ON GENERALIZED INDEPENDENT SUBSETS OF TREES

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ABSTRACT. A natural generalization of the widely discussed independent (or "internally stable") subsets of graphs is to consider subsets of vertices where no two elements have distance less or equal to a fixed number k ("k-independent subsets"). In this paper we give asymptotic results on the average number of k-independent subsets for trees of size n, where the trees are taken from a so-called simply generated family. This covers a lot of interesting examples like binary trees, general planted plane trees and others.

1. INTRODUCTION

A subset I of a graph G is called independent if no two adjacent vertices of G occur in I. In a previous paper ([5]) the total number of independent subsets in certain families of trees of size n has been studied. Recently ([6]) Meir and Moon investigated the number of maximal independent subsets.

An interesting and, as it turns out, difficult generalization of the problem studied in [5] is to consider subsets that obey the following more restrictive conditions:

We call a subset J of G *k-independent* if no two vertices of J have distance $\leq k$.

Thus for $k = 1$ we gain the usual notion of independence.

Besides of their purely graph-theoretical interest, independent subsets of graphs are also important in applications like statistical models that occur in Solid State Physics. Our notion means that we fix a certain minimum distance between selected points of a graph in this model.

To get a first insight what will happen for $k > 1$ let us consider the example of the chain:

Denoting vertices that are selected for k-independent subset by   and the other vertices by   , we may produce all situations by the formal rule

FIGURE 1.

Thus the generating function of the numbers $y_{n,m}^{(k)}$ of k-independent subsets of size m in the chain of n vertices is given by

$$\sum_{n\geq 1, m\geq 0} y_{n,m}^{(k)}\, x^n\, z^m = \frac{1}{1-x}\left(x + \frac{xz}{1-x-x^{k+1}z}\right). \tag{1.1}$$

(Compare [2], [3] for this technique.) If we are interested in the total number of k-independent subsets we have to evaluate at $z = 1$, and get, for an asymptotic estimate, the dominating singularity x_k as the smallest positive solution of the equation

$$1 - x - x^{k+1} = 0. \tag{1.2}$$

From that, it is easy to see that x_k behaves for k getting large as

$$x_k = 1 - \frac{\log k - \log\log k}{k} + O\left(\frac{\log\log k}{k\log k}\right), \tag{1.3}$$

and

$$y_n^{(k)} = \sum_{m\ge 0} y_{n,m}^{(k)} \sim D^{(k)}\, x_k^{-n}, \quad n\to\infty,\ k \text{ fixed}, \tag{1.4}$$

where the constants $D^{(k)}$ behave like

$$D^{(k)} \sim \frac{k}{\log^2 k}, \quad k\to\infty. \tag{1.5}$$

It is the purpose of this paper to give enumerative results on k-independent subsets for a wide class of trees, that contains instances as binary trees, Motzkin trees or general planted plane trees:

Let $\mathcal{F}$ denote a family of rooted trees and $\mathcal{F}_n$ the subset of $\mathcal{F}$ of trees of size $|T| = n$ (i.e. with n vertices). Such a family is called simply generated family of trees if the generating function

$$Y(x) = \sum_{n\ge 1} y_n x^n \tag{1.6}$$

of the sequence $y_n = |\mathcal{F}_n|$ satisfies a functional equation of the type

$$Y = x\varphi(Y)\ , \tag{1.7}$$

where

$$\varphi(t) = 1 + \varphi_1 t + \varphi_2 t^2 + \cdots \tag{1.8}$$

is a power series with non-negative coefficients. The coefficients of $\varphi(t)$ can be used to define weights $\omega(T)$ of rooted trees T such that $\omega(T) > 0$ if and only if $T\in\mathcal{F}$. Let $n_j(T)$ denote the number of vertices of T incident with j edges leading away from the root, then the weight $\omega(T)$ can be defined by ($\varphi_0 = 1$)

$$\omega(T) = \prod_{j\ge 0} \varphi_j^{n_j(T)}\ . \tag{1.9}$$

Now it is easy to see that

$$\sum_{n\ge 1}\left(\sum_{|T|=n}\omega(T)\right)x^n$$

is the solution of (1.7) analytic around $x = 0$. Thus

$$y_n = \sum_{|T|=n}\omega(T)\ . \tag{1.10}$$

For example, we have $\varphi(t) = 1 + t^2$ for binary trees, $\varphi(t) = 1 + t + t^2$ for Motzkin trees, $\varphi(t) = 1/(1-t)$ for plane trees, and $\varphi(t) = e^t$ for non-planar labelled trees.

In section 2 of this paper we set up well-suited generating functions for the number of k-independent subsets of cardinality m in trees of size n.

In section 3 we prove, as our main result, an asymptotic formula for the total number of k-independent subsets in trees of size n (Theorem 1) and asymptotic result on the average size of k-independent subsets (Theorem 2).

In section 4 we apply the general results to several types of trees.

In a final section we give some concluding remarks.

2. Generating Functions

In this section we adopt the method of immediate translation of combinatorial constructions into functional relations that is widely used in recent combinatorial enumeration (compare [2] or [3]). Given a certain simply generated family of trees it is convenient to distinguish the following types of k-independent subsets:

$\mathcal{A}^{(k)}$ ... the k-independent subsets containing the root of the tree

$\mathcal{B}^{(k)}$ ... the k-independent subsets *not* containing the root of the tree

$\mathcal{C}^{(k)}$ ... the k-independent subsets which do not contain a vertex of level $\le \lfloor \frac{k}{2} \rfloor$ of the tree (where the root shall have level 1, the immediate successors of the root have level 2, ...)

The reason for the special importance of the families $\mathcal{C}^{(k)}$ is that a k-independent subset of a tree may contain at most one vertex from the levels $1, 2, \dots, \lfloor \frac{k}{2} \rfloor$ of the tree.

We start by a combinatorial decomposition of the family $\mathcal{C}^{(k)}$: For this purpose we introduce the families $\mathcal{L}_j(\mathcal{U})$ of subsets of vertices that do not contain any vertex from the level $1, 2, \dots, j$ and for which the subtrees beyond are chosen from a certain family $\mathcal{U}$:

FIGURE 2

In order to describe the combinatorial structure of $\mathcal{C}^{(k)}$ we start with the instance $k = 2j$, even:

A subset in $\mathcal{C}^{(2j)}$ has either exactly one vertex from level $j+1$ or it has $j+1$ free levels. In the first case the subset is of the type

FIGURE 3

whereas in the second case we have the structure

FIGURE 4

This combinatorial decomposition translates into the language of generating functions in the following way:

$$\begin{aligned} C^{(2j)}(x,z) = & A^{(2j)}(x,z) \cdot x\,\varphi'(L_{j-1}) \cdot \\ & x\,\varphi'(L_{j-2}) \cdots x\,\varphi'(L_1) x \varphi'(L_0)|_{u=C^{(2j)}(x,z)} \\ & + x\,\varphi(C^{(2j)}(x,z)) \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} L_0(x,u) &= u \\ L_{i+1}(x,u) &= x\,\varphi(L_i(x,u)), \ i \ge 0. \end{aligned} \tag{2.2}$$

Here the coefficient $[x^n z^m]\, C^{(2j)}(x,z)$ is the weighted sum

$$\sum_{|T|=n} \omega(T)\, c_m^{(2j)}(T)$$

of the numbers $c_m^{(2j)}(T)$ of subsets of vertices of size m and type $\mathcal{C}^{(2j)}$ in the tree T. Similarly $A^{(2j)}(x,z)$ is the generating function for the subsets of type $\mathcal{A}^{(2j)}$, and $L_i(x,u)$ the generating function for weighted objects of type $\mathcal{L}^{(i)}(\mathcal{U})$.

