# IRREGULARITIES OF CONTINUOUS DISTRIBUTIONS

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Abstract. Cet article considère un analogue de l'irrégularité des ensembles discrets. Si $X$ est un espace compact et $x : [0,1] \to X$ est une fonction continue (interprétée comme équation d'un mouvement), la discrépance mesure la différence entre le temps de séjour de $x(t)$ dans certains sous-ensembles et le volume de ces sous-ensembles. Le but essentiel de cet article est de démonstrer des estimations infeérieures de la discrépance par une fonction de la longueur de l'arc de $x(t)$, $0 \le t \le 1$. De plus on démontre que les estimations sont optimales à des facteurs logarithmiques près.

## 1. Introduction

Let $x : [0,1] \to \mathbf{R}^K/\mathbf{Z}^K$ be a continuous function with finite arc length

$$s(x) = \int_0^1 |dx(t)| \ . \tag{1}$$

($\mathbf{R}^K/\mathbf{Z}^K$ denotes the $K$-dimensional unit torus. As usual it can be identified with the $K$-dimensioanl unit cube $[0,1)^K$.) Such a function can be interpreted as a particle's movement on $\mathbf{R}^K/\mathbf{Z}^K$. It is of some interest to consider a measure for the irregularity of the distribution behaviour of this particle, the discrepancy

$$D^{(\mathcal{R})}(x) = \sup_R \left| \int_0^1 \chi_R(x(t))\, dt - \lambda_K(R) \right| \ , \tag{2}$$

where the supremum is taken over all rectangles $R \subseteq \mathbf{R}^K/\mathbf{Z}^K$ of the form $R = [a_1,b_1]\times\cdots\times[a_K,b_K]$ with $\max_{1\le k\le K}(b_k-a_k) \le 1$. ($\chi_M$ denotes the characteristic function of a set $M$ and $\lambda_K$ the $K$-dimensional Lebesgue measure. For details see Section 2.1.) In other words, $D^{(\mathcal{R})}(x)$ measures the maximal difference between the time of the particles stay in a rectangle $R$ and the volume of $R$. R. J. Taschner [10] was the first who noticed that the discrepancy cannot be too small. He proved that for $K \ge 2$

$$D^{(\mathcal{R})}(x) > c_K\, s(x)^{-(1+1/(K-1))} \tag{3}$$

provided that $s(x) \ge 1$. (Another more general proof can be found in [5].)

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It is interesting to see that the discrepancy must be essentially larger if the supremum in (2) is taken over all rectangles in arbitrary position. Here we have

$$D^{(\mathcal{R}')}(x) > c_K\, s(x)^{-\left(\frac{1}{2}+1/(K-1)\right)} . \tag{4}$$

The same lower bound holds if we use balls instead of rectangles in arbitrary position or if we consider a particle's movement on the $K$-dimensional unit sphere and the discrepancy with respect to spherical caps. It should be stated that the problem to find bounds for the discrepancy for continuous functions on the sphere was proposed by R.F. Tichy [11].

In the first part of this paper (4) is proved by an application of J. Beck's Fourier transform method [2] and it is shown that the lower bounds (3) and (4) are optimal despite of logarithmic factors of $s(x)$. The next part deals with convex bodies and gives a continuous analogon to J. Beck's solution of a problem of K. F. Roth [1]. In the last section some problems and results concerning the behaviour of the discrepancy are discussed if one consideres a function $x : [0,\infty) \to \mathbf{R}^K/\mathbf{Z}^K$.

It should be noted that the discrepancy of a continuous function, introduced by E. Hlawka [7], is an analogon to the discrepancy of sequences which measures the irregularity of a point distribution. An excellent survey of this subject including recent results with complete proofs can be found in J. Beck and W. Chen's book *Irregularities of Distribution* [2].

## 2. Torus and Sphere

**2.1. Definitions and Results.**

Let $\mathcal{R}^K = \{\prod_{i=1}^K [a_i, b_i] \,|\, 0 < b_i - a_i \le 1,\, i = 1,\dots,K\}$ be the set of all $K$-dimensional rectangles and $\overline{\mathcal{R}}^K = \mathcal{R}^K/\mathbf{Z}^K$ the set of rectangles contained in the unit torus $\mathbf{R}^K/\mathbf{Z}^K$. If $x : [0,1] \to \mathbf{R}^K/\mathbf{Z}^K$ is a continuous function with finite arclegth $s(x) = \int_0^1 |dx(t)|$ the discrepancy of $x(t)$ with respect of $\overline{\mathcal{R}}^K$ is defined by

$$D^{(\mathcal{R})}(x) = \sup_{R\in\overline{\mathcal{R}}^K} \left| \int_0^1 \chi_R(x(t))\, dt - \lambda_K(R) \right| , \tag{5}$$

where $\lambda_K$ is the Lebesgue measure on $\mathbf{R}^K/\mathbf{Z}^K$ (i. e. the Haar measure on $\mathbf{R}^K/\mathbf{Z}^K$).

Set

$$\Delta_K^{(\mathcal{R})}(s) = \inf_{x\in C_s^K} D^{(\mathcal{R})}(x) , \tag{6}$$

where $C_s^K$ is the set of all continuous functions $x : [0,1] \to \mathbf{R}^K/\mathbf{Z}^K$ with $s(x) = s$. Then we can formulate

**Theorem 1.** *If $s \ge 1$, then we have for $K \ge 2$*

$$\Delta_K^{(\mathcal{R})}(s) \gg s^{-(1+1/(K-1))} \tag{7}$$

*and on the other hand*

$$\Delta_2^{(\mathcal{R})} \ll s^{-2} \tag{8}$$

*and for $K \ge 3$*

$$\Delta_K^{(\mathcal{R})}(s) \ll s^{-(1+1/(K-1))} (\log(s+1))^{K-1} , \tag{9}$$

*where the constants implied by $\ll$ or $\gg$ are only depending on the dimension $K$.*

*Remark 1.* If $K=1$ it is easy to see that $\Delta_1^{(R)}(s)=0$ for $s\geq 1$. (cf. [3])

Now let $\mathcal{Q}^K$ denote the set of all cubes $Q\in\mathbf{R}^K$ in arbitrary position such that the length of the sides are $\leq K^{-1/2}$ and $\mathcal{B}^K$ the set of balls $B\in\mathbf{R}^K$ with diameter $\leq 1$. Again we can define discrepancies

$$\begin{aligned} D^{(\mathcal{Q})}(x) &= \sup_{Q\in\mathcal{Q}/\mathbf{Z}^K}\left|\int_0^1 \chi_Q(x(t))\,dt-\lambda_K(Q)\right| \\ D^{(\mathcal{B})}(x) &= \sup_{B\in\mathcal{B}/\mathbf{Z}^K}\left|\int_0^1 \chi_B(x(t))\,dt-\lambda_K(B)\right| \end{aligned}\tag{10}$$

and

$$\begin{aligned} \Delta_K^{(\mathcal{Q})}(s) &= \inf_{x\in C_s^K} D^{(\mathcal{Q})}(x) \\ \Delta_K^{(\mathcal{B})}(s) &= \inf_{x\in C_s^K} D^{(\mathcal{B})}(x)\ . \end{aligned}\tag{11}$$

In comparision to Theorem 1 there is a significant difference.

**Theorem 2.** *If $s\geq 1$, then we have for $K\geq 2$*

$$\left.\begin{array}{l}\Delta_K^{(\mathcal{Q})}(s)\\ \Delta_K^{(\mathcal{B})}(s)\end{array}\right\}\gg s^{-\left(\frac{1}{2}+1/(K-1)\right)}\tag{12}$$

*and on the other hand*

$$\left.\begin{array}{l}\Delta_K^{(\mathcal{Q})}(s)\\ \Delta_K^{(\mathcal{B})}(s)\end{array}\right\}\ll s^{-\left(\frac{1}{2}+1/(K-1)\right)}\left(\log(s+1)\right)^{1/2}\ ,\tag{13}$$

*where the constants implied by $\ll$ or $\gg$ are only depending on the dimension $K$.*

*Remark 2.* Since every cube is also a rectangle (12) implies (4).

*Remark 3.* Since the torus $\mathbf{R}^K/\mathbf{Z}^K$ can be identified with the unit cube $U_0^K=[0,1)^K$ it is also possible to consider the set of cubes $\mathcal{Q}_0^K$ in arbitrary position contained in $U_0^K$ and the set of balls $\mathcal{B}_0$ contained in $U_0^K$. Then it can be shown by a trunctation technique that the corresponding $\Delta_K^{(\mathcal{Q}_0)}(s)$ and $\Delta_K^{(\mathcal{B}_0)}(s)$ satisfy

$$\left.\begin{array}{l}\Delta_K^{(\mathcal{Q}_0)}(s)\\ \Delta_K^{(\mathcal{B}_0)}(s)\end{array}\right\}\gg s^{-\left(\frac{1}{2}+1/(K-1)+\varepsilon\right)}\ ,\tag{14}$$

where $\varepsilon>0$ is arbitrary but fixed and the constants implied by $\gg$ are also depending on $\varepsilon$. (For details see section 3.)

