The Asymptotic Number of Leftist Trees

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Abstract

It is shown that the number of leftist trees of size n in a simply generated familiy of trees is asymptotically given by $\sim \alpha c^n n^{-3/2}$ with certain constants $\alpha > 0$ and $c > 1$. Furthermore it is proved that the number of leaves in leftist trees of with n nodes satisfies a central limit theorem.

1 Introduction

Let T denote a rooted ordered tree T, i.e. its *branches* and *subbranches* etc. are ordered from left to right. The *left branch length* $\mathrm{LBL}(T)$ of T is defined by the distance $d(r,a)$ from the root r to the leftmost *leaf* a of T. (The distance $d(v,w)$ of two nodes $v,w \in V(T)$ denotes the number of edged of the unique path containing v and w.) Furthermore, for every vertex $v \in V(T)$, let T_v denote the subtree of T rooted by v. Finally, $L(T)$ denotes the set of leaves of T.

A rooted ordered tree T is called *leftist tree* if for all $v \in V(T)$

$$\mathrm{LBL}(T_v) = \min_{w \in L(T_v)} d(v,w).$$

Binary leftist trees have been introduced by Crane [4], see also Knuth [13, pp. 150–152, 157–158]. Later Kemp [8, 9] derived asymptotic formulae for the number of binary leftist trees and for various other parameters of binary leftist trees. Recently, Kemp [10, 11] generalized the notion of leftist trees to simply generated families of trees and obtained asymptotic expansions for the numbers of leftist trees with m leaves (as $m \to \infty$). Futhermore he constructed a bijection between 2-3-leftist trees and binary trees.

The purpose of this paper is to derive asymptotic expansions for the numbers of leftist trees of size n (which generalize [8]). Furthermore we prove various distributional results, e.g. that the number of leaves (in leftist trees of size n) satisfy a central limit theorem (as $n \to \infty$).

In section 2 we recall the notion of simply generated families of trees and present various relations for generating functions. With help of this terminology we will state our main results in section 3. Finally, in section 4 we provide some analytic tools which will be used for the proofs (also presented in section 4).

2 Simply Generated Leftist Trees

Let $\mathcal{F} = \mathcal{F}(\phi)$ denote a simply generated family of (rooted) trees (introduced by Meir and Moon [14]). This means that there is a (formal) power series

$$\phi(t) = c_0 + c_1 t + c_2 t^2 + \cdots \tag{1}$$

of non-negative numbers $c_0 > 0$, $c_1 \geq 0$, $c_2 \geq 0, \ldots$, which are used to define a weight $\omega(T)$ for any planted plane (rooted) tree T by

$$\omega(T) = \prod_{i \geq 0} c_i^{D_i(T)}, \tag{2}$$

where $D_i(T)$ denotes the number of nodes in T with out-degree i ($D_0(T)$ is the number of leaves, for example) such that $\mathcal{F}$ consists of all trees with weight $\omega(T) > 0$. In order to exclude the degenerate case (where only chains have positive weight) we also assume that there exists $j \geq 2$ with $c_j > 0$. Let $\mathcal{F}_n$ be the set of trees $T \in \mathcal{F}$ with $|T| = n$ (i.e. T consists of n nodes). Then the generating function

$$y(x) = \sum_{n \geq 0} y_n x^n \tag{3}$$

of the numbers

$$y_n = \sum_{|T|=n} \omega(T) = \sum_{T \in \mathcal{F}_n} \omega(T) \tag{4}$$

satisfies the (formal) functional equation

$$y = x\phi(y). \tag{5}$$

Of course, the most interesting cases are where $c_i \in \{0, 1\}$. Then we are counting special kinds of rooted trees. For example, let $c_i = 1$ for all $i \geq 0$, then y_n counts all rooted planted plane trees of size n. From $\phi(t) = 1/(1-t)$ we get $y = x/(1-y)$ and hence

$$y_n = \frac{1}{n}\binom{2n-2}{n-1}. \tag{6}$$

For $\phi(t) = 1 + t^2$ we get binary trees, for $\phi(t) = 1 + t + t^2$ Motzkin trees and so on.

Alternatively, if c_i are not only 0 or 1 then, y_n may be interpreted as a *weighted* number of trees, e.g. if $\phi(t) = 1 + 2t + t^2$ then trees with more vertices of out-degree 1 get a higher weight $\omega(T)$. After normalizing we are considering (more or less) Galton-Watson branching processes. For example, if $\phi(1) = \sum_{i \geq 0} c_i = 1$ then simply generated families of trees can be considered as realizations of a (Galton-Watson) branching process. Here $\omega(T)$ denotes the probability that T occurs (see Athreya and Ney [1]). Instead of counting a parameter (e.g. the number of leaves) we directly get the probabilistic distribution of a parameter. Moreover, if we are only interested in the distribution of discrete parameters in $\mathcal{F}_n$ (e.g. the number of leaves) resp. in the conditional distribution of the parameter of interest in realizations of size n then we can always reduce our problem to this case. In fact, if $\phi(\tau)$ exists for some $\tau > 0$ then set $\overline{c}_i = c_i\tau^i/\phi(\tau)$ and we have $\sum_{i\geq 0} \overline{c}_i = 1$ and the conditional distributions are the same (compare with Kennedy [12]). The corresponding weight $\overline{\omega}(T)$ is the probability that T occurs (in the branching process).

