EMBEDDED TREES AND THE SUPPORT OF THE ISE

Michael Drmota

Institute of Discrete Mathematics and Geometry

Vienna University of Technology

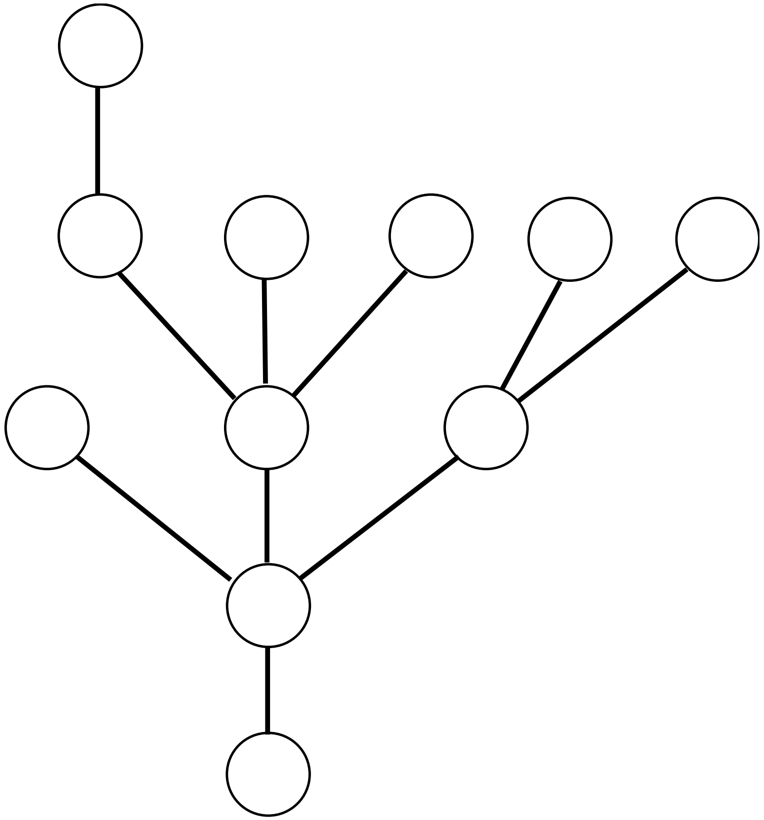
A 1040 Wien, Austria

michael.drmota@tuwien.ac.at

http://www.dmg.tuwien.ac.at/drmota/

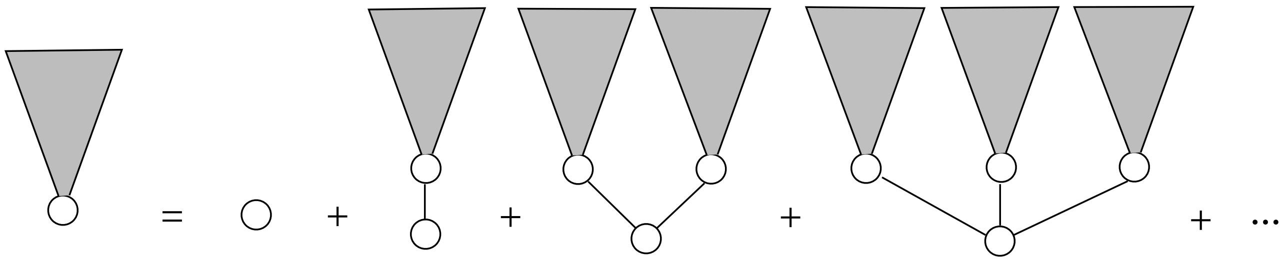
IWOCA 2009, Hradec nad Moravici, June 29 – July 2, 2009

Catalan trees



Catalan trees

p_n = number of Catalan trees with n **edges**; $P(t) = \sum_{n \geq 0} p_n t^n$



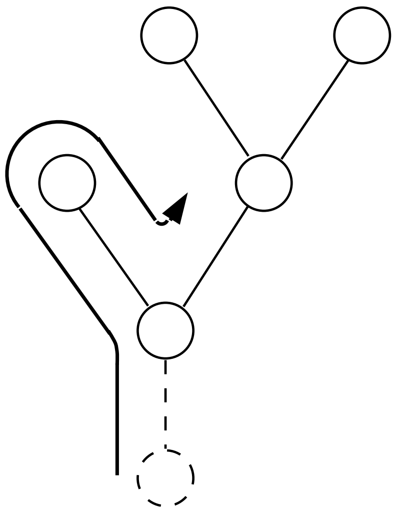
$$P(t) = 1 + tP(t) + t^2 P(t)^2 + t^3 P(t)^3 + \cdots = \frac{1}{1 - tP(t)}$$

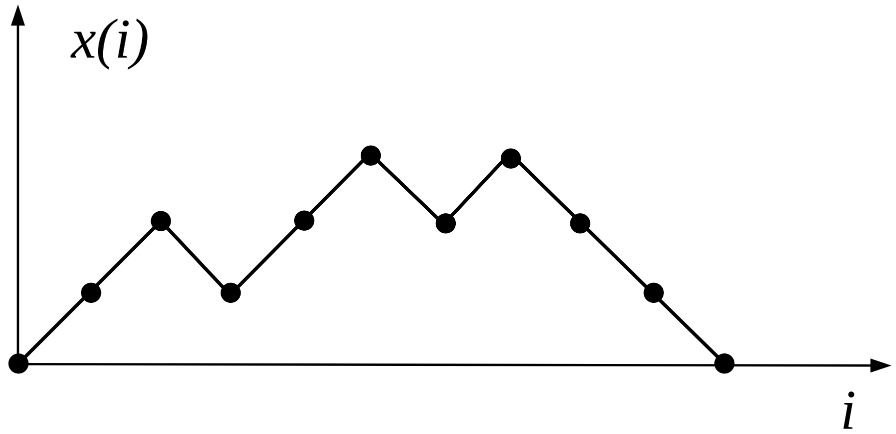
$$P(t) = \frac{1 - \sqrt{1-4t}}{2t} \quad \Longrightarrow \quad p_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi} \cdot n^{3/2}}$$

(Catalan numbers)

Depth-First-Search

Rooted trees and discrete excursions





Bijection between

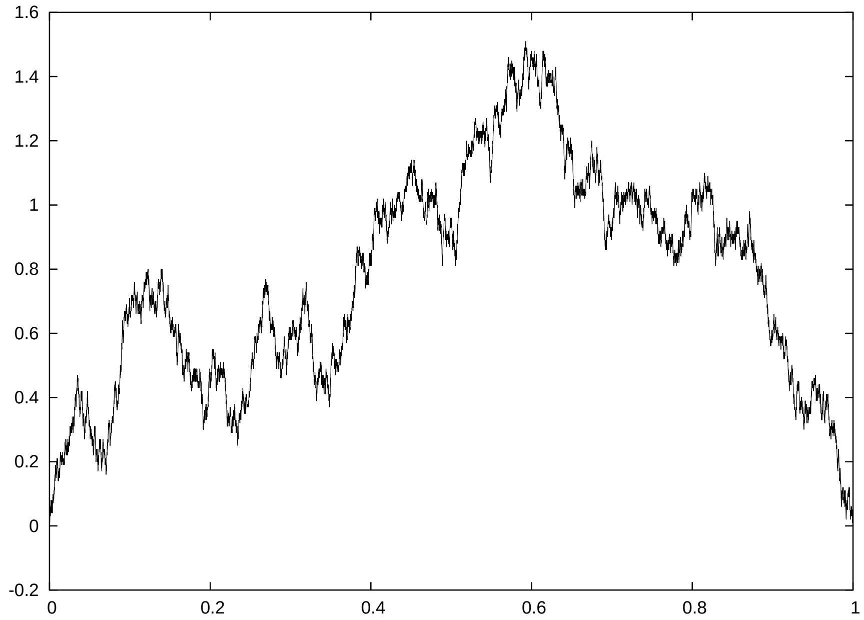
Catalan trees $\longleftrightarrow$ Dyck paths

random trees of size n $\longleftrightarrow$ **random** Dyck paths of length $2n$

Remark: Alternatively we have a **parenthesis representation** (compare with the previous talk).

Depth-First-Search

Brownian excursion $(e(t), 0 \leq t \leq 1)$



Rescaled Brownian motion between 2 zeros.

Random function in $C[0, 1]$.

Depth-First-Search

Kaigh's Theorem

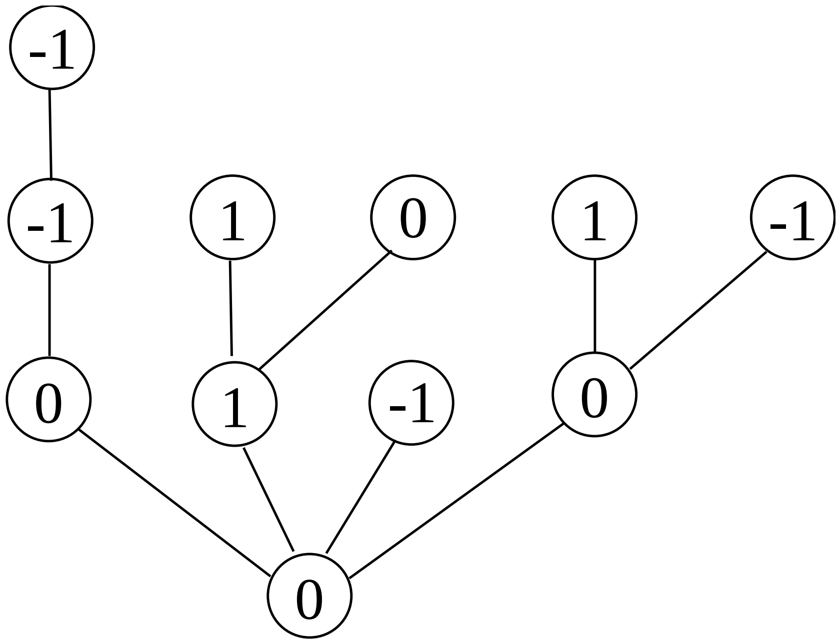
$(X_n(t), 0 \le t \le 2n)$ ... random Dyck path of length $2n$.

$$\Longrightarrow \quad \left(\frac{1}{\sqrt{2n}} X_n(2nt), 0 \le t \le 1\right) \xrightarrow{\text{d}} (e(t), 0 \le t \le 1).$$

Remark. This theorem also holds for more general random walks with independent increments conditioned to be an excursion.