Let $\mathbf{S}^K=\{x\in\mathbf{R}^{K+1}\,|\,|x|=1\}$ denote the $K$-dimensional unit sphere and $\sigma_K$ the normalized surface measure on $\mathbf{S}^K$, i. e. $\sigma_K(\mathbf{S}^K)=1$. Here we consider the set $\mathcal{C}^K$ of all spherical caps $C = \{x \in \mathbf{S}^K \mid |x-m| \le r\}$ ($m \in \mathbf{S}^K$, $r > 0$) and the discrepancy of a continuous function $x : [0,1] \to \mathbf{S}^K$

$$D^{(\mathcal{C})}(x) = \sup_{C \in \mathcal{C}^K} \left| \int_0^1 \chi_C(x(t))\, dt - \sigma_K(C) \right| . \tag{15}$$

If $C_s^K$ denotes the set of all continuous functions $x : [0,1] \to \mathbf{S}^K$ with arclength $s(x) = s$ and

$$\Delta_K^{(\mathcal{C})}(s) = \inf_{x \in C_s^K} D^{(\mathcal{C})}(x) \tag{16}$$

then we have similarly to Theorem 2

**Theorem 3.** *If $s \ge 1$, then we have for $K \ge 2$*

$$\Delta_K^{(\mathcal{C})}(s) \gg s^{-\left(\frac{1}{2}+1/(K-1)\right)} \tag{17}$$

*and on the other hand*

$$\Delta_K^{(\mathcal{C})}(s) \ll s^{-\left(\frac{1}{2}+1/(K-1)\right)} \left(\log(s+1)\right)^{1/2} , \tag{18}$$

*where the constants implied by $\ll$ or $\gg$ are only depending on the dimension $K$.*

*Remark 4.* The lower bounds of Theorem 2 and 3 sharpen the results of [4]:

$$\left.\begin{array}{l} \Delta_K^{(\mathcal{R}')}(s) \\ \Delta_K^{(\mathcal{B})}(s) \\ \Delta_K^{(\mathcal{C})}(s) \end{array}\right\} \gg s^{\left(\frac{1}{2}+1/(K-1)+\varepsilon\right)} \quad \begin{array}{ll} \text{for} & K = 2,3 \\ \text{for} & K \ge 2 \\ \text{for} & K \ge 2 \end{array} \tag{19}$$

($\varepsilon > 0$ arbitrary but fixed.) The technique used in [4] is an application of W. Schmidt's integral equation method [9].

**2.2. Lower Bounds.**

The lower bound (7) of Theorem 1 is due to R. J. Taschner [10] (see also [5]).

For the proof of the lower bounds (12) and (17) of Theorem 2 and 3 we will use J. Beck's Fourier transform technique. If $f \in \mathrm{L}^1(\mathbf{R}^K)$ then the Fourier transform $\widehat{f}$ of $f$ is defined by

$$\widehat{f}(t) = \frac{1}{(2\pi)^{K/2}} \int_{\mathbf{R}^K} e^{-i\langle x,t\rangle} f(x)\, dx , \tag{20}$$

where $\langle x,t\rangle$ denotes the usual inner produkt in $\mathbf{R}^K$. Let $x : [0,1] \to \mathbf{R}^K/\mathbf{Z}^K$ be a continuous function with $s(x) = s$, then we can introduce a measure $z$ on $\mathbf{R}^K$ by defining it on measurable subsets $A \in \mathbf{R}^K$ that are contained in some translated unit cube $y + U_0^K$ by

$$z(A) = \int_0^1 \chi_{A/\mathbf{Z}^K}(x(t))\, dt . \tag{21}$$

Let $\tau$ denote a proper orthogonal transformation (i. e. a rotation) then $\chi_{r,\tau}$ shall denote the characteristic function of the rotated cube $\tau[-r,r]^K = \{\tau x \mid x \in [-r,r]^K\}$ ($r \le K^{-1/2}/2$). (For simplicity we will write $Q(x)$ for the cube $[-x,x]^K$ in the sequel.) Let $M > 0$ be a parameter to be fixed later , and set

$$\mu_M(A) = \lambda_K(A \cap Q(M)) \quad \text{and} \quad z_M(A) = z(A \cap Q(M)) \ . \tag{22}$$

Now consider the function

$$F_{r,\tau} = \chi_{r,\tau} * (dz_M - d\mu_M) \ , \tag{23}$$

where $*$ denotes the convolution operation. More precisely we have

$$\begin{aligned} F_{r,\tau}(x) &= \int\limits_{\mathbf{R}^K} \chi_{r,\tau}(x-y)\,(dz_M(y) - d\mu_M(y)) \\ &= \int\limits_0^1 \chi_{(\tau Q(r)+x)\cap Q(M)/\mathbf{Z}^K}(x(t))\,dt - \lambda_K\left((\tau Q(r)+x)\cap Q(M)\right) \ . \end{aligned} \tag{24}$$

Let $\mathbf{T}$ be the group of proper orthogonal transformations in $\mathbf{R}^K$ and let $d\tau$ be the volume element of the normalized Haar measure on $\mathbf{T}$. Introducing the notation

$$\Phi(q) = \frac{1}{q} \int\limits_q^{2q} \int\limits_{\mathbf{T}} \int\limits_{\mathbf{R}^K} |F_{r,\tau}(x)|^2\,dx\,d\tau\,dr \tag{25}$$

we derive by using the Parseval—Plancherel identity and the convolution relation $\widehat{f*g} = \hat{f}\cdot\hat{g}$ that

$$\Phi(q) = \int\limits_{\mathbf{R}^K} \left( \frac{1}{q} \int\limits_q^{2q} \int\limits_{\mathbf{T}} |\hat{\chi}_{r,\tau}(t)|^2\,d\tau\,dr \right) \left| (\widehat{dz_M - d\mu_M})(t) \right|^2 dt \ . \tag{26}$$

For brevity set

$$\varphi_q(t) = \frac{1}{q} \int\limits_q^{2q} \int\limits_{\mathbf{T}} |\hat{\chi}_{r,\tau}(t)|^2\,d\tau\,dr \tag{27}$$

and

$$\psi(t) = (\widehat{dz_M - d\mu_M})(t) = \frac{1}{(2\pi)^{K/2}} \int\limits_{\mathbf{R}^K} e^{-i\langle x,t\rangle}(dz_M - d\mu_M)(x) \ . \tag{28}$$

We will need two Lemmata.

**Lemma 1.** *(*[2,p. 134]*) If $0 < q < p$, then*

$$\frac{\varphi_p(t)}{\varphi_q(t)} \gg \left(\frac{p}{q}\right)^{K-1} \tag{29}$$

*uniformly in all $t \in \mathbf{R}^K$.*

**Lemma 2.** *If $s(x) = s \ge 1$, then*

$$\Phi\left(\frac{c_K}{s^{1/(K-1)}}\right) \gg s^{-2K/(k-1)}\, M^K \ , \tag{30}$$

*where $c_K$ is a proper constant only depending on the dimension $K$.*

*Proof.* Identify the unit torus $\mathbf{R}^K/\mathbf{Z}^K$ with the unit cube $U_0^K = [0,1)^K$. Now subdivide $U_0^K$ into $N^K$ cubes $Q_{(m_i)} = \prod_{i=1}^K [m_i/N, (m_i+1)/N)$ $(0 \le m_i < N$, $i = 1, \ldots, K)$. Furthermore subdivide the interval $[0,1]$ into $[s(N+1)]$ intervals $[t_j, t_{j+1}]$ $(j = 0, \ldots, [s(N+1)]-1)$, $0 = t_0 < t_1 < \cdots < t_{[s(N+1)]} = 1$, such that the arclength $\int_{t_j}^{t_{j+1}} d|x(t)| < 1/N$ $(j = 0, \ldots, [s(N+1)]-1)$. Trivially, the number of cubes $Q_{(m_i)}$ such that $x([t_j, t_{j+1}]) \cap Q_{(m_i)}$ is non-void is less or equal $2^K$. Therefore there are at most $2^K[s(N+1)]$ cubes $Q_{(m_i)}$ such that $x([0,1]) \cap Q_{(m_i)}$ is non void. Now choose the minimal $N$ such that $2^K s(N+1) \le N^K/2$. Hence there are at least $N^K/2$ cubes $Q_{(m_i)}$ such that $x([t_j, t_{j+1}]) \cap Q_{(m_i)}$ is void. Now consider the subcube $C_{(m_i)} = \prod_{i=1}^K [(m_i + 1/4)/N, (m_i + 3/4)/N)$ of such a cube $Q_{(m_i)}$. If $r < (4NK^{1/2})^{-1}$ and $x \in C_{m_i)}$ we have

$$F_{r,\tau}(x) = -(2r)^K \tag{31}$$

and therefore

$$\Phi(q) \gg M^K q^{2K} \ , \tag{32}$$

if $q < (8NK^{1/2})^{-1}$. Since $N \gg\ll s^{1/(K-1)}$, (32) implies (30). Thus Lemma 2 is proved.

