A Uniform Law of the Iterated Logarithm for Brownian Motion on Compact Riemannian Manifolds

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1. INTRODUCTION

Let X be a compact connected C^∞-Riemannian manifold (without boundary) and μ the normalized surface measure on X. As a starting point we consider the heat equation on X

$$\frac{\partial u}{\partial t} = \Delta u, \qquad u = u(t,x), \tag{1.1}$$

where Δ denotes the Laplace-Beltrami operator on X. The heat equation (1.1) describes a diffusion process (Brownian motion), which can be interpreted as a Markov process on X. Applying a well-known theorem (see Yosida [14]) the solution $u(t,x)$ of the initial value problem

$$\frac{\partial u}{\partial t} = \Delta u, \qquad u(0,x) = u_o(x) \tag{1.2}$$

is unique if $u_o(x)$ is a C^∞-function on X and the solution of (1.2) is given by

$$u(t,x) = \int_X p(t,x,y) u_o(y) dy, \tag{1.3}$$

where p(t,x,y) are transition densities of a Markov process.

We are interested in the distribution behaviour of the trajectories x(t) of a diffusing particle starting at a given point $x(0) = w \in X$. A measure for the distribution of the continuous curve is the discrepancy defined by

$$D_T(x(t)) = \sup_B \left| \frac{1}{T} \int_0^T 1_B(x(t))\, dt - \mu(B) \right| \tag{1.4}$$

where the supremum is taken over all geodesic balls $B = B(x_o, r) \quad = \{x \in X : d(x, x_o) \leq r\}$ (d denoting the geodesic distance), with characteristic function 1_B and surface measure $\mu(B) = \int_B dx$. A continuous function $x(t)$ is called uniformly distributed on X if

$$\lim_{T\to\infty} D_T(x(t)) = 0. \tag{1.5}$$

In the special case of the one-dimensional torus $X = R/Z$ several interesting estimates of the discrepancy $D_T(x(t))$ can be found in E. Hlawka [7]; detailed references on the theory of uniform distribution can be found in the monographs of L. Kuipers and H. Niederreiter [10] and E. Hlawka [8].

In section 3 we shall prove a metric estimate for the discrepancy $D_T(x(t))$ with respect to the Wiener measure μ_w on X, which is induced by the Brownian motion (cf. [9]). Let $\mathcal{C}_w$ denote the space of all continuous functions $x : [0, \infty) \to X$ with $x(0) = w \in X$. Then for fixed $0 < t_1 < t_2 < .... < t_n$ and a Borel set $E \subseteq X^n$ the Wiener measure μ_w is defined by
(1.6)

$$\mu_w(\{x \in \mathcal{C}_w : (x(t_1), ..., x(t_n)) \in E\}) =$$
$$= \int_E p(t_n - t_{n-1}, x_{n-1}, x_n) \ldots p(t_2 - t_1, x_1, x_2) p(t_1, w, x_1)\, dx_1 ... dx_n$$

and the σ-algebra on $\mathcal{C}_w$ is generated by the sets $\{x \in \mathcal{C}_w : (x(t_1), ..., x(t_n) \in E\}$. Our main theorem reads as follows:

THEOREM. Let X be a compact connected Riemannian manifold without boundary. Then for μ_w-almost all functions $x(t)$ with $x(0) = w \in X$ we have

$$\limsup_{T\to\infty} \frac{T D_T(x(t))}{\sqrt{2T \log\log T}} = \alpha \tag{1.7}$$

with a certain positive constant α.

In section 2 we shall establish several auxiliary results for the proof. Our main tool is a general law of the iterated logarithm (cf. W. Philipp [11]). Furthermore we use an infinite series expansion for the transition density of the Brownian motion and prove a simple fact of global differential geometry.

REMARK 1. In the special case of the one-dimensional torus O. Stackelberg [12] has shown the result of our theorem with the constant $\alpha = \sqrt{3}/12$. In [4] the authors have stated the result for the d-dimensional sphere without proof. In [5] compact homogeneous spaces are investigatedby means of ergodic theory.

REMARK 2. In the case of the d-dimensional torus the discrepancy defined by (1.4) is different from the usual concept, where the supremum is taken over all d-dimensional intervals (cf. [8, 10]). However, by verbally the same proof the theorem holds as stated above for the usual definition of the discrepancy on the torus.

2. AUXILIARY RESULTS.

Using a deep asymptotic estimate for the dimensions of the eigenspaces of the Laplacian Δ (cf. [2, 6]) we deduce the following expansion for the heat kernel. For a detailed discussion of the heat kernel we also refer to [1].

