lambda s} = 1-\lambda s+O(s^2)$ $(\lambda \to 0)$ we derive $I_1(\lambda, r) = A(r)+O(\lambda\sqrt{\sinh(2r)})$ and $I_2(\lambda, r) = \lambda B(r)+O(\lambda^2\sqrt{\sinh(2r)})$ with

$$A(r) = \frac{1}{\sqrt{2}}\int_0^{\pi}\sqrt{\sin(s)\sqrt{\sin(s)^2+\sinh(2r)^2}+\sin(s)^2\cosh(2r)^2}\, ds$$

and

$$\begin{aligned}B(r) = \frac{1}{\sqrt{2}}\int_0^{\pi/2}&(\pi-2s)\times\\ &\times\sqrt{\sin(s)\sqrt{\sin(s)^2+\sinh(2r)^2}-\sin(s)^2\cosh(2r)^2}\, ds.\end{aligned} \tag{4.9}$$

Since

$$\begin{aligned}&\sqrt{\sin(s)\sqrt{\sin(s)^2+\sinh(2r)^2}+\sin(s)^2\cosh(2r)^2}\\ &\qquad = \sqrt{\sinh(2r)}\sqrt{\sin(s)+\sin(s)^2}+O(e^{-2r}),\end{aligned}$$

we have

$$A(r) = \sqrt{\sinh(2r)}\sqrt{2}\int_0^{\pi/2} \sqrt{sin(s) + \sin(s)^2}\, ds + O(e^{-2r})$$

for $r \to \infty$. The situation for $B(r)$ is a little bit more difficult, here we have (after substituting s by $\pi/2 - s$)

$$\begin{aligned}&\sqrt{\cos(s)\sqrt{\cos(s)^2 + \sinh(2r)^2} - \cos(s)^2\cosh(2r)^2}\\ &\qquad = \sqrt{\sinh(2r)}\sqrt{\cos(s) - \cos(s)^2}\left(1 + O\left(\frac{e^{-4r}}{1-\cos(s)}\right)\right)\end{aligned}$$

for $r \to \infty$ and $s \geq c\, e^{-2r}$ for some $c > 0$. Therefore we have to split the integration in (4.9) and estimate seperately to get

$$B(r) = \sqrt{2}\sqrt{\sinh(2r)}\int_0^{\pi/2} s\sqrt{\cos(s) - \cos(s)^2}\, ds + O(e^{-2r}).$$

Now it is easy to reach the expansion (4.7) by using

$$\frac{1}{1-e^{-\lambda\pi}} = \frac{1}{\lambda\pi}\left(1 - \frac{\lambda\pi}{2} + O(\lambda^2)\right) \qquad (\lambda \to 0).$$

5. Proof of Lemma 1 and Lemma 2

Proof of Lemma 1. We will always assume that r, u, v are sufficiently small. This assumption will imply a constant C that is used in the formulation of Lemma 1.

First consider the case $\lambda u \leq \lambda v \leq c_1$, where c_1 is chosen in a way that $J_1(x) >><<$ x for $0 \leq x \leq c_1/2$. This can be done since $\cos(xu) = 1 + O(x^2u^2)$ $(x \to 0)$ and (4.8) inply

$$J_1(x) = \frac{x}{2} + O(x^3). \tag{5.1}$$

In this case (4.4) gives (r is sufficiently small)

$$\varphi_v(\lambda) >><< v^4 \text{ and } \varphi_u(\lambda) >><< u^4. \tag{5.2}$$

Thus

$$\frac{\varphi_v(\lambda)}{\varphi_u(\lambda)} >> \frac{v}{u}\left(\frac{v}{u}\right)^3 \geq \frac{v}{u}.$$

Next consider the case $c_1 \leq \lambda u \leq \lambda v$. Since

$$\frac{1}{y}\int_y^{2y} \sin(\lambda r - 1/4)^2\, dr = \frac{1}{\lambda y}\int_{\lambda y}^{2\lambda y} \sin(u - 1/4)^2\, du >><< 1$$

for $\lambda y \geq c_1$, (4.5) implies

$$\varphi_v(\lambda) >><< \frac{v}{\lambda^3} \text{ and} \varphi_u(\lambda) >><< \frac{u}{\lambda^3} \tag{5.3}$$

for sufficiently large $\lambda \geq c_1/u \geq c_1/C$. But this can be guaranteed by choosing C properly small. Thus (3.4) holds.

The case $0 < \lambda u \leq c_1 \leq \lambda v$ can be proved by combining (5.2) and (5.3) to estimate

$$\frac{\varphi_v(\lambda)}{\varphi_u(\lambda)} >> \frac{v}{u}\frac{1}{(\lambda u)^3} >> \frac{v}{u}.$$

Thus the proof of Lemma 1 is complete.