Now let $\mathcal{LF} = \mathcal{LF}(\phi)$ denote the set of simply generated trees in $\mathcal{F}(\phi)$ which are leftist trees. Our aim is to find asymptotic expansions for

$$b_n = \sum_{T\in\mathcal{LF},|T|=n} \omega(T).$$

However, it turns out that we cannot do this directly. We have to take into account the number of leaves. Therefore we also introduce the numbers

$$t_{n,m} = \sum_{T\in\mathcal{LF},|T|=n,|L(T)|=m} \omega(T).$$

Note that the numbers

$$\ell_m = \sum_{n\geq 1} t_{n,m} = \sum_{T\in\mathcal{LF},|L(T)|=m} \omega(T).$$

are not always well defined. (They represent the *number* of trees in $\mathcal{LF}$ with m leaves.) The set $\{T \in \mathcal{LF} : |L(T)| = m\}$ is finite if and only if $c_1 = 0$.

In [10] the following relation for the generating function was proved.

Proposition 1 *Let $\mathcal{F} = \mathcal{F}(\phi)$ a simply generated family of trees and $\mathcal{LF} = \mathcal{LF}(\phi)$ the corresponding famliy of simply generated leftist trees. Then the generating function*

$$L(x,u) = \sum_{n,m\geq 0} t_{n,m}x^n u^m$$

satisfies the functional equation

$$L(x,u) = c_0xu + L\left(x, xu\frac{\phi(L(x,u)) - c_0}{L(x,u)}\right). \tag{7}$$

It is the only solution of this equation with $L(x,0) = L(0,u) = 0$.

Corollary 1 *The generating function $B(x) = L(x,1)$ of the numbers b_n satisfies the functional equation*

$$B(x) = c_0x + L\left(x, x\frac{\phi(B(x)) - c_0}{B(x)}\right). \tag{8}$$

Corollary 2 *Suppose that $c_1 = 0$. Then the generating function $L(u) = L(1, u)$ of the numbers ℓ_m satisfies the functional equation*

$$L(u) = c_0 u + L\left(u \frac{\phi(L(u)) - c_0}{L(u)}\right). \tag{9}$$

In [10] Kemp used this relation (9) to prove an asymptotic formula for ℓ_m.

Theorem 1 *Suppose that the power series $\phi(t) = c_0 + c_1 t + c_2 t^2 + \cdots$ has non-negative coefficients $c_0 > 0$, $c_1 = 0$, $c_2 \geq 0$, $c_3 \geq 0, \ldots$, such that there exist $j \geq 2$ with $c_j > 0$ and has a positive radius of convergence R such that there exists η with $0 < \eta < R$ and*

$$\phi'(\eta) = 1.$$

Then the power series $L(u)$ has positive radius of convergence ρ_L satisfying

$$1 + \eta - \phi(\eta) \leq \rho_L < 1$$

such that $L(\rho_L) \leq 1$ and $\theta(L(\rho_L)) \leq 1$, where $\theta(t) = (\phi(t) - c_0)/t$. We also have

$$\rho_L \theta'(L(\rho_L)) L'(\rho_L \theta(L(\rho_L))) = 1$$

and a local expansion of the form

$$L(u) = L(\rho_L) - C_L \sqrt{1 - \frac{u}{\rho_L}} + \mathcal{O}(|\rho_L - u|) \qquad (u \to \rho_L)$$

with

$$C_L := \left(\frac{2\rho_L \theta'(L(\rho_L)) + \theta(L(\rho_L))}{\theta''(L(\rho_L)) + \rho_L^2 \theta'(L(\rho_L))^3 L''(\rho_L)\theta(L(\rho_L))}\right)^{1/2}.$$

Let $\overline{d}$ denote the greatest common divisor $\overline{d} = \gcd(\{j - 1 : j \geq 2, c_j > 0\})$ then the only singularities of $L(u)$ on the circle of convergence $|u| = \rho_L$ are $u_k = \rho_L e^{2\pi k i/\overline{d}}$, $0 \leq k < \overline{d}$. So we have

$$\ell_m = \overline{d} \frac{C_L}{2\sqrt{\pi}} m^{-\frac{3}{2}} \rho_L^{-m} \left(1 + O(m^{-1})\right), \quad (m \equiv 1 \bmod t, m \to \infty)$$

and $\ell_m = 0$ otherwise.

If $c_1 = 0$ we can use a variation of this approach (i.e. an application of Corollary 3 with x, u interchanged) to obtain a bivariate asymptotic expansion for $t_{n,m}$ and a-posteriori an asymptotic formula for

$$b_n = \sum_{m \geq 1} t_{n,m}.$$

However, we will use an approach (also working for $c_1 > 0$) based on the relation of Corollary 1, see section 4.