Embedded Trees

Integer labels, root has label 0, adjacent labels differ at most by 1:

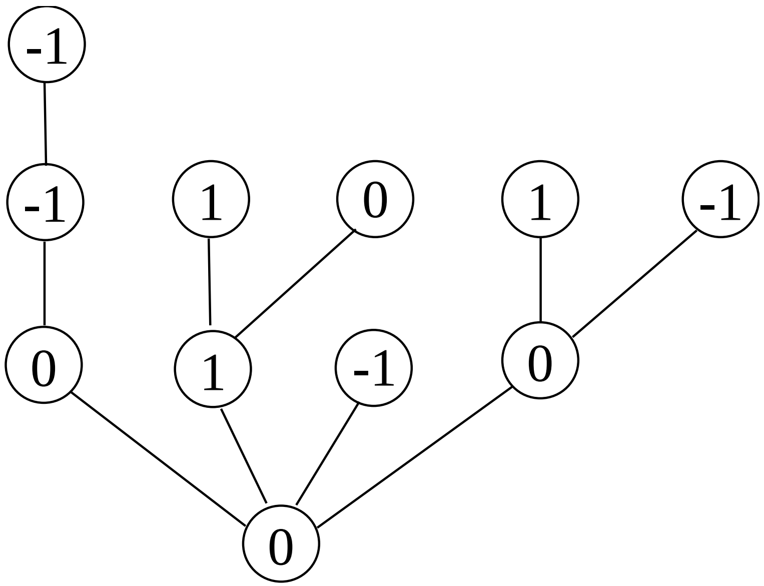


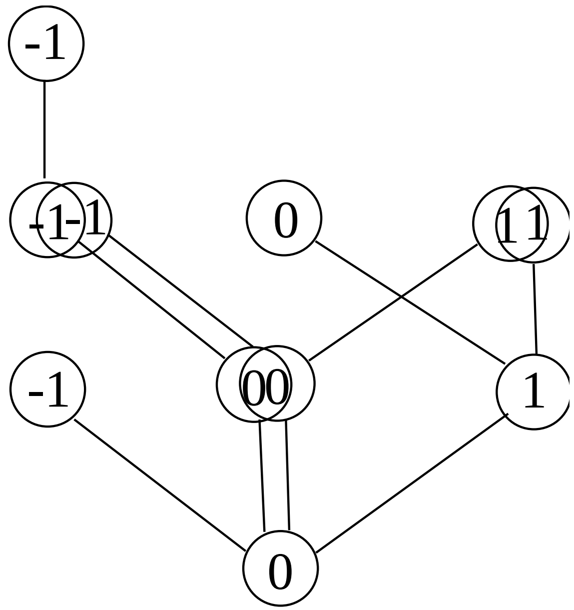
$$u_n = 3^n p_n = \frac{3^n}{n+1}\binom{2n}{n},$$

(the number of embedded trees with n edges)

Embedded Trees

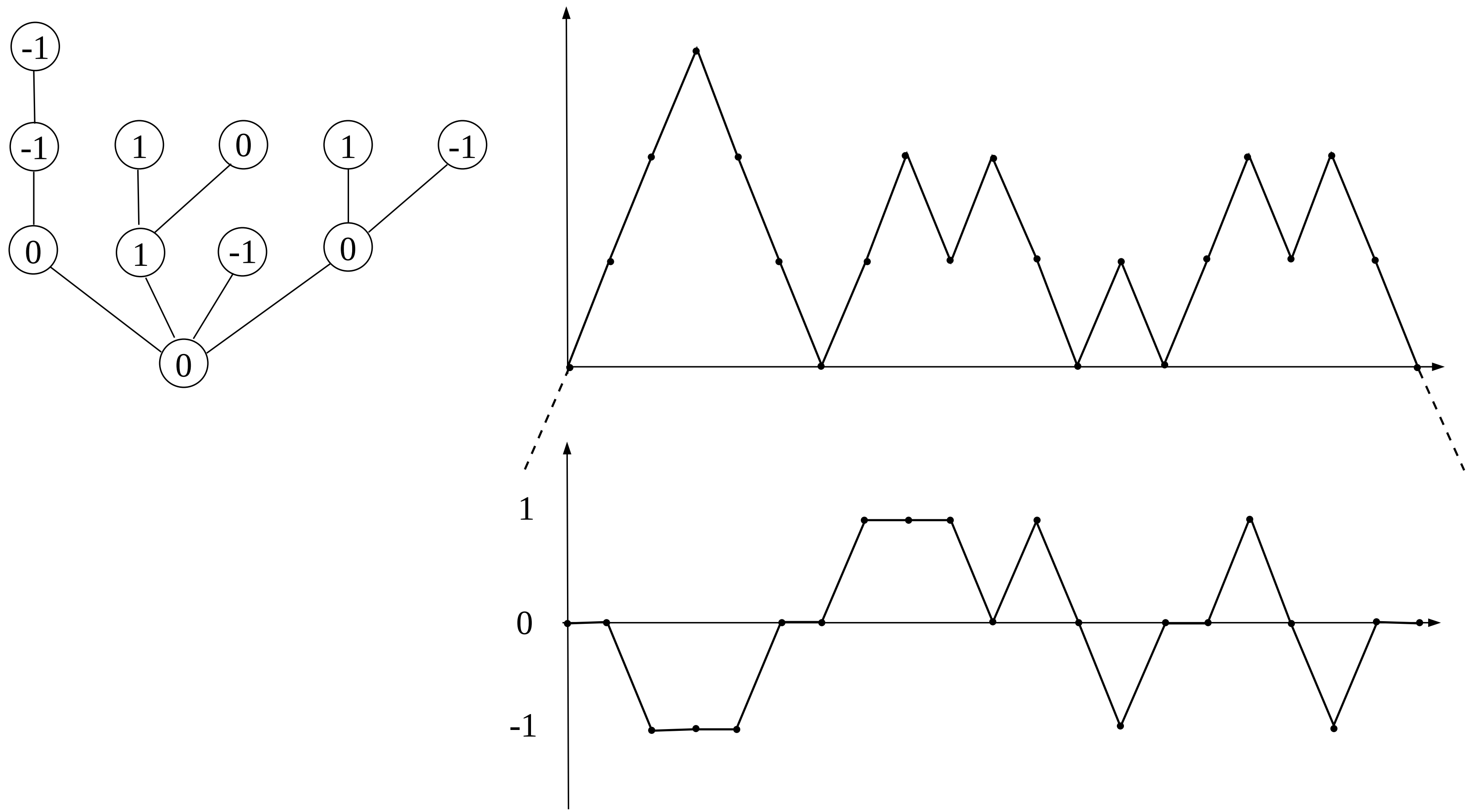
Interpretation as embedding





Brownian Snake

Discrete Brownian snake



Brownian Snake

Theorem (Chassaing+Marckert)

Consider Catalan trees and labels given by independent increments following a distribution η with $\mathbb{E}\,\eta = 0$.

$W_n(s)$ ... discrete Brownian snake corresponding to these trees and labels

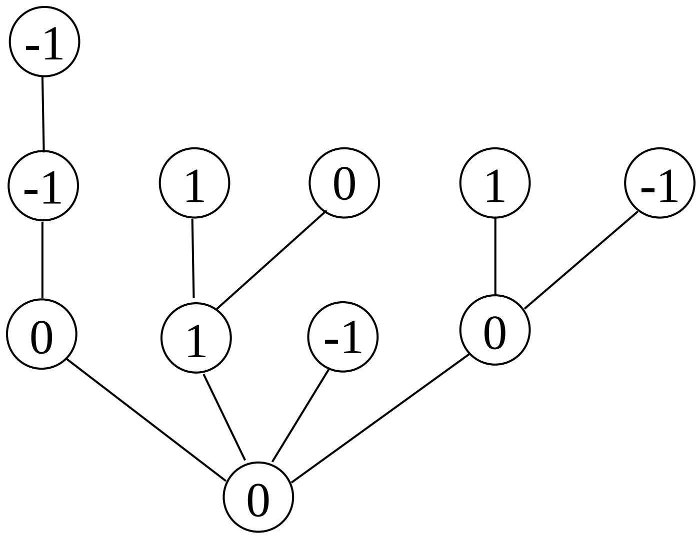
$$\Longrightarrow \quad \boxed{\left(\frac{\gamma}{n^{1/4}} W_n(2nt), 0 \le t \le 1\right) \xrightarrow{\text{d}} (W(t), 0 \le t \le 1)}$$