LEMMA 1. Let $(\varphi_i(x))_{l=0}^{\infty}$ be a complete system of orthonormal eigenfunctions of $-\Delta$ corresponding to the eigenvalues $O = \lambda_o < \lambda_1 \leq \lambda_2 \leq \ldots; \varphi_o(x) \equiv 1$. Then the probability densities $p(t,x,y)$ of the Brownian motion (see (1.3)) are given by

$$p(t,x,y) = \sum_{i=0}^{\infty} \varphi_i(x)\varphi_i(y)e^{-\lambda_i t} \qquad (\text{for } t > 0). \tag{2.1}$$

Proof. As it is well-known (cf. [2, 13]) the Operator $-\Delta$ is selfadjoint and $\lim_{i\to\infty}\lambda_i = \infty$; thus the existence of a complete orthonomal system of eigenfunctions φ_i is evident. In the following we make use of the asymptotic relation (cf. [1, 2, 6])

$$\sum_{\lambda_i \leq t} 1 \sim C_N t^{N/2}, \tag{2.2}$$

where C_N is a constant only depending on the dimension N of X. Let L be an arbitrary positive integer. Then for $t > 0$

$$\begin{aligned}
\left|\sum_{i=0}^{\infty}\lambda_i^L\varphi_i(x)\varphi_i(y)e^{-\lambda_i t}\right| &\leq \sum_{i=0}^{\infty}\lambda_i^L||\varphi_i||_\infty^2 e^{-\lambda_i t} \leq c(X)\sum_{i=0}^{\infty}\lambda_i^{L+K}e^{-\lambda_i t} \leq \\
&\leq c(X)\sum_{n=0}^{\infty}\sum_{n<\lambda_i\leq n+1}\lambda_i^{L+K}e^{-\lambda_i t} \leq \\
&\leq c(X)\sum_{n=0}^{\infty}(n+1)^{L+K}e^{-nt}C_N(n+1)^{N/2} \leq \\
&\leq c(X)C_N\sum_{n=0}^{\infty}(n+1)^{L+K+N/2}e^{-nt} < \infty,
\end{aligned}$$

where we have used the inequality

$$||\varphi_i||_\infty \leq c(X)\lambda_i^k \quad \text{for } i \geq 1$$

with constants $c(X)$ and $k = k(X)$ only depending the manifold X (cf. [6] p. 102, Thm. 8, [13], p.256). Thus we observe that the infinite series given in (2.1) and all its derivatives with respect to t are absolutely convergent.

By the orthogonality of the eigenfunctions φ_i we see that

$$\int_X \tilde{p}(t,x,y)\varphi_i(x)dx = e^{-\lambda_i t}\varphi_i(y), \tag{2.3}$$

where $\tilde{p}$ denotes the infinite series on the left-hand side of (2.1). Thus the integral on the left is a solution of (1.1) with the initial condition $u(0,x) = \varphi_i(x)$. Obviously, (2.3) is also valid for finite linear combinations of the eigenfunctions $\varphi_i(x)$. (2.2) and the observation thereafter yields

$$\tilde{p}(t,x,.) \in C(X) \text{ for all } x \in X, \quad t > 0$$

and $\tilde{p}(t,x,.)$ can be uniquely extended to a continuous linear map $C(X) \to C(X)$.

To see that the series (2.1) is actually the heat kernel $p(t,x,y)$ in (1.3) we note that $p(t,x,y)$ is a probability density. Hence

$$f \to \int_X p(t,x,y)f(x)dx$$

is a continuous linear map which coincides on finite linear combinations of the eigenfunctions $\varphi_i(x)$ with the map

$$f \to \int_X \tilde{p}(t,x,y)f(x)dx.$$

Thus $p(t,x,y) \equiv \tilde{p}(t,x,y)$ and the proof of Lemma 1 is complete.

LEMMA 2. Let X be an N-dimensional compact Riemanian manifold and let $B(\xi,r) = \{x \in X : d(x,\xi) \leq r\}$ be the geodesic ball with center ξ and radius r; d denotes the usual geodesic metric and μ the surface measure on X. Then there exists a constant K only depending on X such that $(r_2 > r_1)$

$$\mu(B(\xi,r_2)) - \mu(B(\xi,r_1)) \leq K(r_2^N - r_1^N). \tag{2.4}$$

Proof. First we note that by the Rinow-Hopf-Theorem (cf. [3])

$$B(\xi,r) = \exp B_\xi(r),$$

where $B_\xi(r)$ denotes the ball of radius r in the tangent space T_ξ and exp the exponential map, which defines a local diffeomorphism of $U \subseteq T_\xi$ onto some open set $V \subseteq X$. Furthermore, by the compactness of X there exists a sufficently large radius R such that for all $\xi \in X \exp_\xi$ is surjective on X. Now we make use of the explicit formula for the Jacobian determinant of $\exp_\xi$ (cf. [3], pp. 253). Using the Jacobi equation (cf. [3], pp. 173) we conclude from the compactness of X that the Jacobi determinant of the exponentional map is bounded by some constant K_1 in every ball $B_\xi(r)$ with $r \leq R$. K_1 can be chosen independently of ξ.

