THE DEGREE DISTRIBUTION OF RANDOM PLANAR GRAPHS

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joint work with Omer Giménez and Marc Noy

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Contents

Random Planar Graphs

"History"

$\mathcal{R}_n$ ... **labelled planar graphs** with n vertices with uniform distribution

X_n ... number of **edges** is a random planar graph with n vertices

Denise, Vasconcellos, Welsh (1996)

$$\boxed{\mathbb{P}\{X_n > \frac{3}{2}n\} \to 1, \quad \mathbb{P}\{X_n < \frac{5}{2}n\} \to 1}.$$

(Note that $0 \le e \le 3n$ for all planar graphs.)

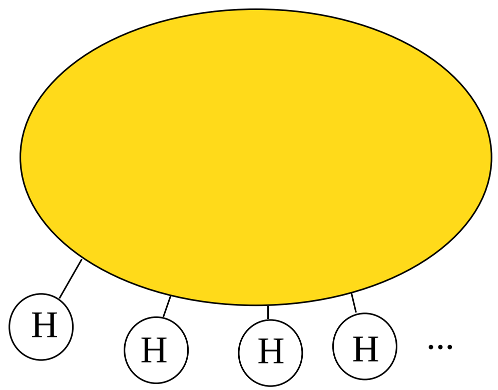
Random Planar Graphs

"History"

McDiarmid, Steger, Welsh (2005)

$$\boxed{\mathbb{P}\{H \text{ appears in } \mathcal{R}_n \text{ at least } \alpha n \text{ times}\} \to 1}$$

H ... any fixed planar graph, $\alpha > 0$ sufficiently small.



Random Planar Graphs

Consequences:

$$\boxed{\mathbb{P}\{\text{There are} \geq \alpha n \text{ vertices of degree } k\} \to 1}$$

$k > 0$ a given integer, $\alpha > 0$ sufficiently small.

$$\boxed{\mathbb{P}\{\text{There are} \geq C^n \text{ automorphisms}\} \to 1}$$

for some $C > 1$.

Random Planar Graphs

Further Results:

$$\boxed{\mathbb{P}\{\mathcal{R}_n \text{ is connected}\} \ge \gamma > 0}$$

[McDiarmid+Reed]

$$\boxed{\mathbb{E}\,\Delta_n = \Theta(\log n)}$$

Δ_n ... maximum degree in $\mathcal{R}_n$

Random Planar Graphs

The number of planar graphs

[Bender, Gao, Wormald (2002)]

b_n ... number of **2-connected** labelled planar graphs

$$\boxed{b_n \sim c \cdot n^{-\frac{7}{2}} \gamma_2^n \, n!}, \quad \gamma_2 = 26.18...$$

[Gimenez+Noy (2005)]

g_n .... number of all labelled planar graphs

$$\boxed{g_n \sim c \cdot n^{-\frac{7}{2}} \gamma^n \, n!}, \quad \gamma = 27.22...$$

Random Planar Graphs

Precise distributional results

[Gimenez+Noy (2005)]

- X_n satisfies a **central limit theorem**:

$$\mathbb{E}\, X_n \sim 2.21... \cdot n, \quad \mathbb{V}\, X_n \sim c \cdot n.$$

$$\mathbb{P}\{|X_n - 2.21... \cdot n| > \varepsilon n\} \leq e^{-\alpha(\varepsilon)\cdot n}$$

- **Connectedness:**

$$\mathbb{P}\{\mathcal{R}_n \text{ is connected}\} \to e^{-\nu} = 0.96...$$

$$\text{number of components of } \mathcal{R}_n =: C_n \to 1 + Po(\nu).$$

Random Planar Graphs

Degree Distribution

Theorem [D.+Giménez+Noy]

