IRREGULARITIES OF DISTRIBUTION AND CONVEX SETS

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Abstract. Let $K \geq 2$, $A \subseteq \mathbf{R}^K/\mathbf{Z}^k$ a convex body contained in the K-dimensional torus $\mathbf{R}^K/\mathbf{Z}^k$ satisfying some differentiability conditions for the boundary ∂A, and $\{\mathbf{p}_1, \dots, \mathbf{p}_N\} \subseteq \mathbf{R}^K/\mathbf{Z}^k$ a finite point distribution. In this paper it is shown that there always exist some $r \leq 1$ and some $\mathbf{x} \in \mathbf{R}^K/\mathbf{Z}^k$ such that $\left|\sum_{\mathbf{p}_n \in rA+\mathbf{x}} 1 - N\lambda_k(rA)\right| \gg N^{1/2-1/(2K)}$ $(N \geq 1)$, where the constant implied by $\gg$ only depends on A. This theorem generalizes and sharpens a theorem for $K = 2$ by J. Beck (see [BC]) and is interesting insofar because it does not remain true for convex bodies with edges. For example in the case of cubes or rectangles there always exist point distributions such that $C(\log N)^{K-1}$ is an upper bound. However, if rotations are allowed, too, then the lower bound $CN^{1/2-1/(2K)}$ remains true for all convex bodies ([BC]).

1. Introduction

The theory of irregularities of distribution tries to quantify the irregularity of counting measures induced by point distributions with respect to continuous measures. In this paper we will discuss point distributions in compact sets and in the torus.

Let $K \geq 2$ and $X \subseteq \mathbf{R}^K$ a compact set with non-void interior $X^o \neq \emptyset$ and $\lambda_k(X) = 1$, where λ_k denotes the K-dimensional Lebesgue measure, $\mathcal{M}$ a system of measurable sets $S \subseteq \mathbf{R}^K$, and $\mathcal{P}_N = \{\mathbf{p}_1, \dots, \mathbf{p}_N\} \subseteq X$ N points contained in X. Furthermore let $z(\mathcal{P}_N, S) = \sum_{\mathbf{p}_n \in S} 1$ denote the number of points $\mathbf{p}_n \in \mathcal{P}_N$ in $S \in \mathcal{M}$, and $\mu(S) = \lambda_k(S \cap X)$ the Lebesgue measure of $S \cap X$. Using these abbreviations set

$$\Delta(\mathcal{P}_N, S) = |z(\mathcal{P}_N, S) - N\,\mu(S)|, \tag{1.1}$$

$$D(\mathcal{P}_N, \mathcal{M}) = \sup_{S \in \mathcal{M}} \Delta(\mathcal{P}_N, S), \quad \text{and} \tag{1.2}$$

$$\Omega(N, \mathcal{M}) = \inf_{\mathcal{P}_N \subseteq X} D(\mathcal{P}_N, \mathcal{M}). \tag{1.3}$$

Similarly we can handle the modulo 1 case. Let $\overline{\mathcal{M}}$ be a system of sets $\overline{S} \subseteq \mathbf{R}^K/\mathbf{Z}^k$ and $\overline{\mathcal{P}_N} = \{\overline{\mathbf{p}}_1, \dots, \overline{\mathbf{p}}_N\} \subseteq \mathbf{R}^K/\mathbf{Z}^k$ N points contained in the K-dimensional torus $\mathbf{R}^K/\mathbf{Z}^k$. Set $\overline{z}(\overline{\mathcal{P}_N}, \overline{S}) = |\{1 \leq n \leq N \mid \overline{\mathbf{p}}_n \in \overline{S}\}|$ the number of points $\overline{\mathbf{p}}_n \in \overline{\mathcal{P}_N}$ in $\overline{S} \in \overline{\mathcal{M}}$, and $\overline{\mu}(\overline{S}) = \overline{\lambda_k}(\overline{S})$, where $\overline{\lambda_k}$ is the Haar measure of the compact group $< \mathbf{R}^K/\mathbf{Z}^k, + >$, which is of course the reduction modulo 1 of the K-dimensional Lebesgue measure. As above set