Let $B(\xi, r_2)$ and $B(\xi, r_1)$ be two geodesic balls in X with $0 < r_1 < r_2$. Then we have

$$\mu(B(\xi, r_2)) - \mu(B(\xi, r_1)) \leq K_1 \int\limits_{B_\xi(r_2)\setminus B_\xi(r_1)} dx = K(r_2^N - r_1^N)$$

where dx denotes the Euclidean volume element in the tangent space T_ξ. Thus the proof of Lemma 2 is complete.

In the following we state a general law of the iterated logarithm. The proof runs along the same lines (with only trivial modifications) as the proofs of Theorems 1.3.1. and 1.3.2 in the monograph [11] of W. Philipp.

LEMMA 3. Let $\mathcal{A} = \{x_n(\gamma) : n = 1, 2, ...; \gamma \in \Gamma\}$ be a family of random variables uniformly bounded by 1. Let

$$S_N^2(\gamma) = S_N^2(x_n(\gamma)) = E\big(\sum_{n \leq N} x_n(\gamma) - Ex_n(\gamma)\big)^2 \tag{2.5}$$

be the variance of the partial sum, and assume that

$$\lim_{N\to\infty} \frac{1}{N} S_N^2(\gamma) = \sigma^2(\gamma) = \sigma^2(x_n(\gamma)) < 1 \tag{2.6}$$

uniformly for $\gamma \in \Gamma$.

Suppose that for each non-negative integer θ there is a subclass $\Gamma_\theta \subseteq \Gamma$ which defines a subfamily $\mathcal{A}_\theta \subseteq \mathcal{A}$ whose elements we denote by $\{x_n(\theta); n = 1, 2, ...\}$. Suppose further that for each θ the family $\mathcal{A}_\theta$ satisfies the following conditions.

- **i)** $1 \leq$ **card**$\mathcal{A}_\theta \leq \exp(c_1\theta)$ **for some constant** $c_1 > 0$ **(exp** $x = e^x$).
- **ii) Each process $\{x_n(\theta); n = 1, 2, ...\} \in \mathcal{A}_\theta$ satisfies the mixing conditions (P denoting the probability)**

$$|P(AB) - P(A)P(B)| \leq P(A)P(B)\psi(M) \tag{2.7}$$

with $\psi(M) = e^{-\lambda M}$ for $M > 1$ ($\lambda > 0$ a fixed constant), for any event $A \in \mathcal{M}_{0,L}$ and any event $B \in \mathcal{M}_{L+M,\infty}$, where $\mathcal{M}_{S,T}$ denotes the σ-field generated by the variables x_i with $S \leq n \leq T$.

iii) For each random variable $x_n(\gamma)$ there are two sequences of random variables $\bar{x}_n(0), \bar{x}_n(1), ...,$ and $x_n(0), x_n(1), ...,$ such that for any $s \geq 0$

$$\bar{y}_n(s) = \sum_{\theta=0}^{s} \bar{x}_n(\theta) \leq x_n(\gamma) \leq \sum_{\theta=0}^{s} x_n(\theta) = y_n(s) \tag{2.8}$$

and

$$0 \leq E(y_n(s) - \bar{y}_n(s)) \leq \exp(-c_2 s) \text{ for } s \geq 1$$

with c_2 a positive constant. For each θ both $\{\bar{x}_n(\theta); n = 1, 2, ...\}$ and $\{x_n(\theta); n = 1, 2, ...\})$ are members of $\mathcal{A}_\theta$.

iv) Using notation (2.5)

$$S_N^2(\theta) = O(\theta^{-6} N) \tag{2.9}$$

unifomly in $\theta \leq \log N$.

v) Each process $\{\bar{y}_n(s); n = 1, 2, ...\}$, and $\{y_n(s); n = 1, 2, ...\}$ satisfies the mixing condition (2.7) with $\psi(M)$ defined in ii).

vi) $S_N^2(\bar{y}_n(s)) = N\sigma^2(\bar{y}_n(s)) + O(1)$ and $S_N^2(y_n(s)) = N\sigma^2(y_n(s)) + O(1)$ uniformly in s and N.

Then, for any $\epsilon > 0$, there almost always is a $N_0(\epsilon, \gamma)$ such that

$$\frac{\sum_{n=1}^{N}(x_n(\gamma) - Ex_n(\gamma))}{\sqrt{2N \log\log N}} < \sigma(\gamma) + \epsilon, \qquad \text{for all } \gamma \in \Gamma \quad N > N_0 \tag{2.10}$$

Furthermore

$$P\left(\limsup_{N\to\infty} \frac{\sum_{n=1}^{N}(x_n(\gamma) - Ex_n(\gamma))}{\sqrt{2N \log\log N}} \geq \sigma(\gamma)\right) = 1, \qquad \text{for all } \gamma \in \Gamma. \tag{2.11}$$

REMARK 3. It was observed by O. Stackelberg [12] that the results of W. Philipp [11] can be stated in the above form. The only additional modification we have made is our definition of $\psi(M)$ which is necessary in order to guarantee the mixing condition for M=1. By an inspection of W. Philipp's proof it is obvious, that our definition of $\psi(M)$ is also sufficient.