We will further discuss the asymptotic number of leftist trees of type λ, i.e. $\mathrm{LBL}(T) = \lambda$. For this purpose we introduce corresponding generating functions $L_\lambda(x, u)$. They satisfy the following relations, see [10].

Proposition 2 *The generating functions $L_\lambda(x,u)$, $\lambda \ge 1$, satisfy the functional equations*

$$L_\lambda(x,u) = L_{\lambda-j}\left(x, (c_0x)^{-1}L_{j+1}(x,u)\right) \tag{10}$$

for all $j = 0, 1, \ldots, \lambda - 1$.

Especially we have

$$\begin{aligned} L_1(x,u) &= c_0xu, \\ L_2(x,u) &= c_0x^2u\frac{\phi(L(x,u)) - c_0}{L(x,u)} = c_0x^2u\theta(L(x,u)) \end{aligned}$$

and inductively

$$\begin{aligned} L_{\lambda+1}(x,u) &= L_2\left(x, (c_0x)^{-1}L_\lambda(x,u)\right) \\ &= xL_\lambda(x,u)\theta(L(x,(c_0x)^{-1}L_\lambda(x,u))). \end{aligned}$$

3 Asymptotic Properties of Leftist Trees

Theorem 2 *Suppose that the radius of convergence of $\theta(t) = (\phi(t) - c_0)/t$ is large enough such that there exist $b > 0$ and $\rho > 0$ with*

$$b = c_0\rho + L(\rho, \rho\theta(b)), \tag{11}$$

$$1 = L_u(\rho, \rho\theta(b))\rho\theta'(b). \tag{12}$$

Then ρ is the radius of convergence of $B(x)$ and $b = B(\rho)$.

Let d denote the greatest common divisor $d = \gcd(\{j : j \ge 1, c_j > 0\})$. Then the number of leftist trees of size n in $\mathcal{LF}$ is given by

$$b_n = dCn^{-\frac{3}{2}}\rho^{-n}\left(1 + \mathcal{O}(n^{-1})\right) \quad (n \equiv 1 \bmod d, n \to \infty) \tag{13}$$

where

$$C = \left(\frac{c_0 + L_x(\rho,\rho\theta(b)) + L_u(\rho,\rho\theta(b))\rho\theta'(b)}{2\pi(L_u(\rho,\rho\theta(b))\theta''(b) + L_{uu}(\rho,\rho\theta(b))\rho\theta'(b)^2)}\right)^{1/2},$$

and $b_n = 0$ otherwise.

Remark The existence of $b > 0$ and $\rho > 0$ satisfying (11) and (12) is surely guaranteed if $\phi(t)$ is an entire function or if $\lim_{t\to R-}\phi(t) = \infty$ (where $R > 0$ denotes the radius of convergence of $\phi(t)$ which coincides with that of $\theta(t)$), see the remark following the proof of Lemma 2. Thus Theorem 2 covers all cases with $c_i \in \{0, 1\}$, i.e., all counting problems.

Theorem 3 *Let $\phi(t)$, ρ, and b be as in Theorem 2 and define μ and σ^2 by*

$$\mu := \frac{F_1}{\rho F_2}$$

and by

$$\begin{aligned}\sigma^2 &:= \mu+\mu^2\\ &+ \frac{F_2^2(F_{33}F_{11}-F_{13}^2)-2F_1F_2(F_{33}F_{12}-F_{13}F_{23})+F_1^2(F_{33}F_{22}-F_{23}^2)}{\rho F_2^3F_{33}},\end{aligned}$$

where

$$\begin{aligned}
F_1 &= c_0\rho+L_u(\rho,\rho\theta(b))\rho\theta(b),\\
F_2 &= c_0+L_x(\rho,\rho\theta(b))+L_u(\rho,\rho\theta(b))\theta(b),\\
F_{11} &= L_{uu}(\rho,\rho\theta(b))\rho^2\theta(b)^2,\\
F_{12} &= c_0+L_{xu}(\rho,\rho\theta(b))\rho\theta(b)\\
&\quad +L_{uu}(\rho,\rho\theta(b))\rho\theta(b)^2+L_u(\rho,\rho\theta(b))\theta(b),\\
F_{13} &= L_u(\rho,\rho\theta(b))\rho\theta'(b)+L_{uu}(\rho,\rho\theta(b))\rho^2\theta(b)\theta'(b),\\
F_{22} &= L_{xx}(\rho,\rho\theta(b))+2L_{xu}(\rho,\rho\theta(b))\theta(b)\\
&\quad +L_{uu}(\rho,\rho\theta(b))\theta(b)^2,\\
F_{23} &= L_u(\rho,\rho\theta(b))\theta'(b)+L_{xu}(\rho,\rho\theta(b))\rho\theta'(b)\\
&\quad +L_{uu}(\rho,\rho\theta(b))\rho\theta(b)\theta'(b),\\
F_{33} &= L_u(\rho,\rho\theta(b))\rho\theta''(b)+L_{uu}(\rho,\rho\theta(b))\rho^2\theta'(b)^2.
\end{aligned}$$

Then we have

$$\sum_{m\ge 0} m\,\frac{t_{n,m}}{b_n}=\mu n+\mathcal{O}(1)$$

and

$$\sum_{m\ge 0}(m-\mu n)^2\frac{t_{n,m}}{b_n}=\sigma^2 n+\mathcal{O}(1)$$

as $n\to\infty$.