In order to gain the first term of (2.1) from Figure 3 it should be noticed that the product of derivatives of φ corresponds to the choice of the path from the root to the unique node in level $j+1$ that is contained in the $2j$-independent subset.

Equality (2.1) may be rewritten as

$$\begin{aligned} C^{(2j)}(x,z) = A^{(2j)}(x,z) \; &\frac{\partial L_j(x,u)}{\partial u}\bigg|_{u=C^{(2j)}(x,z)} \\ &+ x\,\varphi\left(C^{(2j)}(x,z)\right) \end{aligned} \tag{2.3}$$

In a similar manner we find for $k = 2j+1$, odd, that

$$\begin{aligned} C^{(2j+1)}(x,z) = A^{(2j+1)}(x,z) \; &\frac{\partial L_j(x,u)}{\partial u}\bigg|_{u=x\,\varphi(C^{(2j+1)}(x,z))} \\ &+ x\,\varphi\left(C^{(2j+1)}(x,z)\right). \end{aligned} \tag{2.4}$$

Having established this functional equation for $C^{(k)}(x,z)$ we turn to the families $\mathcal{A}^{(k)}$ and $\mathcal{B}^{(k)}$:
For $\mathcal{A}^{(k)}$ and $k = 2j$, even we have:

FIGURE 5

so that

$$\begin{aligned} A^{(2j)}(x,z) &= xz\,\varphi(L_j)|_{u=C^{(2j)}(x,z)} \\ &= z\,L_{j+1}|_{u=C^{(2j)}(x,z)}. \end{aligned} \tag{2.5}$$

Similarly for $k = 2j+1$, odd,

$$\begin{aligned} A^{(2j+1)}(x,z) &= xz\,\varphi(L_j)|_{u=x\,\varphi(C^{(2j+1)}(x,z))} \\ &= z\,L_{j+2}|_{u=C^{(2j+1)}(x,z)}. \end{aligned} \tag{2.6}$$

For $\mathcal{B}^{(k)}$, $k = 2j$ we decompose the set with regard to the first level in the tree from which a vertex is chosen and get (analogously to the equation for $\mathcal{C}^{(k)}$):

$$\begin{aligned} B^{(2j)}(x,z) = \Big[&A^{(2j)}(x,z)\, x\,\varphi'(L_{j-1}) \\ &+ A^{(2j)}(x,z)\, x\,\varphi'(L_{j-1})\, x\,\varphi'(L_{j-2}) \\ &+ \dots \\ &+ A^{(2j)}(x,z)\, x\,\varphi'(L_{j-1}) \cdots x\,\varphi'(L_1) \Big]_{u=C^{(2j)}(x,z)} \\ &+ C^{(2j)}(x,z) \end{aligned} \tag{2.7}$$

as well as

$$
\begin{aligned}
B^{(2j+1)}(x,z) = \Big[&A^{(2j+1)}(x,z)\, x\,\varphi'(L_{j-1}) \\
&+ A^{(2j+1)}(x,z)\, x\,\varphi'(L_{j-1})\, x\,\varphi'(L_{j-2}) \\
&+ \dots \\
&+ A^{(2j+1)}(x,z)\, x\,\varphi'(L_{j-1}) \cdots x\,\varphi'(L_1)\Big]_{u=x\,\varphi(C^{(2j+1)}(x,z))} \\
&+ C^{(2j+1)}(x,z).
\end{aligned} \tag{2.8}
$$

For the following it is convenient to introduce the notations

$$
\begin{aligned}
F^{(k)}(x,u) &= \begin{cases} 0 & \text{for } k=0 \\ x\,\varphi(L_j(x,u))\,\dfrac{\partial L_j}{\partial u}(x,u) & \text{for } k=2j,\ j\ge 1 \\ x\,\varphi(L_j(x,x\varphi(u)))\,\dfrac{\partial L_j}{\partial u}(x,x\varphi(u)) & \text{for } k=2j+1,\ j\ge 0 \end{cases} \\
G^{(k)}(x,u) &= \begin{cases} x\,\varphi(L_j(x,u)) & \text{for } k=2j \\ x\,\varphi(L_j(x,x\varphi(u))) & \text{for } k=2j+1 \end{cases} \\
H^{(k)}(x,u) &= \begin{cases} x\,\varphi(L_j(x,u)) \displaystyle\sum_{i=1}^{j-1}\prod_{l=1}^{i} x\,\varphi'(L_{j-l}(x,u)) & \text{for } k=2j \\ x\,\varphi(L_j(x,x\varphi(u))) \displaystyle\sum_{i=1}^{j-1}\prod_{l=1}^{i} x\,\varphi'(L_{j-l}(x,x\varphi(u))) & \\ & \text{for } k=2j+1 \end{cases}
\end{aligned} \tag{2.9}
$$

so that the functional equations reduce to

$$
\begin{aligned}
C^{(k)} &= z\,F^{(k)}(x,C^{(k)}) + x\,\varphi(C^{(k)}) \\
A^{(k)} &= z\,G^{(k)}(x,C^{(k)}) \\
B^{(k)} &= z\,H^{(k)}(x,C^{(k)}) + C^{(k)}
\end{aligned} \tag{2.10}
$$

for $k \ge 1$.