3. PROOF OF THE THEOREM

Let Γ be the set of all quadruples $\gamma = (\xi_1, r_1, \xi_2, r_2)$ with $\xi_1, \xi_2 \in X$ and $r_1, r_2 > 0$. Then for every $\gamma \in \Gamma$ and for any positive integer n we define the random variables (δ a fixed positive and sufficiently small constant).

$$x_n(\gamma) = \delta \int_n^{n+1} \big(1_{B(\xi_2,r_2)}(x(t)) - 1_{B(\xi_1,r_1)}(x(t))\big)\, dt, \qquad x(t) \in \mathcal{C}_w. \tag{3.1}$$

In order to apply the general Lemma 3 we have to verify the mixing conditions (2.7) for the special random variables (3.1) on the probability space $(\mathcal{C}_w, \mu_w)$.

LEMMA 4. We have

$$|\mu_w(A)\mu_w(B) - \mu_w(AB)| \le \mu_w(A)\mu_w(B)e^{-\lambda M}$$

for $M > 1$, $\lambda > 0$ a fixed constant and $A \in \mathcal{M}_{0,L}$, $\quad B \in \mathcal{M}_{M+L,\infty}$ (see (2.7)).

Proof. First we note that $\mu_w(AB) = \mu_w(A)\mu_w(B|A)$. By the Markov property of Brownian motion (cf. [9]) we derive

$$\begin{aligned}\mu_w(B|A) &= \int_X p\big(L, w, \xi|A\big)\mu_w\big(B|x(L) = \xi\big)\, d\xi = \\ &= \int_X \int_X p\big(L, w, \xi|A\big)\mu_w\big(B|x(L+M) = \eta\big)p\big(M, \eta, \xi\big)\, d\xi\, d\eta.\end{aligned} \tag{3.2}$$

By Lemma 1 we obtain

$$|1 - p(M, \eta, \xi)| = \left|\sum_{i=1}^{\infty} \varphi_i(\eta)\varphi_i(\xi)e^{-M\lambda_i}\right| \le \text{const } e^{-\lambda_1 M} \tag{3.3.}$$

Hence by (3.2)

$$\begin{aligned}&\big|\mu_w(AB) - \mu_w(A)\mu_w(B)\big| \le \\ &\le \mu_w(A) \int_X \int_X p\big(L, w, \xi|A\big)\mu_w(B|x(L+M) = \eta)\, \big|p\big(L+M, w, \eta\big) - p\big(M, \eta, \xi\big)\big|\, d\xi\, d\eta \le \\ &\le \mu_w(A)\mu_w(B)e^{-\lambda M},\end{aligned}$$

for a suitabke constant $\lambda > 0$, which proves Lemma 4.

LEMMA 5. Let $x_n(\gamma)$ be the random variables defined in (3.1) and $S_N^2(\gamma)$ as in (2.5). Then for $N \to \infty$

$$S_N^2(\gamma) = O(1) + \sigma^2(\gamma)N$$

with $0 \le \sigma^2(\gamma) < 1$, where $O(1)$ is absolutely bounded uniformly in $\gamma \in \Gamma$.

Proof. Set
$$f_\gamma(x(t)) = \delta(1_{B(\xi_2,r_2)}(x(t)) - 1_{B(\xi_1,r_1)}(x(t)))$$
for $\gamma \in \Gamma$ and $x(t) \in \mathcal{C}_w$. First we prove

$$\int_1^{N+1} E(f_\gamma(x(t)))dt = \delta N(\mu_1 - \mu_2) + O(1), \tag{3.4}$$

where $\mu_i = \mu(B(\xi_i, r_i)) = \mu(B_i) \quad (i = 1, 2)$ and E denotes the expectation with respect to Wiener measure. We have

$$\int_1^{N+1} E(f_\gamma(x(t)))dt = \delta \int_1^{N+1} \int_{B_2} \sum_{n=0}^{\infty} \varphi_n(w)\varphi_n(y)e^{-\lambda_n t}\, dy\, dt -$$
$$- \delta \int_1^{N+1} \int_{B_1} \sum_{n=0}^{\infty} \varphi_n(w)\varphi_n(y)e^{-\lambda_n t}\, dy\, dt =$$
$$= O(1) + \delta N(\mu_2 - \mu_1) + \delta \int_{B_2} \sum_{n=1}^{\infty} \varphi_n(w)\varphi_n(y)\frac{e^{-\lambda_n(N+1)} - e^{-\lambda_n}}{\lambda_n}\, dy -$$
$$-\delta \int_{B_1} \sum_{n=1}^{\infty} \varphi_n(w)\varphi_n(y)\frac{e^{-\lambda_n(N+1)} - e^{-\lambda_n}}{\lambda_n}\, dy = \delta N(\mu_2 - \mu_1) + O(1),$$