Proof of Lemma 2. If $u = \frac{1}{2}\text{arsinh}\sqrt{\mu(X)/(2\pi N)}$ then $\mu(B(0,2u)) = \mu(X)/(2N)$ and of course $\mu(B(0,u)) \geq c/N$ for some $c > 0$. Thus we have for all $x \in X$ and $r \in [u, 2u]$

$$\left|\sum_{n=1}^{N} 1_{B(x,r)}(p_n) - N\,\mu_X(B(X,R))\right| \geq c\sum_{n=1}^{N} 1_{B(x,r)}(p_n)$$

and so

$$\begin{aligned}\Phi(u) &\geq \frac{c}{N^2}\frac{1}{u}\int_u^{2u}\int_D \left(\sum_{n=1}^{N} 1_{B(x,r)}(p_n)\right)^2 d\mu(x)\,dr \\ &\geq \frac{c}{N^2}\frac{1}{u}\int_u^{2u}\sum_{n=1}^{N}\int_D 1_{B(x,r)}(p_n)\,d\mu(x)\,dr \\ &\geq \frac{c}{N^2}N\,\mu(B(0,u)) >> \frac{1}{N^2}.\end{aligned}$$

This proves Lemma 2.

6. Generalizations and Further Problems

As it has been indicated in the Introduction Theorem 1 can be proved by a proper inbedding to the Euclidian plane. The reason why this is possible is that hyperbolic circles in Klein's model of the hyperbolic plane are Euclician circles (with another center) in $D = \{z \in \mathbf{C} : |z| < 1\}$.

But now let us define the distance between $\zeta_{\mathcal{P}_N}$ and μ_X using ohter test-sets than discs. Let $Q(r)$ $(r > 0)$ be the convex hull of the four points $R, iR, -R, -iR \in D$ with $R = \text{artanh}(r)$. Then we can define the distance between $\zeta_{\mathcal{P}_N}$ and μ_X by

$$\Delta(\mathcal{P}_N, Q) = \sup_{r>0, g\in G} |\zeta_{\mathcal{P}_N}(g\cdot Q(r)) - \mu_X(g\cdot Q(r))| \tag{6.1}$$

and formulate

Theorem 2. *Let $\mathcal{P}_N \subseteq X$ be a point distribution, then $\Delta(\mathcal{P}_N, Q)$ is bounded below by*

$$\Delta(\mathcal{P}_N, Q) >> N^{-3/4}. \tag{6.2}$$

On the other hand, there exsits a point distribution $\mathcal{P}_N \subseteq X$ such that

$$\Delta(\mathcal{P}_N, Q) << N^{-3/4}(\log N)^{1/2}. \tag{6.3}$$

The constants implied by >> and << are independent of N and the point distribution $\mathcal{P}_N$.

The author can only prove this theorem by using the hyperbolic Fourier transform. Let

$$\hat{\omega}_r(\lambda) = \int\limits_B \widehat{1_{Q(r)}}(\lambda, b)\, db \tag{6.4}$$

be the sherical transform (see [3]) of the characteristic function $1_{Q(r)}(z)$, then $\hat{\omega}_r(\lambda)$ is the hyperbolic Fourier transform of the radial function

$$\omega_r(z) = \int\limits_{\mathrm{SO}(2)} 1_{\tau Q(r)}(z)\, d\tau, \tag{6.5}$$

where $d\tau$ is the volume element of the Haar measure of SO(2). Thus

$$\begin{aligned}
&\int\limits_D \left[\int_{\mathrm{SO}(2)} \left(1_{\tau Q(r)} * (d\zeta_{\mathcal{P}_N} - d\mu_X)\right)(z)\, d\tau\right]^2 dz \\
&\quad = \int\limits_D \left[(\omega_r * (d\zeta_{\mathcal{P}_N} - d\mu_X))(z)\right]^2\, dz \\
&\quad = \int\limits_B \int\limits_{\mathbf{R}^+} \hat{\omega}_r(\lambda)^2 \left|(d\widehat{\zeta_{\mathcal{P}_N} - s}\mu_X)(\lambda, b)\right|^2 \frac{\lambda}{2\pi} \tanh\left(\frac{\lambda\pi}{2}\right)\, d\lambda\, db.
\end{aligned}$$

Therefore we can use the same ideas as in the proof of Theorem 1. We only need an analogon of Lemma 1. It can be shown that

$$\hat{\omega}_r(\lambda) = \int\limits_r^R \frac{8}{\pi}\, \frac{\sinh(2r)}{\cosh(2s)}\, \frac{1}{\sqrt{\cosh(2s)^2 - \cosh(2r)^2}}\, \hat{\chi}_s(\lambda)\, ds, \tag{6.6}$$

where $R = \frac{1}{2}\mathrm{artanh}(\sqrt{2}\tanh(2r))$ and $\hat{\chi}_s(\lambda)$ is defined in (4.1). Hence it is possible to prove a similar estimate to (3.4) by using the asymptoric expansions of Proposition 1.

It would be interesting to prove a theorem similar to Theorem 1 and 2 for a more general class of convex test sets but it seems that the technical effort to do this is enormous.