Furthermore, if $\sigma^2\ne 0$ *then the numbers* $t_{n,m}$ *(of leftist trees of size* n *with* m *leaves in* $\mathcal{LF}$*) satisfy a central limit, i.e.*

$$\lim_{n\to\infty}\sum_{m\le \mu n+\xi\sigma\sqrt{n}}\frac{t_{n,m}}{b_n}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi}e^{-t^2/2}\,dt.$$

uniformly for all real ξ*. More precisely, we have*

$$t_{n,m}=b_n\frac{\overline{d}}{\sqrt{2\pi\sigma^2 n}}\left(\exp\left(-\frac{(m-\mu n)^2}{2\sigma^2 n}\right)+\mathcal{O}(n^{-1/2})\right)$$

uniformly for all $n,m\ge 0$ *with* $n\equiv 1 \bmod d$ *and* $m\equiv 0\bmod \overline{d}$.

Note that by Lemma 1 and 2, $L(x,u)$ is analytic for $|x|\le b+\eta$, $|u|\le\rho\theta(b)+\eta$ for some $\eta>0$. Thus, all constants in Theorem 2 and 3 are well defined.

Similar relations hold for leftist trees of type λ.

Theorem 4 *Let $\phi(t)$, ρ, and b be as in Theorem 2 and define μ and σ^2 as in Theorem 3. Then the number $b_n^{(\lambda)}$ of leftist trees of size n and type λ in $\mathcal{LF}$ is given by*

$$b_n^{(\lambda)} = p_\lambda d\, C n^{-\frac{3}{2}} \rho^{-n} \left(1 + \mathcal{O}(n^{-1})\right) \quad (n \equiv 1 \bmod d, n \to \infty)$$

where

$$p_\lambda = L_{\lambda-1}(\rho, \rho\theta(b))\rho\theta'(b).$$

If $c_1 > 0$ then there exists $D > 0$ with

$$p_\lambda \sim D \cdot (c_1\rho)^\lambda \quad (\lambda \to \infty)$$

and if $c_1 = 0$ then there exists $D_1 > 0, 0 < D_2 < 1$ with

$$p_\lambda \sim D_1 \cdot D_2^{J^\lambda} \quad (\lambda \to \infty),$$

where $J = \min\{j \geq 2 : c_j > 0\}$.

Theorem 5 *Let $\phi(t)$, ρ, and b as in Theorem 2 and define μ and σ^2 as in Theorem 3. If $\sigma^2 \neq 0$ then the numbers $t_{n,m}^{(\lambda)}$ of leftist trees of size n with m leaves of type λ in $\mathcal{LF}$ satisfy a central limit, i.e.*

$$\lim_{n\to\infty} \sum_{m \leq \mu n + \xi\sigma\sqrt{n}} \frac{t_{n,m}^{(\lambda)}}{b_n^{(\lambda)}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-t^2/2}\, dt.$$

uniformly for all real ξ. More precisely, we have

$$t_{n,m}^{(\lambda)} = b_n^{(\lambda)} \frac{\overline{d}}{\sqrt{2\pi\sigma^2 n}} \left(\exp\left(-\frac{(m-\mu n)^2}{2\sigma^2 n} \right) + \mathcal{O}(n^{-1/2}) \right)$$

uniformly for all $n, m \geq 0$ with $n \equiv 1 \bmod d$ and $m \equiv 0 \bmod \overline{d}$.

If m_n is sequence with $|m_n - \mu n| = \mathcal{O}(\sqrt{n})$ and $m \equiv 0 \bmod t$ then we also have

$$\lim_{n\to\infty, n\equiv 1 \bmod d} \frac{t_{n,m}^{(\lambda)}}{t_{n,m}} = p_\lambda.$$

Hence, p_λ may be considered as the probability that a leftist tree has type λ.

Finally we want to mention that the results of [9] for binary leftist trees concerning the average left branch length, the average right branch length, the average external path length, the average external free path length, the average left path length, the average number of root-free paths, the average number of leaves in the subtrees of the root, and the average right path length can be generalized to the case of simply generated leftist trees.

4 Analytic Tools and Proofs

One basic tool for our analysis is the following theorem, see [6].

Theorem 6 *Let us suppose that $F(x,y,u)$ is an analytic function in x, u, and y such that $F(0,y,u) \equiv 0$, that $F(x,0,u) \not\equiv 0$, and that all Taylor coefficients of F around 0 are real and non-negative. Then the unique solution $y = y(x,u)$ of the functional equation*

$$y = F(x,y,u) \tag{14}$$

with $y(0,u) = 0$ is analytic around 0 and has non-negative Taylor coefficients around 0.