which follows immediately from the estimates in the proof of Lemma 1. Next we have

$$S_N^2(\gamma) = E\left(\int_1^{N+1} f_\gamma(x(t)) - E(f_\gamma(x(t)))\, dt\right)^2 =$$
$$= E\left(\int_1^{N+1} f_\gamma(x(t))\, dt - \delta N(\mu_2 - \mu_1) + O(1)\right)^2 \tag{3.5}$$
$$= E\left(\int_1^{N+1} (f_\gamma(x(t)) - \delta(\mu_2 - \mu_1))\, dt\right)^2 + O(1).$$

Now we obtain

$$E\left(\int_1^{N+1} (f_\gamma(x(t)) - \delta(\mu_2 - \mu_1))\, dt\right)^2 =$$
$$= 2\int_1^{N+1} \int_1^{t} E\Big(f_\gamma(x(t)) - \delta(\mu_2 - \mu_1)(f_\gamma(x(s)) - \delta(\mu_2 - \mu_1)\Big)\, ds\, dt = \tag{3.6}$$

$$= 2\delta(\mu_1 - \mu_2) \int_1^{N+1} \int_X f_\gamma(x)(t-1)p(t,w,x)\, dx\, dt$$

$$+ 2\delta(\mu_1 - \mu_2) \int_1^{N+1} \int_1^t \int_X f_\gamma(x) p(s,w,x)\, dx\, ds\, dt +$$

$$+ N^2\delta^2(\mu_2 - \mu_1)^2 + 2 \int_1^{N+1} \int_1^t E\big(f_\gamma(x(t)) f_\gamma(x(s))\big)\, ds\, dt.$$

Using (3.3) we get

$$2\delta(\mu_1 - \mu_2) \int_1^{N+1} \int_X f_\gamma(x)(t-1)p(t,w,x)\, dx\, dt$$

$$+ 2\delta(\mu_1 - \mu_2) \int_1^{N+1} \int_1^t \int_X f_\gamma(x) p(s,w,x)\, dx\, ds\, dt =$$

$$= -2\delta^2 N^2 (\mu_2 - \mu_1)^2 + 2\delta N(\mu_1 - \mu_2) \int_X \sum_{n=1}^{\infty} f_\gamma(x)\varphi_n(w)\varphi_n(x) \frac{e^{-\lambda_n}}{\lambda_n}\, dx \, + \, O(1).$$

To compute the third integral we split it:

$$\int_1^{N+1} \int_1^t E\big(f_\gamma(x(t)) f_\gamma(x(s))\big)\, ds\, dt = \int_1^2 \int_1^t + \int_2^{N+1} \int_1^{t-1} + \int_2^{N+1} \int_{t-1}^t = I + II + III.$$

Obviously, $I = O(1)$.

In the sequel all constants $C_1, C_2, ...$ and all O-constants are uniformly bounded by an absolute constant only depending on the manifold X. For the computation of II we set

$$A_{\gamma,n,m} = \int_X \int_X f_\gamma(z) f_\gamma(y) \varphi_n(y) \varphi_n(z) \varphi_m(w) \varphi_m(y)\, dy\, dz$$

and obtain

$$II = \int_2^{N+1} \int_1^{t-1} \int_X \int_X f_\gamma(z) f_\gamma(y) p(t-s,z,y) p(s,w,z)\, dz\, dy\, ds\, dt =$$

$$= \int_2^{N+1} \int_1^{t-1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{\gamma,n,m} e^{-\lambda_n (t-s)} e^{-\lambda_m s}\, ds\, dt =$$

$$= \int_2^{N+1} \Bigg(\sum_{\substack{n,m \\ \lambda_m = \lambda_n}} A_{\gamma,n,m} e^{-\lambda_n t}(t-2) +$$

$$+ \sum_{\substack{n,m \\ \lambda_m \neq \lambda_n}} A_{\gamma,n,m} \frac{e^{(t-1)(\lambda_n - \lambda_m)} - e^{\lambda_n - \lambda_m}}{\lambda_n - \lambda_m} e^{-\lambda_n t} \Bigg))\, dt,$$

the second sum being convergent by (2.3) and

$$\frac{e^{(t-1)u} - e^u}{u} < (t-1)e^{(t-1)u} - e^u.$$

For II we get

$$A_{\gamma,0,0} \frac{1}{2}(N-1)^2 + \sum_{\substack{1 \le n,m \\ \lambda_n = \lambda_m}} A_{\gamma,n,m} \left(\frac{e^{-2\lambda_n}}{\lambda_n^2}(2\lambda_n + 1) - \frac{e^{-\lambda_n (N+1)}}{\lambda_n^2}(\lambda_n(N+1)+1) \right) +$$