Now set $q = c_K s^{-1/(K-1)}$ and $p = (5K^{1/2})^{-1}$, then we can use (29) and (28) to conclude

$$\begin{aligned} \Phi(p) &= \int_{\mathbf{R}^K} \varphi_p(t) |\psi(t)|^2 \, dt \gg s\, \Phi(q) \\ &\gg s^{1-2K/(K-1)}\, M^K = s^{-(1+2/(K-1))}\, M^K \ . \end{aligned} \tag{33}$$

Since $F_{r,\tau}(x) = 0$ for $x \notin Q(M+1)$, (33) implies that either

$$\frac{1}{p} \int_p^{2p} \int_{\mathbf{T}} \int_{Q(M+1)\setminus Q(M-1)} |F_{r,\tau}(x)|^2 \, dx\, d\tau\, dr \gg s^{-(1+2/(K-1))}\, M^K \tag{34}$$

or

$$\frac{1}{p} \int_p^{2p} \int_{\mathbf{T}} \int_{Q(M-1)} |F_{r,\tau}(x)|^2 \, dx\, d\tau\, dr \gg s^{-(1+2/(K-1))}\, M^K \ . \tag{35}$$

Since $F_{r,\tau}(x)$ is absolutely bounded by 1, (34) cannot be true if $M$ is large enough, e. g. $M = \overline{c}_K s^{1+2/(K-1)}$ for some proper chosen constant $\overline{c}_K$. Therefore (35) implies that there is an $r \in [p, 2p]$, a $\tau \in \mathbf{T}$, and an $x \in Q(M-1)$ with

$$\begin{aligned} |F_{r,\tau}(x)| &= \left| \int_0^1 \chi_{(\tau Q(r)+x)/\mathbf{Z}^K}(x(t))\, dt - \lambda_K\left(\tau Q(r) + x\right) \right| \\ &\gg s^{-\left(\frac{1}{2} + 1/(K-1)\right)} \ . \end{aligned} \tag{36}$$

Thus

$$\Delta_K^{(\mathcal{Q})}(s) \gg s^{-\left(\frac{1}{2}+1/(K-1)\right)} \ . \tag{37}$$

The second part of (12), $\Delta_K^{(\mathcal{B})}(s) \gg s^{-(1/2+1/(K-1))}$, can be proved similarly. Instead of (22) set

$$\overline{F}_r = \overline{\chi}_r * (dz_M - d\mu_M) \ , \tag{38}$$

where $\overline{\chi}_r$ denotes the characteristic function of the ball $B(r) = \{x \in \mathbf{R}^K \mid |x| \le r\}$. Now consider

$$\begin{aligned} \overline{\Phi}(q) &= \frac{1}{q} \int_q^{2q} \int_{\mathbf{R}^K} |\overline{F}_r(x)|^2 \, dx \, dr \\ &= \int_{\mathbf{R}^K} \frac{1}{q} \int_q^{2q} |\hat{\overline{\chi}}_r(t)|^2 \, dr \, |(\widehat{dz_M - d\mu_M})(t)|^2 \, dt \\ &= \int_{\mathbf{R}^K} \overline{\varphi}_q(t) |\psi(t)|^2 \ dt \ . \end{aligned} \tag{39}$$

Again we have

**Lemma 3.** *If $0 < q < p$, then*

$$\frac{\overline{\varphi}_p(t)}{\overline{\varphi}_q(t)} \gg \left(\frac{p}{q}\right)^{K-1} \tag{40}$$

*uniformly in all $t \in \mathbf{R}^K$.*

*Proof.* (40) is an immediate consequence of formula (201)

$$\frac{1}{y} \int_y^{2y} |\hat{\overline{\chi}}_r(t)|^2 \, dr \gg\ll \frac{y^{K-1}}{|t|^{K+1}} \quad \text{for} \quad y|t| > c_K \tag{41}$$

and formula (203)

$$\frac{1}{y} \int_y^{2y} |\hat{\overline{\chi}}_r(t)|^2 \, dr \gg\ll y^{2K} \quad \text{for} \quad y|t| \le c_K \tag{42}$$

of [2, p. 228].

**Lemma 4.** *If $s(x) = s \ge 1$, then*

$$\overline{\Phi}\left(\frac{c_K}{s^{1/(K-1)}}\right) \gg s^{-2K/(k-1)} \, M^K \ , \tag{43}$$

*where $c_K$ is a proper constant only depending on the dimension $K$.*

The proof of Lemma 4 is verbally the same as the proof of Lemma 2. Hence we can deduce the bound $\Delta_K^{(\mathcal{B})}(s) \gg s^{-(1/2+1/(K-1)}$ as before. Therefore the proof of (12) in Theorem 2 is finished.

Now let $x : [0,1] \to \mathbf{S}^K$ be a continuous function with $s(x) = s \geq 1$. Define two measures on $\mathbf{R}^{K+1}$, one by $\sigma_0(A) = \sigma_K(A \cap \mathbf{S}^K)$ and a second by

$$z_0(A) = \int_0^1 \chi_{A\cap\mathbf{S}^K}(x(t))\,dt \ . \tag{44}$$

Set

$$G_r = \overline{\chi}_r * (dz_0 - d\sigma_0) \tag{45}$$

and

$$\begin{aligned}\Psi(q) &= \frac{1}{q}\int_q^{2q} \int_{\mathbf{R}^{K+1}} |G_r(x)|^2\,dx\,dr \\ &= \int_{\mathbf{R}^{K+1}} \overline{\varphi}_q(t)(t)\left|(\widehat{dz_0 - d\sigma_0})(t)\right|^2\,dt \ .\end{aligned} \tag{46}$$

Then we can prove

**Lemma 5.** *If $s(x) = s \geq 1$, then*

$$\Psi\left(\frac{c_K}{s^{1/(K-1)}}\right) \gg s^{-\frac{2K+1}{K-1}} \ , \tag{47}$$

*where $c_K$ is a proper chosen constant only depending on the dimension $K$.*

*Proof.* It is well known that $\mathbf{S}^K$ can be represented by $Y : (0,\infty) \times [0,\pi]^{K-1} \times [0,2\pi) \to \mathbf{R}^{K+1}$:

$$\begin{aligned} y_1 &= \rho\,\sin\varphi_1\,\sin\varphi_2\cdots\sin\varphi_{K-2}\,\sin\varphi_{K-1}\,\sin\varphi_K \\ y_2 &= \rho\,\sin\varphi_1\,\sin\varphi_2\cdots\sin\varphi_{K-2}\,\sin\varphi_{K-1}\,\cos\varphi_K \\ y_3 &= \rho\,\sin\varphi_1\,\sin\varphi_2\cdots\sin\varphi_{K-2}\,\cos\varphi_{K-1} \\ &\ \vdots \\ y_K &= \rho\,\sin\varphi_1\,\cos\varphi - 2 \\ y_{K+1} &= \rho\,\cos\varphi_1 \ , \end{aligned} \tag{48}$$

where $\rho$ is set equal to 1. Now consider the images $R_{(m_i)} = Y(\{1\} \times B_{(m_i)})$ of the $4N^K$ cubes

$$B_{(m_i)} = \prod_{j=1}^{K-1}\left[\frac{\pi}{4}\left(1+2\frac{m_j}{N}\right), \frac{\pi}{4}\left(1+2\frac{m_j+1}{N}\right)\right) \times \left[\pi\frac{m_K}{2N}, \pi\frac{m_K+1}{2N}\right) \tag{49}$$

$$0 \leq m_1, \dots, m_{K-1} < N,\ 0 \leq m_K < 4N \ .$$

As in Lemma 2 it is possible to choose $N \gg\ll s^{1/(K-1)}$ such that for at least $2N^K$ sets $R_{(m_i)}$ the intersection $x([0,1]) \cap R_{(m_i)}$ is void. For these $R_{(m_i)}$ consider the images $\overline{R}_{(m_i)} = Y(C_{(m_i)})$ of the rectangles

$$C_{(m_i)} = \left[1, 1+\frac{1}{8N}\right] \times \prod_{j=1}^{K-1}\left[\frac{\pi}{4}\left(1+2\frac{m_j+\frac{1}{4}}{N}\right), \frac{\pi}{4}\left(1+2\frac{m_j+\frac{3}{4}}{N}\right)\right) \tag{50}$$

$$\times \left[\frac{\pi(m_K+1/4)}{2N}, \frac{\pi(m_K+3/4)}{2N}\right) \; .$$

If $\rho \in \left[\frac{\pi}{16N}, \frac{\pi}{8N}\right]$ and $y \in C_{(m_i)}$ (for such $R_{(m_i)}$) we have similarly to (31)

$$|G_r(x)| \gg N^{-K} \; . \tag{51}$$

This implies (47).

Now it is easy to verify the lower bound (17) in Theorem 3. By Lemma 3 we have

$$\frac{\overline{\varphi}_1(t)}{\overline{\varphi}_q(t)} \gg s^{K/(K-1)} \; , \tag{52}$$

where $q = c_K s^{-1/(K-1)}$. (Note that we are working in $\mathbf{R}^{K+1}$.) Thus

$$\Psi(1) \gg s^{K/(K-1)}\,\psi(q) \gg s^{-\left(\frac{1}{2}+1/(K-1)\right)} \; . \tag{53}$$

### 2.3. Upper Bounds.

The essential ideas of this section can be stated in two simple Lemmata. (We postpone the proofs at the end of this section.)