Furthermore, assume that the region of convergence of $F(x,y,u)$ is large enough such that for some $u_0 > 0$ there exist non-negative solutions $x = x_0$ and $y = y_0$ of the system of equations

$$\begin{aligned} y &= F(x,y,u_0), \\ 1 &= F_y(x,y,u_0). \end{aligned}$$

with $F_x(x_0,y_0,u_0) \neq 0$ and $F_{yy}(x_0,y_0,u_0) \neq 0$. Then there exist functions $f(u)$, $g(x,u)$, $h(x,u)$ which are analytic around $x = x_0$, $u = u_0$ such that $y(x,u)$ is analytic for $|x| < x_0$ and $|u| \leq u_0$, and has a representation of the form

$$y(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{f(u)}} \tag{15}$$

locally around $x = x_0$, $u = u_0$. We have $f(u_0) = x_0$, $g(f(u),u) = y(f(u),u)$, and

$$h(f(u),u) = \sqrt{\frac{2f(u)F_x(f(u),g(f(u),u),u)}{F_{yy}(f(u),g(f(u),u),u)}}.$$

If u is real and close to u_0 then $f(u)$ is the radius of convergence of the power series $x \mapsto y(x,u)$. Moreover, if $\arg(x - f(u)) \neq 0$ then (15) provides a local analytic continuation of $y(x,u)$.

For the next step we need the following theorem by Flajolet and Odlyzko [7] about singularity analysis.

Theorem 7 (Transfer Lemma) *Let $y(x)$ be analytic in a region $|x| < R + \varepsilon$, $|\arg(x - R)| > \phi$ ($\varepsilon > 0$, $0 \leq \phi < \frac{\pi}{2}$) such that*

$$y(x) = \mathcal{O}\left(\frac{1}{(1 - x/R)^\alpha}\right) \tag{16}$$

for some real α. Then

$$y_n = [x^n]y(x) = \mathcal{O}\left(R^{-n}n^{\alpha-1}\right) \qquad (n \to \infty). \tag{17}$$

Corollary 3 *Suppose that a power series $y(x,u)$ in two variables with non-negative coefficients $y_{n,m} = [x^n u^m]y(x,u)$ has a representation of the form (15) locally around $x = x_0 > 0$ and $u = u_0 > 0$ such that for u in a neighbourhood of u_0 $R(u) = |f(u)|$ is the radius of convergence of the power series $x \mapsto y(x,u)$ and $x = f(u)$ is the only singularity on the circle $|x| = |f(u)|$. Moreover, suppose that $y(x,u)$ has an analytic continuation to the region $|x| \le R(u) + \varepsilon$, $\arg(x - R(u)) \ne 0$, for some $\varepsilon > 0$. Furthermore, define μ and σ^2 by*

$$\mu(u) = -\frac{uf'(u)}{f(u}$$

and by

$$\sigma(u)^2 = \mu(u) - \mu(u)^2 - \frac{uf''(u)}{f(u)}.$$

Then

$$[x^n]y(x,u) = \sum_{n\ge 0} y_{n,m}u^m = \frac{h(R(u)),u)}{2\sqrt{\pi}} n^{-3/2} R(u)^{-n}(1 + \mathcal{O}(n^{-1}) \tag{18}$$

uniformly for u in a neighbourhood of u_0,

$$\sum_{m\ge 0} m\, \frac{y_{n,m}u_0^m}{[x^n]y(x,u_0)} = n \cdot \mu(u_0) + \mathcal{O}(1) \tag{19}$$

and

$$\sum_{m\ge 0} (m - \mu(u_0)\, n)^2 \frac{y_{n,m}u_0^m}{[x^n]y(x,u_0)} = n \cdot \sigma(u_0)^2 + \mathcal{O}(1) \tag{20}$$

as $n \to \infty$.

If $\sigma(u_0)^2 \ne 0$ then we have

$$\frac{1}{[x^n]y(x,u_0)} \sum_{m \le \mu(u_0)n + \xi\sigma(u_0)\sqrt{n}} y_{n,m}u_0^m = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-t^2/2}\, dt + o(1) \tag{21}$$

uniformly for all $\xi \in \mathbf{R}$.

Moreoever, if the radius of convergence $R(u)$ of the power series $x \mapsto y(x,u)$ is greater than $R(u_0)$ for all $u \ne u_0$ with $|u| = u_0$ then $y_{n,m}$ is asymptotically given by

$$\begin{aligned} y_{n,m} &= u_0^{-m} \frac{h(R(u_0)),u_0)}{2\sqrt{\pi}} n^{-3/2} R(u_0)^{-n} \times \\ &\quad \times \frac{1}{\sqrt{2\pi\sigma(u_0)^2 n}} \left(\exp\left(-\frac{(m - \mu(u_0)n)^2}{2\sigma(u_0)^2 n} \right) + \mathcal{O}(n^{-1/2}) \right) \end{aligned} \tag{22}$$

uniformly for all $m \ge 0$ as $n \to \infty$.

