THE JOINT DISTRIBUTION OF q-ADDITIVE FUNCTIONS

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ABSTRACT. It is proved that the joint limiting distribution of q_1-additive and q_2-additive functions for coprime q_1, q_2 is independently normal if the second moments grow sufficiently fast. For the sum-of-digits function we also provide a local limit theorem. The proofs use an extensions of methods by Bassily and Katai [1] and by Kim [18] combined with Baker's theorem on linear forms of logarithms.

1. INTRODUCTION

Let $q > 1$ be a given integer. A real-valued function f, defined on the non-negative integers, is said to be q-additive if $f(0) = 0$ and

$$f(n) = \sum_{j\ge 0} f(a_{q,j}(n)q^j) \quad \text{for} \quad n = \sum_{j\ge 0} a_{q,j}(n)q^j,$$

where $a_{q,j}(n) \in E_q := \{0,1,\ldots,q-1\}$. A special q-additive function is the sum of digits function

$$s_q(n) = \sum_{j\ge 0} a_{q,j}(n).$$

The statistical behaviour of the sum of digits fucntion and, more generally, for q-additive function has been very well studied by several authors.

The most general result concerning the *mean value* of q-additive functions is due to Manstavičius [21] (extending earlier work of Coquet [3]). Let

$$m_{k,q} := \frac{1}{q}\sum_{c\in E_q} f(cq^k), \qquad m_{2;k,q}^2 := \frac{1}{q}\sum_{c\in E_q} f^2(cq^k)$$

and

$$M_q(x) := \sum_{k=0}^{[\log_q x]} m_{k,q}, \qquad B_q^2(x) = \sum_{k=0}^{[\log_q x]} m_{2;k,q}^2.$$

Then

$$\frac{1}{x}\sum_{n<x} (f(n) - M_q(x))^2 \le cB_q^2(x), \tag{1.1}$$

which implies

$$\frac{1}{x}\sum_{n<x} f(n) = M_q(x) + O(B_q(x)).$$

For the sum-of-digits function $s_q(n)$ much more precise results are known, e.g. Delange [5] proved (for integral x) that

$$\frac{1}{x}\sum_{n<x} s_q(n) = \frac{q-1}{2}\log_q x + \gamma(\log_q x),$$

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where γ is a continuous, nowhere differentiable and periodic function with period 1. (Higher moments of $a_q(n)$ were considered by Kirschenhofer [19] and by Kennedy and Cooper [17] (for the variance) and by Grabner, Kirschenhofer, Prodinger and Tichy [12].)

There also exist distributional results for q-additive functions. In 1972 Delange [4] proved an analogue to the Erdős-Wintner theorem. There exists a distribution function $F(y)$ such that, as $x \to \infty$

$$\frac{1}{x} \# \{ n < x \,|\, f(n) < y \} \to F(y) \tag{1.2}$$

if and only if the two series $\sum_{k \ge 0} m_{k,q}$, $\sum_{k \ge 0} m_{2;k,q}^2$ converge. This theorem is generalized by Kátai [16] who proved that there exists a a distribution function $F(y)$ such that, as $x \to \infty$

$$\frac{1}{x} \# \{ n < x \,|\, f(n) - M_q(x) < y \} \to F(y)$$

if and only if the series $\sum_{k \ge 0} m_{2;k,q}^2$ converges.

The most general theorem known concering a central limit theorem is again due to Manstavičius [21]. Suppose that, as $x \to \infty$,

$$\max_{cq^j < x} |f(cq^j)| = o(B_q(x))$$

and that $D_q(x) \to \infty$, where

$$D_q^2(x) = \sum_{k=0}^{\log_q x} \sigma_{k,q}^2 \quad \text{and} \quad \sigma_{k,q}^2 := \frac{1}{q} \sum_{c \in E_q} f^2(cq^k) - m_{k,q}^2.$$

Then, as $x \to \infty$,

$$\frac{1}{x} \# \left\{ n < x \,\middle|\, \frac{f(n) - M_q(x)}{D_q(x)} < y \right\} \to \Phi(y),$$

where Φ is the normal distribution function.

Similar distribution results for the sum of digits function of number systems related to substitution automata were considered by Dumont and Thomas [8]. For number systems whose bases satisfy linear recurrences we refer to [6].

Furthermore, Bassily and Kátai [1] studied the distribution of q-additive functions on polynomial sequences.

Theorem 1. *Let f be a q-additive function such that $f(cq^j) = \mathcal{O}(1)$ as $j \to \infty$ and $c \in E_q$. Assume that $\frac{D_q(x)}{(\log x)^\eta} \to \infty$ as $x \to \infty$ for some $\eta > 0$ and let $P(x)$ be a polynomial with integer coefficients, degree r, and positive leading term. Then, as $x \to \infty$,*

$$\frac{1}{x} \# \left\{ n < x \,\middle|\, \frac{f(P(n)) - M_q(x^r)}{D_q(x^r)} < y \right\} \to \Phi(y)$$

and

$$\frac{1}{\pi(x)} \# \left\{ p < x \,\middle|\, \frac{f(P(p)) - M_q(x^r)}{D_q(x^r)} < y \right\} \to \Phi(y).$$

This result relies on the fact that suitably modified centralized moments converge, compare with Lemma 4. Note also that this theorem was only stated (and proved) for $\eta = \frac{1}{3}$. However, a short inspection of the proof shows that $\eta > 0$ is sufficient.

2. Joint Distributions

It is a natural question to ask, whether there are analogue results for the joint distribution of q_ℓ-additive functions $f_\ell(n)$ (if $q_1, q_2, \dots, q_d > 1$ are pairwisely comprime integers). For example, Hildebrand [14] announced that one always has

$$\frac{1}{x}\#\{n < x \,|\, f_\ell(n) < y_\ell, 1 \le \ell \le d\} \to F_1(y) \cdots F_d(y)$$

if f_ℓ satisfies (1.2) for all $\ell = 1, 2, \dots, d$ and that there is a joint central limit theorem of the form

$$\frac{1}{x}\#\left\{n < x \,\middle|\, \frac{f_\ell(n) - M_{q_\ell}(x)}{D_{q_\ell}(x)} < y_\ell, 1 \le \ell \le d\right\} \to \Phi(y_1)\Phi(y_2)\cdots\Phi(y_d)$$

if $B_{q_\ell}(x) \to \infty$ and $B_{q_\ell}(x^\eta) \sim B_{q_\ell}(x)$ for every $\eta > 0$ as $x \to \infty$. (Note that the sum of digits function $s_q(n)$ is not covered by this result.)

In this paper we will first extend the above result of Bassily and Kátai to the joint distibution of q_ℓ-additive functions f_ℓ $(1 \le \ell \le d)$ on specific polynomial sequences if $q_1, q_2, \dots, q_d$ are pairwisely coprime.

Theorem 2. *Let $q_1, q_2, \dots, q_d > 1$ be pairwisely coprime integers and Let f_ℓ, $1 \le \ell \le d$ be q_ℓ-additive function such that $f_\ell(cq_\ell^j) = \mathcal{O}(1)$ as $j \to \infty$ and $c \in E_\ell$. Assume that $\frac{D_{q_\ell}(x)}{(\log x)^\eta} \to \infty$ as $x \to \infty$, $1 \le \ell \le d$, for some $\eta > 0$ and let $P_\ell(x)$ be polynomials with integer coefficients of different degrees r_ℓ and positive leading term, $1 \le \ell \le d$. Then, as $x \to \infty$,*

$$\frac{1}{x}\#\left\{n < x \,\middle|\, \frac{f_\ell(P_\ell(n)) - M_{q_\ell}(x^{r_\ell})}{D_{q_\ell}(x^{r_\ell})} < y_\ell, 1 \le \ell \le d\right\} \to \Phi(y_1)\Phi(y_2)\cdots\Phi(y_d)$$

and

$$\frac{1}{\pi(x)}\#\left\{p < x \,\middle|\, \frac{f_\ell(P_\ell(p)) - M_{q_\ell}(x^{r_\ell})}{D_{q_\ell}(x^{r_\ell})} < y_\ell, 1 \le \ell \le d\right\} \to \Phi(y_1)\Phi(y_2)\cdots\Phi(y_d).$$

This theorem contains an *unnatural condition*, namely that one has to consider polynomials $P_\ell(x)$ with different degrees r_ℓ. It seems that this condition is not necessary. However, this is the crux of the matter. By using a variation of Bassily and Kátai's proof (combined with Baker's theorem on linear forms of logarithms) we could handle the case $d = 2$ with linear polynomals $P_\ell(x) = A_\ell x + B_\ell$.