$$\overline{\Delta}(\overline{\mathcal{P}_N}, \overline{S}) = \left|\overline{z}(\overline{\mathcal{P}_N}, \overline{S}) - N\,\overline{\mu}(S)\right|, \tag{1.4}$$

$$\overline{D}(\overline{\mathcal{P}_N}, \overline{\mathcal{M}}) = \sup_{\overline{S}\in\overline{\mathcal{M}}} \overline{\Delta}(\overline{\mathcal{P}_N}, \overline{S}), \quad \text{and} \tag{1.5}$$

$$\overline{\Omega}(N, \overline{\mathcal{M}}) = \inf_{\overline{\mathcal{P}_N}\subseteq\mathbf{R}^K/\mathbf{Z}^k} \overline{D}(\overline{\mathcal{P}_N}, \overline{\mathcal{M}}). \tag{1.6}$$

In view of these notations one main question in the theory of irregularities of distribution is to find the correct order of magnitude of $\Omega(N,\mathcal{M})$ resp. that of $\overline{\Omega}(N,\overline{\mathcal{M}})$ for $N\to\infty$.

Example 1. Let $\overline{\mathcal{M}} = \{r[0,1]^K + \mathbf{x}\,|\,0<r<1, \mathbf{x}\in\mathbf{R}^K/\mathbf{Z}^k\}$ be the set of all cubes of volume <1 with edges parallel to the axes. Then we have

$$(\log N)^{\frac{K-1}{2}} \ll \overline{\Omega}(N,\overline{\mathcal{M}}) \ll (\log N)^{K-1}, \tag{1.7}$$

where the constants implied by $\ll$ only depend on the dimension K (see [BC]). It is conjectured that $(\log N)^{K-1}$ is the correct order of magnitude. But until yet this conjecture is verified only for $K=2$.

Example 2. Now let $\overline{\mathcal{M}}$ be the set of all balls contained in $\mathbf{R}^K/\mathbf{Z}^k$ with diameter <1. Here the situation is completely different (see [BC]). We have

$$N^{\frac{1}{2}-\frac{1}{2K}} \ll \overline{\Omega}(N,\overline{\mathcal{M}}) \ll N^{\frac{1}{2}-\frac{1}{2K}}(\log N)^{\frac{1}{2}}. \tag{1.8}$$

The same bounds can be obtained in the Euclidian case for $\Omega(N,\mathcal{M})$, where $\mathcal{M}$ is the set of all balls with diameter <1 contained in $\mathbf{R}^K$ and X as above. Of course the constants implied by $\ll$ depend on X, too.

Example 3. Let $A\subseteq\mathbf{R}^K/\mathbf{Z}^k$ be a convex body with diameter <1 and let $\overline{\mathcal{M}} = \{r(\tau A)+\mathbf{x}\,|\,0<r\le 1, \tau\in\mathbf{T}, \mathbf{x}\in\mathbf{R}^K/\mathbf{Z}^k\}$ be the system of sets similar to A. ($\mathbf{T}$ denotes the set of all rotations in $\mathbf{R}^K/\mathbf{Z}^k$.) Then we have the same bounds as in (1.8) (see [BC]). Again we get an analogue result for $\Omega(N,\mathcal{M})$ in the Euclidian case.

Example 4. Let $K=2$ and $A\subseteq\mathbf{R}^2/\mathbf{Z}^2$ a convex set such that the boundary curve Γ of A is twice continuously differentiable and the minimal curvature of Γ is positive. Then we have (see [BC])

$$N^{\frac{1}{4}}(\log N)^{-\frac{1}{2}} \ll \overline{\Omega}(N,\overline{\mathcal{M}}) \ll N^{\frac{1}{4}}(\log N)^{\frac{1}{2}}. \tag{1.9}$$

Comparing the results of Examples 1 and 3 in the case of cubes it turns out that the order of magnitude of $\overline{\Omega}(N,\overline{\mathcal{M}})$ drastically changes when roation is allowed. On the other hand if we consider ellipses in the 2-dimensional case Examples 3 and 4 provide that the order of magnitude of $\overline{\Omega}(N,\overline{\mathcal{M}})$ does not change if rotation is allowed or not. $\overline{\Omega}(N,\overline{\mathcal{M}})$ behaves like in the case of balls as Example 2 shows. Inspecting the assumptions used above is seems that differentiability and curvature properties of the boundary of the convex set A are essential to explain this phenomenon.