Now let $\omega : [0,1] \to X$ be a continuous function with finite arc-length $s(\omega)$, define a measure on X by

$$\zeta_\omega(A) = \int\limits_0^1 1_{A\cap X}(\omega(t))\, dt, \tag{6.7}$$

and the distance between ζ_ω and μ_X by

$$\Delta_c(\omega, B) = \sup_{r>0, y\in H} \left|\zeta_\omega(B(y,r)) - \mu_X(B(y,r))\right| \tag{6.8}$$

or by

$$\Delta_c(\omega, Q) = \sup_{r>0, g\in G} \left|\zeta_\omega(g\cdot Q(r)) - \mu_X(g\cdot Q(r))\right|. \tag{6.9}$$

Combining the methods of [2] and the concept developed in this paper one can prove

Theorem 3. *If $\omega : [0,1] \to X$ is a continuous function with finite arclength $s = s(\omega) \geq 1$, then we have*

$$\left.\begin{array}{l}\Delta_c(\omega, B)\\ \Delta_c(\omega, Q)\end{array}\right\} >> s^{-3/2}. \tag{6.10}$$

On the other hand, if $s \geq 1$, then there exist continuous functions $\omega_B, \omega_Q : [0,1] \to X$ with $s(\omega_B) = s(\omega_Q) = s$ with

$$\Delta_c(\omega_B, B) << s^{-3/2}(\log s)^{1/2} \tag{6.11}$$

and

$$\Delta_c(\omega_Q, Q) << s^{-3/2}(\log s)^{1/2}. \tag{6.12}$$

The constants implied by $>>$ and $<<$ are independent of ω and $s = s(\omega) \geq 1$.

Another generalization where it seems to be impossible to find a proper imbedding to the Euclidian plane if following situation. Let $\mathcal{P} = \{p_1, p_2, \dots\}$ be an infinte discrete subset of H and

$$D_R(\mathcal{P}, B) = \sup_{0<r\leq R, x\in H} \left| \sum_{n=1}^{\infty} 1_{B(x,r)}(p_n) - \mu(B(x,r)) \right| \tag{6.13}$$

the distance between the counting measure

$$\zeta(A) = \sum_{n=1}^{\infty} 1_A(p_n) \tag{6.14}$$

and the Haar measure μ.

If we want to use the hyperbolic Fourier transform to find an estimate for $D_R(\mathcal{P}, B)$ we have to restrict these two measures to

$$\zeta_M(A) = \zeta(A \cap B(0,M)) \qquad \text{and} \qquad \mu_M(A) = \mu(A \cap B(0,M)). \tag{6.15}$$

Here it can be proved that

$$\begin{aligned}\Xi(M,R) &= \int_0^R \int_D \left|(\chi_r * (d\zeta_M - d\mu_M))(z)\right|^2 \, d\mu(z)\, dr \\ &= \int_{\mathbf{R}^+} \int_0^R \hat{\chi}_r(\lambda)^2 \, dr \int_B \left|(\widehat{d\zeta_M - d\mu_M})(\lambda, b)\right|^2 db \, \frac{\lambda}{2\pi} \tanh\left(\frac{\lambda\pi}{2}\right) d\lambda \\ &>> \frac{e^{2(M+R)}}{R^3}\end{aligned} \tag{6.16}$$

by using the estimate

$$\min_{0\leq\lambda\leq\Lambda} \int_0^R \hat{\chi}_r(\lambda)^2 \, dr >> \frac{\sinh(2R)}{\Lambda^3} \qquad (R \to \infty) \tag{6.17}$$

and some explicit hyperbolic Fourier transforms. It is a pity that (6.16) only gives a trivial lower bound for $D_R(\mathcal{P}, B)$. In addition it is impossible to improve (6.16) essentially because there exists a point distribution $\mathcal{P}_N \subseteq B(0, M)$, $N = [\pi(\sinh(M))^2]$, such that

$$\Xi(M, R) << e^{2(M+R)}. \tag{6.18}$$

(The mean value of all possible point distributions can be estimated by $e^{2(M+R)}$.) This is a little bit surprising since it is only possible to find a point distribution $\mathcal{P} \subseteq H$ by probabilistic methods as in [1] such that

$$D_R(\mathcal{P}, B) << e^R \sqrt{R}. \tag{6.19}$$

Maybe (6.19) is not essentially sharp. (6.18) gives reason to conjecture that there are point distributions with a smaller upper bound of $D_R(\mathcal{P}, B)$. This would be a significant difference to the results known for the Euclidian plane (compare with [1]). If we could restate those results for the hyperbolicv plane e^R would be the "natural" lower bound for $D_R(\mathcal{P}, B)$. It seems that the solution of this problem efforts other methods as used in this paper.

References

[1] Beck, J., and W. Chen, *Irregularities of Distribution*, Cambridge University Press, Cambridge—New York, 1987.

[2] Drmota, M., *Irregularities of continuous distributions*, Ann. Inst. Fourier (3) **39** (1989), 501–527.

[3] Helgason, S., *Topics in Harmonic Anaslysis on Homogeneous Spaces*, Birkhäuser, Boston—Basel—Stuttgart, 1981.

Michael Drmota
Department for Discrete Mathematics
Technical University of Vienna
Wiedner Hauptstrasse 8–10/118
A-1040 Vienna,AUSTRIA