$$+ \sum_{1 \le m} (A_{\gamma,0,m} + A_{\gamma,m,0}) \left(\frac{N-1}{\lambda_m} e^{-\lambda_m} - \frac{e^{-N\lambda_m} - e^{-\lambda_m}}{\lambda_m^2} \right) +$$

$$+ \sum_{\substack{1 \le n,m \\ \lambda_n \neq \lambda_m}} A_{\gamma,n,m} \left(\frac{e^{-\lambda_m - N\lambda_n} - e^{-\lambda_m - \lambda_n}}{\lambda_n(\lambda_n - \lambda_m)} - \frac{e^{-\lambda_n - N\lambda_m} - e^{-\lambda_n - \lambda_m}}{\lambda_m(\lambda_n - \lambda_m)} \right)$$

$$= O(1) + A_{\gamma,0,0} \left(\frac{N^2}{2} - N \right) - N \sum_{1 \le m} (A_{\gamma,0,m} + A_{\gamma,m,0}) \frac{e^{-\lambda_m}}{\lambda_m},$$

where we used the mean-value theorem to obtain

$$\frac{\lambda_m e^{-\lambda_m - N\lambda_n} - \lambda_n e^{-\lambda_n - N\lambda_m}}{\lambda_n \lambda_m (\lambda_n - \lambda_m)} =$$

$$= \frac{e^{-\lambda_n - \lambda_m}}{\lambda_n \lambda_m} \left(\lambda_m (N-1) e^{-(N-1)\tilde{\lambda}} - e^{-(N-1)\lambda_m} \right)$$

$$\lambda_1 \le \tilde{\lambda} \in \big(\min(\lambda_n, \lambda_m), \max(\lambda_n, \lambda_m) \big)$$

and see that $\sum_{\substack{1 \le n,m \\ \lambda_n \neq \lambda_m}} = O(1).$

For the last double integral in (3.6) we obtain using (3.3)

$$\begin{aligned} III &= \int_2^{N+1}\int_{t-1}^{t}\int_X\int_X f_\gamma(z)f_\gamma(y)p(t-s,z,y)p(s,w,z)\,dz\,dy\,ds\,dt = \\ &= O(1) + \int_2^{N+1}\int_{t-1}^{t}\int_X\int_X f_\gamma(z)f_\gamma(y)p(t-s,z,y)\,dz\,dy\,ds\,dt = \\ &= O(1) + \int_2^{N+1}\int_0^{1}\int_X\int_X f_\gamma(z)f_\gamma(y)p(u,z,y)\,dz\,dy\,du\,dt = \\ &= O(1) + C_1 N. \end{aligned}$$

Combining (3.5), (3.6) and the results for the three integrals I,II, III yields

$$S_N^2(\gamma) = O(1) + \sigma^2(\gamma)N \text{ with } \sigma^2(\gamma) = O(\delta^2\mu(\text{supp } f_\gamma)), \tag{3.7}$$

where supp denotes the support of f_γ. Thus the desired limit $\sigma^2(\gamma) = \lim\limits_{N\to\infty} S_N^2(\gamma)/N$ exists uniformly in γ. Choosing δ sufficiently small we immediately obtain

$$\sigma^2(\gamma) < 1.$$

In order to complete the proof of our main theorem we have to verify all conditions of Lemma 3 for the special random variables in (3.1). Condition (2.6) is already satisfied because of Lemma 6. Now we have to define a suitable subclass Γ_θ of Γ and a subfamily $\mathcal{A}_\theta \subseteq \mathcal{A} = \{x_n(\gamma), n = 1, 2, ... \text{ with } \gamma \in \Gamma\}$.

Let ξ be a fixed point in X and let R be sufficiently large such that the exponential mapping $\exp_\xi : B_\xi(R) \to X$ is surjective (cf. proof of Lemma 2). Since $\exp_\xi$ is a C^∞-mapping, we see that

$$d(\exp_\xi \eta_1, \exp_\xi \eta_2) \le K_o|\eta_1 - \eta_2| \text{ for all } \eta_1, \eta_2 \in B_\xi(R), \tag{3.8}$$