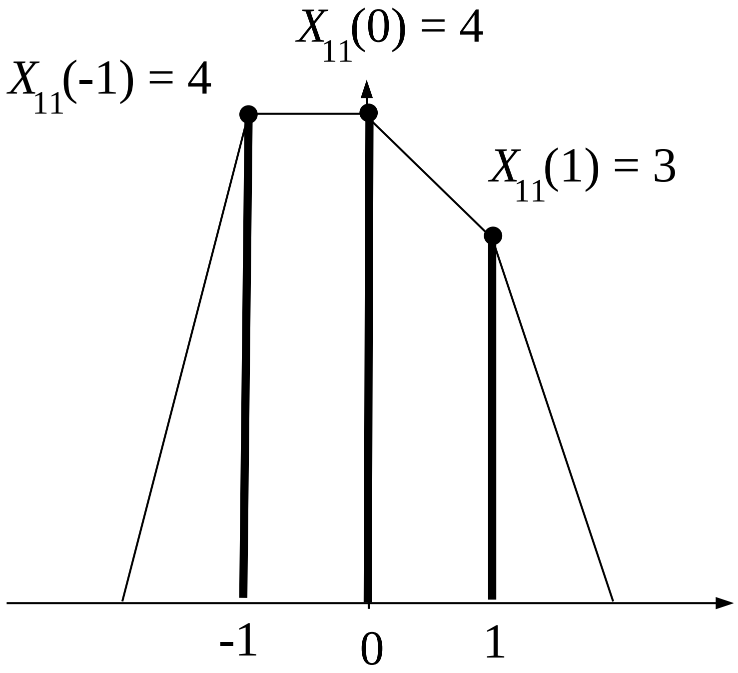
with $\gamma = \sqrt{2}(\mathbb{V}\mathrm{ar}\,\eta)^{-\frac{1}{2}}$.

$W(t)$ ... Brownian snake

Continuous Limits

Value distribution





Continuous Limits

Theorem (Aldous)

Consider Catalan trees with labels given by independent increments following a distribution η with $\mathbb{E}\,\eta = 0$.

$X_n(j)$ ... number of nodes with label j

$$\Longrightarrow \quad \boxed{\frac{1}{n+1}\sum_{j\in\mathbb{Z}} X_n(j)\delta_{\gamma j\, n^{-1/4}} \xrightarrow{\mathrm{d}} \mu_{\mathrm{ISE}}}$$

with $\gamma = \sqrt{2}\,(\mathbb{V}\mathrm{ar}\,\eta)^{-\frac{1}{2}}$.

Continuous Limits

Theorem (Devroye+Janson)

Suppose additionally that η is integer valued and aperiodic.

$(X_n(j))_{j\in\mathbb{Z}}$ ... profile corresponding to η
$(X_n(j)$ ... number of nodes with label $j)$
$(X_n(t), -\infty < t < \infty)$ ... the linearly interpolated process:

$$\Longrightarrow \quad \boxed{\left(n^{-3/4}X_n(n^{1/4}t),\, -\infty < t < \infty\right) \xrightarrow{\mathrm{d}} (\gamma f_{\mathrm{ISE}}(\gamma t),\, -\infty < t < \infty)}$$

Integrated SuperBrownian Excursion (ISE)

Occupation measure of the Brownian snake: random measure

$$\mu_{\text{ISE}}(A) = \int_0^1 \mathbf{1}_A(W(t))\,dt$$

Density of the ISE: random density

$$f_{\text{ISE}}(s) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 \mathbf{1}_{[s,s+\varepsilon]}(W(t))\,dt$$

Remark. The ISE has **finite support** $[L_{\text{ISE}}, R_{\text{ISM}}]$
(Its length $R_{\text{ISE}} - L_{\text{ISE}}$ is a random variable.)

Continuous Limits

$[L_{\mathrm{ISE}}, R_{\mathrm{ISE}}]$ ... (random) support of the ISE

Theorem

M_n ... largest label of a random embedded tree with n edges
m_n ... smallest label of a random embedded tree with n edges

$$\Longrightarrow \quad \boxed{\frac{M_n}{n^{1/4}} \xrightarrow{\mathrm{d}} \gamma^{-1} R_{\mathrm{ISE}}}, \quad \boxed{\frac{m_n}{n^{1/4}} \xrightarrow{\mathrm{d}} \gamma^{-1} L_{\mathrm{ISE}}}.$$

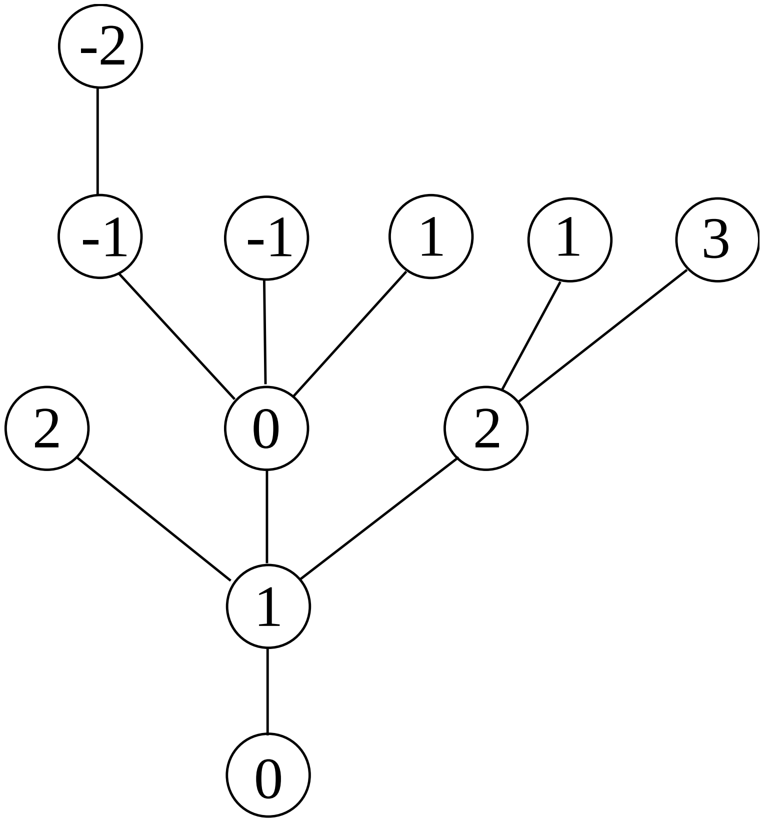
Remark.

$$\left(\frac{M_n}{n^{1/4}}, \frac{m_n}{n^{1/4}}\right) \xrightarrow{\mathrm{d}} \left(\gamma^{-1} R_{\mathrm{ISE}}, \gamma^{-1} L_{\mathrm{ISE}}\right), \qquad \frac{M_n - m_n}{n^{1/4}} \xrightarrow{\mathrm{d}} \gamma^{-1}(R_{\mathrm{ISE}} - L_{\mathrm{ISE}}).$$

Counting of Embedded Trees

Simplified Problem:

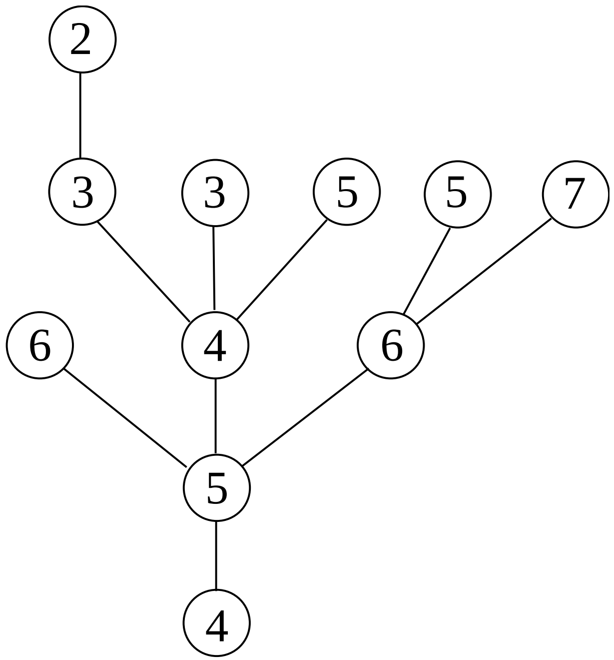
Embedded Trees with increments ± 1



Counting of Embedded Trees

Generating functions

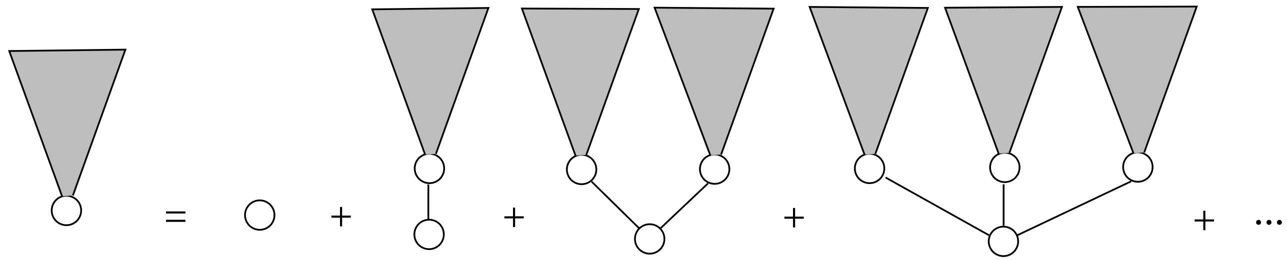
$R_j(t)$ ... generating function of embedded trees, where we assume that the **root is labelled by** j
(and labels of adjacent vertices differ by ± 1)