Theorem 3. *Let $q_1, q_2 > 1$ be coprime integers and Let f_ℓ be q_ℓ-additive function such that $f_\ell(cq_\ell^j) = \mathcal{O}(1)$ as $j \to \infty$ and $c \in E_\ell$, $\ell = 1, 2$. Assume that $\frac{D_{q_\ell}(x)}{(\log x)^\eta} \to \infty$ as $x \to \infty$, $\ell = 1, 2$, for some $\eta > 0$. Let $P_\ell(x) = A_\ell x + B_\ell$, $\ell = 1, 2$, be arbitrary linear polynomials with integer coefficients and positive leading terms A_ℓ coprime to q_ℓ. Then, as $x \to \infty$,*

$$\frac{1}{x}\#\left\{n < x \,\middle|\, \frac{f_\ell(P_\ell(n)) - M_{q_\ell}(x)}{D_{q_\ell}(x)} < y_\ell, \ell = 1, 2\right\} \to \Phi(y_1)\Phi(y_2).$$

For the sum-of-digits functions we can also prove a local version of Theorem 3.

Theorem 4. *Let $q_1, q_2 > 1$ be coprime integers and set $d = \gcd(q_1 - 1, q_2 - 1)$. Then, as $x \to \infty$*

$$\frac{1}{x}\#\{n < x \,|\, s_{q_1}(n) = k_1, s_{q_2}(n) = k_2\}$$
$$= d\prod_{\ell=1}^{2}\left(\frac{1}{\sqrt{2\pi\frac{q_\ell^2-1}{12}\log_{q_\ell} x}}\exp\left(-\frac{\left(k_\ell - \frac{q-1}{2}\log_{q_\ell} x\right)^2}{2\frac{q_\ell^2-1}{12}\log_{q_\ell} x}\right)\right) + o\left((\log x)^{-1}\right)$$

uniformly for all integers $k_1, k_2 \ge 0$ with $k_1 \equiv k_2 \bmod d$.

Note that $s_{q_\ell}(n) \equiv n \bmod (q_\ell - 1)$. Thus we always have $s_{q_1}(n) \equiv s_{q_2}(n) \bmod d$ and consequently

$$\#\{n < x \,|\, s_{q_1}(n) = k_1, s_{q_2}(n) = k_2\} = 0$$

if $k_1 \not\equiv k_2 \bmod d$.

There are some other results indicating that the q_ℓ-ary digital expansions are *asymptotically independent* for different bases q_ℓ, e.g. Kim[1] [18] showed that for all integers $c_1, \dots, c_d$

$$\frac{1}{x}|\{n < x : s_{q_j}(n) \equiv c_j \bmod m_j \ (1 \le j \le d)\}| = \frac{1}{m_1 m_2 \cdots m_d} + \mathcal{O}(x^{-\delta})$$

with

$$\delta = \frac{1}{120 d^2 q^2 m^2},$$

where $q_1, \dots, q_d > 1$ are pairwisely coprime integers and $m_1, \dots, m_d$ are positive integers such that

$$\gcd(q_j - 1, m_j) = 1 \qquad (1 \le j \le d);$$

$q = \max\{q_1, \dots, q_d\}$, $m = \max\{m_1, \dots, m_d\}$ and the $\mathcal{O}$-constant depends only on d and q. (This results shapens a result by Bésineau [2] and solves a conjecture of Gelfond [11].)

Drmota and Larcher [7] used a variation of Kim's method to prove that d-dimensional sequence $(\alpha_1 s_{q_1}(n), \alpha_2 s_{q_2}(n), \dots, \alpha_d s_{q_d}(n))_{n \ge 0}$ is uniformly distributed modulo 1 if and only if $\alpha_1, \alpha_2, \dots, \alpha_d$ are irrational. (Grabner, Liardet and Tichy [13] could prove a similar theorem by ergodic means.)

Another problem has been considered by Senge and Straus [27]. They proved that if q_1 and q_2 are coprime and c is any given positive constant then there are only finitely many $n \ge 0$ such that

$$s_{q_1}(n) \le c \text{ and } s_{q_2}(n) \le c.$$

This result was later generalized and sharpended by Stewart [28], Schlickewei [23, 24] and by Pethő and Tichy [22]. The proofs use Baker's method on linear forms of logarithms and the p-adic version of Schmidt's subspace theorem by Schlickewei applied to S-unit equations.

One would get a much deeper insight into all these results if one could prove a local version of Theorem 2, e.g. asymptotic expansions or general estimates for the numbers

$$\frac{1}{x}\#\{n < x \,|\, s_q(n^2) = k\}$$

of for

$$\frac{1}{\pi(x)}\#\{p < x \,|\, s_q(p) = k\}$$

(and of course multivariate versions.) It seems that problems of this kind are extremely difficult, e.g. it is an open question whether there are infinitely primes p with even sum-of-digits function $s_2(p)$. The best known results concering these questions are due to Fouvry and Mauduit [9, 10] who proved that

$$\frac{1}{x}\#\{n < x \,|\, n \in \mathbf{P} \vee (n = n_1 \cdot n_2 \wedge n_1, n_2 \in \mathbf{P}), s_q(n) \equiv 0 \bmod 2\} \ge c > 0$$

for some constant $c > 0$. ($\mathbf{P}$ denotes the set of primes.)

Theses questions are also related to two other conjectures of Gelfond [11], namely that $s_q(P(n))$ and $s_q(p)$ are uniformly distributed modulo m.

Remark Schmidt [26] and Schmid [25] discussed the joint distribution of $s_2(k_\ell n)$ for different odd integers k_ℓ, $1 \le \ell \le d$. (The distribution modulo m was investigated by Solinas [29].) It is surely possible to extend their result to the joint

[1] For the sake of shortness we restrict to the sum-of-digits function $s_q(n)$

distribution of $f_\ell(P_\ell(n))$, $1 \le \ell \le d$, where f_ℓ are q_ℓ-additive functions, P_ℓ are (certain) integer polynomials, and $q_\ell > 1$ arbitrary integers (e.g. all of them are equal). However, we will not discuss this question here.

3. Proof of the Theorem 2

As already mentioned, Theorem 2 is a direct generalization of Bassily and Kátai's result of [1]. Therefore we can proceed as in [1].

The first two Lemmata on exponential sums are stated in [1], a proof can be also found in [15].

Lemma 1. *Let $f(y)$ be a polynomial of degree k of the form*

$$f(y) = \frac{a}{b}y^k + \alpha_1 y^{k-1} + \cdots + \alpha_k$$

with $\gcd(a,b) = 1$. *Let τ be a positive number satisfying*

$$\tau \ge 2^{3(k-2)}$$

and

$$(\log x)^\tau < b < x^k (\log x)^{-\tau}.$$

Then, as $x \to \infty$

$$\frac{1}{x}\sum_{n<x} e(f(n)) = \mathcal{O}\left((\log x)^{-\tau}\right).$$

Lemma 2. *Let $f(y)$ be an in Lemma 1 and τ_0, τ arbitrary positive numbers satisfying*

$$\tau \ge 2^{6k}\tau_0$$

and

$$(\log x)^\tau < b < x^k (\log x)^{-\tau}.$$

Then, as $x \to \infty$

$$\frac{1}{\pi(x)}\sum_{p<x} e(f(p)) = \mathcal{O}\left((\log x)^{-\tau_0}\right).$$

The third lemma is proved in [1] with help of Lemmata 1 and 2 and the inequality of Erdős-Turán.

Lemma 3. *Let $0 < \Delta < 1$ and*

$$U_{b,q,\Delta} := [0,\Delta] \cup \bigcup_{b=1}^{q-1} \left[\frac{b}{q} - \Delta, \frac{b}{q} + \Delta\right] \cup [1-\Delta, 1].$$

Then for every $\varepsilon > 0$ and aribitray $\lambda > 0$ we have uniformly for $N^\varepsilon < j < rN - N^\varepsilon$ and $0 < \Delta < 1/(2q)$, as $x \to \infty$

$$\frac{1}{x}\#\left\{n < x \,\middle|\, \left\{\frac{P(n)}{q^{j+1}}\right\} \in U_{b,q,\Delta}\right\} \ll \Delta + (\log x)^{-\lambda}$$

and

$$\frac{1}{\pi(x)}\#\left\{p < x \,\middle|\, \left\{\frac{P(p)}{q^{j+1}}\right\} \in U_{b,q,\Delta}\right\} \ll \Delta + (\log x)^{-\lambda}.$$

We will also make use of the following limiting relations for *centralized moments* for q-additive functions, see [1].