The main purpose of this paper is to generalize Example 4 to the K-dimensional case by using an aymptotic expansion for the Fourier transform of the characteristic function of convex bodies by E. Hlawka ([Hl1], [Hl2]). We will prove the following theorems:

Theorem 1. *Let $K \geq 2$ and $A \subseteq \mathbf{R}^K/\mathbf{Z}^k$ a convex body such that the boundary ∂A can be mapped (uniquely) onto the unit sphere (using parallel pairs of support hyperplanes) by a bijective $6K$ times continuously differentiable function and that the product $L = r_1 \cdots r_K$ of all radii of the main curvature is uniformly bounded below by a positive constant, i.e.* $\inf L > 0$. *Let $\mathcal{M} = \{rA + \mathbf{x} \mid \frac{1}{2} \leq r \leq 1, \mathbf{x} \in \mathbf{R}^K\}$ be a system of sets homothetic to A and $X \subseteq \mathbf{R}^K$ a compact set with non-empty interior $X^o \neq \emptyset$ and Lebesgue measure $\lambda_k(X) = 1$. Then $\Omega(N, \mathcal{M})$ is bounded by*

$$N^{\frac{1}{2}-\frac{1}{2K}} \ll \Omega(N, \mathcal{M}) \ll N^{\frac{1}{2}-\frac{1}{2K}}(\log N)^{\frac{1}{2}} \quad (N > 1). \tag{1.10}$$

A similar theorem holds with rexpect to the unit torus $\mathbf{R}^K/\mathbf{Z}^k$.

Theorem 2. *Let $K \geq 2$ and $A \subseteq \mathbf{R}^K/\mathbf{Z}^k$ a convex body with diameter < 1 satisfying the the same assumptions as in Theorem 1, and let $\mathcal{M} = \{rA+\mathbf{x} \mid \frac{1}{2} \leq r \leq 1, \mathbf{x} \in \mathbf{R}^K\}$ be a system of sets homothetic to A. Then $\overline{\Omega}(N, \overline{\mathcal{M}})$ is bounded by*

$$N^{\frac{1}{2}-\frac{1}{2K}} \ll \overline{\Omega}(N, \overline{\mathcal{M}}) \ll N^{\frac{1}{2}-\frac{1}{2K}}(\log N)^{\frac{1}{2}} \quad (N > 1). \tag{1.11}$$

The lower bound of Theorem 2 is a special case of

Theorem 3. *Let $K \geq 2$ and $A \subseteq \mathbf{R}^K$ a convex body satisfying the the same assumptions as in Theorem 1. Then for any discrete point set $\mathcal{P} = \{\mathbf{p}_1, \mathbf{p}_2, \dots\} \subseteq \mathbf{R}^K$ and for all $R \geq 1$ there exists some r with $\frac{R}{2} \leq r \leq R$ and some $\mathbf{x} \in \mathbf{R}^K$ such that*

$$\left| \sum_{\mathbf{p}_n \in rA+\mathbf{x}} 1 - \lambda_k(rA) \right| \gg R^{\frac{K-1}{2}} \qquad (R \geq 1), \tag{1.12}$$

where the constant implied by $\gg$ only depends on A.