$$\boxed{R_j(t) = R_0(t) \equiv R(t)}$$

Counting of Embedded Trees

Generating functions



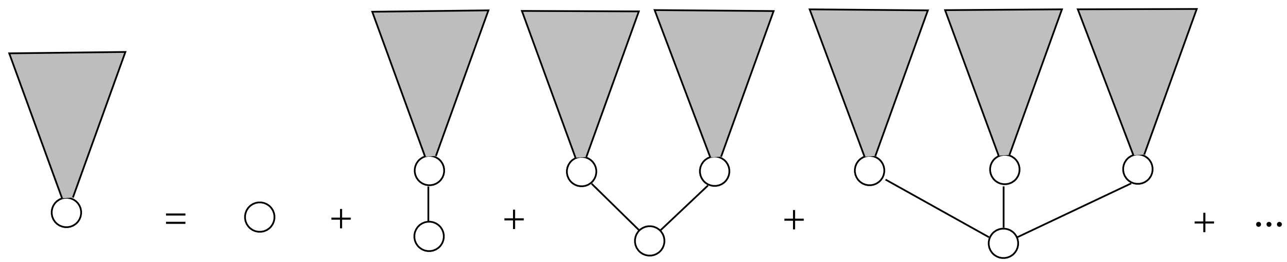
$$\boxed{R_j(t) = \frac{1}{1 - t(R_{j-1}(t) + R_{j+1}(t))}}$$

$$\Longrightarrow \quad R(t) = \frac{1}{1 - 2tR(t)} = \frac{1 - \sqrt{1-8t}}{4t} = \sum_{n \geq 0} \boxed{\frac{2^n}{n+1}\binom{2n}{n}} t^n.$$

Counting of Embedded Trees

Generating functions

$R_j^{[0]}(t)$ ... generating function of embedded trees, where we assume that the **root is labelled by** j and **all labels are non-negative**



$$R_j^{[0]}(t) = \frac{1}{1 - t(R_{j-1}^{[0]}(t) + R_{j+1}^{[0]}(t))}$$

$$R_{-1}^{[0]}(t) = R_{-2}^{[0]}(t) = \ldots = 0.$$

Counting of Embedded Trees

Generating functions (A **MIRACLE** occurs)

$$\boxed{R_j^{[0]}(t) = R(t)\frac{u_j u_{j+4}}{u_{j+1}u_{j+3}}}$$

where

$$u_j = u_j(t) = 1 - Z(t)^{j+1}$$

and

$$Z(t) = \frac{1-(1-8t)^{1/4}}{1+(1-8t)^{1/4}}$$

is the solution of the equation

$$Z + \frac{1}{Z} + 2 = \frac{1}{tR(t)}$$

(Bouttier, Di Francesco, and Guitter)

Counting of Embedded Trees

Maximum label

$$\mathbb{P}\{M_n \le j\} = \frac{[t^n]\, R_j^{[0]}(t)}{\frac{2^n}{n+1}\binom{2n}{n}}$$

$$[t^n]\, R_j^{[0]}(t) = \frac{1}{2\pi i}\int_\gamma R_j^{[0]}(t) t^{-n-1}\, dt$$

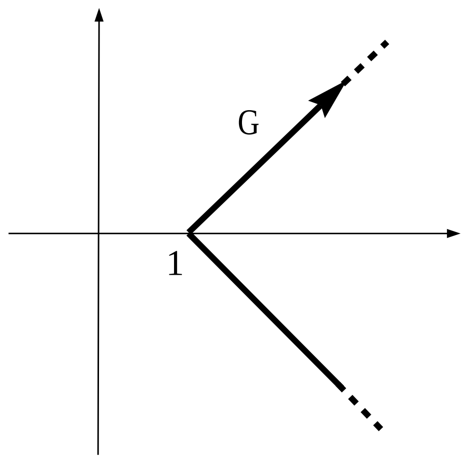
Counting of Embedded Trees

Maximum label

$$\mathbb{P}\{R_{\mathsf{ISE}} > \lambda\} = \frac{12}{i\sqrt{\pi}} \int_\Gamma \frac{v^5 e^{v^4}}{\sinh^2(\lambda v)} dv$$

(Mireille Bousquet-Mélou)

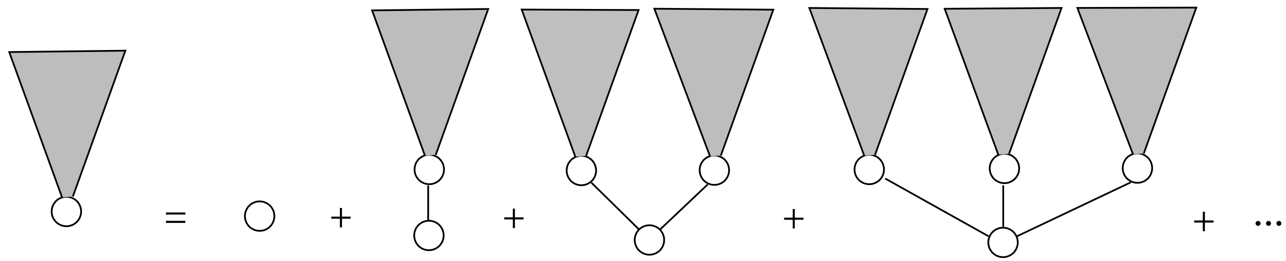
$$\Gamma = \{1 - te^{-i\pi/4}, t \in (-\infty, 0]\} \cup \{1 + te^{i\pi/4}, t \in [0, \infty)\},$$



Counting of Embedded Trees

Generating functions

$R_j^{[0,L]}(t)$ ... generating function of embedded trees, where we assume that the **root is labelled by** j and **where all labels are bounded between** 0 **and** L



$$\boxed{R_j^{[0,L]}(t) = \frac{1}{1 - t(R_{j-1}^{[0,L]}(t) + R_{j+1}^{[0,L]}(t))}}$$

$$R_{-1}^{[0,L]}(t) = R_{L+1}^{[0,L]}(t) = 0.$$

Counting of Embedded Trees

Generating functions (A second **MIRACLE** occurs)

$$R_j^{[0,L]}(t) = \frac{\theta_1\left(\frac{2}{L+6},q\right)^3}{\theta_1\left(\frac{1}{L+6},q\right)^2\theta_1\left(\frac{4}{L+6},q\right)} \frac{\theta_1\left(\frac{j+1}{L+6},q\right)\theta_1\left(\frac{j+5}{L+6},q\right)}{\theta_1\left(\frac{j+2}{L+6},q\right)\theta_1\left(\frac{j+4}{L+6},q\right)}$$

where $q = q(t)$ be determined by the equation

$$t = \frac{\theta_1\left(\frac{1}{L+6},q\right)^4\theta_1\left(\frac{4}{L+6},q\right)}{\theta_1\left(\frac{2}{L+6},q\right)^5}$$

and $\theta_1(u;q)$ is the **Jacobi theta function**

$$\theta_1(u;q) = 2i\sin(\pi u)\prod_{j\ge 1}\left(1-2q^j\cos(2\pi u)+q^{2j}\right).$$

Counting of Embedded Trees

Properties of the Jacobi theta function

$$\theta_1(u+1;q) = -\theta_1(u;q)$$

$$\theta_1(u+\tau;q) = -q^{-\frac{1}{2}}e^{-2\pi i u}\theta_1(u;q),$$

where

$$q = e^{2\pi i \tau}.$$

(Bouttier, Di Francesco, and Guitter)

Counting of Embedded Trees

Properties of the Jacobi theta function

Lemma

$$\left(\frac{\theta_1(2\alpha)}{\theta_1(\alpha)}\right)^2\theta_1(u)\theta_1(u+2\alpha)\theta_1(u+4\alpha)$$
$$=\frac{\theta_1(4\alpha)}{\theta_1(2\alpha)}\theta_1(u+\alpha)\theta_1(u+2\alpha)\theta_1(u+3\alpha)$$
$$+\theta_1(u-\alpha)\theta_1(u+3\alpha)\theta_1(u+4\alpha)+\theta_1(u)\theta_1(u+\alpha)\theta_1(u+5\alpha)$$

Proof

Both sides transform in the same way under the transformations $u \mapsto u+1$ and $u \mapsto u+\tau$ (where $q = e^{2\pi i\tau}$). Hence, the ratio is an elliptic function. The possible poles $(0, -2\alpha, -4\alpha, \ldots)$ have vanishing residues. Thus, the ratio is constant which is actually 1 (by setting $u = -\alpha$).