Lemma 4. *Let f be a q-additive function such that $f(cq^j) = \mathcal{O}(1)$ as $j \to \infty$ and $c \in E_q$ and let $P(x)$ be a polynomial with integer coefficients, degree r, and positive leading term. Furthermore, suppose that for some $\eta > 0$ we have. $D_q(x^r)/(\log x)^\eta \to 0$ as $x \to \infty$. Define f_1 for $n < x^r$ by*

$$f_1(n) = \sum_{(\log_q x)^\eta \le j \le r \log_q x - (\log_q x)^\eta} f(a_{q,j}(n) q^j)$$

and set

$$M_{q,1}(x^r) := \sum_{(\log_q x)^\eta \le k \le r \log_q x - (\log_q x)^\eta} m_{k,q},$$

$$D^2_{q,1}(x^r) := \sum_{(\log_q x)^\eta \le k \le r \log_q x - (\log_q x^\eta_)} \sigma^2_{k,q}.$$

Then, ax $x \to \infty$

$$\frac{1}{x} \# \sum_{n<x} \left(\frac{f_1(P(n)) - M_{q,1}(x^r)}{D_{q,1}(x^r)} \right)^k \to \int_{-\infty}^{\infty} z^k \, d\Phi(z)$$

and

$$\frac{1}{\pi(x)} \# \sum_{p<x} \left(\frac{f_1(P(p)) - M_{q,1}(x^r)}{D_{q,1}(x^r)} \right)^k \to \int_{-\infty}^{\infty} z^k \, d\Phi(z)$$

In [1] this property is only proved for $\eta = \frac{1}{3}$. However, as already mentioned, it is also true for any $\eta > 0$.

Proposition 1. *Let* $N_\ell = [\log_{q_\ell} x]$, $1 \le \ell \le d$, *let* $\lambda > 0$ *be an arbitrary constant and* h_ℓ, $1 \le \ell \le d$, *positive integers. Furthermore, let* $P_\ell(x)$, $1 \le \ell \le d$, *be integer polynomials with non-negative leading terms and different degrees* $r_\ell \ge 1$. *Then for integers*

$$N_\ell^\eta \le k_1^{(\ell)} < k_2^{(\ell)} < \cdots k_{h_\ell}^{(\ell)} \le r_\ell N_\ell - N_\ell^\eta \quad (1 \le \ell \le d) \tag{3.1}$$

(with some $\eta > 0$*) we have, as* $x \to \infty$

$$\frac{1}{x} \# \left\{ n < x \,|\, a_{q_\ell, k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 0 \le j \le h_\ell, 1 \le \ell \le d \right\}$$
$$= \frac{1}{q_1^{h_1} q_2^{h_2} \cdots q_d^{h_d}} + \mathcal{O}\left((\log x)^{-\lambda} \right) \tag{3.2}$$

and

$$\frac{1}{\pi(x)} \# \left\{ p < x \,|\, a_{q_\ell, k_j^{(\ell)}}(P_\ell(p)) = b_j^{(\ell)}, 0 \le j \le h, 1 \le \ell \le d \right\}$$
$$= \frac{1}{q_1^{h_1} q_2^{h_2} \cdots q_d^{h_d}} + \mathcal{O}\left((\log x)^{-\lambda} \right) \tag{3.3}$$

uniformly for $b_j^{(\ell)} \in E_{q_\ell}$ *and* $k_j^{(\ell)}$ *in the given range, where the implicit constant of the error term may depend on* q_ℓ, *on the polynomials* P_ℓ, *on* h_ℓ *and on* λ.

Proof. We follow [1]. Let $f_{b,q,\Delta}(x)$ be defined by

$$f_{b,q,\Delta}(x) := \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} \mathbf{1}_{[\frac{b}{q}, \frac{b+1}{q}]}(\{x+z\}) \, dz,$$

where $\mathbf{1}_A$ denotes the characteristic function of the set A and $\{x\} = x - [x]$ the fractional part of x. The Fourier coefficients of the Fourier series $f_{b,q,\Delta}(x) = \sum_{m \in \mathbf{Z}} d_{m,b,q,\Delta} e(mx)$ are given by

$$d_{0,b,q,\Delta} = \frac{1}{q}$$

and for $m \ne 0$ by

$$d_{m,b,q,\Delta} = \frac{e\left(-\frac{mb}{q}\right) - e\left(-\frac{m(b+1)}{q}\right)}{2\pi i m} \cdot \frac{e\left(\frac{m\Delta}{2}\right) - e\left(-\frac{m\Delta}{2}\right)}{2\pi i m \Delta}.$$

Note that $d_{m,b,q,\Delta} = 0$ if $m \ne 0$ and $m \equiv 0 \bmod q$ and that

$$|d_{m,b,q,\Delta}| \le \min\left(\frac{1}{\pi |m|}, \frac{1}{\Delta \pi m^2} \right).$$

By definition we have $0 \le f_{b,q,\Delta}(x) \le 1$ and

$$f_{b,q,\Delta}(x) = \begin{cases} 1 & \text{if } x \in \left[\frac{b}{q} + \Delta, \frac{b+1}{q} - \Delta\right], \\ 0 & \text{if } x \in [0,1] \setminus \left[\frac{b}{q} - \Delta, \frac{b+1}{q} + \Delta\right]. \end{cases}$$

So if we set

$$t(y_1, \dots, y_d) := \prod_{\ell=1}^{d} \prod_{j=1}^{h_\ell} f_{b_j^{(\ell)}, q_\ell, \Delta}\left(\frac{y_\ell}{q_\ell^{k_j^{(\ell)}+1}}\right)$$

then we get for $\Delta < 1/(2q)$

$$\left| \#\left\{n < x \,|\, a_{q_\ell, k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 0 \le j \le h_\ell, 1 \le \ell \le d\right\} - \sum_{n<x} t(P_1(n), \dots, P_d(n)) \right|$$

$$\le \sum_{\ell=1}^{d} \sum_{j=1}^{h_\ell} \#\left\{n < x \,\middle|\, \left\{\frac{P_\ell(n)}{q_\ell^{k_j^{(\ell)}+1}}\right\} \in U_{b_j^{(\ell)}, q_\ell, \Delta}\right\} \ll \Delta x + x(\log x)^{-\lambda}$$

and

$$\left| \#\left\{p < x \,|\, a_{q_\ell, k_j^{(\ell)}}(P_\ell(p)) = b_j^{(\ell)}, 0 \le j \le h_\ell, 1 \le \ell \le d\right\} - \sum_{p<x} t(P_1(p), \dots, P_d(p)) \right|$$

$$\le \sum_{\ell=1}^{d} \sum_{j=1}^{h_\ell} \#\left\{n < x \,\middle|\, \left\{\frac{P_\ell(p)}{q_\ell^{k_j^{(\ell)}+1}}\right\} \in U_{b_j^{(\ell)}, q_\ell, \Delta}\right\} \ll \Delta \pi(x) + \pi(x)(\log x)^{-\lambda},$$

where $U_{b_j^{(\ell)}, q_\ell, \Delta}$ is given in Lemma 3.

For convenience, let $\mathbf{m}_\ell = (m_1^{(\ell)}, \dots, m_{h_\ell}^{(\ell)})$ denote h_ℓ-dimensional integer vectors and $\mathbf{v}_\ell = \left(q_\ell^{-k_1^{(\ell)}-1}, \dots, q_\ell^{-k_{h_\ell}^{(\ell)}-1}\right)$, $1 \le \ell \le d$. Furthermore set

$$T_{\mathbf{m}_1, \dots, \mathbf{m}_d} := \prod_{\ell=1}^{d} \prod_{j=1}^{h_\ell} d_{m_j^{(\ell)}, b_j^{(\ell)}, q_\ell, \Delta}.$$

Then $t(P_1(n), \dots, P_d(n))$ has Fourier series expansion

$$t(y_1, \dots, y_d) = \sum_{\mathbf{m}_1, \dots, \mathbf{m}_d} T_{\mathbf{m}_1, \dots, \mathbf{m}_d} e\left(\mathbf{m}_1 \cdot \mathbf{v}_1 y_1 + \dots + \mathbf{m}_d \cdot \mathbf{v}_d y_d\right).$$

Thus, we are led to consider the exponential sums

$$S_1 = \sum_{\mathbf{m}_1, \dots, \mathbf{m}_d} T_{\mathbf{m}_1, \dots, \mathbf{m}_d} \sum_{n<x} e\left(\mathbf{m}_1 \cdot \mathbf{v}_1 P_1(n) + \dots + \mathbf{m}_d \cdot \mathbf{v}_d P_d(n)\right) \tag{3.4}$$

and

$$S_2 = \sum_{\mathbf{m}_1, \dots, \mathbf{m}_d} T_{\mathbf{m}_1, \dots, \mathbf{m}_d} \sum_{p<x} e\left(\mathbf{m}_1 \cdot \mathbf{v}_1 P_1(p) + \dots + \mathbf{m}_d \cdot \mathbf{v}_d P_d(p)\right). \tag{3.5}$$