2. Proof of Theorem 1

The upper bound of Theorem 1 follows from Example 3. So we only have to concentrate on the lower bound. The essential tool for proving it is the following asymptotic expansion of E. Hlawka [Hl1]

Lemma 1. *Let $K \geq 2$ and $\mathbf{0} \in A \subseteq \mathbf{R}^K/\mathbf{Z}^k$ a convex body satisfying the the same assumptions as in Theorem 1. Let $H(\mathbf{u})$ denote the support function*

$$H(\mathbf{u}) = \sup_{\mathbf{x} \in A} \mathbf{u} \cdot \mathbf{x}$$

and $L(\mathbf{u}) = L = r_1 \cdots r_K$ the product of the radii of the main curvature corresponding to the the point $\mathbf{u}$ on the unit sphere $|\mathbf{u}| = 1$. Then we have

$$\begin{aligned}\int_B e^{i\mathbf{t}\mathbf{x}} = \frac{(2\pi)^{\frac{K-1}{2}}}{|\mathbf{t}|^{\frac{K+1}{2}}} \Bigg(&\frac{\sqrt{L(\frac{\mathbf{t}}{|\mathbf{t}|})}}{H(\frac{\mathbf{t}}{|\mathbf{t}|})} e^{i|\mathbf{t}|H(\frac{\mathbf{t}}{|\mathbf{t}|}) - i\frac{(K+1)\pi}{4}} \\ &+ \frac{\sqrt{L(-\frac{\mathbf{t}}{|\mathbf{t}|})}}{H(-\frac{\mathbf{t}}{|\mathbf{t}|})} e^{-i|\mathbf{t}|H(-\frac{\mathbf{t}}{|\mathbf{t}|}) + i\frac{(K+1)\pi}{4}} + \mathcal{O}\left(|\mathbf{t}|^{-\frac{1}{2}}\right)\Bigg)\end{aligned} \tag{2.1}$$

uniformly for $\mathbf{t} \in \mathbf{R}^K$, $|\mathbf{t}| \to \infty$.

Remark. It should be noted that the error term $\mathcal{O}\left(|\mathbf{t}|^{-\frac{1}{2}}\right)$ can be sharpened to $\mathcal{O}\left(|\mathbf{t}|^{-1}\right)$ by assuming that the mapping between the boundary ∂A and the unit sphere is $6(K+2)$ times continuously differentiable (see [Hl2]).

Now set

$$\varphi_q(\mathbf{t}) = \frac{1}{q} \int_q^{2q} |\hat{\chi}_r(\mathbf{t})|^2 \, dr, \tag{2.2}$$

where $\hat{\chi}_r$ is the Fourier transform of the characteristic function χ_r of rA. Lemma 1 enables us to prove

Lemma 2. *Let* $0 < p < q$. *Then we have*

$$\frac{\varphi_q(\mathbf{t})}{\varphi_p(\mathbf{t})} \gg \left(\frac{q}{p}\right)^{K-1} \tag{2.3}$$

uniformly for $\mathbf{t} \in \mathbf{R}^K$.

We postpone the proof of Lemma 2 to the end of this section.

In order to prove Theorem 1 we introduce the function

$$\begin{aligned} F_r(\mathbf{x}) &= \left(\chi_r * (dz - d\mu)\right)(\mathbf{x}) \\ &= \sum_{\mathbf{p}_n \in r(-A)+\mathbf{x}} 1 - N\lambda_k\left((r(-A)+\mathbf{x}) \cap X\right) \end{aligned} \tag{2.4}$$

which is of course related to $\Omega(N, \mathcal{M})$. We only have to interchange the rôles of A and $-A$. (But this is a technical detail we will not mention in the sequel.) Following the ideas of J. Beck (see [BC]) we get by Parseval's identity

$$\begin{aligned} \Phi(q) &= \frac{1}{q} \int_q^{2q} \int_{\mathbf{R}^K} |F_r(\mathbf{x})| \, d\mathbf{x} \, dr \\ &= \frac{1}{q} \int_q^{2q} \int_{\mathbf{R}^K} |\hat{F}_r(\mathbf{t})| \, d\mathbf{t} \, dr \\ &= \int_{\mathbf{R}^K} \varphi_q(\mathbf{t}) |\widehat{dz - d\mu}(\mathbf{t})|^2 \, d\mathbf{t} \end{aligned} \tag{2.5}$$

where

$$\hat{F}_r(\mathbf{t}) = \frac{1}{(2\pi)^{\frac{K}{2}}} \int_{\mathbf{R}^K} F_r(\mathbf{x}) e^{-i\mathbf{t}\mathbf{x}} \, d\mathbf{x}$$

denotes the Fourier transform of F_r. First it is easy to obtain a trivial bound for $\Phi(cN^{-1/K})$.