3. Asymptotic Results

In this section we will prove the following two main results for simply generated families of trees where $\varphi(t)$ from (1.8) has the properties

i) $\varphi(t)$ has positive radius of convergence R
ii) $\varphi_j > 0$ for some $j \ge 2$
iii) there exists a τ with $0 < \tau < R$ such that $\tau\varphi'(\tau) = \varphi(\tau)$

and

iv)

$$\lim_{t\to R-} \frac{1}{\varphi'(t)} \le \frac{1}{2\,\varphi'(\tau)}.$$

Theorem 1. *The average total number $y_n^{(k)}$ of k-independent subsets of vertices in simply generated trees of size n fulfills the following asymptotic relation: Let $d = gcd\{j \mid \varphi_j > 0\}$. Then*

$$y_n^{(k)} \sim D^{(k)} \cdot \left(\frac{x_0}{x_k}\right)^n, \quad n \to \infty,\ n \equiv 1 (mod\, d) \tag{3.1}$$

(and $y_n^{(k)} = 0$ otherwise), where (x_k, u_k) is the solution with smallest positive x of the system

$$\begin{aligned} u &= F^{(k)}(x,u) + x\,\varphi(u) \\ 1 &= \frac{\partial F^{(k)}}{\partial u}(x,u) + x\,\varphi'(u) \end{aligned} \tag{3.2}$$

with $F^{(k)}(x,u)$ defined in (2.9), and where $D^{(k)}$ is given by

$$\begin{aligned} D^{(k)} = &\left(\frac{\varphi''(u_0)}{\varphi(u_0)} \frac{x_k\left(\frac{\partial F^{(k)}}{\partial u}(x_k,u_k) + \varphi(u_k)\right)}{\frac{\partial^2 F^{(k)}}{\partial u^2}(x_k,u_k) + x_k\,\varphi''(u_k)}\right)^{1/2} \times \\ &\times \left(\frac{\partial G^{(k)}}{\partial u}(x_k,u_k) + \frac{\partial H^{(k)}}{\partial u}(x_k,u_k) + 1\right). \end{aligned} \tag{3.3}$$

The constants x_k resp. $D^{(k)}$ have the following asymptotic behaviour for $k \to \infty$:

$$x_k = x_0(1 - r_k) \tag{3.4}$$

where

$$\begin{aligned} \sqrt{r_k} = &\frac{2}{\mu_0}\left(\frac{\log k}{k} - \frac{\log\log k}{k}\right) + \frac{\log\frac{16}{27} + 2\log\mu_0}{\mu_0}\frac{1}{k} \\ &+ O\left(\frac{1}{k\,\log\log k}\right) \end{aligned} \tag{3.5}$$

with

$$\mu_0 = x_0\sqrt{\frac{\varphi(u_0)\varphi''(u_0)}{2}}, \tag{3.6}$$

and

$$D^{(k)} = \frac{3}{8}\sqrt{\frac{3}{2}}\,\frac{1}{\mu_0}\,(\log k)^{3/2} + O\left((\log k)^{1/2}\log\log k\right). \tag{3.7}$$

Remark 1. The reader should take notice of the fact that the asymptotic behaviour of the singularities x_k differs significantly from the behaviour of x_k for the chain (which was studied in the introduction). The reason is that we are now considering families that contain vertices of degree greater than 1 (i.e. $\varphi''(t) > 0$ for $t > 0$), which constitutes a different analytic behaviour of the generating functions.

As an immediate consequence of Theorem 1 we have

Corollary 1. *The quantities $y_n^{(k)}$ from Theorem 1 obey for $k \to \infty$*

$$\lim_{\substack{n\to\infty \\ n\equiv 1 (\mathrm{mod}\ d)}} \frac{1}{n}\log y_n^{(k)} = \frac{4}{\mu_0^2}\frac{\log^2 k}{k^2} + O\left(\frac{\log k\,\log\log k}{k^2}\right). \tag{3.8}$$

The second main result is

Theorem 2. *The average size $m_n^{(k)}$ of k-independent subsets of vertices in simply generated trees of size n fulfills*

$$m_n^{(k)} \sim E^{(k)} \cdot n, \quad n \to \infty,\ n \equiv 1 (mod\, d), \tag{3.9}$$

where the constants

$$E^{(k)} = \frac{F^{(k)}(x_k, u_k)}{x_k \left(\frac{\partial F^{(k)}}{\partial x}(x_k, u_k) + \varphi(u_k) \right)} \tag{3.10}$$

satisfy (for $k \to \infty$)

$$E^{(k)} = \frac{4}{\mu_0^2} \frac{\log k - \log\log k}{k^2} + O\left(\frac{1}{k^2}\right). \tag{3.11}$$

Remark 2. It is remarkable that the ratio between the total number $\alpha_n^{(k)}$ of k-independent subsets (in trees of size n) that contain the root and the total number $\beta_n^{(k)}$ of k-independent subsets that do not contain the root satisfies an asymptotic relation similar to the ratio between $m_n^{(k)}$ and n:

$$\frac{\alpha_n^{(k)}}{\beta_n^{(k)}} \sim \overline{E}^{(k)}, \quad n \to \infty, n \equiv 1 (\operatorname{mod} d), \tag{3.12}$$

where the constants

$$\overline{E}^{(k)} = \frac{\frac{\partial G^{(k)}}{\partial u}(x_k, u_k)}{\frac{\partial H^{(k)}}{\partial u}(x_k, u_k) + 1} \tag{3.13}$$

satisfy (for $k \to \infty$)

$$\overline{E}^{(k)} = \frac{4}{\mu_0^2} \frac{\log k - \log\log k}{k^2} + O\left(\frac{1}{k^2}\right). \tag{3.14}$$

(This is an easy consequence of equations (3.23), (3.24), and (3.58).)

Proof of Theorem 1. In the first part of the proof we deduce that the quantities $y_n^{(k)}$ obey an asymptotic relation of type (3.1): Let $d = \gcd\{j \mid \varphi_j > 0\}$. In the following we will only consider the case $d = 1$ (which will imply uniqueness of singularities on the circle of convergence). For $d > 1$ all calculations follow the same lines and the results remain unchanged since for $n \equiv 1 (\operatorname{mod} d)$ the occurring factors d cancel.

For $k = 0$ (3.2) reduces to the system

$$\begin{aligned} u &= x\,\varphi(u) \\ 1 &= x\,\varphi'(u), \end{aligned} \tag{3.15}$$

so that x_0 is the radius of convergence of the generating function $Y(x)$ of the numbers y_n of weighted trees of size n and $u_0 = \tau < R$ from condition iii) at the beginning of this section.

Now let $k \geq 1$. Our next aim is to use Theorem 1 of [7] for the functional equation

$$u = F^{(k)}(x,u) + x\,\varphi(u) \tag{3.16}$$

(From (2.10) we know that $c^{(k)}(x) = C^{(k)}(x,1)$ fulfills (3.16).) For this purpose let $\mathcal{D}_k$ denote the interior of the set of points such that the series $F^{(k)}(x,u) + x\,\varphi(u)$ converges absolutely and $\mathcal{S}_k$ the set of points (x,u) in $\mathcal{D}_k$ with positive coordinates that fulfill (3.2).