Elliptic Functions

Weierstrass $\wp$-function

$$\boxed{\wp(z;\tau) = \frac{1}{z^2} + \sum_{(m_1,m_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(z - m_1\tau - m_2)^2} - \frac{1}{(m_1\tau + m_2)^2} \right)},$$

where $\Im(\tau) > 0$ and $z \notin \mathbb{Z} + \tau\mathbb{Z}$

$$\boxed{\wp(z+1;\tau) = \wp(z+\tau;\tau) = \wp(z;\tau)}$$

Elliptic Functions

Relation to Jacobi's θ_1-function

$$\frac{\theta_1'''(0,q)}{3\,\theta_1'(0,q)} - \frac{\theta_1''(u,q)}{\theta_1(u,q)} + \left(\frac{\theta_1'(u,q)}{\theta_1(u,q)}\right)^2 = \wp(u;\tau)$$

Elliptic Functions

Eisenstein series

$$G_k(\tau) = \sum_{(m_1,m_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m_1 + m_2\tau)^k}.$$

$$G_4(\tau) = \frac{5}{8}\left(\frac{\theta_1'''(0,q)}{\theta_1'(0,q)}\right)^2 + \frac{3}{8}\frac{\theta_1^{(5)}(0,q)}{\theta_1'(0,q)},$$

Support of the ISE

Main Result

$$\boxed{\mathbb{P}\{R_{\mathsf{ISE}} \le \lambda_1,\, -L_{\mathsf{ISE}} \le \lambda_2\} = \frac{20}{3i\pi^{5/2}} \int_\Gamma \wp\left(-\frac{i\lambda_1}{\pi}v;\, -\frac{i(\lambda_1+\lambda_2)}{\pi}v\right)\, v^5\, e^{\frac{5}{9}v^4}\, dv}$$

Corollary

$$\boxed{\mathbb{P}\{R_{\mathsf{ISE}} - L_{\mathsf{ISE}} \le \lambda\} = -\frac{20}{3\pi^{7/3}} \int_0^\lambda \int_\Gamma \frac{\partial}{\partial\tau}\wp\left(-\frac{is}{\pi}v;\, -\frac{i\lambda}{\pi}v\right)\, v^6\, e^{\frac{5}{9}v^4}\, dv\, ds}$$

Support of the ISE

Proof

$$\mathbb{P}\{M_n \le k,\, m_n \ge -j\} = \frac{[t^n]\, R_j^{[0,j+k]}(t)}{\frac{2^n}{n+1}\binom{2n}{n}}.$$

$$[t^n]\, R_j^{[0,j+k]}(t) = \frac{1}{2\pi i}\int_\gamma R_j^{[0,j+k]}(t)t^{-n-1}\, dt,$$

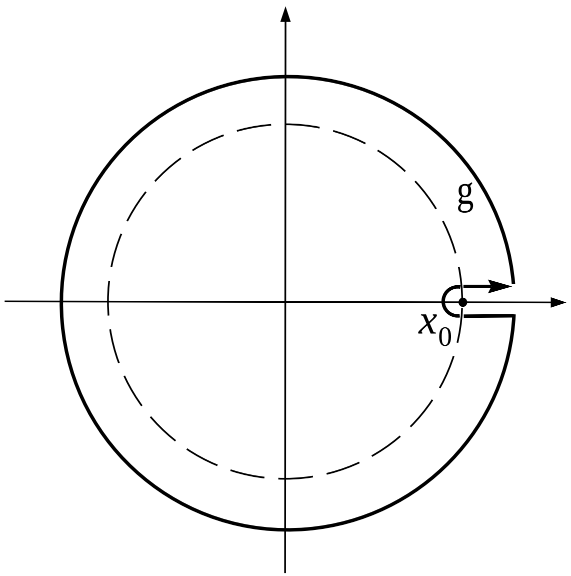
Support of the ISE

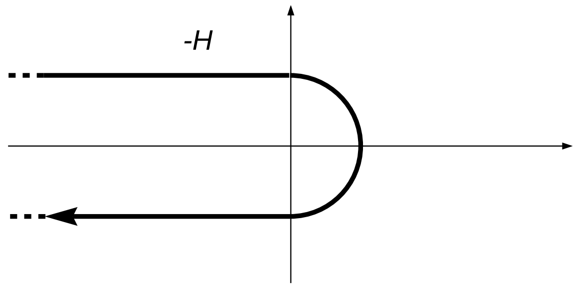
$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where $x_0 = \frac{1}{8}$, $c > 0$,

$$\gamma_1 = \left\{ x = x_0 \left(1 - c\frac{i + n^{1/4} - s}{n} \right) : 0 \leq s \leq n^{1/4} \right\},$$

$$\gamma_2 = \left\{ x = x_0 \left(1 - c\frac{1}{n} e^{-i\varphi} \right) : -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \right\},$$

$$\gamma_3 = \left\{ x = x_0 \left(1 + c\frac{i + s}{n} : \right) 0 \leq s \leq n^{1/4} \right\},$$





Support of the ISE

Lemma 1

Suppose that $|1-q| \geq c/L$ *for some constant* $c > 0$ *(where* $q = e^{2\pi i\tau}$*). Then*

$$\boxed{\frac{\theta_1\left(\frac{1}{L+6},q\right)^4 \theta_1\left(\frac{4}{L+6},q\right)}{\theta_1\left(\frac{2}{L+6},q\right)^5} = \frac{1}{8}\left(1 - \frac{25}{(L+6)^4}G_4(\tau) + O\left(\frac{1}{L^6|1-q|^6}\right)\right)}.$$

Remark.

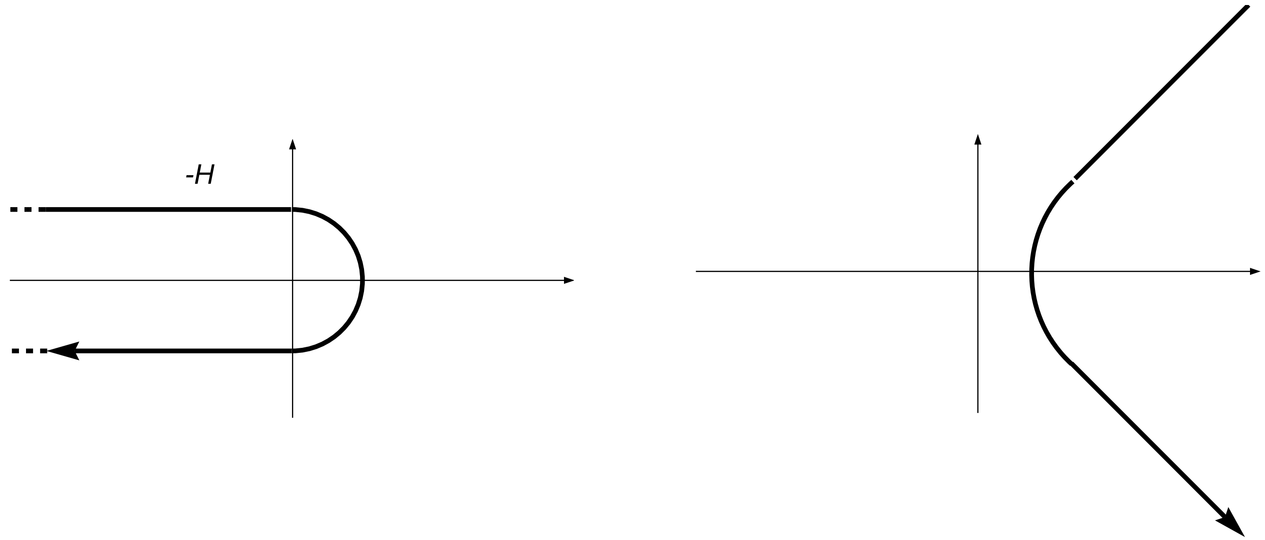
$$G_4(\tau) = \frac{\pi^4}{45}\tau^{-4} + O\left(\tau^{-3}\right)$$

uniformly for $\tau \to 0$ with $\varepsilon \leq \arg(\tau) \leq \pi - \varepsilon$ (for any $\varepsilon > 0$)

Support of the ISE

Corollary

$$t = \frac{\theta_1\left(\frac{1}{L+6}, q\right)^4 \theta_1\left(\frac{4}{L+6}, q\right)}{\theta_1\left(\frac{2}{L+6}, q\right)^5} = \frac{1}{8}\left(1 - \frac{5\pi^4}{9L^4\tau^4} + O(\ldots)\right)$$



Contour for $1/\tau^4$ and for i/τ.

Support of the ISE

Lemma 2

Suppose that $|1-q| \geq c/L$ *for some constant* $c > 0$ *(where* $q = e^{2\pi i\tau}$*). Then*

$$\boxed{R_j^{[0,L]}(t) = 2\left(1 - \frac{3}{(L+6)^2}\wp\left(\frac{j+1}{L+6};\tau\right) + O\left(\frac{1}{L^4|1-q|^4}\right)\right)}$$

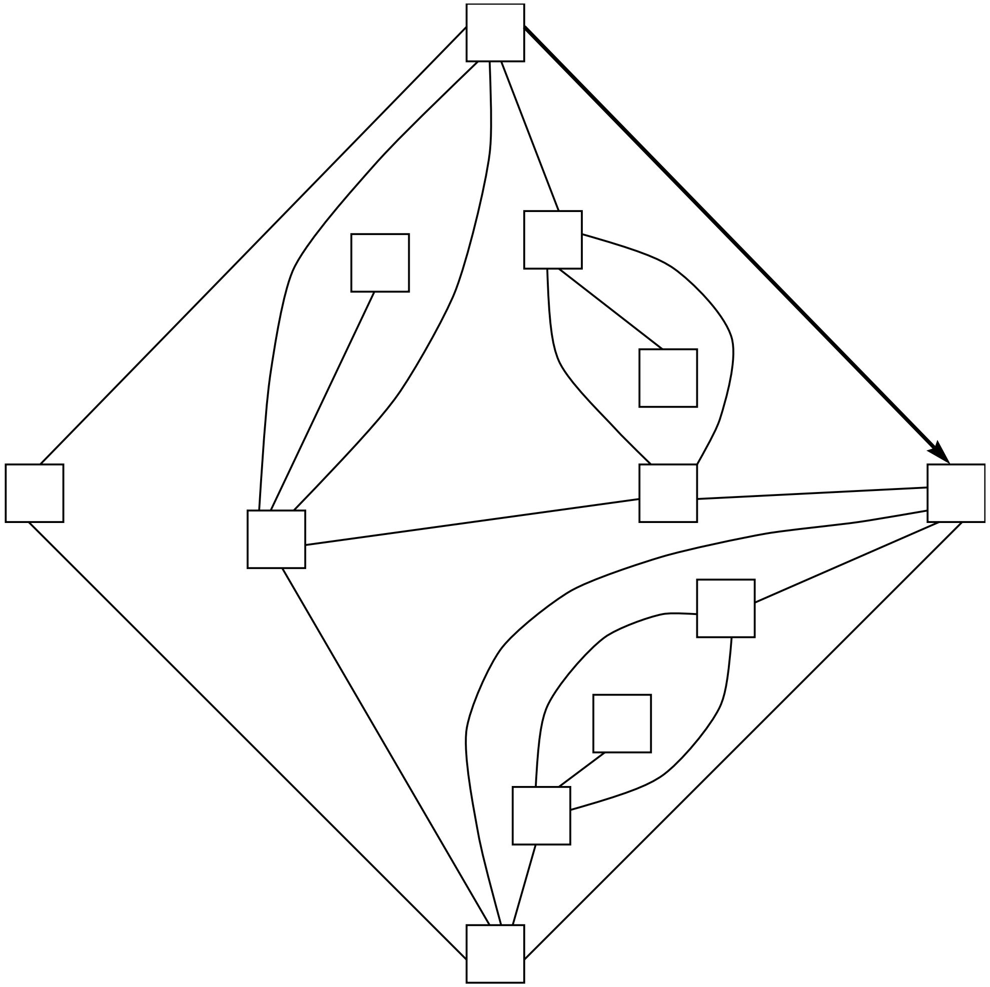
uniformly for $\varepsilon \leq j/L \leq 1-\varepsilon$ *(for any* $\varepsilon > 0$*).*