**Lemma 6.** *Let $X$ be a convex body in $\mathbf{R}^K$ and $x_1, \dots, x_N$ $N$ points in $X$. Then there is a permutation $\pi$ of $\{1, \dots, N\}$ such that*

$$\sum_{i=1}^{N-1} |x_{\pi(i+1)} - x_{\pi(i)}| \ll N^{1-1/K} \; . \tag{54}$$

*If $x_1, \dots, x_N$ are on the boundary $\partial X$ of $X$, then there is a permutation $\rho$ of $\{1, \dots, N\}$ such that*

$$\sum_{i=1}^{N-1} \delta(x_{\rho(i+1)}, x_{\rho(i)}) \ll N^{1-1/(K-1)} \; , \tag{55}$$

*where $\delta(\cdot,\cdot)$ denotes the geodesic distance on $\partial X$. The constants implied by $\ll$ in (54) and (55) are only depending on the diameter diam$(X)$ of $X$.*

*Remark 5.* It should be noted that the estimates (54) and (55) are best possible. For every $N \geq 1$ there are $N$ points $x_1, \dots, x_N$ in $X$ such that for every permutation $\pi$ of $\{1, \dots, N\}$

$$\sum_{i=1}^{N-1} |x_{\pi(i+1)} - x_{\pi(i)}| \gg N^{1-1/K)} \; . \tag{56}$$

(For example take an orthogonal lattice in $X$.) Similarly it can be shown that (55) is optimal, too.

**Lemma 7.** *Let $X$ be a convex body in $\mathbf{R}^K$ with $\lambda_K(X)=1$ and let $\sigma$ denote the normalized surface measure on $\partial X$.*

*If $x_1,\dots,x_N$ are $N$ points in $X$, then for every $\varepsilon>0$ there is a continuous function $x:[0,1]\to X$ with $s(x)\ll N^{1-1/K}$ such that*

$$\left|\int\limits_0^1 \chi_A(x(t))\,dt-\lambda_K(A)\right| \le \left|\frac{1}{N}\sum_{n=1}^N \chi_A(x_n)-\lambda_K(A)\right|+\varepsilon \tag{57}$$

*holds uniformly in all measurable subsets $A\subseteq X$.*

*If $x_1,\dots,x_N$ are on $\partial X$, then for every $\varepsilon>0$ there is a continuous function $x:[0,1]\to\partial X$ with $s(x)\ll N^{1-1/(K-1)}$ such that*

$$\left|\int\limits_0^1 \chi_C(x(t))\,dt-\sigma(C)\right| \le \left|\frac{1}{N}\sum_{n=1}^N \chi_C(x_n)-\sigma(C)\right|+\varepsilon \tag{58}$$

*holds uniformly in all measurable subsets $C\subseteq\partial X$.*

**Corollary.** *Let $\mathcal{D}$ be a system of measurable subsets of convex body $X\subseteq\mathbf{R}^K$ with $\lambda_K(X)=1$. If we set*

$$\Omega^{(\mathcal{D})}(N)=\inf_{\{x_1,\dots,x_N\}\subseteq X}\ \sup_{D\in\mathcal{D}}\left|\frac{1}{N}\sum_{n=1}^N \chi_D(x_n)-\lambda_K(D)\right| \tag{59}$$

*and*

$$\Delta^{(\mathcal{D})}(s)=\inf_{x\in C_s}\ \sup_{D\in\mathcal{D}}\left|\int\limits_0^1 \chi_D(x(t))\,dt-\lambda_K(D)\right|\ , \tag{60}$$

*where $C_s$ is the set of all continuous functions $x:[0,1]\to X$ with $s(x)=s$, then there is a constant $c>0$ such that*

$$\Delta^{(\mathcal{D})}(s)\le\Omega^{(\mathcal{D})}(N) \tag{61}$$

*for $s=c\,N^{1-1/K}$. A similar statement is true for the boundary $\partial X$ of $X$.*

*Remark 6.* This Corollary has two interesting consequences. The first one, that will be used here, is that an upper bound for $\Omega(N)$, the smallest discrepancy of sequences, yields an upper bound for $\Delta(s)$, the smallest discrepancy of continuous functions. On the other hand, a lower bound for functions in terms of the arc length $s$ gives a lower bound for sequences in terms of the number $N$. For example, consider $N$ points on $x_1,\dots,x_n$ on the sphere $\mathbf{S}^K$. Then Theorem 3 and the above Corollary imply that there is a sherical cap $C\subseteq\mathbf{S}^K$ such that

$$\left|\frac{1}{N}\sum_{n=1}^N \chi_C(x_n)-\sigma_K(C)\right| \gg N^{-\left(\frac{1}{2}+1/(2K)\right)}\ . \tag{62}$$

(Compare with [2, Theorem 24C].) Therefore the lower bounds for the discrepancy of functions proved here are in some sense more general than the corresponding results for sequences [2].

Now consider the $K$-dimensional torus $\mathbf{R}^K/\mathbf{Z}^K$ which can be identified with $[0,1)^K$. Every subset of $\mathbf{R}^K/\mathbf{Z}^K$ can be considered as a subset of $[0,1)^K$, and a continuous function $x:[0,1]\to[0,1)^K$ can be interpreted as a continuous function $x:[0,1]\to\mathbf{R}^K/\mathbf{Z}^K$. Hence we can use Lemma 7 to construct well distributed continuous functions $x:[0,1]\to\mathbf{R}^K/\mathbf{Z}^K$, too.

It is well known (cf. [2]) that

$$\begin{aligned}\Omega^{(\mathcal{R})}(N) &\ll \frac{(\log N)^{K-1}}{N}\\ \left.\begin{array}{r}\Omega^{(Q)}(N)\\ \Omega^{(B)}(N)\\ \Omega^{(C)}(N)\end{array}\right\} &\ll N^{-\left(\frac12+1/(2K)\right)}(\log N)^{1/2}\ .\end{aligned}\tag{63}$$

Thus the above Corollary implies the upper bounds (9), (13), and (18) for the discrete values $s=cN^{1-1/K}$. But $\Delta_K^{(\cdot)}(s)$ is a decreasing function in s, since $C_{s_1}^K\subseteq C_{s_2}^K$ for $s_1\le s_2$. Therefore (9), (13), and (18) are satisfied for all $s\ge 1$ for proper chosen constants. A proof for the upper bound (8) can be found in [3] (or cf. Remark 7 in section 4).

*Proof of Lemma 6.* First assume that $X$ is a ball which center is the origin of $\mathbf{R}^K$. For $K=1$ (54) is trivial. Now suppose that $K\ge 2$ and let $X_1=X\cap\{x_1=0\}$ be the intersection with the hyperplane $\{x_1=0\}$. Set $M=[N^{1/K}]$ and let $Z\subseteq X_1$ be the set of all points $z\in X_1$ such that $M\cdot z$ is an integer point. Trivially $|Z|\ll M^{K-1}\ll N^{1-1/K}$. For every $i=1,\dots,N$ let $z(x_i)$ be a point $z\in Z$ such that the distance between $x_i$ and the line $l(z)=\{y\in X\mid\langle y-z,x\rangle=0\text{ for all }x\in\{x_1=0\}\}$ is minimal and let $x_i^0\in l(z(x_i))$ be the corresponding point. (This means that $|x_i,x_i^0|$ is minimal on $l(z(x_i))$.) Let $z_1,\dots,z_{|Z|}$ be a numeration of $Z$, $k_j$ be the number of points $x_i^0$ on $l(z_j)$, and $I_j$ the set of indices $i\in\{1,\dots,N\}$ with $x_i^0\in l(z_j)$. If $I_j$ is non-void, consider the points $x_{i_1}^0,\dots,x_{i_{k_j}}^0\in l(z_j)$ in a way that the first coordinates $x_{i,1}^0=x_{i,1}$ are ordered, i. e. $x_{i_1,1}^0\le x_{i_2,1}^0\le\cdots\le x_{i_{k_j},1}^0$. Now define a permutation $\pi$ by $\pi\left(\sum_{l=1}^{j-1}k_l+m\right)=i_m$ $(m=1,\dots,k_j)$. Thus we have

$$\begin{aligned}\sum_{i=1}^{N-1}|x_{\pi(i+1)}-x_{\pi(i)}| &= \sum_{j=1}^{|Z|}\sum_{i=\sum_{l=1}^{j-1}k_l+1}^{\sum_{l=1}^{j}k_i}|x_{\pi(i+1)}-x_{\pi(i)}|\\ &\le 2|Z|\mathrm{diam}(X)+\sum_{j=1}^{|Z|}k_j\frac{\sqrt{K-1}}{M}\\ &\ll N^{1-1/K}\ .\end{aligned}\tag{64}$$

Now consider $N$ points on the surface $\partial X$. Since the case $K=2$ is trivial we can assume $K>2$. It is well known that for every positive integer $L$ there are $L$ points $z_1,\dots,z_L$ on $X_1\cap\partial X$ such that the maximal geodesic distance to the next point $\delta=\max_{1\le i\le L}\min_{j\ne i}\delta(x_i,x_j)\ll L^{-1/(K-2)}$. If $L=[N^{1-1/(K-1)}]$ then $\delta\ll N^{-1/(K-1)}$. The set $\partial X\cap\{x_2=\cdots=x_K=0\}$ consists of two points $\underline{z}$ and $\overline{z}$ which can be considered as the poles of the sphere $\partial X$. Let $l(z_j)$ $(j=1,\dots,L)$

be the meridian joining $\underline{z}$ and $\overline{z}$ and containing $z_j$. Now the construction of a permutation $\rho$ can be managed as before to get the estimate

$$\begin{aligned}\sum_{i=1}^{N-1} \delta(x_{\rho(i+1)}, x_{\rho(i)}) &\ll L + N \cdot N^{-1/(K-1)} \\ &\ll N^{1-1/(K-1)} \; .\end{aligned} \tag{65}$$

It should be noted that the constants implied by $\ll$ in (64) and (65) only depend on the diameter $\mathrm{diam}(X)$.