Lemma 3. *Let* $X \subseteq \mathbf{R}^K$ *a compact set with non-empty interior* $X^o \neq \emptyset$ *and Lebesgue measure* $\lambda_k(X) = 1$. *Then there exists some constant* $c > 0$ *such that*

$$\Phi\left(cN^{-\frac{1}{K}}\right) \gg 1 \qquad (N \geq 1). \tag{2.6}$$

Proof. Assume that $q \leq r \leq 2q$, where $q = \frac{1}{2}(2\lambda_k(A)N)^{-1/K}$. Then

$$\frac{1}{2^{K+1}N} \leq \lambda_k(rA) \leq \frac{1}{2N}$$

implies

$$\left| \sum_{\mathbf{p}_n \in rA+\mathbf{x}} 1 - N\lambda_k\left((rA+\mathbf{x}) \cap X\right) \right| \geq \frac{1}{2} \sum_{\mathbf{p}_n \in rA+\mathbf{x}} 1$$

and

$$\begin{aligned}
&\int_{\mathbf{R}^K} \left| \sum_{\mathbf{p}_n \in rA+\mathbf{x}} 1 - N\lambda_k\left((rA+\mathbf{x}) \cap X\right) \right|^2 d\mathbf{x} \\
&\geq \frac{1}{4} \int_{\mathbf{R}^K} \left(\sum_{\mathbf{p}_n \in rA+\mathbf{x}} 1 \right)^2 d\mathbf{x} \\
&\geq \frac{1}{4} \int_{\mathbf{R}^K} \left(\sum_{\mathbf{p}_n \in rA+\mathbf{x}} 1 \right) d\mathbf{x} \\
&= \frac{1}{4} N\lambda_k(A) \\
&\geq \frac{1}{2^{K+3}},
\end{aligned}$$

which proves $\Phi(q) \gg 1$. $\square$

Now we can apply Lemma 2 with $p = cN^{-1/K}$ and $q = \frac{1}{2}$ to obtain

$$\begin{aligned}
\Phi\left(\frac{1}{2}\right) &= \int_{\mathbf{R}^K} \varphi_q(\mathbf{t}) |\widehat{dz - d\mu}(\mathbf{t})|^2 \, d\mathbf{t} \\
&\gg N^{1-\frac{1}{K}} \Phi(p) \\
&\gg N^{1-\frac{1}{K}}.
\end{aligned} \tag{2.7}$$

Hence there exists some r with $\frac{1}{2} \leq r \leq 1$ and some $\mathbf{x} \in \mathbf{R}^K$ such that

$$|F_r(\mathbf{x})| \geq N^{\frac{1}{2} - \frac{1}{2K}}$$

which proves Theorem 1.

Proof of Lemma 2. It is sufficient to show that there is come costant $C > 0$ such that

$$\frac{1}{y} \int_y^{2y} |\hat{\chi}_r(\mathbf{t})|^2 \, d\mathbf{t} \gg\ll \begin{cases} \dfrac{y^{K-1}}{|\mathbf{t}|^{K+1}} & \text{for } y|\mathbf{t}| \geq C \\ y^{2K} & \text{for } y|\mathbf{t}| < C. \end{cases} \tag{2.8}$$

Set

$$f(\mathbf{t}) = \frac{1}{(2\pi)^{\frac{K}{2}}} \int_A e^{-i\mathbf{t}\mathbf{x}}. \tag{2.9}$$