Recall that

$$R_n^{[0,L]}(t) = \frac{\theta_1\left(\frac{2}{L+6},q\right)^3}{\theta_1\left(\frac{1}{L+6},q\right)^2\theta_1\left(\frac{4}{L+6},q\right)} \frac{\theta_1\left(\frac{n+1}{L+6},q\right)\theta_1\left(\frac{n+5}{L+6},q\right)}{\theta_1\left(\frac{n+2}{L+6},q\right)\theta_1\left(\frac{n+4}{L+6},q\right)}$$

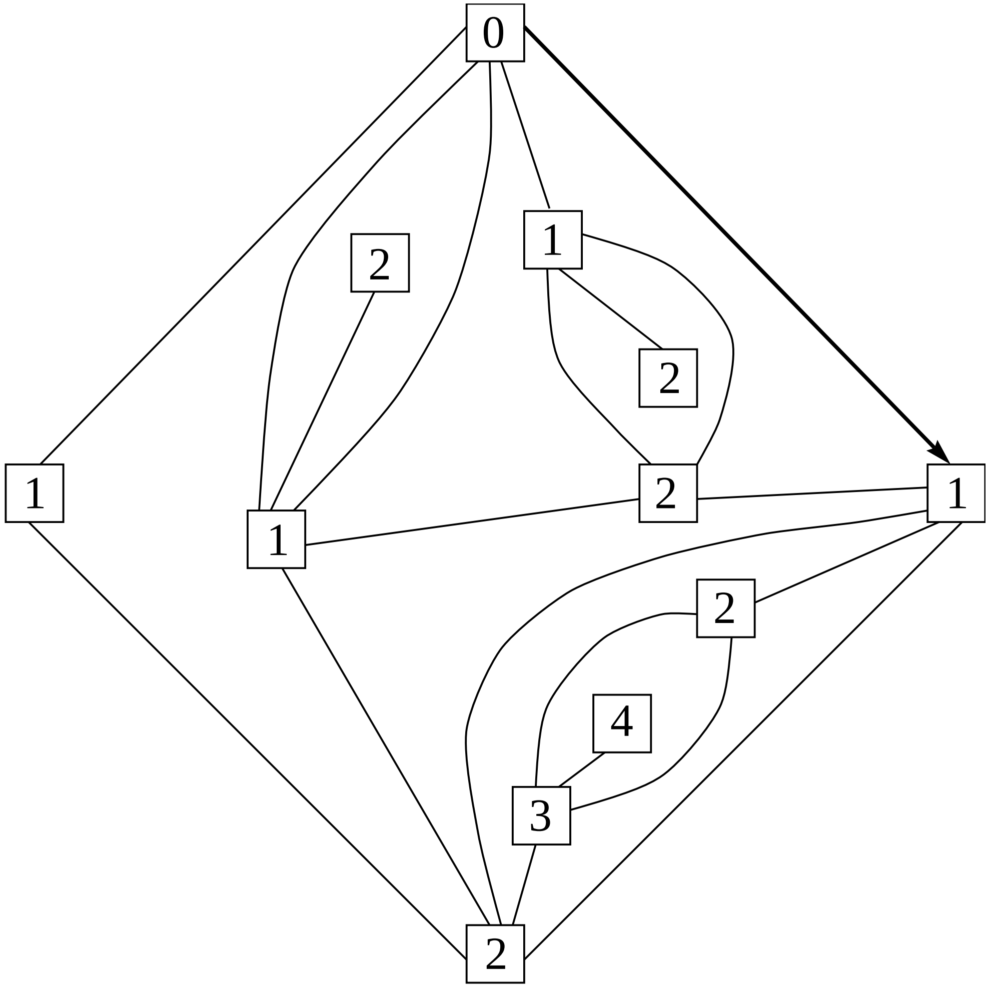
Quadrangulations

Schaeffer bijection: start with a quadrangulation



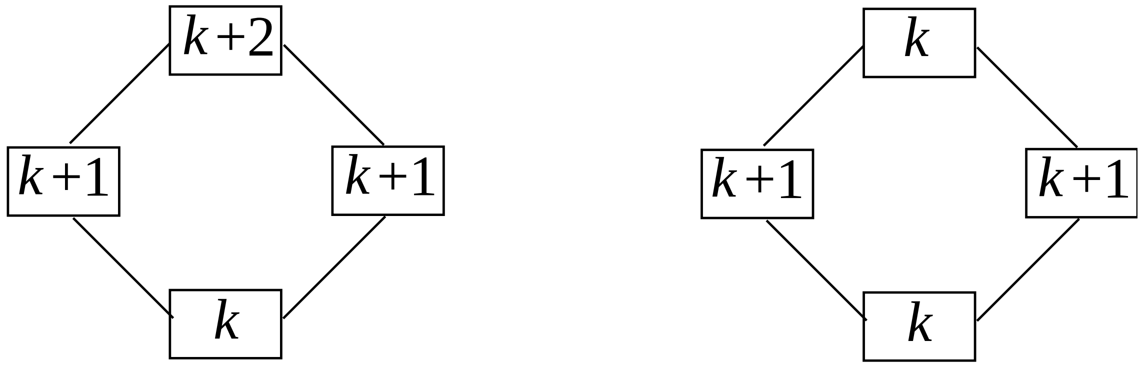
Quadrangulations