First $d = \gcd\{j \,|\, \varphi_j > 0\} = 1$ implies that there exist arbitrarily large $l > j > i$ with $\gcd(j-i, l-i) = 1$ and $y_i y_j y_l > 0$. From the combinatorial interpretation it follows that the same is true for the coefficients of $c^{(k)}(x)$. Furthermore all these coefficients are non-negative. From (2.9) we can conclude that the coefficients of $F^{(k)}(x,u) + x\,\varphi(u)$ are non-negative, too. Therefore in order to apply Theorem 1 of [7] it suffices to check that $\mathcal{S}_k$ is not empty. Observe that

$$\mathcal{D} = \left\{(x,u) \,|\, 0 < u < R,\ 0 < x < \frac{u}{\varphi(u)}\right\} \subseteq \mathcal{D}_k.$$

We will show that there is a solution (ξ,ρ) of (3.2) in $\mathcal{D}$. Let $u \in (0,R)$. Since $F^{(k)}(0,u) + 0 \cdot \varphi(u) = 0$, $F^{(k)}\left(\frac{u}{\varphi(u)}, u\right) + \frac{u}{\varphi(u)}\varphi(u) > u$, and the mapping $x \to F^{(k)}(x,u) + x\,\varphi(u)$ is monotone, there exists a unique $x_1^{(k)}(u)$ satisfying

$$0 < x_1^{(k)}(u) < \frac{u}{\varphi(u)}$$

and

$$u = F^{(k)}\left(x_1^{(k)}(u), u\right) + x_1^{(k)}(u)\,\varphi(u).$$

Furthermore the implicit function theorem shows that $x_1^{(k)}$ is a C^∞-function in $(0,R)$. In addition we have

$$F^{(k)}\left(\frac{u_0}{2\,\varphi(u_0)}, u_0\right) \leq \frac{u_0}{2\,\varphi(u_0)}\,\varphi(u_0)\left(\frac{u_0}{2\,\varphi(u_0)}\,\varphi'(u_0)\right)^{\lfloor k/2 \rfloor} \leq \frac{u_0}{2},$$

so that

$$x_1^{(k)}(u_0) \geq \frac{u_0}{2\,\varphi(u_0)} = \frac{x_0}{2} = \frac{1}{2\,\varphi'(u_0)}. \tag{3.17}$$

In the same way we can define a C^∞-function $x_2^{(k)}$ in $(R^{(k)}, R)$ with some $R^{(k)} < u_0$ satisfying

$$0 < x_2^{(k)}(u) < \frac{1}{\varphi'(u)}$$

and

$$1 = \frac{\partial F^{(k)}}{\partial u}(x_2^{(k)}(u), u) + x_2^{(k)}(u)\,\varphi'(u).$$

Since

$$1 = \frac{\partial F^{(k)}}{\partial u}(x_1^{(k)}, u) + x_1^{(k)}\,\varphi'(u) + \left(\frac{\partial F^{(k)}}{\partial x}(x_1^{(k)}, u) + \varphi(u)\right)\frac{dx_1^{(k)}}{du},$$

and

$$\frac{\partial F^{(k)}}{\partial x}(x_1^{(k)},u)+\varphi(u)\ge\varphi(u)>0$$

we have $\frac{dx_1^{(k)}}{du}(u)=0$ if and only if $x_2^{(k)}(u)$ exists and $x_1^{(k)}(u)=x_2^{(k)}(u)$ or if there exists a solution $(\xi,\rho)\in\mathcal{D}$ with $\rho=u$ and $\xi=x_1^{(k)}(u)$. Now suppose that $\mathcal{D}\cap\mathcal{S}_k$ is void. Then $\frac{dx_1^{(k)}}{du}(u)>0$ for $u\in(0,R)$ and therefore (3.17) implies that

$$\lim_{u\to R-}x_1^{(k)}(u)>x_1^{(k)}(u_0)\ge\frac{1}{2\,\varphi'(u_0)}. \tag{3.18}$$

On the other hand we know that

$$\liminf_{u\to R-}x_2^{(k)}(u)\le\lim_{u\to R-}\frac{1}{\varphi'(u)}\le\frac{1}{2\,\varphi'(\tau)}=\frac{1}{2\,\varphi'(u_0)}.$$

But, since $\lim_{u\to0+}x_1^{(k)}(u)=0$, the last inequality combined with (3.18) is a contradiction to the assumption that there is no common solution (ξ,ρ) of (3.2) in $\mathcal{D}$.

Therefore we can use Theorem 1 of [7] to conclude that there is a unique solution (x_k,u_k) of (3.2), and x_k is the radius of convergence of $c^{(k)}(x)$ and the only singularity of the circle $|x|=x_k$. Furthermore $c^{(k)}(x_k)=u_k$.

Since

$$\frac{\partial^2F^{(k)}}{\partial u^2}(x_k,u_k)+x_k\varphi''(u_k)\ge x_k\varphi''(u_k)>0$$

$c^{(k)}(x)$ has the following asymptotic behaviour about x_k:

$$c^{(k)}(x)=u_k-c_1^{(k)}(x_k-x)^{1/2}+c_2^{(k)}(x_k-x)+O\left((x_k-x)^{3/2}\right), \tag{3.19}$$

where the positive constant $c_1^{(k)}$ is determined by

$$\left(c_1^{(k)}\right)^2=\frac{2\left(\frac{\partial F^{(k)}}{\partial x}(x_k,u_k)+\varphi(u_k)\right)}{\frac{\partial^2F^{(k)}}{\partial u^2}(x_k,u_k)+x_k\,\varphi''(u_k)}. \tag{3.20}$$

Let $a^{(k)}(x)=A^{(k)}(x,1)$ and $b^{(k)}(x)=B^{(k)}(x,1)$. Then, from (2.10),

$$\begin{aligned}a^{(k)}(x)&=G^{(k)}\left(x,c^{(k)}(x)\right)\\&=a_0^{(k)}-a_1^{(k)}(x_k-x)^{1/2}+a_2^{(k)}(x_k-x)+O\left((x_k-x)^{3/2}\right)\end{aligned} \tag{3.21}$$

as well as

$$\begin{aligned}b^{(k)}(x)&=H^{(k)}\left(x,c^{(k)}(x)\right)+c^{(k)}(x)\\&=b_0^{(k)}-b_1^{(k)}(x_k-x)^{1/2}+b_2^{(k)}(x_k-x)+O\left((x_k-x)^{3/2}\right),\end{aligned} \tag{3.22}$$

where

$$a_1^{(k)}=\frac{\partial G^{(k)}}{\partial x}(x_k,u_k)\cdot c_1^{(k)} \tag{3.23}$$

and

$$b_1^{(k)} = \left(\frac{\partial H^{(k)}}{\partial x}(x_k, u_k) + 1 \right) \cdot c_1^{(k)}. \tag{3.24}$$

Thus, from Darboux's theorem [4] the coefficient of $a^{(k)}(x) + b^{(k)}(x)$ (which gives the total number of k-independent subsets) fulfills

$$[x^n] \left(a^{(k)}(x) + b^{(k)}(x) \right) \sim \frac{1}{2} \sqrt{\frac{x_k}{\pi}} \left(a_1^{(k)} + b_1^{(k)} \right) x_k^{-n} \, n^{-3/2}. \tag{3.25}$$

Since

$$y_n = [x^n] \, Y(x) \sim \frac{1}{2\sqrt{\pi}} \left(\frac{2\,\varphi(u_0)}{\varphi''(u_0)} \right)^{1/2} x_0^{-n} \, n^{-3/2} \tag{3.26}$$

we finally get (3.1) and (3.3).