where d denotes the geodesic distance on X and $|\cdot|$ the distance in the tangent space T_ξ. Now we choose a fixed basis of orthogonal vectors of length 2 R in the tangent space T_ξ. For every non-negative integer θ we define the class Ξ_θ to be the class of all points η in T_ξ such that all coordinates of η have dyadic digit representation $\pm 0, a_1, ..., a_\theta (a_i = 0, 1)$. Note that with our choice of the basis in T_ξ the coordinates of any point in $B_\xi(R)$ have absolute value < 1. Let A_θ be the class of all dyadic rationals r of the form $r = 1$ or $r = 0, a_1 ... a_\theta$ and let $\rho = \sup_{x,y\in X} d(x, y)$ denote the diameter of X. Then Γ_θ is defined to be the class of all quadruples $\gamma = (\xi_1, r_1, \xi_2, r_2)$ with $\xi_1, \xi_2 \in \exp_\xi \Xi_\theta$ and $\frac{r_1}{\rho}, \frac{r_2}{\rho} \in A_\theta$. Furthermore we define $\mathcal{A}_\theta = x_n(\gamma), n = 1, 2, ... \text{ with } \gamma \in \Gamma_\theta\}$ for all $\theta = 0, 1, 2, ...$ Then we obtain

$$\text{card } \mathcal{A}_\theta = \text{card } \Gamma_\theta \le 2^{N\theta+1}2^\theta 2^{N(\theta+1)}2^\theta = 2^{4\theta+2N}. \tag{3.9}$$

Thus an estimate of the form (i) in Lemma 3 is satisfied. Obviously the mixing condition (2.7) is guaranteed by Lemma 4.

In the following we define the approximations $\bar{y}_n(s)$ and $y_n(s)$ (cf. (2.8)). For any $\eta \in X, \theta \in N_o$ there exists a point $\eta_\theta \in \exp_\xi \Xi_\theta$ such that

$$d(\eta, \exp_\xi \rho) \leq K_o \sqrt{N} 2^{-\theta} \tag{3.10}$$

(cf. (3.8)). Hence for arbitrary r with $0 < r < R$

$$B(\eta, r) \supseteq B(\eta_\theta, \max(0, r - K_o\sqrt{N}2^{-\theta})). \tag{3.11}$$

$$\text{Set } r_\theta = \left[2^\theta \frac{r}{\rho}\right] \frac{\rho}{2^\theta},$$

i.e. $r_\theta/\rho \in A_\theta$ and $r - \rho 2^{-\theta} \leq r_\theta \leq r$, $[\quad]$ denoting the integral part. Then we have by Lemma 2, (3.10) and (3.11)

$$\begin{aligned} &\mu(B(\eta, r)) - \mu(B(\eta_\theta, r_\theta)) \leq \\ \leq&\mu(B(\eta, r)) - \mu(B(\eta, \max(0, r_\theta - d(\eta, \eta_\theta)))) \leq \\ \leq&KNr^N(r - r_\theta + d(\eta, \eta_\theta)) \leq \\ \leq&KNr^N(\rho 2^{-\theta} + K_o\sqrt{N}2^{-\theta}) \leq \\ \leq&K_1 2^{-\theta} \end{aligned} \tag{3.12}$$

with a constant K_1 only depending on the manifold X. By verbally the same arguments we see that for arbitrary r $(0 < r < R)$ and $\eta \in X, \theta \in N_o$ there exists a point $\bar{\eta}_\theta \in \exp_\xi \Xi_\theta$ and a radius $\bar{r}_\theta$ with $\bar{r}_\theta/\rho \in A_\theta$ such that

$$\mu(B(\bar{\eta}_\theta, \bar{r}_\theta)) - \mu(B(\eta, r)) \leq K_2 2^{-\theta}. \tag{3.13}$$

Now let $\gamma = (\xi_1, r_1, \xi_2, r_2)$ be a fixed quadruple in Γ. For any $\theta \in N_o$ let $\gamma_\theta = (\bar{\xi}_{1,\theta}, \bar{r}_{1,\theta}, \xi_{2,\theta}, r_{2,\theta})$ and $\bar{\gamma}_\theta = (\xi_{1,\theta}, r_{1,\theta}, \bar{\xi}_{2,\theta}, \bar{r}_{2,\theta})$ where $\xi_{i,\theta}, r_{i,\theta}$ $(i = 1,2)$ are the approximations of ξ_i, r_i as considered in (3.12); $\bar{\xi}_{i,\theta}, \bar{r}_{i,\theta}$ are the approximations as considered in (3.13). Then we define

$$\begin{aligned} \bar{x}_n(2\theta) &= \int_{n-1}^{n} 1_{B(\xi_{2,\theta}, r_{2,\theta})}(x(t)) - 1_{B(\xi_{2,\theta-1}, r_{2,\theta-1})}(x(t))\, dt \\ \bar{x}_n(2\theta+1) &= \int_{n-1}^{n} 1_{B(\bar{\xi}_{1,\theta}, \bar{r}_{1,\theta})}(x(t))\, dt \\ x_n(2\theta) &= \int_{n-1}^{n} 1_{B(\bar{\xi}_{2,\theta}, \bar{r}_{2,\theta})}(x(t)) - 1_{B(\bar{\xi}_{x,\theta-1}, \bar{r}_{2,\theta-1})}(x(t))\, dt \\ x_n(2\theta+1) &= \int_{n-1}^{n} 1_{B(\xi_{1,\theta}, r_{1,\theta})}(x(t)) - 1_{B(\xi_{1,\theta-1}, r_{1,\theta-1})}(x(t))\, dt. \end{aligned} \tag{3.14}$$