Schaeffer bijection: calulate the distance to the root vertex



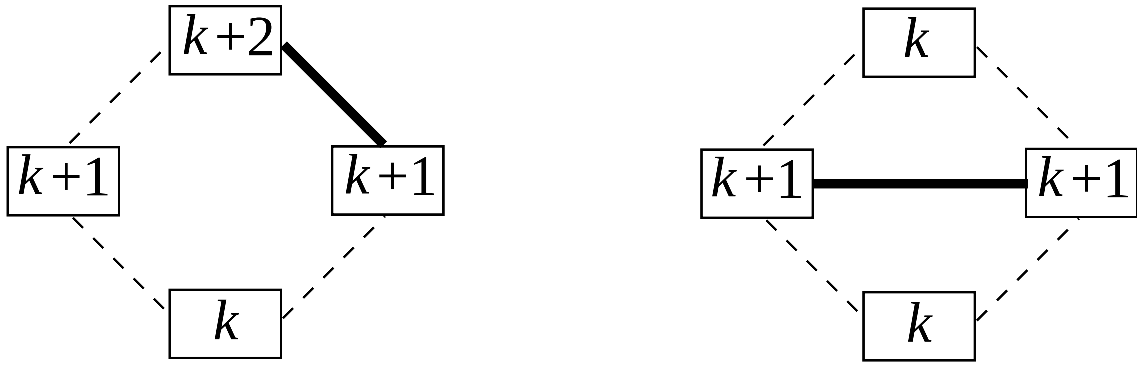
Quadrangulations

Schaeffer bijection: there are only two possible constellations



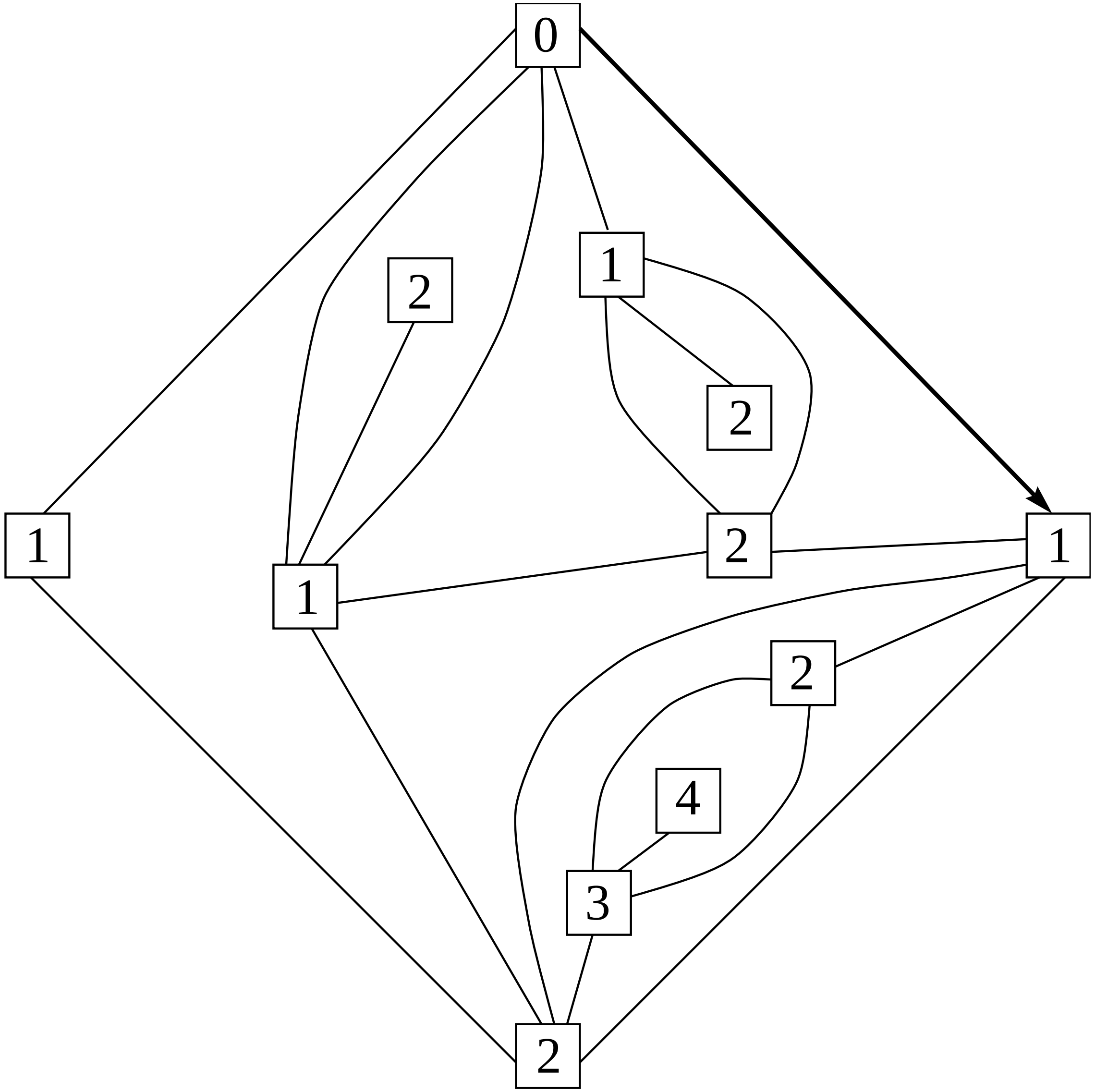
Quadrangulations

Schaeffer bijection: include **fat** edges



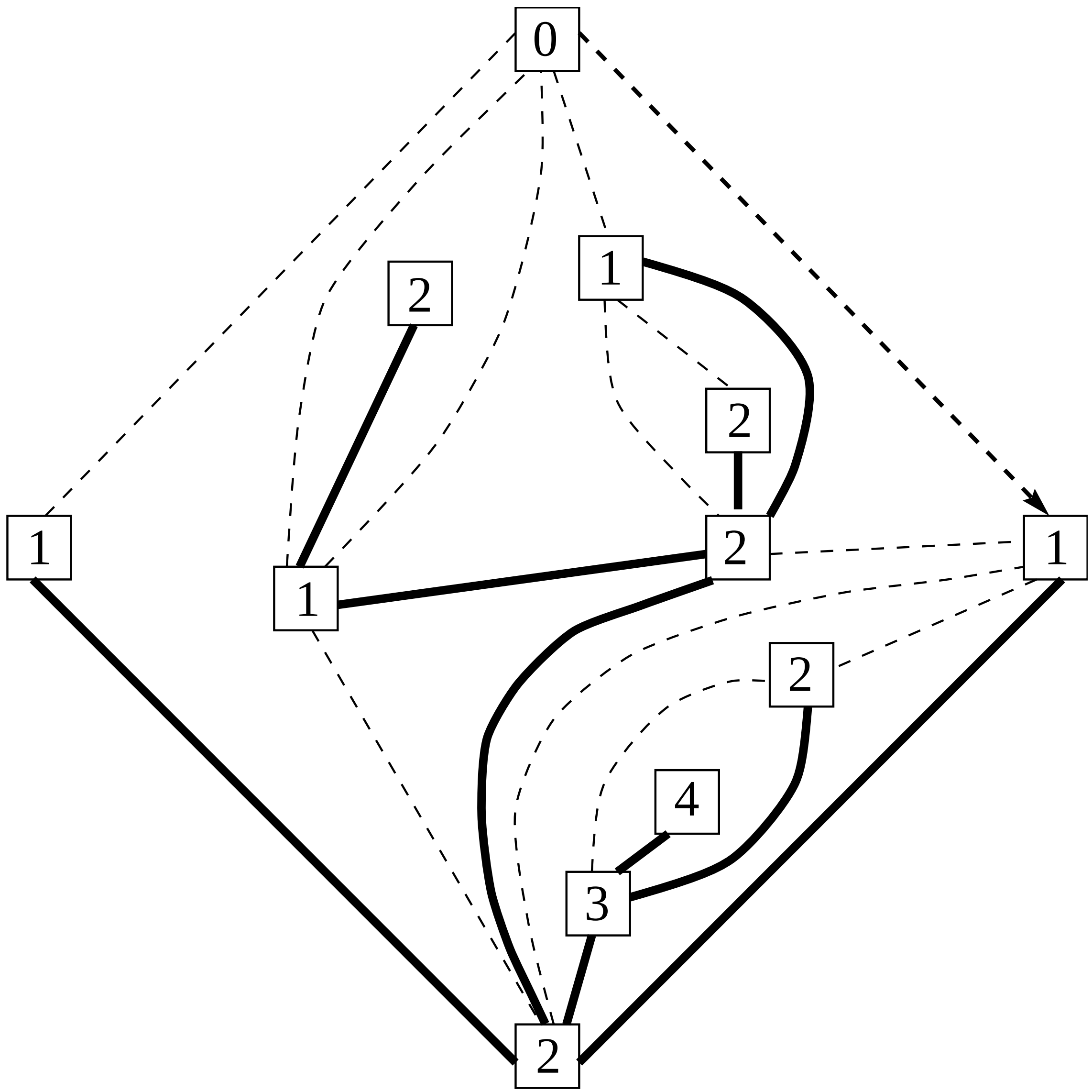
Quadrangulations

Schaeffer bijection: include **fat** edges



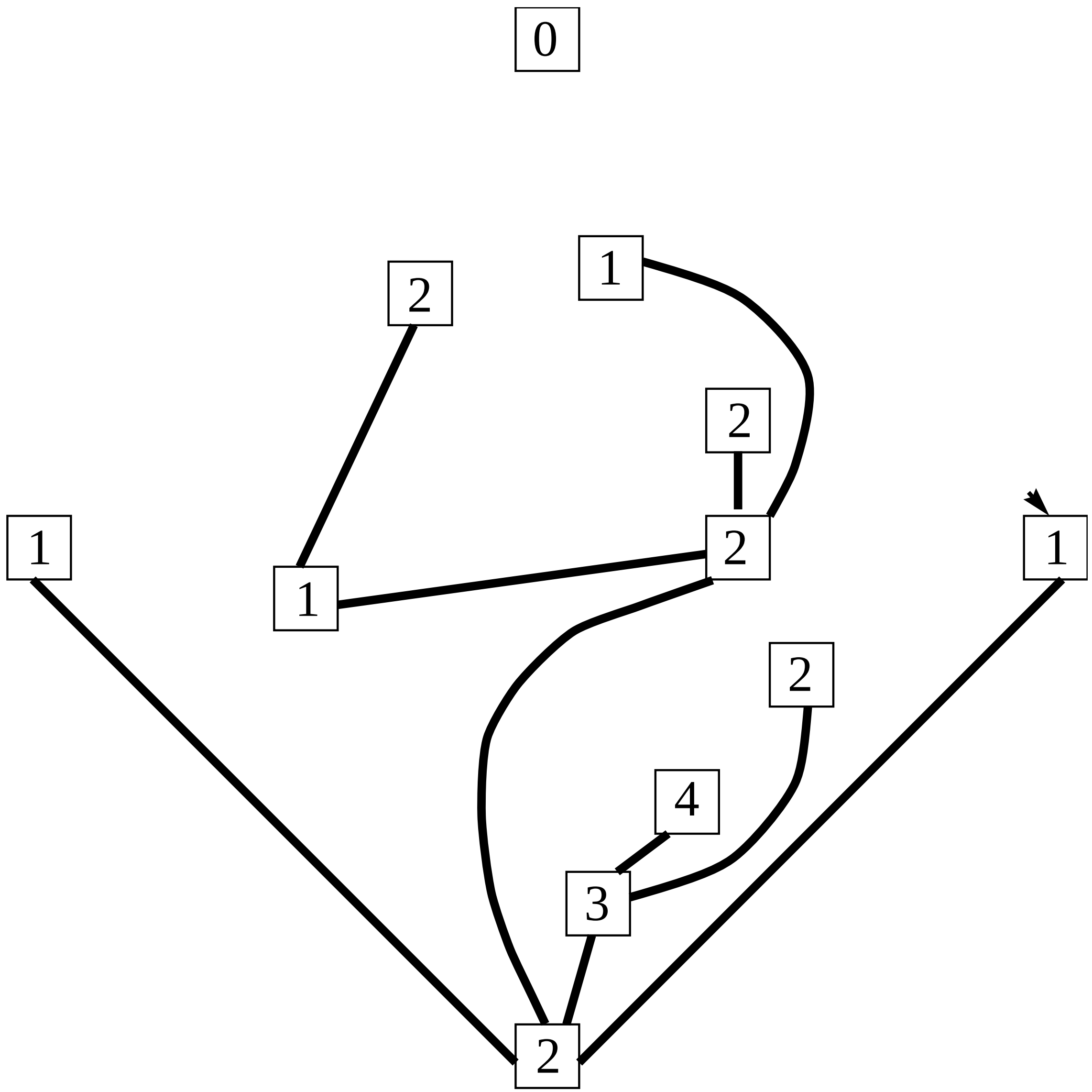
Quadrangulations

Schaeffer bijection: include **fat** edges



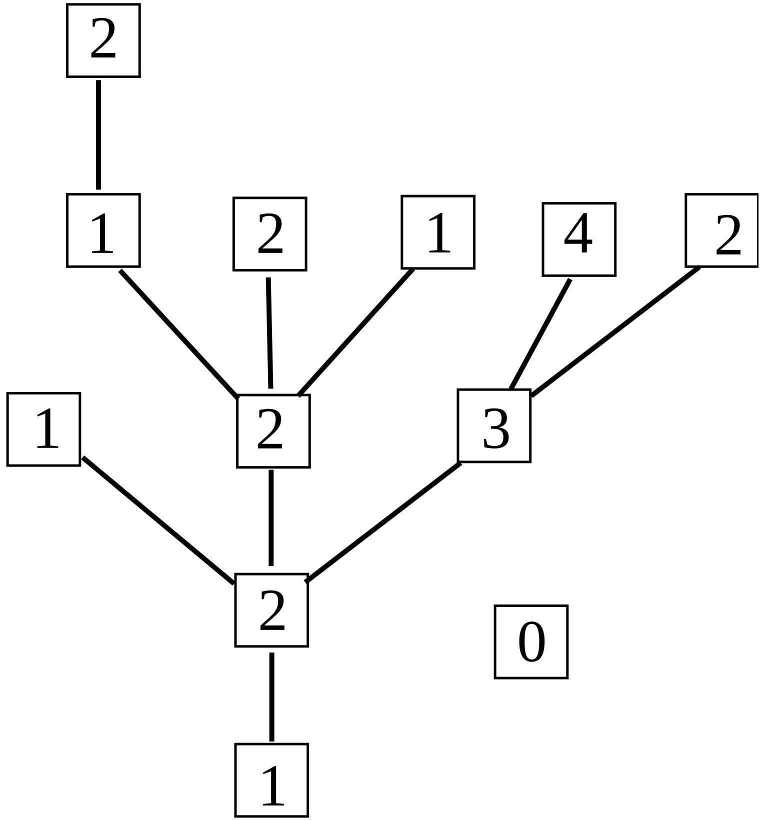