Now we turn our attention to the asymptotic behaviour of the singularity x_k of $c^{(k)}(x)$ for $k \to \infty$: We confine our discussion to the instance $k = 2j$ even (k odd may be treated in an analogue manner).

First let us motivate the asymptotic expansion of $\sqrt{r_{2j}}$.
If $x = x_0(1-r)$ ($r \geq 0$ then the solution of $\overline{u} = x\,\varphi(\overline{u})$ has an expansion of the form

$$\overline{u}(x) = u_0 - \frac{1}{\nu_0}\sqrt{r} + O(r), \qquad (r \to \infty) \tag{3.27}$$

where

$$\nu_0 = \sqrt{\frac{\varphi''(u_0)}{2\,\varphi(u_0)}}. \tag{3.28}$$

Let $u = u_0 - s$ with $0 \leq s \leq (1-\varepsilon)\nu_0^{-1} \cdot \sqrt{r}$ ($0 < \varepsilon < 1/2$ small but fixed). This guarantees that $x\,\varphi(u) < u$ and $x\,\varphi'(u) < 1$ for $r \to 0+$. Moreover we have $\overline{u} > u$ and

$$\begin{aligned} \alpha = x\,\varphi'(\overline{u}) &= 1 - x_0\sqrt{2\,\varphi(u_0)\,\varphi''(u_0)}\,\sqrt{r} + O(r) \\ &= 1 - 2\mu_0\sqrt{r} + O(r) \qquad (r \to \infty). \end{aligned} \tag{3.29}$$

The crucical part of the proof is to find a correct asymptotic expansion for $L_i(x,u)$. We know that $\lim_{i\to\infty} L_i(x,u) = \overline{u}(x)$. Writing $L_i = \overline{u} + w_i$ and using $L_{i+1} = x\,\varphi(L_i)$ we get

$$w_{i+1} = \alpha\, w_i + \beta\, w_i^2 + O(w_i^3), \tag{3.30}$$

with $w_0 = \overline{u} - u = s - \sqrt{r}/\nu_0 + O(r) = O(\sqrt{r})$ and $\beta = x\,\varphi''(\overline{u})/2$. It is easy to derive $w_i = O(w_0\,\alpha^i)$ as a first estimate. To get a better one, set $q_i = \alpha^i/w_i$, following a method of DeBruijn [1]. This yields

$$q_{i+1} = q_i - \beta\,\alpha^{i-1} + O(w_i \alpha^i) \tag{3.31}$$

and therefore

$$q_i = \frac{1}{w_0} - \frac{\beta}{\alpha}\,\frac{1-\alpha^i}{1-\alpha} + O\left(w_0\,\frac{1-\alpha^{2i}}{1-\alpha^2} \right). \tag{3.32}$$

For w_i this translates to

$$\begin{aligned} w_i &= \frac{w_0\,\alpha^i}{1-\frac{w_0\beta}{\alpha}\,\frac{1-\alpha^i}{1-\alpha}+O\left(w_0^2\,\frac{1-\alpha^{2i}}{1-\alpha^2}\right)} \\ &= \frac{w_0\,\alpha^i}{\frac{\sqrt{r}+\nu_0 s}{2\sqrt{r}}+\frac{\sqrt{r}-\nu_0 s}{2\sqrt{r}}\,\alpha^i+O(\sqrt{r})} \qquad (r\to\infty) \end{aligned} \tag{3.33}$$

with O-constants uniform with respect to i. Next we have to discuss

$$\begin{aligned} \frac{\partial L_i}{\partial u}(x,u) &= x\,\varphi'(L_0)\,x\,\varphi'(L_1)\cdots x\,\varphi'(L_{i-1}) \\ &= \alpha^i\prod_{l=0}^{i-1}\left(1+\frac{\varphi''(\overline{u})}{\varphi'(\overline{u})}\,w_l+O(w_l^2)\right) \\ &= \alpha^i\,\exp\left(\frac{\varphi''(\overline{u})}{\varphi'(\overline{u})}\sum_{l=0}^{i-1}w_l\right)(1+O(\sqrt{r})). \end{aligned} \tag{3.34}$$

Since

$$\begin{aligned} \frac{\varphi''(\overline{u})}{\varphi'(\overline{u})}\sum_{l=0}^{i-1}w_l &= \frac{\varphi''(u_0)}{\varphi'(u_0)}\,w_0\int_0^i\frac{\alpha^x}{A+B\alpha^x}\,dx\cdot\big(1+O(\sqrt{r})\big) \\ &= -2\,\log(A+B\alpha^i)+O(\sqrt{r}), \end{aligned} \tag{3.35}$$

where

$$A=\frac{\sqrt{r}+\nu_0\,s}{2\sqrt{r}}\quad\text{and}\quad B=\frac{\sqrt{r}-\nu_0\,s}{2\sqrt{r}}, \tag{3.36}$$

we get

$$\frac{\partial L_i}{\partial u}(x,u)=\frac{\alpha^i}{(A+B\alpha^i)^2}\,\big(1+O(\sqrt{r})\big)\qquad(r\to 0+). \tag{3.37}$$

and finally

$$F^{(2j)}(x,u)=\frac{u_0\,\alpha^i}{(A+B\alpha^i)^2}\,\big(1+O(\sqrt{r})\big)\qquad(r\to 0+), \tag{3.38}$$

where the O-constant can be chosen independently of i. Using the expansion

$$u-x\,\varphi(u)=u_0(r-\nu_0^2\,s^2)(1+O(\sqrt{r})) \tag{3.39}$$

we may conclude that $\alpha^j=O(r)$ for $x=x_{2j}$ and $u=u_{2j}$ and that

$$F^{(2j)}(x,u)=\frac{u_0}{A^2}\,\alpha^j\,\big(1+O(\sqrt{r})\big)\qquad(r\to 0+). \tag{3.40}$$

Furthermore we will need

$$\frac{\partial F^{(2j)}}{\partial u}(x,u)=F^{(2j)}(x,u)\left(\frac{\partial L_{j+1}}{\partial u}\,\frac{1}{L_{j+1}}+\sum_{i=1}^{j-1}\frac{\varphi''(L_i)}{\varphi'(L_i)}\,\frac{\partial L_i}{\partial u}\right). \tag{3.41}$$

Proceeding as above we get

$$\sum_{i=1}^{j-1} \frac{\varphi''(L_i)}{\varphi'(L_i)} \frac{\partial L_i}{\partial u} = \frac{2\nu_0}{\sqrt{r}+\nu_0\, s} \left(1+O(\sqrt{r})\right), \tag{3.42}$$