Let us consider for a moment just the first sum S_1. If $\mathbf{m}_1, \dots, \mathbf{m}_d$ are all zero then

$$T_{\mathbf{m}_1, \dots, \mathbf{m}_d} \sum_{n<x} e\left(\mathbf{m}_1 \cdot \mathbf{v}_1 P_1(n) + \dots + \mathbf{m}_d \cdot \mathbf{v}_d P_d(n)\right) = \frac{x + O(1)}{q_1^{h_1} \cdots q_d^{h_d}}$$

which provids the leading term. Furthermore, if there exists ℓ and j with $m_j^{(\ell)} \ne 0$ and $m_j^{(\ell)} \equiv 0 \bmod q_\ell$ then $T_{\mathbf{m}_1, \dots, \mathbf{m}_d} = 0$. So it remains to consider the case where there exists ℓ and j with $m_j^{(\ell)} \not\equiv 0 \bmod q_\ell$. Here the exponent is of the form

$$\mathbf{m}_1 \cdot \mathbf{v}_1 P_1(n) + \dots + \mathbf{m}_d \cdot \mathbf{v}_d P_d(n) = \frac{a_1}{b_1} P_1(n) + \dots + \frac{a_d}{b_d} P_d(n)$$

in which we assume that $\gcd(a_\ell, b_\ell) = 1$, $1 \le \ell \le d$. The first observation is that for any ℓ for which there exists j with $m_j^{(\ell)} \not\equiv 0 \bmod q_\ell$ there exists $\eta_\ell > 0$ (only depending on q_ℓ) such that

$$b_\ell \ge q_\ell^{\eta_\ell k_s^{(\ell)}}$$

if $m_s^{(\ell)} \ne 0$, $m_s^{(\ell)} \not\equiv 0 \bmod q_\ell$ and $m_{s+1}^{(\ell)} = m_{s+1}^{(\ell)} = \cdots = m_{h_\ell}^{(\ell)} = 0$, compare with [1]. For the reader's convenience we repeat the argument. Suppose that the prime factorisation of q_ℓ is given by $q_\ell = p_1^{e_1} \cdots p_k^{e_k}$. If $m_s^{(\ell)} \not\equiv 0 \bmod q_\ell$ then there exists t such that $m_s^{(\ell)} \not\equiv 0 \bmod p_t^{e_t}$. Now we have

$$b_\ell \left(m_s^{(\ell)} + q_\ell^{k_s^{(\ell)} - k_{s-1}^{(\ell)}} m_{s-1}^{(\ell)} + \cdots q_\ell^{k_s^{(\ell)} - k_1^{(\ell)}} m_1^{(\ell)} \right) = a_\ell q_\ell^{k_s^{(\ell)}+1}.$$

Hence $b_\ell \equiv 0 \bmod p_t^{k_s^{(\ell)} e_t}$ and consequently $b_\ell \ge p_t^{k_s^{(\ell)} e_t} \ge q_\ell^{\eta_\ell k_s^{(\ell)}}$. Note that we also have $b_\ell \le q_\ell^{\eta_\ell k_{h_\ell}^{(\ell)}}$.

Now let D denote the set of $\ell \in \{1, 2, \ldots, d\}$ such that there exists j with $m_j^{(\ell)} \not\equiv 0 \bmod q_\ell$. Since all degrees r_ℓ are different there exists a unique ℓ_0 with $r_{\ell_0} = \max\{r_\ell \,|\, \ell \in D\}$. We now want to apply Lemma 1 with $k = r_{\ell_0}$ and $b = b_{\ell_0}$. If $k_j^{(\ell)}$ are contained in the range (3.1) then for every $\tau > 0$ there exists $x_0(\tau)$ such that for $x \ge x_0(\tau)$

$$(\log x)^\tau < b_{\ell_0} < x^{r_{\ell_0}} (\log x)^{-\tau}.$$

Consequently, we can apply Lemma 1 and obtain

$$\begin{aligned} &\frac{1}{x} \# \left\{ n < x \,|\, a_{q_\ell, k_j^{(\ell)}}(P(n)) = b_j^{(\ell)}, 0 \le j \le h_\ell, 1 \le \ell \le d \right\} \\ &\qquad = \frac{1}{q_1^{h_1} q_2^{h_2} \cdots q_d^{h_d}} + O\left((\log x)^{-\lambda} \sum_{\mathbf{m} \ne \mathbf{0}} |T_{\mathbf{m}_1, \ldots, \mathbf{m}_d}| \right) + O\left(\Delta + (\log x)^{-\lambda}\right), \end{aligned}$$

where $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_d)$. Since

$$\sum_{\mathbf{m} \ne \mathbf{0}} |T_{\mathbf{m}_1, \ldots, \mathbf{m}_d}| \le (2 + 2\log(1/\Delta))^{h_1 + \cdots + h_d}$$

it is possible to choose $\Delta = (\log x)^{-\lambda_1}$ for a sufficiently large constant λ_1 such that (3.2) holds.

The proof of (3.3) runs along the same lines. □

Corollary 1. *Let $N_\ell = [\log_{q_\ell} x]$, $1 \le \ell \le d$, and $\lambda, \eta > 0$. Then for integers $k_j^{(\ell)}$ satisfying*

$$N_\ell^\eta \le k_j^{(\ell)} < r_\ell N_\ell - N_\ell^\eta \quad (1 \le j \le h_\ell,\ 1 \le \ell \le d)$$

and $b_j^{(\ell)} \in E_{q_\ell}$, we uniformly have, as $x \to \infty$

$$\begin{aligned} &\frac{1}{x} \# \left\{ n < x \,|\, a_{q_\ell, k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 0 \le j \le h_\ell, 1 \le \ell \le d \right\} \\ &\qquad = \prod_{\ell=1}^{d} \left(\frac{1}{x} \# \left\{ n < x \,|\, a_{q_\ell, k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 0 \le j \le h_\ell \right\} \right) + \mathcal{O}\left((\log x)^{-\lambda} \right) \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{\pi(x)} \# \left\{ p < x \,|\, a_{q_\ell, k_j^{(\ell)}}(P_\ell(p)) = b_j^{(\ell)}, 0 \le j \le h_\ell, 1 \le \ell \le d \right\} \\ &\qquad = \prod_{\ell=1}^{d} \left(\frac{1}{\pi(x)} \# \left\{ p < x \,|\, a_{q_\ell, k_j^{(\ell)}}(P_\ell(p)) = b_j^{(\ell)}, 0 \le j \le h_\ell \right\} \right) + \mathcal{O}\left((\log x)^{-\lambda} \right). \end{aligned}$$

Proof. If there exists ℓ and j_1, j_2 with $k_{j_1}^{(\ell)} = k_{j_2}^{(\ell)}$ but $b_{j_1}^{(\ell)} \neq b_{j_2}^{(\ell)}$ then both sides are zero.

So it remains to consider the case, where for every ℓ the integers $k_j^{(\ell)}$, $1 \le j \le h_\ell$, are different, and without loss of generality we can assume that they are increasing. Hence we can directly apply Proposition 1. □

Corollary 2. *For any choice of integers k_ℓ, $1 \le \ell \le d$, we have, as $x \to \infty$*

$$\frac{1}{x}\sum_{n<x}\prod_{\ell=1}^{d}\left(\frac{f_{\ell,1}(P_\ell(n)) - M_{q_\ell,1}(x^{r_\ell})}{D_{q_\ell,1}(x^{r_\ell})}\right)^{k_\ell} - \prod_{\ell=1}^{d}\left(\frac{1}{x}\sum_{n<x}\left(\frac{f_{\ell,1}(P_\ell(n)) - M_{q_\ell,1}(x^{r_\ell})}{D_{q_\ell,1}(x^{r_\ell})}\right)^{k_\ell}\right) \to 0.$$

and

$$\frac{1}{\pi(x)}\sum_{p<x}\prod_{\ell=1}^{d}\left(\frac{f_{\ell,1}(P_\ell(p)) - M_{q_\ell,1}(x^{r_\ell})}{D_{q_\ell,1}(x^{r_\ell})}\right)^{k_\ell} - \prod_{\ell=1}^{d}\left(\frac{1}{\pi(x)}\sum_{p<x}\left(\frac{f_{\ell,1}(P_\ell(p)) - M_{q_\ell,1}(x^{r_\ell})}{D_{q_\ell,1}(x^{r_\ell})}\right)^{k_\ell}\right) \to 0.$$