Then we have

$$
\begin{aligned}
\hat{\chi}_r(\mathbf{t}) &= r^k f(r\mathbf{t}) \\
&= \frac{1}{(2\pi)^{\frac12}} \frac{r^{\frac{K-1}{2}}}{|\mathbf{t}|^{\frac{K+1}{2}}} \left(\frac{\sqrt{L(\frac{\mathbf{t}}{|\mathbf{t}|})}}{H(\frac{\mathbf{t}}{|\mathbf{t}|})} e^{-ir|\mathbf{t}|H(\frac{\mathbf{t}}{|\mathbf{t}|})+i\frac{(K+1)\pi}{4}} \right. \\
&\qquad\qquad \left. + \frac{\sqrt{L(-\frac{\mathbf{t}}{|\mathbf{t}|})}}{H(-\frac{\mathbf{t}}{|\mathbf{t}|})} e^{ir|\mathbf{t}|H(-\frac{\mathbf{t}}{|\mathbf{t}|})-i\frac{(K+1)\pi}{4}} + \mathcal{O}\left((r|\mathbf{t}|)^{-\frac12}\right)\right) \qquad (2.10) \\
&= \frac{1}{(2\pi)^{\frac12}} \frac{r^{\frac{K-1}{2}}}{|\mathbf{t}|^{\frac{K+1}{2}}} \left(g(r,\mathbf{t}) + \mathcal{O}\left((r|\mathbf{t}|)^{-\frac12}\right)\right).
\end{aligned}
$$

Let $C_1 > 0$ be sufficiently small such that

$$f(r\mathbf{t}) \gg\ll \lambda_k(A) \gg\ll 1$$

for $r|\mathbf{t}| \le C_1/2$. Hence

$$\frac{1}{y}\int_y^{2y} |\hat{\chi}_r(\mathbf{t})|^2\, d\mathbf{t} \gg\ll y^{2K}$$

for $y|\mathbf{t}| \le C_1$. Furthermore it is an easy exercise to obtain

$$\frac{1}{y}\int_y^{2y} |g(r,\mathbf{t})|^2\, dr \gg\ll 1$$

uniformly for $\mathbf{t} \in \mathbf{R}^K$ since $L(\mathbf{u})$ and $H(\mathbf{u})$ are uniformly bounded by $0 < L_1 \le L(\mathbf{u}) \le L_2$ and by $0 < H_1 \le H(\mathbf{u}) \le H_2$, where L_1, L_2, H_1, H_2 depend on A. Thus

$$\frac{1}{y}\int_y^{2y} \left| g(r,\mathbf{t}) + \mathcal{O}\left((r|\mathbf{t}|)^{-\frac12}\right)\right|^2 dr \gg\ll 1$$

uniformly for $y|\mathbf{t}| \ge C_2$, where $C_2 > 0$ is sufficiently large. Hence

$$\frac{1}{y}\int_y^{2y} |\hat{\chi}_r(\mathbf{t})|^2\, d\mathbf{t} \gg\ll \frac{r^{K-1}}{|\mathbf{t}|^{K+1}}$$

for $y|\mathbf{t}| \ge C_2$. Finally by continuity we have

$$\min_{C_1 \le y|\mathbf{t}| \le C_2} \frac{1}{y}\int_y^{2y} |f(r\mathbf{t})|^2\, dr = \min_{\substack{C_1 \le V \le C_2 \\ |\mathbf{u}|=1}} \frac{1}{V}\int_V^{2V} |f(v\mathbf{u})|^2\, dv > 0$$

and so we get (2.8) for $C = C_2$. $\square$

3. Proof of Theorems 2 and 3

Again the upper bound of Theorem 2 follows from Example 3. Furthermore the lower bound follows from Theorem 3 by identifying the K-dimensional torus $\mathbf{R}^K/\mathbf{Z}^k$ with $\left[0, N^{1/K}\right)^K \subseteq \mathbf{R}^K$ and expanding the identified point distribution $\mathcal{P}_N = \{\mathbf{p}_1, \ldots, \mathbf{p}_N\} \subseteq \left[0, N^{1/K}\right)^K$ periodically (with period $N^{1/K}$) to a discrete set $\mathcal{P} = \mathcal{P}_N + N^{1/K}\mathbf{Z}^K \subseteq \mathbf{R}^K$.