(where we have used $\alpha^j = O(r)$). Thus the second equation $1 - x\,\varphi'(u) = \frac{\partial F^{(2j)}}{\partial u}(x,u)$ of (3.2) reduces to

$$x_0\varphi''(u_0)\, s = u_0(r-\nu_0^2 s)\frac{2\nu_0}{\sqrt{r}+\nu_0\, s} \left(1+O(\sqrt{r})\right) \tag{3.43}$$

or to

$$s = \frac{1}{2\nu_0}\sqrt{r} + O(r). \tag{3.44}$$

Therefore the first equation of $u - x\,\varphi(u) = F^{(2j)}(x,u)$ of (3.2) can be rewritten as

$$\frac{3}{4}\, r = \frac{16}{9}\,\alpha^j \left(1+O(\sqrt{r})\right). \tag{3.45}$$

Since

$$\alpha^j = (1-2\mu_0\sqrt{r}+O(r))^j = \exp(2\mu_0 j\sqrt{r}+O(jr)) \tag{3.46}$$

it is apparent that the main part of an asymptotic expansion for $\sqrt{r_{2j}}$ must be of the form

$$\frac{1}{\mu_0}\left(\frac{\log}{j} - \frac{\log\log j}{j}\right) + \frac{C}{j}, \tag{3.47}$$

where C can be evaluated by the equation

$$\frac{27}{64}\frac{1}{\mu_0^2} = \exp(-2\mu_0 C).$$

This gives the remarkable constant

$$C = \frac{\log\frac{64}{27} + 2\,\log\mu_0}{2\,\mu_0}. \tag{3.48}$$

The above considerations give a strong motivation that the main terms of an asymptotic expansion for $\sqrt{r_{2j}}$ and s_{2j} should be

$$\sqrt{\overline{r}_{2j}} = \frac{1}{\mu_0 j}\left(\log j - \log\log j + \frac{3}{2}\log\frac{4}{3} + \log\,\mu_0\right) \tag{3.49}$$

and

$$\overline{s}_{2j} = \frac{1}{2\nu_0}\sqrt{\overline{r}_{2j}}. \tag{3.50}$$

Now we set

$$\sqrt{r} = \sqrt{\overline{r}_{2j}} + \frac{C_1}{j\,\log\log j}, \qquad x = x_0(1-r) \tag{3.51}$$

and

$$s = \overline{s}_{2j} + \frac{C_2\,\log j}{j\,\log\log j}, \qquad u = u_0 - s, \tag{3.52}$$

(where C_1 and C_2 are absolutely bounded by 1). As above it can be shown that

$$u - x\,\varphi(u) - F^{(2j)}(x,u) = \frac{3}{2}\,\frac{u_0 C_1}{\mu_0}\,\frac{\log^2 j}{j^2 \log\log^2 j}\,\left(1 + O\left(\frac{1}{\log\log j}\right)\right) \tag{3.53}$$

and

$$1 - x\,\varphi'(u) - \frac{\partial F^{(2j)}}{\partial u}(x,u) = u_0\,\frac{(C_1 + 2\nu_0 C_2)\log j}{j\,\log\log j}\,\left(1 + O\left(\frac{1}{\log\log j}\right)\right). \tag{3.54}$$

Therefore there are two positive constants $\overline{C}_1, \overline{C}_2$ with $\overline{C}_2 > \frac{1}{2\nu_0}\overline{C}_1$ such that for sufficiently large j

$$\begin{aligned} u - x\,\varphi(u) - F^{(2j)}(x,u) > 0 \text{ for } C_1 = \overline{C}_1 \quad \text{and } |C_2| \le \overline{C}_2 \\ u - x\,\varphi(u) - F^{(2j)}(x,u) < 0 \text{ for } C_1 = -\overline{C}_1 \text{ and } |C_2| \le \overline{C}_2 \end{aligned}$$

and

$$\begin{aligned} 1 - x\,\varphi'(u) - \frac{\partial F^{(2j)}}{\partial u}(x,u) > 0 \text{ for } C_2 = \overline{C}_2 \quad \text{and } |C_1| \le \overline{C}_1 \\ 1 - x\,\varphi'(u) - \frac{\partial F^{(2j)}}{\partial u}(x,u) < 0 \text{ for } C_2 = -\overline{C}_2 \text{ and } |C_1| \le \overline{C}_1. \end{aligned}$$

Thus there must be a common solution $(\sqrt{r}, s) = (\sqrt{r_{2j}}, s_{2j})$ with

$$\sqrt{r_{2j}} \in \left[\sqrt{\overline{r}_{2j}} - \frac{\overline{C}_1}{j\,\log\log j}, \sqrt{\overline{r}_{2j}} + \frac{\overline{C}_1}{j\,\log\log j}\right] \tag{3.55}$$

and

$$s_{2j} \in \left[\overline{s}_{2j} - \frac{\overline{C}_2\,\log j}{j\,\log\log j}, \overline{s}_{2j} + \frac{\overline{C}_2\,\log j}{j\,\log\log j}\right]. \tag{3.56}$$

This gives the asymptotic expansion for x_{2j}. It should be remarked that it is now possible to find a better asymptotic expansion for s_{2j} by bootstrapping, namely

$$s_{2j} = \frac{1}{2\nu_0}\,\sqrt{r_{2j}} + O\left(\frac{\log^2 j}{j^2}\right). \tag{3.57}$$

This will be used in the sequel.

For the expansion of $D^{(2j)}$ we need the following list of expansions:

$$\begin{aligned} \frac{\partial L_i}{\partial x}(x_{2j}, u_{2j}) &= \frac{\varphi(u_0)}{(3+\alpha^i)^2}\left((9 + \alpha^{i-1})\frac{1-\alpha^i}{1-\alpha} + 6\,i\,\alpha^i\right) \times \\ &\quad \times \left(1 + O\left(\frac{\log j}{j}\right)\right) \\ \frac{\partial F^{(2j)}}{\partial x}(x_{2j}, u_{2j}) &= \frac{3}{4}\,u_0\sqrt{r_{2j}}\,j\,\sqrt{\frac{\varphi(u_0)\,\varphi''(u_0)}{2}}\,\left(1 + O\left(\frac{1}{\log j}\right)\right) \\ \frac{\partial^2 F^{(2j)}}{\partial u^2}(x_{2j}, u_{2j}) &= x_0\,\varphi''(u_0)\,\left(1 + O\left(\frac{\log j}{j}\right)\right) \\ \frac{\partial G^{(2j)}}{\partial u}(x_{2j}, u_{2j}) &= \frac{3}{4}\,r_{2j}\,\left(1 + O\left(\frac{\log j}{j}\right)\right) \\ \frac{\partial H^{(2j)}}{\partial u}(x_{2j}, u_{2j}) &= \frac{3}{4}\,\mu_0\,j\,\sqrt{r_{2j}}\,\left(1 + O\left(\frac{1}{\log j}\right)\right) \end{aligned} \tag{3.58}$$

.

(The calculations are quite similar to the ones from above.) Thus (3.7) is proved. The case $k = 2j+1$ can be treated along the same lines.