Proof. First, by the transfer lemma (Theorem 7) it follows that

$$[x^n]y(x,u) = \sum_{n\ge 0} y_{n,m}u^m = \frac{h(R(u)),u)}{2\sqrt{\pi}} n^{-3/2}R(u)^{-n}(1+\mathcal{O}(n^{-1}))$$

uniformly for u in a neighbourhood of u_0. Now, (21) follows from Bender [2]. The asymptotic expansion (22) can be deduced from Bender and Richmond [3] or directly by using saddle point approximation of the Cauchy integral representing $y_{n,k}$ from $y_n(u) = [x^n]y(x,u)$, compare with [5].

Now observe that

$$\sum_{m\ge 0} y_{n,m}u_0^m = [x^n]y_u(x,u_0).$$

Now by differentiating (15) setting $u = u_0$ we get

$$y_u(x,u_0) = h(f(u_0),u_0)\frac{-\mu(u_0)}{2\sqrt{1-x/f(u_0)}} + C_0(x-f(u_0)) + \mathcal{O}\left(\sqrt{1-x/f(u_0)}\right).$$

Thus, Theorem 7 and (18) imply (19). The proof of (20) runs along the same lines. □

We first want to apply Theorem 6 to the functional equation

$$L(x,u) = c_0xu + L\left(x, xu\theta(L(x,u))\right),$$

where $\theta(x) = (\phi(x)-c_0)/x$. At a first glance it seems that this is impossible since $L = L(x,u)$ does not occur as a *usual variable.* We have to take into account the following two observations.

Lemma 1 *Suppose that the power series $B(x) = L(x,1)$ converges for some real $x = x_0 > 0$ then for every $0 < \eta_1 < 1$ there exists $\eta_2 > 0$ such that the power series $L(x,u)$ converges for for $|u| \le 1-\eta_1$ and $|x| \le x_0 + \eta_2$.*

Proof. First consider the case $c_1 = 0$. Here it follows by induction that $t_{n,m} > 0$ can only occur if $m \ge n-m$, i.e. $m \ge n/2$. Now for every $0 < \eta_1 < 1$ there surely exists $\eta_2 > 0$ with

$$(1+\eta_2)^2(1-\eta_1) = 1.$$

Hence

$$\begin{aligned} L(x_0(1+\eta_2), 1-\eta_1) &= \sum_{n,m} t_{n,m}x_0^n(1+\eta_2)^n(1-\eta_1)^m \\ &\le \sum_{n,m} t_{n,m}x_0^n\left((1+\eta_2)^2(1-\eta_1)\right)^m \\ &= B(x_0). \end{aligned}$$

Now suppose that $c_1 > 0$. First of all we note that $x_0 < 1/c_1$. Namely, if $c_1 > 0$ then $\mathcal{LF}$ contains all chains C_n of length $n \ge 0$ with weights $\omega(C_n) = c_0 c_1^n$. Now, if $x_0 \ge 1/c_1$ then

$$B(x_0) \ge \sum_{n\ge 0} c_0 c_1^n x_0^n \ge \sum_{n \ge 0} c_0$$

contradicts the assumption that $B(x_0)$ converges.

In a next step we show that there exist $\eta_2 > 0$ and $\eta > 0$ such that the series

$$\sum_n \sum_{m \le \eta n} t_{n,m} x_0^n (1+\eta_2)^n \tag{23}$$

converges. For this purpose we consider the corresponding simply generated family of trees $\mathcal{F}$ with corresponding numbers $y_{n,m}$ of trees with n nodes and m leaves. We trivially have $t_{n,m} \le y_{n,m}$. Let $\tau = \tau(u)$ be the solution of the equation $\tau\phi'(\tau) = \phi(\tau) + c_0(u-1)$, where $u > 0$ is (sufficiently) small. If ϕ has positive radius of convergence then $\tau(u)$ exists for (sufficiently) small u and we have $\lim\limits_{u\to 0+} \tau(u) = 0$. For any fixed $0 < u \le 1$ it follows by Theorem 6 that

$$y(x,u) = \sum_{n,m} y_{n,m} x^n u^m$$

has a representation of the form

$$y(x,u) = g(x,u) - h(x,u)\sqrt{1 - x\phi'(\tau(u))}$$

locally around $x_0(u) = 1/\phi'(\tau(u))$. ($y = y(x,u)$ satisfies the functional equation $y = x\phi(y) + xc_0(u-1)$.) Hence, by the transfer lemma (Theorem 7) we have

$$\sum_{m\ge 1} y_{n,m} u^m \le C(u) n^{-3/2} \phi'(\tau(u))^n.$$

Furthermore, since $0 < u \le 1$, this also gives

$$\sum_{m \le \eta n} y_{n,m} \le C(u) n^{-3/2} \phi'(\tau(u))^n u^{-\eta n}.$$

Since $\lim\limits_{u\to 0} \phi'(\tau(u)) = c_1$ and $x_0 < 1/c_1$ there exists $u > 0$, $\eta > 0$, and $\eta_2 > 0$ such that

$$x_0(1+\eta_2)\phi'(\tau(u))u^{-\eta} < 1.$$

Hence, the series in (23) converges.