Proof. In order to demonstrate, how this property can be derived we consider the case $d = 2$ and $k_1 = k_2 = 2$. Set $A_\ell = [(\log_{q_\ell} x)^\eta]$ and $B_\ell = [\log_{q_\ell} x - (\log_{q_\ell} x)^\eta]$ and observe that

$$f_{\ell,1}(P_\ell(n)) - M_{q_\ell,1}(x^{r_\ell}) = \sum_{j=A_1}^{B_1}\sum_{b\in E_{q_\ell}}\left(f_\ell(bq_\ell^j)\delta(a_{q_\ell,j}(P_\ell(n)),b) - \frac{m_{j,q_\ell}}{q_\ell}\right),$$

where $\delta(x,y)$ denotes the Kronecker delta. Hence we have

$$\begin{aligned}
&\frac{1}{x}\sum_{n<x}\left(\frac{f_{1,1}(P_1(n)) - M_{q_1,1}(x^{r_1})}{D_{q_1,1}(x^{r_1})}\right)^2\left(\frac{f_{2,1}(P_2(n)) - M_{q_2,1}(x^{r_2})}{D_{q_2,1}(x^{r_2})}\right)^2 \\
&= \sum_{j_1=A_1}^{B_1}\sum_{j_2=A_1}^{B_1}\sum_{j_3=A_2}^{B_2}\sum_{j_4=A_2}^{B_2}\sum_{b_1\in E_{q_1}}\sum_{b_2\in E_{q_1}}\sum_{b_3\in E_{q_2}}\sum_{b_4\in E_{q_2}}\frac{1}{D_{q_1,1}^2(x^{r_1})D_{q_2,1}^2(x^{r_2})}\times \\
&\times\frac{1}{x}\sum_{n<x}\left(f_1(b_1q_1^{j_1})\delta(a_{q_1,j_1}(P_1(n)),b_1) - \frac{m_{j_1,q_1}}{q_1}\right)\times \\
&\times\left(f_1(b_2q_1^{j_2})\delta(a_{q_1,j_2}(P_1(n)),b_2) - \frac{m_{j_2,q_1}}{q_1}\right)\times \\
&\times\left(f_2(b_3q_2^{j_3})\delta(a_{q_2,j_3}(P_2(n)),b_3) - \frac{m_{j_3,q_2}}{q_2}\right)\times \\
&\times\left(f_2(b_4q_2^{j_4})\delta(a_{q_2,j_4}(P_2(n)),b_4) - \frac{m_{j_4,q_2}}{q_2}\right)
\end{aligned}$$

By Corollary 1 it follows that

$$\begin{aligned}
&\frac{1}{x}\sum_{n<x}\left(f_1(b_1q_1^{j_1})\delta(a_{q_1,j_1}(P_1(n)),b_1)-\frac{m_{j_1,q_1}}{q_1}\right)\times\\
&\qquad\times\left(f_1(b_2q_1^{j_2})\delta(a_{q_1,j_2}(P_1(n)),b_2)-\frac{m_{j_2,q_1}}{q_1}\right)\times\\
&\qquad\quad\times\left(f_2(b_3q_2^{j_3})\delta(a_{q_2,j_3}(P_2(n)),b_3)-\frac{m_{j_3,q_2}}{q_2}\right)\times\\
&\qquad\qquad\times\left(f_2(b_4q_2^{j_4})\delta(a_{q_2,j_4}(P_2(n)),b_4)-\frac{m_{j_4,q_2}}{q_2}\right)\\
&=f_1(b_1q_1^{j_1})f_1(b_2q_1^{j_2})f_2(b_3q_2^{j_3})f_2(b_4q_2^{j_4})\times\\
&\quad\times\frac{1}{x}\#\Big\{n<x|a_{q_1,j_1}(P_1(n))=b_1,a_{q_1,j_2}(P_1(n))=b_2,\\
&\qquad\qquad a_{q_2,j_3}(P_2(n))=b_3,a_{q_2,j_4}(P_2(n))=b_4\Big\}\\
&\quad-f_1(b_1q_1^{j_1})f_1(b_2q_1^{j_2})f_2(b_3q_2^{j_3})\times\\
&\qquad\times\frac{1}{x}\#\Big\{n<x|a_{q_1,j_1}(P_1(n))=b_1,a_{q_1,j_2}(P_1(n))=b_2,a_{q_2,j_3}(P_2(n))=b_3\Big\}\frac{m_{j_4,q_2}}{q_2}\\
&\qquad\mp\cdots+\frac{m_{j_1,q_1}}{q_1}\frac{m_{j_2,q_1}}{q_1}\frac{m_{j_3,q_2}}{q_2}\frac{m_{j_4,q_2}}{q_2}\\
&=\left(f_1(b_1q_1^{j_1})f_1(b_2q_1^{j_2})\frac{1}{x}\#\left\{n<x|a_{q_1,j_1}(P_1(n))=b_1,a_{q_1,j_2}(P_1(n))=b_2\right\}\right)\times\\
&\qquad\times\left(f_2(b_3q_2^{j_3})f_2(b_4q_2^{j_4})\frac{1}{x}\#\left\{n<x|a_{q_2,j_3}(P_2(n))=b_3,a_{q_2,j_4}(P_2(n))=b_4\right\}\right)\\
&\quad-\left(f_1(b_1q_1^{j_1})f_1(b_2q_1^{j_2})\frac{1}{x}\#\left\{n<x|a_{q_1,j_1}(P_1(n))=b_1,a_{q_1,j_2}(P_1(n))=b_2\right\}\right)\times\\
&\qquad\quad\times\left(f_2(b_3q_2^{j_3})\frac{1}{x}\#\left\{n<x|a_{q_2,j_3}(P_2(n))=b_3\right\}\right)\frac{m_{j_4,q_2}}{q_2}\\
&\qquad\mp\cdots+\left(\frac{m_{j_1,q_1}}{q_1}\frac{m_{j_2,q_1}}{q_1}\right)\left(\frac{m_{j_3,q_2}}{q_2}\frac{m_{j_4,q_2}}{q_2}\right)+O\left((\log x)^{-\lambda}\right)\\
&=\left(\frac{1}{x}\sum_{n<x}\left(f_1(b_1q_1^{j_1})\delta(a_{q_1,j_1}(P_1(n)),b_1)-\frac{m_{j_1,q_1}}{q_1}\right)\times\right.\\
&\qquad\qquad\left.\left(f_1(b_2q_1^{j_2})\delta(a_{q_1,j_2}(P_1(n)),b_2)-\frac{m_{j_2,q_1}}{q_1}\right)\right)\times\\
&\quad\times\left(\frac{1}{x}\sum_{n<x}\left(f_2(b_3q_2^{j_3})\delta(a_{q_2,j_3}(P_2(n)),b_3)-\frac{m_{j_3,q_2}}{q_2}\right)\times\right.\\
&\qquad\qquad\left.\left(f_2(b_4q_2^{j_4})\delta(a_{q_2,j_4}(P_2(n)),b_4)-\frac{m_{j_4,q_2}}{q_2}\right)\right)\\
&\qquad+O\left((\log x)^{-\lambda}\right)
\end{aligned}$$

So we directly obtain the proposed result with an error term of the form $O((\log x)^{-\lambda+4-4\eta})$. □

By combining Lemma 4, Corollary 2, and the Frechet-Shohat theorem it follows that, as $x\to\infty$

$$\frac{1}{x}\#\left\{n<x\,\middle|\,\frac{f_{\ell,1}(P_\ell(n))-M_{q_\ell,1}(x^{r_\ell})}{D_{q_\ell,1}(x^{r_\ell})}<y_\ell,1\le\ell\le d\right\}\to\Phi(y_1)\Phi(y_2)\cdots\Phi(y_d)$$

and

$$\frac{1}{\pi(x)}\#\left\{p<x \,\middle|\, \frac{f_{\ell,1}(P_\ell(p))-M_{q_\ell,1}(x^{r_\ell})}{D_{q_\ell,1}(x^{r_\ell})}<y_\ell, 1\le \ell\le d\right\}\to \Phi(y_1)\Phi(y_2)\cdots\Phi(y_d).$$

Since

$$M_{q_\ell}(x^{r_\ell})-M_{q_\ell,1}(x^{r_\ell})=O((\log x)^\eta)$$

and

$$D_{q_\ell}(x^{r_\ell})-D_{q_\ell,1}(x^{r_\ell})=O((\log x)^\eta)$$

it also follows that

$$\max_{n<x}\left|\frac{f_\ell(P_\ell(n))-M_{q_\ell}(x^{r_\ell})}{D_{q_\ell}(x^{r_\ell})}-\frac{f_{\ell,1}(P_\ell(n))-M_{q_\ell,1}(x^{r_\ell})}{D_{q_\ell,1}(x^{r_\ell})}\right|\to 0$$

as $x\to\infty$. Consequently we finally obtain the limiting relations stated in Theorem 2.

4. Proof of the Theorem 3

The proof of Theorem 3 is similar to the proof of Theorem 2, i.e., we will prove an analogue to Proposition 1. However, the proof requires an additional ingredience, namely a proper version of Baker's theorem on linear forms. More precisely, we will use the following version due to Waldschmidt [30].