Proof of Theorem 2. Using the interpretation of the coefficient

$$[x^n z^m]\left(A^{(k)}(x,z) + B^{(k)}(x,z)\right)$$

as a weighted sum of the number of k-independent subsets of size m in trees of size n, it is easy to see that

$$[x^n]\left(A^{(k)}(x,1) + B^{(k)}(x,1)\right) = m_n^{(k)}\, y_n^{(k)}\, y_n. \tag{3.59}$$

Since

$$\begin{aligned}
\frac{\partial A^{(k)}}{\partial z} &= G^{(k)} + z\,\frac{\partial G^{(k)}}{\partial u}\,\frac{\partial C^{(k)}}{\partial z}\\
\frac{\partial B^{(k)}}{\partial z} &= H^{(k)} + z\,\frac{\partial H^{(k)}}{\partial u}\,\frac{\partial C^{(k)}}{\partial z}\\
\frac{\partial C^{(k)}}{\partial z} &= F^{(k)} + \left(z\,\frac{\partial F^{(k)}}{\partial u} + x\,\varphi'(C^{(k)})\right)\,\frac{\partial C^{(k)}}{\partial z}\\
\frac{\partial A^{(k)}}{\partial x} &= z\,\frac{\partial G^{(k)}}{\partial x} + z\,\frac{\partial G^{(k)}}{\partial u}\,\frac{\partial C^{(k)}}{\partial x}\\
\frac{\partial B^{(k)}}{\partial x} &= z\,\frac{\partial H^{(k)}}{\partial x} + z\,\frac{\partial H^{(k)}}{\partial u}\,\frac{\partial C^{(k)}}{\partial x}\\
\frac{\partial C^{(k)}}{\partial x} &= z\,\frac{\partial F^{(k)}}{\partial x} + \varphi(C^{(k)}) + \left(z\,\frac{\partial F^{(k)}}{\partial u} + x\,\varphi'(C^{(k)})\right)\,\frac{\partial C^{(k)}}{\partial x}
\end{aligned} \tag{3.60}$$

we can conclude that

$$\begin{aligned}
\frac{\partial A^{(k)}}{\partial z} + \frac{\partial B^{(k)}}{\partial z} = G^{(k)} + H^{(k)}&\\
&- \frac{F^{(k)}}{z\,\frac{\partial F^{(k)}}{\partial x} + \varphi(C^{(k)})}\left(z\,\frac{\partial G^{(k)}}{\partial x} + z\,\frac{\partial H^{(k)}}{\partial x}\right)\\
&+ \frac{F^{(k)}}{z\,\frac{\partial F^{(k)}}{\partial x} + \varphi(C^{(k)})}\left(\frac{\partial A^{(k)}}{\partial x} + \frac{\partial B^{(k)}}{\partial x}\right).
\end{aligned} \tag{3.61}$$

Using the fact that

$$\begin{aligned}
\frac{\partial A^{(k)}}{\partial x}(x,1) + \frac{\partial B^{(k)}}{\partial x}(x,1) &= \frac{d}{dx}\left(a^{(k)}(x) + b^{(k)}(x)\right)\\
&= \frac{a_1^{(k)} + b_1^{(k)}}{2}\,(x_k - x)^{-1/2}\\
&\quad - (a_2^{(k)} + b_2^{(k)}) + O\left((x_k - x)^{1/2}\right)
\end{aligned} \tag{3.62}$$

and that $F^{(k)}$, $G^{(k)}$, and $H^{(k)}$ are regular at (x_k, u_k) we get the local expansion

$$\frac{\partial A^{(k)}}{\partial z}(x,1) + \frac{\partial B^{(k)}}{\partial z}(x,1) = d_1^{(k)}(x_k - x)^{-1/2} + d_2^{(k)} + O\left((x_k - x)^{1/2}\right) \tag{3.63}$$

with

$$d_1^{(k)} = \frac{F^{(k)}(x_k, u_k)}{\frac{\partial F^{(k)}}{\partial x}(x_k, u_k) + \varphi(u_k)} \cdot \frac{a_1^{(k)} + b_1^{(k)}}{2}. \tag{3.64}$$

Thus, as a consequence of Darboux's theorem,

$$m_n^{(k)}\, y_n^{(k)}\, y_n \sim d_1^{(k)} \frac{1}{\sqrt{x_k \pi}}\, x_k^{-n} n^{-1/2} \qquad (n \to \infty) \tag{3.65}$$

and with (3.25)

$$m_n^{(k)} \sim \frac{F^{(k)}(x_k, u_k)}{x_k \left(\frac{\partial F^{(k)}}{\partial x}(x_k, u_k) + \varphi(u_k)\right)} \cdot n \tag{3.66}$$

which proves (3.10). (3.11) follows from

$$\begin{aligned} F^{(k)}(x_k, u_k) &= \frac{3}{4}\, u_0\, r_k \left(1 + O\left(\frac{\log k}{k}\right)\right) \\ \frac{\partial F^{(k)}}{\partial x}(x_k, u_k) &= \frac{3}{4}\, u_0\, \frac{k}{2} \sqrt{r_k} \sqrt{\frac{\varphi(u_0)\, \varphi''(u_0)}{2}} \left(1 + O\left(\frac{1}{\log k}\right)\right) \\ \sqrt{r_k} &= \frac{1}{2\mu_0 k} (\log k - \log\log k) \left(1 + O\left(\frac{1}{\log k}\right)\right). \end{aligned} \tag{3.67}$$

This completes the proof of Theorem 2.

4. Results for Special Families of Trees

4.1. Planted Plane Trees. This is the instance $\varphi(t) = 1 + t + t^2 + \cdots = 1/(1-t)$. It seems remarkable to note that in this case the functions $L_i(x, u)$ defined in (2.2) may be determined explicitely by using the well-known recurrence relation for continued fractions.

Proposition 1. *For $\varphi(t) = 1/(1-t)$ we have for $n \geq 0$*

$$L_{n+1}(x, u) = \frac{x(d_{n+1}(x) - u\, d_n(x))}{d_{n+2}(x) - u\, d_{n+1}(x)}, \tag{4.1}$$

where

$$d_n(x) = \left(\frac{1 + \sqrt{1-4x}}{2}\right)^n - \left(\frac{1 - \sqrt{1-4x}}{2}\right)^n. \tag{4.2}$$

The proof is easily given by induction.

Next from Proposition 1 one gains explicit forms of all functional equations, so that the asymptotic behaviour of the singularities x_k could be also determined in a more direct way in this instance.