Finally the remaining sum can be estimated as above. We just note that we have to choose $\eta_2 > 0$ small enough that $(1+\eta_2)^{1/\eta}(1-\eta_1) \le 1$.

$$\begin{aligned}
\sum_n \sum_{m \ge \eta n} t_{n,m} x_0^n (1+\eta_2)^n (1-\eta_1)^m &\le \sum_{n,m} t_{n,m} x_0^n \left((1+\eta_2)^{1/\eta}(1-\eta_1)\right)^m \\
&\le B(x_0).
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2 *Suppose that the power series $B(x) = L(x,1)$ converges for some real $x = x_0 > 0$. Then $x_0\theta(B(x_0)) < 1$ and we have*

$$B(x_0) = c_0x_0 + L\left(x_0, x_0\theta(B(x_0))\right).$$

Furthermore, if $\rho > 0$ denotes the radius of convergence $\rho > 0$ of $B(x)$ then the series $B(\rho)$ converges and we have

$$L_u(\rho, \rho\theta(B(\rho))\rho\theta'(B(\rho)) = 1.$$

Proof. If $B(x_0)$ converges then $B(x)$ is analytic for $|x| < x_0$. Hence, there surely exists $0 < r < x_0$ such that $|x|\theta(B(x))| < 1$ for $|x| \le r$. Consequently, by Lemma 1 there exists $\eta > 0$ such that the function $L(x,u)$ is analytic for $|x| \le r + \eta$ and $|u| < |x\theta(B(x))| + \eta < 1$. Hence, both sides of (8) are analytic for $|x| \le r$ and therefore coincide. Especially, we know that $L(x,u)$ is analytic for $|x| \le r + \eta$ and $|u| \le r\theta(B(r)) + \eta$. Hence, $L(x,u)$ may be considered as a known function in this region, and the only unknown function is $B(x)$. Since we know that $L_u(r, r\theta(B(r))r\theta'(B(r)) < 1$ the implicit function theorem implies that $B(x)$ can be continued analytically to the range $|x| < r+\varepsilon$ (for some $\varepsilon > 0$). Observe that for all (real) x with $r < x < r + \varepsilon$ we also have $x\theta(B(x)) < 1$. Namely, if $x\theta(B(x)) \ge 1$ then

$$c_0x + L(x, x\theta(B(x)) \ge c_0x + L(x,1) = c_0x + B(x) > B(x)$$

which is of course a contradiction.

In this way it is always possible to continue $B(x)$ to a larger region $|x| < r'+\varepsilon$ if we know that $B(r')$ converges, that $L_u(r', r'\theta(B(r'))r'\theta'(B(r')) < 1$, and that $B(x)$ satisfies (8) for $|x| \le r'$. Furthermore $r'\theta(B(r')) < 1$.

Thus, if $\rho > 0$ denotes the radius of convergence of $B(x)$ then for all real x with $0 < x < \rho$ we get $\theta(B(x)) < 1/x$. Since $\theta(B(x))$ is strictly increasing and $1/x$ is strictly increasing we thus get $\rho\theta(B(\rho)) \le 1$ and consequently $B(\rho)$ converges. It also follows that $L_u(\rho, \rho\theta(B(\rho))r\theta'(B(\rho)) = 1$ and that $B(x)$ satisfies (8) for $|x| \le \rho$. Since $x_0 \le \rho$ we have proved the lemma. □

Remark Lemma 2 shows that $\theta(B(x))$ stays bounded in the described anlaytic continuations procedure. Thus, if $\theta(t)$ is entire or $\lim_{t\to R-} \theta(t) = \infty$ (where $R > 0$ denotes the radius of convergence of $\theta(t)$) then it is impossible that we leave the region of convergence of $\theta(t)$. Thus, ρ and b satisfying (11) and (12) of Theorem 2 exist in these cases.

Now, the proof of Theorem 2 is almost immediate.

Proof of Theorem 2. Let $\rho > 0$ denote the radius of convergence of $B(x)$. Then by Lemma 2 we have that $B(\rho)$ converges, $\rho\theta(B(\rho)) < 1$ and $L_u(\rho, \rho\theta(B(\rho))r\theta'(B(\rho)) = 1$. Furthermore, by Lemma 1 there exists $\eta > 0$ such that $L(x,u)$ is convergent (and analytic) for $|x| < \rho + \eta$ and $|u| < \rho\theta(B(\rho)) + \eta$.