In order to prove the lower bound of Theorem 3 let $R \geq 1$ be given, set $Q_M = [-M, M]^K$, $z_M(S) = \sum_{\mathbf{p}_n \in S \cap Q_M} 1$, and $\mu_M = \lambda_k(S \cap Q_M)$, where $M \geq 1$ (sufficiently large) will be fixed in the sequel. Furthermore set

$$\Phi_M(q) = \frac{1}{q} \int_q^{2q} \int_{\mathbf{R}^K} |F_{M,r}(\mathbf{x})| \, d\mathbf{x} \, dr, \tag{3.1}$$

where $F_{M,r} = \chi_r * (dz_M - d\mu_M)$ and χ_r is the characteristic functions of rA as in the previous section. (As above we have to interchange the rôles of A and $-A$.)

First we may assume w.l.o.g. that $\sum_{\mathbf{p}_n \in Q_M} 1 \geq \frac{1}{2}(2M)^k$. Indeed, if $\sum_{\mathbf{p}_n \in Q_M} 1 < \frac{1}{2}(2M)^k$ then

$$\int\limits_{\mathbf{R}^K \cap Q_{M-Rd(A)}} \left(\sum_{\mathbf{p}_n \in RA + \mathbf{x}} 1 \right) d\mathbf{x} \leq \frac{1}{2}(2M)^k \lambda_k(RA),$$

where $d(A)$ denotes the diameter of A, implies that there exists some $\mathbf{x} \in Q_{M-Rd(A)}$ such that $RA + \mathbf{x} \subseteq Q_M$ and

$$\sum_{\mathbf{p}_n \in RA + \mathbf{x}} 1 \leq \frac{1}{2} \frac{(2M)^K}{(2(M - Rd(A))^K} \lambda_k(RA) \leq \frac{3}{4} \lambda_k(RA)$$

if M is chosen large enough. Thus (1.12) holds in this case.

Now we can proceed similarly to the proof of Lemma 3 to obtain

$$\Phi_M(c) \gg M^K \qquad (M \geq 1)$$

for some constant $c > 0$. So Parseval's identity and (2.2) yield

$$\Phi_M\left(\frac{R}{2}\right) \gg M^K R^{K-1}. \tag{3.2}$$

Hence there exists some r satisfying $\frac{R}{2} \leq r \leq R$ such that either

$$\int\limits_{Q_{M-Rd(A)}} |F_{M,r}(\mathbf{x})|^2 \, d\mathbf{x} \gg M^K R^{K-1} \tag{3.3}$$

or

$$\int_{Q_{M+Rd(A)}\setminus Q_{M-Rd(A)}} |F_{M,r}(\mathbf{x})|^2\,d\mathbf{x} \gg M^K R^{K-1} \tag{3.4}$$

since $F_{M,r}(\mathbf{x}) = 0$ for $\mathbf{x} \notin Q_{M+Rd(A)}$. If (3.3) holds we are done. Thus we only have to discuss (3.4). Of course (3.4) implies that there is some $\mathbf{x} \in \mathbf{R}^K$ such that

$$|F_{M,r}(\mathbf{x})| \gg R^{\frac{K-2}{2}} M^{\frac{1}{2}}.$$

Now choose $M = cR^{K+2}$, where $c > 0$ is a properly chosen constant such that $|F_{M,r}(\mathbf{x})| \ge 3\lambda_k(A)R^K$. Since $\lambda_k\left((rA+\mathbf{x})\cap Q_M\right) \le \lambda_k(A)R^K$ we have

$$\sum_{p_n\in rA+\mathbf{x}} 1 \ge \sum_{p_n\in (rA+\mathbf{x})\cap Q_M} 1 \ge 2\lambda_k(A)R^K.$$

Thus (1.12) holds in this case, too.

References

[BC] J[redacted]Beck and W. Chen, *Irregularities of Distribution*, Cambridge University Press, Cambridge, 1987.

[Hl1] E[redacted]Hlawka, *Integrale auf konvexen Körpern I*, Monatsh. Math. **54** (1950), 1–36.

[Hl2] E[redacted]Hlawka, *Integrale auf konvexen Körpern II*, Monatsh. Math. **54** (1950), 81–99.

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