For $k = 2$ it is possible to find explicit expressions for the involved generating functions in this way:

Proposition 2. *For $\varphi(t) = 1/(1-t)$ and $k = 2$ we have*

$$c^{(2)}(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2} \tag{4.3}$$

and

$$a^{(2)}(x) + b^{(2)}(x) = \frac{1 + 2x - x^2 - (1+x)\sqrt{1 - 6x + x^2}}{2} \tag{4.4}$$

so that

$$x_2 = 3 - 2\sqrt{2} \tag{4.5}$$

and

$$y_n^{(2)} \sim \frac{2^{1/4}}{\sqrt{\pi}} \left(3\sqrt{2} - 4\right) \left(3 + 2\sqrt{2}\right)^n n^{-3/2}, \quad n \to \infty. \tag{4.6}$$

In the following we give a table of numerical approximates for the singularities x_k gained from the system (3.2) and for $k \geq 5$ of the corresponding values of the main asymptotic term

$$\overline{x}_k = x_0(1 - \overline{r}_k) \tag{4.7}$$

with $\sqrt{\overline{r}_k}$ defined in equation (3.49), as well as the average fractions $E^{(k)}$ of nodes in k-independent subsets and for $k \geq 5$ the corresponding values of approximations

$$\overline{E}^{(k)} = 4\frac{\log k - \log\log k}{k^2 \mu_0^2}. \tag{4.8}$$

(Compare to (3.11)):

k	x_k	$\overline{x}_k$	$E^{(k)}$	$\overline{E}^{(k)}$
0	0.25			
1	0.148148...		0.4	
2	0.171572...		0.207106...	
3	0.186987...		0.185179...	
4	0.198741...		0.108788...	
5	0.206975...	0.111023...	0.103279...	0.181368...
10	0.228105...	0.225505...	0.035215...	0.058742...
100	0.248996...	0.249112...	0.000960...	0.001231...

Table 1. $(\varphi(t) = (1-t)^{-1})$

The different accuracy of the two asymptotic approximations is due to the fact that the three main terms of $\sqrt{\overline{r}_k}$ are known whereas for $E^{(k)}$ we only know use the first two terms.

4.2. Binary Trees. For $\varphi(t) = 1 + t^2$ the table corresponding to the upper one is the following. (Note that in this instance $\pm x_k$ are the singularities of smallest absolute value.)

k	x_k	$\overline{x}_k$	$E^{(k)}$	$\overline{E}^{(k)}$
0	0.5			
1	0.301606 ...		0.390019 ...	
2	0.340170 ...		0.210835 ...	
3	0.368780 ...		0.188892 ...	
4	0.390693 ...		0.113307 ...	
5	0.406770 ...	0.172467 ...	0.107230 ...	0.362736 ...
10	0.448855 ...	0.440610 ...	0.039448 ...	0.117484 ...
100	0.496899 ...	0.497227 ...	0.001508 ...	0.002462 ...

Table 2. $(\varphi(t) = 1 + t^2)$

For $\varphi(t) = 1 + 2t + t^2$, which corresponds to the so-called extended binary trees where endnodes are neglected we have

k	x_k	$\overline{x}_k$	$E^{(k)}$	$\overline{E}^{(k)}$
0	0.25			
1	0.152679 ...		0.386198 ...	
2	0.170093 ...		0.203771 ...	
3	0.181926 ...		0.186064 ...	
4	0.190850 ...		0.125890 ...	
5	0.197690 ...	0.090784 ...	0.119789 ...	0.725473 ...
10	0.217541 ...	0.219589 ...	0.054551 ...	0.234968 ...
100	0.247471 ...	0.247909 ...	0.002641 ...	0.004924 ...

Table 3. $(\varphi(t) = 1 + 2t + t^2)$

4.3. Motzkin Trees. For $\varphi(t) = 1 + t + t^2$ we get

k	x_k	$\overline{x}_k$	$E^{(k)}$	$\overline{E}^{(k)}$
0	0.333333 ...			
1	0.202745 ...		0.387523 ...	
2	0.226692 ...		0.206579 ...	
3	0.243372 ...		0.187946 ...	
4	0.256125 ...		0.123464 ...	
5	0.265855 ...	0.111550 ...	0.117121 ...	0.544105 ...
10	0.293414 ...	0.292061 ...	0.049177 ...	0.176226 ...
100	0.330569 ...	0.330971 ...	0.002103 ...	0.003693 ...

Table 4. $(\varphi(t) = 1 + t + t^2)$

4.4. Non-planar Labelled Trees. For $\varphi(t) = \exp(t)$ we have

k	x_k	$\overline{x}_k$	$E^{(k)}$	$\overline{E}^{(k)}$
0	0.367879...			
1	0.222218...		0.390757...	
2	0.250522...		0.207909...	
3	0.270011...		0.188169...	
4	0.284981...		0.120852...	
5	0.296167...	0.126894...	0.114651...	0.362736...
10	0.327235...	0.324182...	0.045771...	0.117484...
100	0.365490...	0.365839...	0.001629...	0.002462...

Table 5. $(\varphi(t) = \exp(t))$

5. Concluding Remarks

Remark 1. It should be noted that condition iv) on $\varphi(t)$, namely

$$\lim_{t\to R-} \frac{1}{\varphi'(t)} \le \frac{1}{2\,\varphi'(\tau)}, \tag{5.1}$$

is necessary at least for $k = 1$ to guarantee the existence of a solution of system (3.2): Let

$$\varphi(t) = \frac{3}{2}\,t + 10^{-3}\,t^2 + (1-t)^{3/2}. \tag{5.2}$$

Then we have

$$\begin{aligned} &R = 1, \quad u_0 = \tau = 0.99999955\ldots, \\ &\text{whereas there is no } u_1 < 1 = R \text{ satisfying (3.2).} \end{aligned}$$

It is easily seen that inequality (5.1) is not fulfilled in this case.

In fact it can be shown that in any instance where a $u_1 < R$ exists we have $u_1 > u_0 = \tau$. Thus the solutions of (3.2) must not only be looked for in the domain $0 \le x \le x_0$, $0 \le u \le u_0$, as one might expect as a first guess.

Remark 2. A different generalization of the notion of k-independence from the chain to trees would be to forbid with each selected vertex v to select another vertex from the following k levels in the subtree with root v. It is not difficult to see that for this notion the functional equations corresponding to (2.10) read

$$\begin{aligned} A^{(k)} &= z\,L_{k+1}\left(A^{(k)} + B^{(k)}\right) \\ B^{(k)} &= x\,\varphi\left(A^{(k)} + B^{(k)}\right). \end{aligned} \tag{5.3}$$

This leads to the equation

$$u = x\,\varphi(u) + L_{k+1}(x,u) \tag{5.4}$$

fulfilled by $u = A^{(k)}(x,1) + B^{(k)}(x,1)$.

Since $\lim_{k\to\infty} x_k = x_0$ would imply $\lim_{k\to\infty} L_{k+1}(x_k, u_k) = u_0$, it is impossible that the pair (x_k, u_k) of smallest positive solutions of (5.4) and $1 = x\,\varphi'(u) + \frac{\partial L_{k+1}}{\partial u}(x,u)$ converges to (x_0, u_0). So the asymptotic behaviour of the singularities x_k must be totally different for this notion of k-independence.

Remark 3. A natural question might be to consider maximal k-independent subsets. Nevertheless it turns out that the functional equations for this problem are *much more* delicate to be treated. This question will probably be discussed in a forthcoming paper.

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