Hence, all assumptions of Theorem 6 are satisfied and we get that $B(x)$ has a local representation of the form

$$B(x) = g(x) - h(x)\sqrt{1 - \frac{x}{\rho}}$$

locally around $x = \rho$. Obviously, the same local expansion holds for x locally around $x = \rho e^{2\pi i l/d}$, $1 \le l \le d-1$, where $d = \gcd\{i > 0 : c_i > 0\}$. If $|x| = \rho$ and $\arg(x) \ne 2\pi l/d$ for $l = 0, 1, \ldots, d-1$ then it is an easy exercise to show that

$$|L_u(x, x\theta(B(x))x\theta'(B(x))| < L_u(|x|, |x|\theta(B(|x|))|x|\theta'(B(|x|)) = 1.$$

Hence, by the implicit function theorem, $B(x)$ has a local analytic continuation. Consequently, the only singularities on the circle of convergence $|x| = \rho$ are $x = \rho e^{2\pi i l/d}$, $1 = 0, 1, \ldots d-1$.

Now the transfer lemma (Theorem 7) (combined with the arguments used in the proof of its Corollary 3) directly gives (13). □

The proof of Theorem 3 is now also very easy.

Proof of Theorem 3. As in the proof of Theorem 2, $L(x,u)$ (for x close to ρ and u close to 1) can be considered as the *unknown function* in

$$L(x,u) = c_0 x u + F(x, xu\theta(L(x,u)),$$

where $F(x,u) := L(x,u)$ may be considered as a *known function* for this range of x and u. Again all assumptions of Theorem 6 are satisfied and we obtain an representation of the form (15). Now everything follows (almost) automatically by applying Corollary 3. □

As we will see in a moment most parts of Theorems 4 and 5 follow almost directly from the corresponding parts of Theorems 2 and 3. The proofs are completed then with asymptotic expansions for p_λ.

Proofs of Theorems 4 and 5. First observe that $L_\lambda(x,u)$ have the same structure of singularities as $L(x,u)$. We have

$$\begin{aligned} L_\lambda(x,u) &= L_{\lambda-1}(x, xu\theta(L(x,u)) \\ &= L_{\lambda-1}(x, xu\theta(g(x,u) - h(x,u)\sqrt{1 - x/f(u)})) \\ &= g_\lambda(x,u) - h_\lambda(x,u)\sqrt{1 - x/f(u)}) \end{aligned}$$

with

$$g_\lambda(\rho, 1) = L_{\lambda-1}(\rho, \rho\theta(b))$$

and

$$h_\lambda(\rho, 1) = h(\rho, 1) L_{\lambda-1,u}(\rho, \rho\theta(b))\rho\theta'(b) = h(\rho, 1)p_\lambda.$$

Moreover, $L_\lambda(x,u)$ has (of course) the same analytic continuation as $L(x,u)$. Thus, all asmptotic properties of $y_{n,m}^{(\lambda)}$ are precisely the same. Only the constant C has to be replaced by $p_\lambda C$.

Finally, we sketch the proof for the asymptotic expansions for p_λ. First of all, we have $p_\lambda = o(1)$ as $\lambda \to \infty$. Thus, $L_\lambda(\rho,1) = o(1)$ and consequently

$$\theta(L(\rho,(c_0\rho)^{-1}L_\lambda(\rho,1))) \sim \theta(L_\lambda(\rho,1)) \to \theta(0) = c_1.$$

If $c_1 > 0$ we have

$$\begin{aligned} L_{\lambda+1}(\rho,1) &= \rho L_\lambda(\rho,1)\theta(L(\rho,(c_0\rho)^{-1}L_\lambda(\rho,1))) \\ &\sim L_\lambda(\rho,1)(\rho c_1). \end{aligned}$$

Thus we obtain (after a standard bootstrapping procedure)

$$L_\lambda(\rho,1) \sim C_1(\rho c_1)^\lambda.$$

Similarly, we have

$$\begin{aligned} (L_{\lambda+1})_u(\rho,\rho\theta(b)) &= \rho L_{\lambda,u}(\rho,\rho\theta(b))\theta(L(\rho,(c_0\rho)^{-1}L_\lambda(\rho,\rho\theta(b)))) \\ &\quad +\rho L_{\lambda,u}(\rho,\rho\theta(b))\theta'(L(\rho,(c_0\rho)^{-1}L_\lambda(\rho,\rho\theta(b)))) \\ &\quad\quad \times L(\rho,(c_0\rho)^{-1}L_\lambda(\rho,\rho\theta(b)))\frac{L_{\lambda,u}(\rho,\rho\theta(b))}{c_0\rho} \\ &\sim L_{\lambda,u}(\rho,\rho\theta(b))(\rho c_1) \end{aligned}$$

which yields $p_\lambda \sim D(\rho c_1)^\lambda$ as $\lambda \to \infty$ for some constant $D > 0$.

If $c_1 = 0$ we have to take into account that

$$L_{\lambda+1}(\rho,1) = \rho L_\lambda(\rho,1)\theta(L(\rho,(c_0\rho)^{-1}L_\lambda(\rho,1))) \sim \rho c_J L_\lambda(\rho,1)^J,$$

where $J = \min\{j \geq 2 : c_j > 0\}$, and we can proceed along similar lines. $\square$

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