If $X$ is a general convex body let $Y$ denote the smallest closed ball containing $X$. Trivially $\mathrm{diam}(X) \gg\ll \mathrm{diam}(Y)$. Therefore (64) implies (54). For the proof of (55) we need a surjective non-expansive function $p : \partial Y \to \partial X$. Then, if $x_1, \dots, x_N$ are points on $\partial X$, we can choose points $y_1, \dots, y_N$ on $\partial Y$ with $p(y_i) = x_i$. Therefore $\delta(x_i, x_j) \le \delta(y_i, y_j)$ and (65) imply (55). Such a function $p$ can be constructed in the following way. Let $B(r) = \{x \in \mathbf{R}^K \mid |x| \le r\}$ be the closed ball with radius $r$. Then the boundary of $X_r = \frac{R}{r+R} \cdot (X + B(r))$ $(R = \mathrm{diam}(Y)/2)$ converges uniformly to $\partial Y$ as $r \to \infty$. Furthermore there is a canonical surjective non-expansive function $p_r : \partial X_r \to \partial X$ for every $r > 0$. Now let $p(y) = x$ if for every $\varepsilon > 0$ there is a $r > 0$ and a point $y_r \in \partial X_r \cap (y + B(\varepsilon))$ with $p_r(y_r) = x$. This completes the proof of Lemma 6.

*Proof of Lemma 7.* It is no loss of generality to assume that $\varepsilon < 1$ and (by Lemma 6) that

$$s = \sum_{n=1}^{N-1} |x_{n+1} - x_n| \ll N^{1-1/K} \; . \tag{66}$$

Now set

$$x(t) = \begin{cases} x_i \quad \text{for} \quad \dfrac{i-1}{N} \le t \le \dfrac{i-\varepsilon}{N}, \quad 1 \le i < N \\ x_i + (x_{i+1} - x_i)\dfrac{t - \frac{i-\varepsilon}{N}}{\varepsilon/N} \quad \text{for} \quad \dfrac{i-\varepsilon}{N} \le t \le \dfrac{i}{N}, \quad 1 \le i < N \\ x_N \quad \text{for} \quad 1 - \dfrac{1}{N} \le t \le 1 \; . \end{cases} \tag{67}$$

Trivially

$$\left| \frac{1}{N} \sum_{n=1}^{N} \chi_A(x_n) - \int_0^1 \chi_A(x(t))\, dt \right| \le \varepsilon \; . \tag{68}$$

Thus (68) implies (57). The construction for (58) can be managed similarly.

## 3. Convex Bodies

### 3.1. Definitions and Results.

Let $X \subseteq \mathbf{R}^K$ $(K \ge 2)$ be a convex body with $\lambda_K(X) = 1$ and let $x : [0,1] \to X$ be a continuous function with arclength $s(x) = s$. Furthermore let $\mathcal{Q}$ denote the set of all cubes $Q \subseteq \mathbf{R}^K$ in arbitrary position and $\mathcal{Q}_0$ the cubes contained in $X$, respectively let $\mathcal{B}$ denote the set of all balls and $\mathcal{B}_0$ the set of all balls contained in $X$. Now set

$$\Delta^{(\mathcal{Q})}(s) = \inf_{x\in C_s} \sup_{Q\in\mathcal{Q}} \left| \int_0^1 \chi_Q(x(t))\, dt - \lambda_K(Q\cap X) \right| \tag{69}$$

and

$$\Delta^{(\mathcal{Q}_0)}(s) = \inf_{x\in C_s} \sup_{Q\in\mathcal{Q}_0} \left| \int_0^1 \chi_Q(x(t))\, dt - \lambda_K(Q) \right| \ , \tag{70}$$

where $C_s$ denotes the set of all continuous functions $x : [0,1] \to X$ with $s(x) = s$. $\Delta^{(\mathcal{B})}(s)$ and $\Delta^{(\mathcal{B}_0)}(s)$ can be defined similarly. Here we can formulate

**Theorem 4.** *Let $X \subseteq \mathbf{R}^K$ $(K \geq 2)$ be a convex body with $\lambda_K(X) = 1$. If $s \geq 1$, then we have*

$$\left.\begin{array}{l}\Delta^{(\mathcal{Q})}(s)\\ \Delta^{(\mathcal{B})}(s)\end{array}\right\} \gg s^{-\left(\frac{1}{2}+1/(K-1)\right)} \tag{71}$$

*and*

$$\left.\begin{array}{l}\Delta^{(\mathcal{Q})}(s)\\ \Delta^{(\mathcal{B})}(s)\end{array}\right\} \ll s^{-\left(\frac{1}{2}+1/(K-1)\right)} \left(\log(s+1)\right)^{1/2} \ , \tag{72}$$

*where the constants implied by $\ll$ and $\gg$ are only depending on the convex body $X \subseteq \mathbf{R}^K$.*

For $\mathcal{Q}_0$ and $\mathcal{B}_0$ we must use a trunctation method to get

**Theorem 5.** *Let $X \subseteq \mathbf{R}^K$ $(K \geq 2)$ be a convex body with $\lambda_K(X) = 1$. If $s \geq 1$, then we have for arbitrary $\varepsilon > 0$*

$$\left.\begin{array}{l}\Delta^{(\mathcal{Q}_0)}(s)\\ \Delta^{(\mathcal{B}_0)}(s)\end{array}\right\} \gg s^{-\left(\frac{1}{2}+1/(K-1)+\varepsilon\right)} \tag{73}$$

*and*

$$\left.\begin{array}{l}\Delta^{(\mathcal{Q}_0)}(s)\\ \Delta^{(\mathcal{B}_0)}(s)\end{array}\right\} \ll s^{-\left(\frac{1}{2}+1/(K-1)\right)} \left(\log(s+1)\right)^{1/2} \ , \tag{74}$$

*where the constants implied by $\ll$ and $\gg$ are only depending on the convex body $X \subseteq \mathbf{R}^K$ and on $\varepsilon > 0$ in (73).*

It is also interesting to consider segments instead of cubes or balls. Let $\mathcal{S}$ denote the set of all half-spaces $\{y = (y_1, \ldots, y_K) \in \mathbf{R}^K \mid a_1 y_1 + \cdots + a_K y_K \geq b\}$ $(a_1, \ldots, a_K, b \in \mathbf{R})$ and

$$\Delta^{(\mathcal{S})}(s) = \inf_{x\in C_s} \sup_{S\in\mathcal{S}} \left| \int_0^1 \chi_S(x(t))\, dt - \lambda_K(S\cap X) \right| \ , \tag{75}$$

then on gets

**Theorem 6.** *Let $X \subseteq \mathbf{R}^K$ $(K \ge 2)$ be a convex body with $\lambda_K(X) = 1$. If $s \ge 1$, then we have*

$$\Delta^{(\mathcal{S})}(s) \gg s^{-\left(\frac{1}{2}+1/(K-1)\right)} (\log(s+1))^{(1-K^2)/4} \frac{1}{\phi(s)} , \tag{76}$$

*where $\phi(s)$ is an arbitrary positive monotonic function with $\lim_{s\to\infty} \phi(s) = \infty$. Furthermore*

$$\Delta^{(\mathcal{S})}(s) \ll s^{-\left(\frac{1}{2}+1/(K-1)\right)} (\log(s+1))^{1/2} . \tag{77}$$

*The constants implied by $\ll$ and $\gg$ are only depending on the convex body $X \subseteq \mathbf{R}^K$ and on $\phi(s)$ in (76).*

Theorem 6 is a continuous analogon to J. Beck's [1] solution of a problem of K. F. Roth.