Quadrangulations

Schaeffer bijection: delete the dotted edges



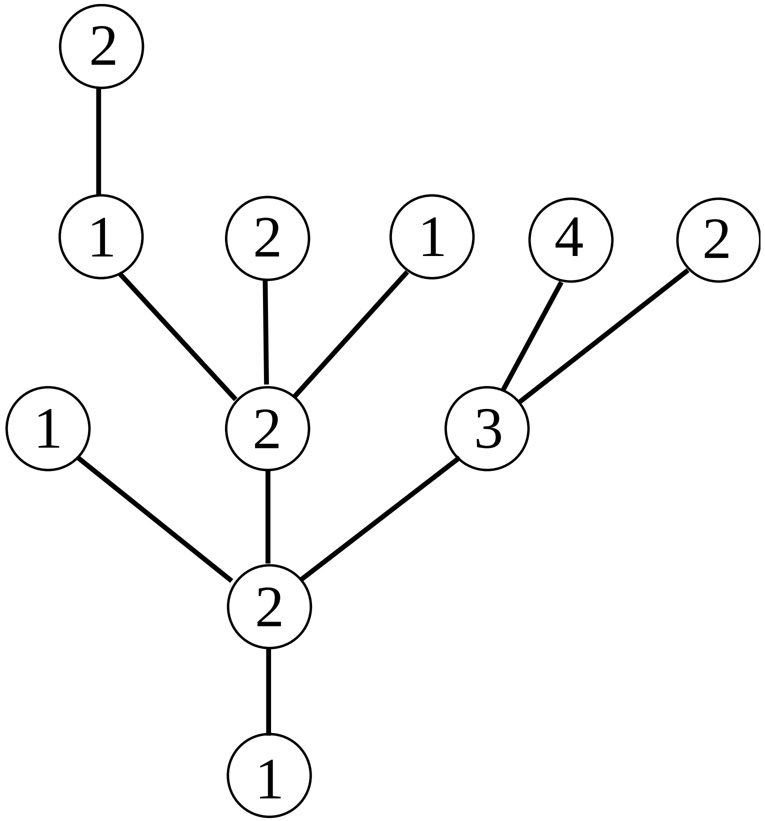
Quadrangulations

Schaeffer bijection: a labelled tree occurs



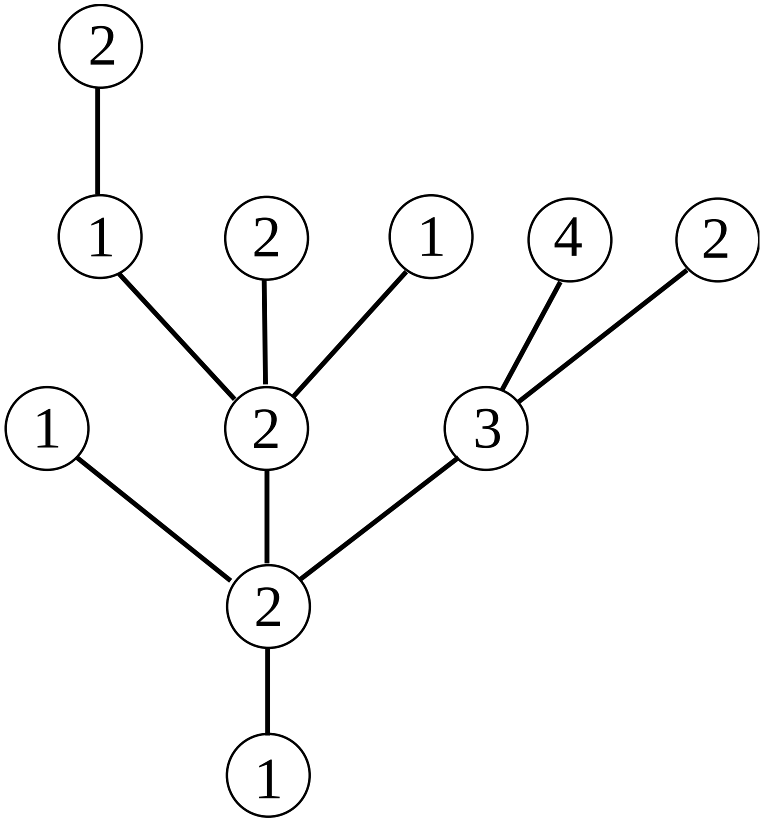
Quadrangulations

Schaeffer bijection: a labelled tree occurs



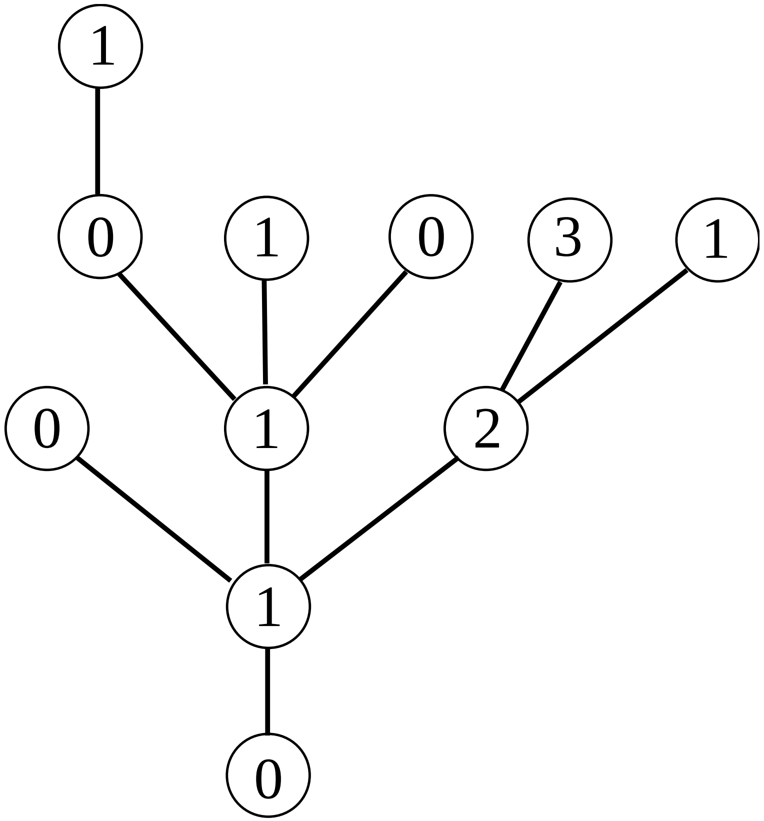
Well-Labelled Trees

Positive labels, root has label 1,
adjacent labels differ at most by 1:



Well-Labelled Trees

Nonnegative labels, root has label 0,
adjacent labels differ at most by 1



Thank You!