Now we set

$$\bar{y}_n = \sum_{\theta=0}^{s} \bar{x}_n(\theta) \text{ and } y_n(s) = \sum_{\theta=0}^{s} x_n(\theta). \tag{3.15}$$

Obviously, $\bar{y}_n(s)$ and $y_n(s)$ are random variables of the class $\mathcal{A}$ and $\bar{x}_n(\theta), x_n(\theta)$ are contained in the class $\mathcal{A}_\theta$. Furthermore we have

$$|E(y_n(s) - \bar{y}_n(s))| \le \int_n^{n+1} \sup_{y\in X} p(t,w,y)\, dt \,\Big(\mu(B(\bar{\eta}_{s-1}\bar{r}_{s-1})) - \mu(B(\eta_{s-1}, r_{s-1}))\Big).$$

Applying Lemma 1 and (3.12), (3.13) we derive

$$|E(y_n(s) - \bar{y}_n(s))| \le e^{-C_2 s}$$

thus proving (2.9). From (3.7) we conclude that

$$S_N^2(\theta) = S_N^2(\gamma_\theta) = O(1) + \sigma^2(\gamma_\theta)N \text{ with } \sigma^2(\gamma_\theta) = O(2^{-\theta}).$$

Hence

$$S_N^2(\theta) = O(\theta^{-6}N) \qquad \text{for } \theta \le \log N, \tag{3.16}$$

proving (iv) in Lemma 3.

All other conditions in Lemma 3 are trivially fullfilled since the class $\mathcal{A}_\theta$ is a subclass of $\mathcal{A}$. Hence the assertion of Lemma 3 yields

$$\limsup_{N\to\infty} \frac{ND_N(x(t))}{\sqrt{N\log\log N}} = \text{const.} \tag{3.17}$$

for μ_w-almost all trajectories $x(t) \in \mathcal{C}_w$.

To see that $\alpha > 0$ we estimate $\sigma^2(\gamma)$ for $\gamma \in \Gamma$ with $v_1 = v_2$ and $f_\gamma \not\equiv 0$. Since for such γ the coefficients $A_{\gamma,0,\mu}$ vanish the computation leading to (3.7) shows that

$$\begin{aligned}
\sigma^2(\gamma) &= \sum_{1\le m} A_{\gamma,m,o}\frac{e^{-\lambda_m}}{\lambda_m} + \int_0^1 \int_X \int_X f_\gamma(z) f_\gamma(y) p(t,y,z) dy\, dz\, dt = \\
&= \int_0^\infty \int_X \int_X f_\gamma(y) f_\gamma(z) p(t,y,z) dy\, dz\, dt = \\
&= \int_0^\infty \int_X \int_X \int_X f_\gamma(y) f_\gamma(z) p(t/2,y,x) p(t/2,x,z)\, dx\, dy\, dz\, dt \; = \\
&= \int_0^\infty \int_X \left(\int_X f_\gamma(y) p(t/2,y,x)\, dy\right)^2 dx\, dt > 0
\end{aligned}$$

where we have used the Chapman-Kolmogoroff equations

$$\int_X p(t,x,y)p(s,z,x)\,dx = p(t+s,z,y)$$

which is an immediate consequence of (2.1) and the orthonormality of φ_i. From

$$S_N^2(x_n(\gamma_1)+x_n(\gamma_2)) \le S_N^2(x_n(\gamma_1)) + S_N^2(x_1(\gamma_2))$$

we see by taking $\gamma_1 = (\xi_1, r_1, \xi_2, 0), \gamma_2 = (\xi_1, 0, \xi_2, r_2)$ that $x_n(\gamma_1)+x_n(\gamma_2) = x_n(\gamma)$. Hence $\lim_{N\to\infty} \frac{1}{N} S_N^2(x_n(\gamma_i)) > 0$ for some $i \in \{1,2\}$ i.e. there exists a geodesic ball $B = \gamma_i$ with $\sigma^2(\gamma_i) > 0$. Since $\alpha = \frac{1}{\delta}\sup_B \sigma(B)$ we have $\alpha > 0$. In order to complete the proof of (1.7) we note that

$$\int_0^T (1_B(x(t))dt - \mu(B))dt = \int_0^{[T]} + \int_{[T]}^T (1_B(x(t))dt - \mu(B))dt.$$

Hence we have

$$\frac{T\ D_T(x(t))}{\sqrt{2T\ \log\log\ T}} = \frac{[T]D_{[T]}(s(t))}{\sqrt{2[T]\log\log[T]}} + O(1), \qquad T\to\infty$$

and the proof of the theorem is complete.

Acknowledgement. We are indebted to the referee for many valuable remarks and comments.

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