**3.2. Lower Bounds.**
The proof of (71) is quite similar to the proofs of (12) and (17). Let $x : [0,1] \to X$ be a continuous function with $s(x) = s \ge 1$. Define two measures

$$\mu(A) = \lambda_K(A \cap X) \quad \text{and} \quad z(A) = \int_0^1 \chi_A(x(t))\, dt \tag{78}$$

and set

$$F_{r,\tau} = \chi_{r,\tau} * (dz - d\mu) \tag{79}$$

as in (22). Again we can conclude for

$$\begin{aligned} \Phi(q) &= \frac{1}{q} \int_q^{2q} \int_{\mathbf{T}} \int_{\mathbf{R}^K} |F_{r,\tau}(x)|^2 \, dx \, d\tau \, dr \\ &= \int_{\mathbf{R}^K} \left( \frac{1}{q} \int_q^{2q} \int_{\mathbf{T}} |\hat{\chi}_{r,\tau}(t)|^2 \, d\tau \, dr \right) |(\widehat{dz - d\mu})(t)|^2 \, dt \\ &= \int_{\mathbf{R}^K} \varphi_q(t) \, |\psi(t)|^2 \, dt \end{aligned} \tag{80}$$

that there is a constant $c$ with

$$\Phi\left( \frac{c}{s^{1/(K-1)}} \right) \gg s^{-2K/(K-1)} . \tag{81}$$

Therefore Lemma 1 yields

$$\Phi(1) \gg s^{1-2K/(K-1)} = s^{-(1+2/(K-1))} , \tag{82}$$

which implies $\Delta^{(\mathcal{Q})}(s) \gg s^{-(1/2+1/(K-1))}$. The second part of (71) can be shown similarly.

We omit the proof of (73) since the major ideas are also included in the simpler proof of (76). In addition we generalize [1] to higher dimensions.

Let $x(t)$ be a continuous function as above. Set $n = c\, s^{1/(K-1)}$, where the constant $c$ will be fixed in the sequel. Now $y(t) = n\, x(t/T)$ with $T = n^K$ is a continuous function $y : [0,T] \to Y$ with $Y = n\, X$. The arclength of $y(t)$ can be evaluated by $s(y) = n\, s = s'$. Now (76) can be restated in the following way: There is a halfspace $S$ with

$$\left| \int_0^T \chi_S(y(t))\, dt - \lambda_K(S \cap Y) \right| \gg \frac{T^{\frac{1}{2}-1/(2K)}}{(\log T)^{(K^2-1)/4}\, \phi(T)} \ . \tag{83}$$

Set

$$z(A) = \int_0^T \chi_A(y(t))\, dt\,, \quad \mu(A) = \lambda_K(A \cap Y)\,, \tag{84}$$

$$R = \{x = (x_1, \dots , x_K) \in \mathbf{R}^K \mid |x_1| \le m, \dots , |x_{K-1}| \le m, |x_K| \le \frac{1}{100}\}, \tag{85}$$

where $m = n(\log n)^{K/2}\phi(n)^{2K-1}$,

$$E(x) = e^{-|x|^2/r^2} \ , \tag{86}$$

where $r = n(\log n)^{(K-1)/2}\phi(n)^{2K}$, and

$$F_\tau = E \cdot (\chi_{\tau R} * (dz - d\mu)) \ , \tag{87}$$

where $\tau$ is the class of all proper orthogonal transformations $\tau'$ that create the same image $\tau' R$ of $R$, i. e. $\tau' R$ is independent of the choice $\tau' \in \tau$. Let $\mathbf{T}$ denote the homogeneous space consisting of all these classes. Since $R \cap Y$ is the set-theoretic difference of two parallel segmets $S_1 \cap Y$, $S_2 \cap Y$, we have

$$\begin{aligned} \int_{\mathbf{R}^K} F_\tau(x)^2\, dx &= \int_{\mathbf{R}^K} E(x)^2\, dx \cdot \left| \int_0^1 \chi_{\tau R + x}(y(t))\ dt - \mu(\tau R + x) \right|^2 \\ &\quad + T^2 \int_{|x| \ge m/2} E(x)^2\, dx \\ &\ll r^K \Delta^2 + T^2 r^K e^{-\left(\frac{m/2}{r}\right)^2} \ , \end{aligned} \tag{88}$$

where

$$\Delta = \sup_{S \in \mathcal{S}} |z(S) - \mu(S)| \ . \tag{89}$$

Thus

$$\int_{\mathbf{T}} \int_{\mathbf{R}^K} F_\tau(x)^2\, dx\, d\tau \ll r^K \Delta^2 + O(1) \ , \tag{90}$$

where $d\tau$ is the volume element of the homogeneous space $\mathbf{T}$ induced by the Haar measure of the compact group of all proper orthogonal transformations. Therefore (83) follows from

$$\int_{\mathbf{T}} \int_{\mathbf{R}^K} |\hat{F}_\tau(t)|^2\, dt\, d\tau \gg \frac{n^{2K-1} r^{K-1}}{m^{K-1}} \ . \tag{91}$$

As in [1] it can be shown that for a proper constant $c_0$

$$\int_{T(c_0)} |\psi(t)|^2\, dt \le \frac{1}{2} \int_{Q(100)} |\psi(t)|^2\, dt \ , \tag{92}$$

where

$$\psi(t) = (\widehat{dz - d\mu})(t) = \frac{1}{(2\pi)^{K/2}} \int_{\mathbf{R}^K} e^{-i\langle x,t\rangle} (dz - d\mu)(x) \tag{93}$$

and

$$\begin{aligned} T(c_0) = \Bigg\{ t \in Q(100) \,\Bigg|\, & \text{ther are indices } l_1, \dots, l_K \text{ with} \\ & l_i \ge 0,\ i = 1, \dots, K, \text{ and } l_1 + \cdots + l_K \ge 1 \text{ such that} \\ & \left| \frac{\partial^{l_1 + \cdots + l_K}}{\partial t_1^{l_1} \cdots \partial t_K^{l_K}} \psi(t) \right| \ge c_0\, (2n)^{k_1 + \cdots + l_K} |\psi(t)| \Bigg\} \ . \end{aligned} \tag{94}$$

But we need following additional observations for the proof of (92). Set

$$g(x) = \prod_{i=1}^{K} \left( \frac{2}{\pi} \left( \frac{\sin(50\, x_i)}{x_i} \right)^2 \right) \ ,$$

$$G(x) = \int_{\mathbf{R}^K} g(x - y)\, (dz - d\mu)((y) \ , \tag{95}$$

and

$$v(x) = \int_0^T \chi_{Q+x}(y(t))\, dt \ ,$$

where $Q = Q(\frac{1}{100})$. Instead of $|G(x)| \ge v(x)$ (inequality (19) of [1]) is sufficies to prove

$$\int_{\mathbf{R}^K} G(x)^2\, dx \gg \int_{\mathbf{R}^K} v(x)^2\, dx \ . \tag{96}$$

We will need

**Lemma 8.** *If $s(y) = s' = n\, s$ and $n = c\, s^{1/(K-1)}$, then*

$$\int_{\mathbf{R}^K} v(x)^2\, dx \gg c^{K-1}\, n^K \ . \tag{97}$$

*Proof.* Let $t_i$ $(i = 0, \dots, [100\, s'])$ be defined by

$$\int_0^{t_i} |dy(t)| = \frac{i}{100} \ . \tag{98}$$

Therefore $s, t \in [t_i, t_{i+1}]$ implies $|y(t) - y(s)| \leq \frac{1}{100}$. Since

$$v(x)^2 = \int_0^T \int_0^T \chi_{Q+x}(y(t))\, \chi_{Q+x}(y(s))\, ds\, dt \ , \tag{99}$$

Cauchy—Schwarz's inequality yields

$$\begin{aligned} \int_{\mathbf{R}^K} v(x)^2\, dx &= \int_0^T \int_0^T \int_{\mathbf{R}^K} \chi_{Q+x}(y(t))\, \chi_{Q+x}(y(s))\, dx\, ds\, dt \\ &\geq \underset{|y(t)-y(s)| \leq \frac{1}{100}}{\int_0^T \int_0^T} 50^{-K}\, ds\, dt \\ &\geq 50^{-K} \sum_{i=1}^{[100\, s']} (t_i - t_{i-1})^2 + 50^{-K} (T - t_{[100\, s']})^2 \\ &\geq 50^{-K} \frac{T^2}{100\, s' + 1} \geq \frac{T^2}{n\, s} \gg c^{K-1}\, n^K \ . \end{aligned} \tag{100}$$

If we set

$$I = \int_{\mathbf{R}^K} g(x)\, dx = 100^K \ , \tag{101}$$

we get

$$\underline{g} = \inf_{x \in Q} g(x) = \left(\frac{2}{\pi}\right)^K \left(100 \sin \frac{1}{2}\right)^{2K} \geq 10\, I \ . \tag{102}$$

Define $c_1 = I/(10\, I - 1)$. Then $\frac{1}{10} < c_1 < \frac{1}{9}$. Let $R_1$, $R_2$ denote the sets $R_1 = \{x \in \mathbf{R}^K \mid v(x) \geq c_1\}$ and $R_2 = \mathbf{R}^K \setminus R_1$, then we have for $x \in R_1$

$$\begin{aligned} G(x) &= \int_0^T g(x - y(t))\, dt - \int_Y g(x - y)\, dy \\ &\geq \int_0^T \underline{g}\, \chi_{Q+x}(y(t))\, dt - I = \underline{g}\, v(x) - I \\ &\geq 10\, I\, v(x) - I \geq v(x) + (10\, I - 1) c_1 - I \geq v(x) \ . \end{aligned} \tag{103}$$

Since $\int_{R2} v(x)^2\, dx \ll n^K$, Lemma 8 implies that $c$ can be chosen in a way that

$$\int_{R_2} v(x)^2\, dx \leq \frac{1}{2} \int_{\mathbf{R}^K} v(x)^2\, dx \ . \tag{104}$$

Thus (100), (104), and (97) imply

$$\int_{\mathbf{R}^K} G(x)^2\, dx \geq \int_{R_1} G(x)^2\, dx \geq \int_{R_1} v(x)^2\, dx \gg \int_{\mathbf{R}^K} v(x)^2\, dx \ , \tag{105}$$

if $c$ is chosen in this proper way.