Lemma 5. *Let $\alpha_1,\alpha_2,\dots,\alpha_n$ be non-zero algebraic numbers and $b_1,b_2,\dots,b_n$ integers such that*

$$\alpha_1^{b_1}\cdots\alpha_n^{b_n}\ne 1$$

and let $A_1,A_2,\dots,A_n\ge e$ real numbers with $\log A_j\ge h(\alpha_j)$, where $h(\cdot)$ denotes the absolute logarithmic height. Set $d=[\mathbf{Q}(\alpha_1\dots,\alpha_n):\mathbf{Q}]$. Then

$$\left|\alpha_1^{b_1}\cdots\alpha_n^{b_n}-1\right|\ge \exp(-U),$$

where

$$U=2^{6n+32}n^{3n+6}d^{n+2}(1+\log d)(\log B+\log d)\log A_1\cdots\log A_n$$

and

$$B=\max\{2,|b_1|,|b_2|,\dots,|b_n|\}.$$

Corollary 3. *Let $q_1,q_2>1$ be coprime integers and m_1,m_2 integers such that $m_1\not\equiv 0 \bmod q_1$ and $m_2\not\equiv 0 \bmod q_2$. Then there exists a constant $C>0$ such that for all integers $k_1,k_2>1$*

$$\left|\frac{m_1}{q_1^{k_1}}+\frac{m_2}{q_2^{k_2}}\right|\ge \max\left(\frac{|m_1|}{q_1^{k_1}},\frac{|m_2|}{q_2^{k_2}}\right)\cdot e^{-C\log q_1\log q_2\log(\max(k_1,k_2))\cdot\log(\max(|m_1|,|m_2|))}.$$

Proof. Since $q_1,q_2>1$ are coprime integers and $m_1\not\equiv 0 \bmod q_1$, $m_2\not\equiv 0\bmod q_2$ we surely have $m_1q_1^{-k_1}+m_2q_2^{-k_2}\ne 0$. So can apply Lemma 5 for $n=3$, $\alpha_1=q_1$, $\alpha_2=q_2$, $\alpha_3=-m_2/m_1$, $b_1=k_1$, $b_2=-k_2$, $b_3=1$ and directly obtain

$$\begin{aligned}\left|\frac{m_1}{q_1^{k_1}}+\frac{m_2}{q_2^{k_2}}\right| &= |m_1|\cdot q_1^{k_1}\cdot\left|-q_1^{k_1}q_2^{-k_2}\frac{m_2}{m_1}-1\right|\\ &\ge |m_1|q_1^{k_1}e^{-C\log q_1\log q_2\log(\max(k_1,k_2))\cdot\log\max(|m_1|,|m_2|)}.\end{aligned}$$

Since the problem is symmetric it is no loss of generality to assume that $|m_1|q_1^{-k_1}\ge |m_2|q_2^{-k_2}$. □

Finally we will use the following (trivial) lemma on exponential sums.

Lemma 6. *Let α is a real number with $0 < |\alpha| \le \frac{1}{2}$. Then, as $x \to \infty$*

$$\sum_{n<x} e(\alpha n) \ll \frac{1}{|\alpha|}$$

Proposition 2. *Let $P_\ell(x) = A_\ell x + B_\ell$, $\ell = 1, 2$, be linear polynomials with integer coefficients and non-negative leading terms A_ℓ which are coprime to q_ℓ. Set $N_\ell = [\log_{q_\ell} x]$, $\ell = 1, 2$, let $\lambda > 0, \eta > 0$ be an arbitrary constant and let h_1, h_2 be positive integers. Then for integers*

$$N_\ell^\eta \le k_1^{(\ell)} < k_2^{(\ell)} < \cdots k_{h_\ell}^{(\ell)} \le N_\ell - N_\ell^\eta \quad (\ell = 1, 2) \tag{4.1}$$

we have, as $x \to \infty$

$$\begin{aligned} &\frac{1}{x}\# \left\{ n < x \,|\, a_{q_\ell, k_j^{(\ell)}}(A_\ell n + B_\ell) = b_j^{(\ell)}, 0 \le j \le h_\ell, \ell = 1, 2 \right\} \\ &\quad = \frac{1}{q_1^{h_1} q_2^{h_2}} + \mathcal{O}\left((\log x)^{-\lambda}\right) \end{aligned} \tag{4.2}$$

uniformly for $b_j^{(\ell)} \in E_{q_\ell}$ and $k_j^{(\ell)}$ in the given range, where the implicit constant of the error term may depend on q_ℓ, on h_ℓ and on λ.

Proof. The proof runs along the same lines as the proof of Proposition 1. The only problem is to estimate the sum

$$\sum_{(\mathbf{m}_1, \mathbf{m}_1) \ne \mathbf{0}} |T_{\mathbf{m}_1, \mathbf{m}_2}| \cdot \left| \frac{1}{x} \sum_{n<x} e\left((A_1 \mathbf{m}_1 \cdot \mathbf{v}_1 + A_2 \mathbf{m}_2 \cdot \mathbf{v}_2) n\right) \right|,$$

where $\mathbf{m}_\ell = (m_1^{(\ell)}, \ldots, m_{h_\ell}^{(\ell)})$ and $\mathbf{v}_\ell = \left(q_\ell^{-k_1^{(\ell)}-1}, \ldots, q_\ell^{-k_{h_\ell}^{(\ell)}-1}\right)$, $\ell = 1, 2$, such that the integer $k_j^{(\ell)}$ are in the given range (4.1).

Firstly we fix $\Delta = (\log x)^{-\lambda_0}$ with an arbitrary (but fixed) constant $\lambda_0 > 0$. Furthermore, since

$$\sum_{\exists \ell\, \exists j : |m_j^{(\ell)}| > (\log x)^{2\lambda_0}} |T_{\mathbf{m}_1, \mathbf{m}_2}| \ll (\log x)^{-\lambda_0}$$

we can restrict on those $\mathbf{m} \ne \mathbf{0}$, for which $|m_j^{(\ell)}| \le (\log x)^{2\lambda_0}$ for all ℓ, j and for which $m_j^{(\ell)} \not\equiv 0 \bmod q_\ell$ if $m_j^{(\ell)} \ne 0$.

We also note that it is also sufficient to consider just the case where $m_j^{(\ell)} \ne 0$ for all j and $\ell = 1, 2$. (Otherwisely we just reduce h_1 resp. h_2 to a smaller value and use the same arguments.)

Set $\delta = \eta/(h_1 + h_2 - 1)$. Then there exists an integer k with $0 \le k \le h_1 + h_2 - 2$ such that for all j and $\ell = 1, 2$

$$k_{j+1}^{(\ell)} - k_j^{(\ell)} \notin \left[(\log x)^{k\delta}, (\log x)^{(k+1)\delta}\right).$$

So fix k with this property. Before discussing the general case, let us consider two extremal ones.

Firstly suppose that

$$k_{j+1}^{(\ell)} - k_j^{(\ell)} < (\log x)^{k\delta}$$

for all j and $\ell = 1, 2$. Set

$$\overline{m}_\ell = A_\ell \sum_{j=1}^{h_\ell} m_j^{(\ell)} q_\ell^{k_{h_\ell}^{(\ell)} - k_j^{(\ell)}} \qquad (\ell = 1, 2).$$

Then we have $\overline{m}_\ell \not\equiv 0 \bmod q_\ell$ and

$$\log |\overline{m}_\ell| \ll (\log x)^{k\delta}$$

Hence, we can apply Corollary 3 to

$$A_1\mathbf{m}_1\cdot\mathbf{v}_1+A_2\mathbf{m}_2\cdot\mathbf{v}_2=\frac{\overline{m}_1}{q_1^{k_{h_1}^{(1)}+1}}+\frac{\overline{m}_2}{q_2^{k_{h_2}^{(2)}+1}}$$

and obtain

$$|A_1\mathbf{m}_1\cdot\mathbf{v}_1+A_2\mathbf{m}_2\cdot\mathbf{v}_2|\ge\max\left(q_1^{-k_{h_1}^{(1)}-1},q_2^{-k_{h_2}^{(1)}-1}\right)e^{-C\log\log x\,(\log x)^{k\delta}}$$

for some constant $C>0$. Since $|A_1\mathbf{m}_1\cdot\mathbf{v}_1+A_2\mathbf{m}_2\cdot\mathbf{v}_2|\le\frac12$ we get from Lemma 6

$$\begin{aligned}\left|\frac1x\sum_{n<x}e\left((A_1\mathbf{m}_1\cdot\mathbf{v}_1+A_2\mathbf{m}_2\cdot\mathbf{v}_2)n\right)\right|&\ll\frac1x q^{\log_q x-(\log x)^{(h_1+h_2-1)\delta}}e^{C\log\log x\,(\log x)^{k\delta}}\\&=e^{-(\log x)^{(h_1+h_2-1)\delta}/\log q+C\log\log x\,(\log x)^{k\delta}}\\&\ll(\log x)^{-\lambda}\end{aligned}$$

for any given $\lambda>0$.