If $t \in \mathbf{R}^K \setminus \{0\}$ let $\alpha(t) \in \mathbf{T}$ be the class of all proper orthogonal transformations that can be represented by a proper orthogonal matrix $A$ with columns $a_1, \ldots, a_K$ such that $\langle a_1, t\rangle = \ldots = \langle a_{K-1}, t\rangle = 0$. If $\tau \in \mathbf{T}$, let $e_K(\tau) = \pm a_K$ be the last column of a corresponding orthogonal matrix. Set $B(x) = (\sin x)/x$ and $\beta(t,\tau) = B(\frac{1}{100}\langle t, e_K(\tau)\rangle)$. For $|\triangle\tau| = \mathrm{dist}(\tau, \alpha(t))$ we have (similarly to [2])

$$\hat{\chi}_{\tau R}(t+u) = \hat{\chi}_{\tau R}\left(\beta(t,\tau) + O(|u|) + O(m^K\,|\triangle\tau|)\right) \tag{106}$$

and for $t$ not contained in $T(c_0)$

$$|\psi(t+u) - \psi(t)| \le c_0\,|\psi(t)|\,\left(e^{2Kn|u|} - 1\right)\ . \tag{107}$$

Using

$$\left|\int\limits_{\mathbf{R}^K} \hat{E}(-u)\,\hat{\chi}_{\tau R}\,\beta(t,\tau)\,\psi(t)\,du\right| \gg |\psi(t)|\,r^{K-1} \tag{108}$$

and some elementary estimates we can derive

$$\begin{aligned} |\hat{F}_\tau(t)| &\ge c'\,|\psi(t)|\,r^{K-1}\,\left(1 - O\left(\frac{m^{K-1}n}{r}\right) - O\left(\frac{m^K}{r^{K-1}}|\triangle\tau|\right)\right) \\ &\gg |\psi(t)|\,r^{K-1} \end{aligned} \tag{109}$$

for $t$ not contained in $T(c_o)$ and $|\triangle\tau| \le n/(mr)$. Thus we get

$$\begin{aligned} \int\limits_{\mathbf{T}}\int\limits_{\mathbf{R}^K} |\hat{F}_\tau(t)|^2\,dt\,d\tau &\ge \int\limits_{Q(100)\setminus T(c_0)}\ \int\limits_{|\triangle\tau|\le\frac{n}{mr}} |\hat{F}_\tau(t)|^2\,d\tau\,dt \\ &\gg \left(\frac{n}{mr}\right)^{K-1} r^{2K-2} \int\limits_{Q(100)\setminus T(c_0)} |\psi(t)|^2, dt \\ &\gg \left(\frac{n}{mr}\right)^{K-1} r^{2K-2}\,n^K = \frac{n^{2K-1}r^{K-1}}{m^{K-1}} \end{aligned} \tag{110}$$

since we have by (92), (95), (97), and (105)

$$\begin{aligned} \int\limits_{Q(100)\setminus T(c_0)} |\psi(t)|^2\,dt &\gg \int\limits_{Q(100)} |\psi(t)|^2\,dt \\ &\gg \int\limits_{\mathbf{R}^K} |\hat{g}(t)\,\psi(t)|^2\,dt = \int\limits_{\mathbf{R}^K} G(x)^2\,dx \\ &\gg \int\limits_{\mathbf{R}^K} v(x)^2\,dx \gg n^K\ . \end{aligned} \tag{111}$$

This completes the proof of (76).

**3.3Upper Bounds.**

A combination of the results for sequences [2] and the Corollary of Lemma 7 immediately implies the upper bounds (72), (74), and (77).

## 4. Further Problems and Results

It is also interestiing to consider continuous functions $x : [0, \infty) \to \mathbf{R}^K/\mathbf{Z}^K$, where we can define the discrepancy

$$D_T^{(\mathcal{R})} = \sup_{R \in \mathcal{R}} \left| \frac{1}{T} \int_0^T \chi_R(x(t))\, dt - \lambda_K(R) \right| \tag{112}$$

as a function of $T > 0$. (The same can be done with balls, squares, or with caps on the sphere $\mathbf{S}^K$.)

If the arc length

$$s_T(x) = \int_o^T |dx(t)| \tag{113}$$

is finite for all $T > 0$, we can alpply Theorem 1 and 2 (for $K \geq 2$) to get

$$\liminf_{T \to \infty} D_T^{(\mathcal{R})}(x)\, s_T(x)^{1+1/(K-1)} > c_K > 0 \tag{114}$$

and

$$\liminf_{T \to \infty} D_T^{(\mathcal{B})}(x)\, s_T(x)^{\frac{1}{2}+1/(K-1)} > c_K > 0\ . \tag{115}$$

It is urprising that the exponents $1 + 1/(K-1)$ $(K \geq 2)$ and $1/2 + 1/(K-1)$ $(K = 2, 3)$ are not optimal if lim inf is replaced by lim sup.

**Theorem 7.** *([3]) If $\phi(T)$ is an arbitrary monotonically increasing function with*

$$\int_0^\infty \frac{1}{\phi(T)}\, dT < \infty \tag{116}$$

*and $x : [0, \infty) \to \mathbf{R}^K/\mathbf{Z}^K$ (or $\to \mathbf{S}^K$) $(K \geq 1)$ is a continuous function with $\lim_{T \to \infty} s_T(x) = \infty$, then*

$$\limsup_{T \to \infty} D_T^{(\mathcal{A})}(x)\, \phi\left(s_T(x)\right) > 0\ , \tag{117}$$

*where $\mathcal{A}$ is one of the sets $\mathcal{R}$, $\mathcal{Q}$, $\mathcal{B}$, (or $\mathcal{C}$).*

For example we have

$$\limsup_{T \to \infty} D_T^{(\mathcal{R})}(x)\, s_T(x)\, \left(\log s_T(x)\right)^{1+\varepsilon} > 0\ . \tag{118}$$

It is worth remarking that Theorem 7 is best possible for $K = 1$ (cf. [3]).

For every increasing function $\phi(T)$ with

$$\int_o^T \frac{1}{\phi(T)}\, dT = \infty \tag{119}$$

there exists a function $x : [0, \infty) \to \mathbf{R}^K/\mathbf{Z}^K$ with $\lim_{T \to \infty} s_T(x) = \infty$ and

$$\limsup_{T \to \infty} D_T^{(\mathcal{R})}(x)\, \phi\left(s_T(x)\right) = 0\ . \tag{120}$$

It seems that (117) is not optimal for $K \geq 2$. For example, one may concecture that

$$\limsup_{T \to \infty} D_T^{(\mathcal{R})}(x)\, s_T(x)\, \left(\log s_T(x)\right)^{2-K+\varepsilon} > 0\ . \tag{121}$$

For $K = 2$ this would be best possible, since we can prove

**Theorem 8.** *Let $0 < \alpha < 1$ be a irrational number of finite approximation type $\eta < 2$. Then $x(t) = (t, \alpha t)$ satisfies*

$$\limsup_{T\to\infty} D_T^{(\mathcal{R})}(x)\, s_T(x) = \sqrt{1+\alpha^2}\limsup_{T\to\infty} T\, D_T^{(\mathcal{R})}(x) < \infty \ . \tag{122}$$

*Proof.* We will use the inequality of Erdös—Turan—Koksma for functions

$$D_T^{(\mathcal{R})}(t, \alpha t) \le \tag{123}$$

$$C\left(\frac{1}{H} + \sum_{0<\max(|h_1|,|h_2|)\le H} \frac{1}{R(h_1,h_2)}\left|\frac{1}{T}\int_0^T e^{2\pi i(h_1 t + h_2 \alpha t)}\, dt\right|\right)$$

to estimate the discrepancy. ($R(h_1,h_2) = \max(|h_1|,1)\max(|h_2|,1)$) Since

$$\left|\int_0^T e^{2\pi i(h_1+\alpha h_2)t}\, dt\right| \le \frac{1}{\pi |h_1 + h_2\alpha|}\ , \tag{124}$$

we have to estimate

$$\begin{aligned} &2\sum_{h_1=1}^{H}\frac{1}{h_1^2} + 2\sum_{h_2=1}^{H}\frac{1}{\alpha h_2^2} + 2\sum_{h_1,h_2=1}^{H}\frac{1}{h_1h_2(h_1+\alpha h_2)} \\ &+ 2\sum_{h_1,h_2=1}^{H}\frac{1}{h_1h_2|h_1-\alpha h_2|} = S_1+S_2+S_3+S_4\ . \end{aligned} \tag{125}$$