Next suppose that

$$k_{j+1}^{(\ell)}-k_j^{(\ell)}\ge(\log x)^{(k+1)\delta}$$

for all j and $\ell=1,2$. Here we set

$$\overline{m}_\ell=A_\ell m_1^{(\ell)}\quad(\ell=1,2)$$

and obtain

$$\begin{aligned}|A_1\mathbf{m}_1\cdot\mathbf{v}_1+A_2\mathbf{m}_2\cdot\mathbf{v}_2|&\ge\left|\frac{\overline{m}_1}{q_1^{k_1^{(1)}+1}}+\frac{\overline{m}_2}{q_2^{k_1^{(2)}+1}}\right|-\left|\sum_{j_1=2}^{h_1}\frac{m_{j_1}^{(1)}}{q_1^{k_{j_1}^{(1)}+1}}\right|-\left|\sum_{j_2=2}^{h_2}\frac{m_{j_2}^{(2)}}{q_2^{k_{j_2}^{(2)}+1}}\right|\\&\ge\max\left(q_1^{-k_{h_1}^{(1)}-1},q_2^{-k_{h_2}^{(1)}-1}\right)e^{-C(\log\log x)^2}\\&\quad-O\left((\log x)^{2\lambda_0}\max\left(q_1^{-k_{h_1}^{(1)}-1},q_2^{-k_{h_2}^{(1)}-1}\right)e^{-(\log x)^{(k+1)\delta}}\right)\\&\gg\max\left(q_1^{-k_{h_1}^{(1)}-1},q_2^{-k_{h_2}^{(1)}-1}\right)e^{-C(\log\log x)^2}.\end{aligned}$$

Thus, we again have

$$\left|\frac1x\sum_{n<x}e\left((A_1\mathbf{m}_1\cdot\mathbf{v}_1+A_2\mathbf{m}_2\cdot\mathbf{v}_2)n\right)\right|\ll(\log x)^{-\lambda}\tag{4.3}$$

for any given $\lambda>0$.

In general, we assume that for some s_ℓ ($\ell=1,2$)

$$k_{j+1}^{(\ell)}-k_j^{(\ell)}<(\log x)^{k\delta}\quad(j<s_\ell)$$

and

$$k_{s_\ell+1}^{(\ell)}-k_{s_\ell}^{(\ell)}\ge(\log x)^{(k+1)\delta}$$

Here we set

$$\overline{m}_\ell=A_\ell\sum_{j=1}^{s_\ell}m_j^{(\ell)}q_\ell^{k_{s_\ell}^{(\ell)}-k_j^{(\ell)}}\quad(\ell=1,2).$$

Then we have (as in the first case) $\overline{m}_\ell\not\equiv0\bmod q_\ell$ and

$$\log|\overline{m}_\ell|\ll(\log x)^{k\delta}.$$

Furthermore, we can estimate the sums

$$\sum_{j=s_\ell+1}^{h_\ell}\frac{m_j^{(\ell)}}{q_\ell^{k_j^{(\ell)}+1}}=O\left((\log x)^{2\lambda_0}q_\ell^{-(\log x)^{(k+1)\delta}}\right).$$

Thus we get

$$|A_1\mathbf{m}_1\cdot\mathbf{v}_1 + A_2\mathbf{m}_2\cdot\mathbf{v}_2| \ge \left|\frac{\overline{m}_1}{q_1^{k_{s_1}^{(1)}+1}} + \frac{\overline{m}_2}{q_2^{k_{s_2}^{(2)}+1}}\right| - \left|\sum_{j_1=s_1+1}^{h_1}\frac{m_{j_1}^{(1)}}{q_1^{k_{j_1}^{(1)}+1}}\right| - \left|\sum_{j_2=s_2+1}^{h_2}\frac{m_{j_2}^{(2)}}{q_2^{k_{j_2}^{(2)}+1}}\right|$$

$$\ge \max\left(q_1^{-k_{s_1}^{(1)}-1}, q_2^{-k_{s_2}^{(1)}-1}\right) e^{-C\log\log x\,(\log x)^{k\delta}}$$

$$- O\left((\log x)^{2\lambda_0}\max\left(q_1^{-k_{s_1}^{(1)}-1}, q_2^{-k_{s_2}^{(1)}-1}\right) e^{-(\log x)^{(k+1)\delta}}\right)$$

$$\gg \max\left(q_1^{-k_{s_1}^{(1)}-1}, q_2^{-k_{s_2}^{(1)}-1}\right) e^{-C\log\log x\,(\log x)^{k\delta}},$$

which again implies (4.3).

Hence, we finally get

$$\sum_{(\mathbf{m}_1,\mathbf{m}_1)\ne\mathbf{0}} |T_{\mathbf{m}_1,\mathbf{m}_2}|\cdot\left|\frac{1}{x}\sum_{n<x} e\left((A_1\mathbf{m}_1\cdot\mathbf{v}_1 + A_2\mathbf{m}_2\cdot\mathbf{v}_2)n\right)\right|$$
$$= O\left((\log x)^{-\lambda_0}\right) + O\left((\log x)^{4\lambda_0-\lambda}\right),$$

which completes the proof of Proposition 2 □

5. Proof of the Theorem 4

The proof of Theorem 4 relies on a direct application of proper saddle point approximations.

Set

$$a_{k_1k_2} = \#\{n<x \mid s_{q_1}(n) = k_1, s_{q_2}(n) = k_2\}.$$

Then the *empirical characteristic function* is given by

$$\varphi_x(t_1,t_2) = \frac{1}{x}\sum_{n<x} e^{it_1 s_{q_1}(n)+it_2 s_{q_2}(n)}$$
$$= \frac{1}{x}\sum_{k_1,k_2\ge 0} a_{k_1k_2} e^{it_1k_2+it_2k_2},$$

which implies that the numbers $a_{k_1k_2}$ can be determined by

$$a_{k_1k_2} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\varphi_x(t_1,t_2)e^{-it_1k_2-it_2k_2}\,dt_1\,dt_2.$$

We first use Theorem 2 to extract the asymptotic leading term of $a_{k_1k_2}$. In fact, we need a little bit more general property.

Lemma 7. *Set*

$$M_\ell(x) := \frac{q_\ell-1}{2}\log_{q_\ell} x \quad \textit{and} \quad D_\ell(x) := \frac{q_\ell^2-1}{12}\log_{q_\ell} x$$

and let $P(x)$ denote the linear polynomial $P(x) = \mathrm{lcm}(q_1-1,q_2-1)x+B$ for some integer B with $0\le B<\mathrm{lcm}(q_1-1,q_2-1)$. Then, for every $\varepsilon>0$ there exist $x_0 = x_0(\varepsilon)$ such that

$$\left|\frac{1}{x}\sum_{n<x} e^{it_1 s_{q_1}(P(n))+it_2 s_{q_2}(P(n))} - e^{i(t_1M_{q_1}(x)+t_2M_{q_2}(x))-\frac{1}{2}(t_1^2D_{q_1}^2(x)+t_2^2D_{q_2}^2(x))}\right| < \varepsilon$$

for all $x\ge x_0$ and for all t_1,t_2, real.

Proof. First we want to notice that Theorem 2 cannot be directly applied. It may occur that the leading term $A = \mathrm{lcm}(q_1 - 1, q_2 - 1)$ of $P(x)$ is not coprime to q_1 resp. to q_2. However, if $A = q_\ell^{K_\ell}\overline{A}_\ell$ (for some $K_\ell > 0$ and $\overline{A}_\ell$ coprime to q_ℓ) and if B_ℓ has q_ℓ-ary expansion $B_\ell = B_0 + B_1 q_\ell + \cdots + B_{L_\ell} q_\ell^{L_\ell}$ then

$$\begin{aligned} s_{q_\ell}(An+B) &= s_{q_\ell}(q_\ell^{K_\ell}\overline{A}_\ell n + B_0 + B_1 q_\ell + \cdots + B_{L_\ell} q_\ell^{L_\ell}) \\ &= s_{q_\ell}(q_\ell^{K_\ell - 1}\overline{A}_\ell n + B_1 + B_2 q_\ell + \cdots + B_{L_\ell} q_\ell^{L_\ell - 1}) + B_0 \\ &= s_{q_\ell}(q_\ell^{K_\ell - 2}\overline{A} n + B_2 + B_3 q_\ell + \cdots + B_{L_\ell} q_\ell^{L_\ell - 2}) + B_0 + B_1 \\ &\vdots \\ &= s_q(\overline{A}_\ell n + \overline{B}_\ell) + \overline{C}_\ell \end{aligned}$$

for some integers $\overline{B}_\ell, \overline{C}_\ell$. Thus, the joint (normalized) limiting distribution of $(s_{q_1}(An+B), s_{q_2}(An+B))$ is the same as that of $(s_{q_1}(\overline{A}_1 n + \overline{B}_1), s_{q_2}(\overline{A}_2 n + \overline{B}_2))$, and $\overline{A}_\ell$ is coprime to q_ℓ, $\ell = 1, 2$. Hence, we can always apply Theorem 2 for properly chosen linear polynomials $P_\ell(x)$, $\ell = 1, 2$.

By Levi's theorem it now follows from Theorem 2 (and the above remark) that for every fixed t_1, t_2 we have, as $x \to \infty$

$$\frac{1}{x}\sum_{n<x} e^{i(t_1 s_{q_1}(P(n)) + t_2 s_{q_2}(P(n)))/\sqrt{\log x}} - e^{i(t_1 M_1(x) + t_2 M_{q_2}(x))/\sqrt{\log x} - \frac{1}{2}(t_1^2 D_1^2(x) + t_2^2 D_2^2(x))/(\log x)} \to 0. \tag{5.1}$$

Moreover, we can show that this convergence is uniform for all all t_1, t_2. Since $\Phi(y_1)\Phi_2(y)$ is continuous we know that the *normalized empirical distribution function*

$$\tilde{F}_x(y_1, y_2) := \frac{1}{x}\#\{n < x \mid s_{q_\ell}(n) \le M_\ell(n) + y_\ell D_\ell(x),\ \ell = 1, 2\}$$

converges uniformly to $\Phi(y_1)\Phi_2(y)$. Furthermore, the *variances*

$$\frac{1}{x}\sum_{n<x} \frac{(s_{q_\ell}(n) - M_\ell(n))^2}{D_\ell^2(x)}$$

are bounded (compare with (1.1)). Hence we get

$$\int_{\max\{|y_1|,|y_2|\} \ge A} d\tilde{F}_x(y_1, y_2) \ll \frac{1}{A^2}.$$

Thus it follows by elementary means (and by using the definition of the characteristic function) that the convergence in (5.1) ist uniform. □

The proof of Theorem 2 will also make use of the following estimate on exponential sums.

Proposition 3. *Let $q_1, q_2, \ldots, q_d > 1$ be pairwisely coprime integers. Then there exists a constant $c > 0$ such that for all all real numbers $t_1, t_2, \ldots, t_d$*

$$\left| \frac{1}{x}\sum_{n<x} e(t_1 s_{q_1}(n) + t_2 s_{q_2}(n) + \cdots + t_d s_{q_d}(n)) \right| \ll e^{-c \log x \sum_{\ell=1}^{d} \|(q_\ell - 1)t_\ell\|^2},$$

where $\|t\| = \min_{k \in \mathbf{Z}} |t - k|$ denotes the distance to the integers.

A proof of Proposition 3 can be found in [7]. It is more or less a slight generalization of a corresponding estimate of exponential sums presented by Kim [18].

Now we can start with the proof of (Theorem 4).

Proof. For any $K>0$ and integers s_1,s_2 set

$$C_K(s_1,s_2) := \left\{ (t_1,t_2)\in[-\pi,\pi]^2 : \left| t_\ell - \frac{2\pi s_\ell}{q_\ell - 1} \bmod 2\pi \right| \le \frac{K}{\sqrt{\log x}}, \ell = 1,2 \right\}.$$

Furthermore set

$$A_K := [-\pi,\pi]^2 \setminus \bigcup_{s_1=0}^{q_1-2} \bigcup_{s_2=0}^{q_2-2} C_K(s_1,s_2).$$

By Proposition 3 for every $\varepsilon>0$ there exists $K=K(\varepsilon)$ such that

$$\frac{1}{(2\pi)^2}\int_{A_K} |\varphi_x(t_1,t_2)|\, dt_1\, dt_2 \le \frac{\varepsilon}{\log x}.$$

Furthermore, we can choose $K \le c'(-\log\varepsilon)^{\frac12}$ (for some constant $c'>0$). So it remains to consider the integrals

$$\begin{aligned} I_K(s_1,s_2) &:= \frac{1}{(2\pi)^2}\int\limits_{C_K(s_1,s_2)} \left(\frac1x \sum_{n<x} e^{it_1(s_{q_1}(n)-k_1)+it_2(s_{q_2}(n)-k_2)} \right) dt_1\, dt_2 \\ &= e^{-2\pi i\left(k_1\frac{s_1}{q_1-1}+k_2\frac{s_2}{q_2-1}\right)} \frac{1}{(2\pi)^2} \times \\ &\times \int\limits_{C_K(0,0)} \left(\frac1x\sum_{n<x} e^{it_1'(s_{q_1}(n)-k_1)+it_2'(s_{q_2}(n)-k_2)} \right) e^{2\pi i\left(\frac{s_1}{q_1-1}+\frac{s_2}{q_2-1}\right)n}\, dt_1'\, dt_2'. \end{aligned}$$

By Lemma 7 it is easy to evaluate $I_K(0,0)$ asymptotically. For sufficiently large $x\ge x_0(\varepsilon)$ we have

$$\left| \varphi_x(t_1,t_2) - e^{i(t_1M_1(x)+t_2M_2(x))-\frac12(t_1^2D_1^2(x)+t_2^2D_2^2(x))} \right| < \varepsilon$$

for all t_1,t_2, real, and consequently

$$\begin{aligned} I_K(0,0) &= \frac{1}{(2\pi)^2}\int_{C_K(0,0)} e^{it_1(M_1(x)-k_1)+it_2(M_2(x)-k_2)-\frac12(t_1^2D_1^2(x)+t_2^2D_2^2(x))}\, dt_1\, dt_2 \\ &\quad + O\left(\frac{\varepsilon K^2}{\log x}\right) \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it_1(M_1(x)-k_1)+it_2(M_2(x)-k_2)-\frac12(t_1^2D_1^2(x)+t_2^2D_2^2(x))}\, dt_1\, dt_2 \\ &\quad + O\left(\frac{\varepsilon(-\log\varepsilon)}{\log x}\right) \\ &= \prod_{\ell=1}^{2}\left(\frac{1}{\sqrt{2\pi}D_{q_\ell}(x)} \exp\left(-\frac{(k_\ell - M_{q_\ell}(x))^2}{2D_{q_\ell}^2(x)} \right) \right) + O\left(\frac{\varepsilon(-\log\varepsilon)}{\log x}\right). \end{aligned} \tag{5.2}$$

In order to treat the remaining integrals $I_K(s_1,s_2)$ we recall that d and A denote $d=\gcd(q_1-1,q_2-1)$ and $A=\mathrm{lcm}(q_1-1,q_2-1)$. We represent s_1,s_2 by

$$s_\ell = m_\ell\frac{q_\ell-1}{d}+r_\ell \qquad \left(0\le m_\ell<d,\ 0\le r_\ell<\frac{q_\ell-1}{d},\ \ell=1,2\right)$$

and observe that

$$\frac{s_1}{q_1-1}+\frac{s_2}{q_2-1} = \frac{m_1+m_2}{d}+\frac{r_1}{q_1-1}+\frac{r_2}{q_2-1} = \frac{m_1+m_2}{d}+\frac{r_1\frac{q_2-1}{d}+r_2\frac{q_1-1}{d}}{A}.$$

Thus,

$$\zeta := e^{2\pi i\left(\frac{s_1}{q_1-1}+\frac{s_2}{q_2-1}\right)}$$

is always an A-th root of unity and $\zeta=1$ if and only if

$$m_1+m_2=d,\ r_1=0,\ \text{and } r_2=0. \tag{5.3}$$

Thus, if (5.3) is satisfied, i.e., $s_1 = m_1\frac{q_1-1}{d}$ and $s_2 = (d-m_1)\frac{q_2-1}{d}$, we have (recall that $k_1 \equiv k_2 \bmod d$)

$$I_K(s_1,s_2) = e^{-2\pi i \frac{m_1}{d}(k_1-k_2)} I_K(0,0) = I_K(0,0)$$

Hence

$$\sum_{m_1=0}^{d-1} I_K\left(m_1\frac{q_1-1}{d}, (d-m_1)\frac{q_2-1}{d}\right) = dI_K(0,0)$$

which fits (by (5.2) the asymptotic leading term of $a_{k_1k_2}$.

Finally we have to consider the case, where

$$\zeta = e^{2\pi i\left(\frac{s_1}{q_1-1}+\frac{s_2}{q_2-1}\right)} \ne 1.$$

Here we have

$$I_K(s_1,s_2) = e^{-2\pi i\left(k_1\frac{s_1}{q_1-1}+k_2\frac{s_2}{q_2-1}\right)} \times$$
$$\times \sum_{B=0}^{A-1}\zeta^B \int_{C_K(0,0)} \left(\frac{1}{x}\sum_{n'<(x-B)/A} e^{it_1'(s_{q_1}(An'+B)-k_1)+it_2'(s_{q_2}(An'+B)-k_2)}\right) dt_1'\, dt_2'.$$

As above, it follows by Lemma 7 that for sufficiently large $x \ge x_1(\varepsilon)$ (and of course uniformly for all $B = 0, 1, \dots, A-1$)

$$\int_{C_K(0,0)} \left(\frac{1}{x}\sum_{n'<(x-B)/A} e^{it_1'(s_{q_1}(An'+B)-k_1)+it_2'(s_{q_2}(An'+B)-k_2)}\right) dt_1'\, dt_2'$$
$$= \frac{1}{A}\prod_{\ell=1}^{2}\left(\frac{1}{\sqrt{2\pi}D_{q_\ell}(x)}\exp\left(-\frac{(k_\ell - M_{q_\ell}(x))^2}{2D_{q_\ell}^2(x)}\right)\right) + O\left(\frac{\varepsilon\log(-\varepsilon)}{\log x}\right)$$

Thus

$$I_K(s_1,s_2) = O\left(\frac{\varepsilon(-\log\varepsilon)}{\log x}\right).$$

This completes the proof of Theorem 4 □

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