Trivially $S_1 = O(1)$ and $S_2 = O(1)$. Since $h_1 + \alpha h_2 \ge 2\sqrt{\alpha h_1 h_2}$, $S_3 = O(1)$, too. In order to estimate $S_4$ we will split up the sum into three parts

$$\sum_{h_1,h_2=1}^{H} = \sum_{h_1 \le \alpha h_2 - \frac12} + \sum_{|h_1-\alpha h_2|<\frac12} + \sum_{h_1\ge \alpha h_2+\frac12} = R_1+R_2+R_3 \tag{126}$$

and use the fact that

$$\frac{1}{h_1(h_1-\alpha h_2)} = \frac{1}{\alpha h_2(h_1-\alpha h_2)} - \frac{1}{\alpha h_1 h_2}\ . \tag{127}$$

Now it is easy to show that

$$\begin{aligned} \sum_{h_1\ge\alpha h_2+\frac12}\frac{1}{\alpha h_2^2(h_1-\alpha h_2)} &= \frac{\pi^2}{6\alpha}\log H + O(1)\ , \\ \sum_{h_1\ge\alpha h_2+\frac12}\frac{1}{\alpha h_1 h_2^2} &= \frac{\pi^2}{6\alpha}\log H + O(1)\ , \end{aligned} \tag{128}$$

and

$$\sum_{h_1 \le \alpha h_2 - \frac{1}{2}} \left( \frac{1}{\alpha h_1 h_2^2} + \frac{1}{\alpha h_2^2 (\alpha h_2 - h_1)} \right) = O(1) \ .$$

Thus $R_1 + R_3 = O(1)$. Now consider $R_2$. For every $h_2 > 1/(2\alpha)$ there is a unique $h_1$ satisfying $|h_1 - \alpha h_2| < 1/2$. These numbers satisfy $h_1 \gg\ll h_2$ and $|h_1 - \alpha h_2| = \|\alpha h_2\|$ ($\|x\| = \min_{n \in \mathbf{Z}} |x - n|$). $\alpha$ is of approximation type $\eta < 2$. This means that there is a constant $c$ such that $\|\alpha h_2\| \ge c\, h_2^{-\eta}$ for all $h_2 \ge 1$. It is easy to show that $|\, \|\alpha p\| - \|\alpha q\|\, | \ge c(2r)^{-\eta}$ if $0 \le p < q \le r$. Thus

$$\sum_{h=1}^{r} \frac{1}{\|\alpha h\|} \le \frac{1}{c} \sum_{h=1}^{r} \frac{(2r)^\eta}{h} = O\left(r^\eta \log r\right) \tag{129}$$

implies by partial summation that

$$\begin{aligned} R_2 &= \sum_{|h_1 - \alpha h_2| < \frac{1}{2}} \frac{1}{h_1 h_2 \|\alpha h_2\|} \ll \sum_{1/(2\alpha) < h_2 \le H} \frac{1}{h_2^2 \|\alpha h_2\|} \\ &= O\left(\frac{H^\eta \log H}{H^2}\right) + O\left(\sum_{h_2=1}^{H} \frac{1}{h_2^3} h_2^\eta \log h_2\right) \\ &= O(1) \ . \end{aligned} \tag{130}$$

Therefore we can tend $H \to \infty$ in (123) to prove (122), i.e. $D_T^{(\mathcal{R})}(t, \alpha t) = O(1/T)$.

*Remark 7.* Consider $x(t) = (pt, qt)$ $(0 \le\le 1)$, where $p < q$ are distinct prime numbers. Then we can use the inequality of Erdös—Turan—Koksma to get ($H \to \infty$)

$$D^{(\mathcal{R})}(x) \le c\left(\frac{1}{H} + 2 \sum_{1 \le k \le H/q}^{H} \frac{1}{pqk^2}\right) = O\left(\frac{1}{pq}\right) \ . \tag{131}$$

In fact it can be shown that $D^{(\mathcal{R})}(pt, qt) = 1/(4pq)$. By using $p = p_n$ and $q = p_{n+1}$ ($p_n$ is the $n$-th prime number satisfying $p_n \sim n \log n$) we can deduce (10)

$$\Delta_2^{(\mathcal{R})}(s) \le \left(\frac{1}{2} + o(1)\right) s^{-2} \qquad (s \to \infty) \ . \tag{132}$$

It is a pity that this method fails to reach the lower bound (9) in higher dimensions.

*Remark 8.* The example $x(t) = (t, \alpha t)$ of Theorem 7 seems to be optimal in another sense, too. The author conjectures that there is no continuous function $x : [0, \infty) \to \mathbf{R}^2/\mathbf{Z}^2$ with finite arc length $s_T(x)$ (for all $T > 0$) such that

$$\lim_{T \to \infty} T\, D_T^{(\mathcal{R})} = 0 \ . \tag{133}$$

Only in the one dimensional case it is possible to find functions $x(t)$ satisfying (133). But this is the only example known. On the other hand it is easy to construct functions $x(t)$ (for a very general setting of the discrepancy on compact spaces) such that $\limsup_{T \to \infty} T\, D_T^{(\cdot)}(x)$ is arbitrarily small but not 0. (The construction is similar to that of [6].)

The higher dimensional case ($K \ge 3$) seems to be more difficult. Until now the author only knows

**Theorem 9.** *If $K \geq 2$, then there exist continuous functions $x_K : [0,\infty) \to \mathbf{R}^K/\mathbf{Z}^K$ with* $\lim_{T\to\infty} s_T(x_K) = \infty$ *and*

$$\limsup_{T\to\infty} D_T^{(\mathcal{R})}(x_K)\, s_T(x)\, \left(\log s_T(x)\right)^{1-K} < \infty \ . \tag{134}$$

*Proof.* It is well known that there are sequences $(x_n)_{n=1}^{\infty}$, on the torus $\mathbf{R}^K/\mathbf{Z}^K$, with

$$D^{(\mathcal{R})}\left((x_n)_{n=1}^{N}\right) \leq c_k \frac{(\log N)^K}{N} \qquad (N > 1) \ . \tag{135}$$

Set $l_n = [\log n]^K$, $L_N = \sum_{n=1}^{N} l_n$, and $X_n = \{x_i \,|\, L_{n-1} \leq i < L_n\}$. Thus $|X_n| = l_n$ and Lemma 6 imply that there is a sequence $(y_n)_{n=1}^{\infty}$ with $Y_n = \{y_i \,|\, L_{n-1} \leq i < L_n\} = X_n$ and

$$\sum_{i=L_{n-1}}^{L_n-1} |y_{i+1} - y_i| \leq C(\log n)^{K-1} + \sqrt{K} \ . \tag{136}$$

Since $X_n = Y_n$ and $|Y_n| \leq (\log n)^K$ we have

$$D^{(R)}\left((y_n)_{n=1}^{N}\right) \leq (c_K + 1)\, \frac{(\log N)^K}{N} \ . \tag{137}$$

Therefore the continuous function

$$x_K(t) = \begin{cases} y_n & \text{for} \quad t \in [n-1, n-2^{-n}] \\ y_n + 2^n(t - n + 2^{-n})(y_{n+1} - y_n) & \text{for} \quad t \in [n-2^{-n}, n] \end{cases} \tag{138}$$

satisfies

$$D_T^{(\mathcal{R})}(x_K) \leq D^{(\mathcal{R})}\left((y_n)_{n=1}^{[T]}\right) + \frac{2}{T} = O\left(\frac{\left(\log s_T(x_K)\right)^{K-1}}{s_T(x_K)}\right) \ , \tag{139}$$

since

$$s_T(x_K) \leq \sum_{L_{n-1} \leq T} \left(C(\log n)^{K-1} + \sqrt{K}\right) = O\left(\frac{T}{\log T}\right) \ . \tag{140}$$

*Remark 9.* It is apparent from the proof on Theorem 9 that lim sup-bounds for $D_T^{(\mathcal{R})}$ imply lim sup-bounds for the discrepancy of sequences. If one could prove $(\alpha < 0)$

$$\limsup_{T\to\infty} D_T^{(\mathcal{R})}(x)\, s_T(x)\, \left(\log s_T(x)\right)^{\alpha} > c_K > 0 \tag{141}$$

for functions $x : [0,\infty) \to \mathbf{R}^K/\mathbf{Z}^K$, then (141) would imply

$$\limsup_{T\to\infty} D^{(\mathcal{R})}\left((x_n)_{n=1}^{N}\right)\, N\, (\log N)^{\alpha\frac{K}{K-1}} > c_K' > 0 \ . \tag{142}$$

Maybe one can improve Roth's bound [8] by this method.

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