Let $p_{n,k}$ be the probability that a random node in a random planar graph $\mathcal{R}_n$ has degree k. Then the limit

$$\boxed{p_k := \lim_{n\to\infty} p_{n,k}}$$

exists. The probability generating function

$$\boxed{p(w) = \sum_{k\ge 1} p_k w^k}$$

can be explicitly computed; $\boxed{p_k \sim c\, k^{-\frac{1}{2}} q^k}$ for some $c > 0$ and $0 < q < 1$.

p_1	p_2	p_3	p_4	p_5	p_6
0.0367284	0.1625794	0.2354360	0.1867737	0.1295023	0.0861805

Random Planar Graphs

- Implicit equation for $D_0(y,w)$:

$$1+D_0=(1+y\boxed{w})\exp\left(\frac{\sqrt{S}(D_0(t-1)+t)}{4(3t+1)(D_0+1)}-\right.$$
$$\left.-\frac{D_0^2(t^4-12t^2+20t-9)+D_0(2t^4+6t^3-6t^2+10t-12)+t^4+6t^3+9t^2}{4(t+3)(D_0+1)(3t+1)}\right),$$

where $t=t(y)$ satisfies $y+1=\dfrac{1+2t}{(1+3t)(1-t)}\exp\left(-\dfrac{1}{2}\dfrac{t^2(1-t)(18+36t+5t^2)}{(3+t)(1+2t)(1+3t)^2}\right)$.
and $S=(D_0(t-1)+t)(D_0(t-1)^3+t(t+3)^2)$.

- Explicit expressions in terms of $D_0(y,w)$ (**SEVERAL PAGES !!!!**):

$$B_0(y,w),\ B_2(y,w),\ B_3(y,w)$$

- Explict expression for $p(w)$:

$$\boxed{p(w)=-e^{B_0(1,w)-B_0(1,1)}B_2(1,w)+e^{B_0(1,w)-B_0(1,1)}\frac{1+B_2(1,1)}{B_3(1,1)}B_3(1,w)}$$

Random Planar Graphs

Conjecture for maximum degree Δ_n

$$\frac{\Delta_n}{\log n} \to \frac{1}{\log(1/q)} \qquad \text{in probability}$$

and

$$\mathbb{E}\,\Delta_n \sim \frac{\log n}{\log(1/q)}$$

where $q = 0.6734506...$ appear in the asymptotics of $p_k \sim c\,k^{-\frac{1}{2}}q^k$;
$1/\log(1/q) = 2.529464248...$

Degree Distribution

$X_n^{(k)}$ ... **number of vertices of degree** k in a random labelled planar graph of size n

$p_{n,k}$ ... **probability that a random vertex** in a random labelled planar graph of size n **has degree** k

$\widehat{p}_{n,k}$ ... **probability that the root vertex** in a random labelled vertex rooted planar graph of size n **has degree** k

- $\boxed{p_{n,k} = \widehat{p}_{n,k}}$

- $\boxed{\mathbb{E}\, X_n^{(k)} = n\, p_{n,k}}$

Degree Distribution

Generating functions for counting planar graphs

$b_{n,m}$ ... number of **2-connected labelled planar** graphs with n vertices and m edges, $b_n = \sum_m b_{n,m}$

$$B(x,y) = \sum_{n,m} b_{n,m} \frac{x^n}{n!} y^m$$

$c_{n,m}$ ... number of **connected labelled planar** graphs with n vertices and m edges, $c_n = \sum_m c_{n,m}$

$$C(x,y) = \sum_{n,m} c_{n,m} \frac{x^n}{n!} y^m$$

$g_{n,m}$ ... number of **all labelled planar** graphs with n vertices and m edges, $g_n = \sum_m g_{n,m}$

$$G(x,y) = \sum_{n,m} g_{n,m} \frac{x^n}{n!} y^m$$

Degree Distribution

Generating functions for counting planar graphs

$$
\begin{aligned}
G(x,y) &= \exp\left(C(x,y)\right), \\
\frac{\partial C(x,y)}{\partial x} &= \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right), \\
\frac{\partial B(x,y)}{\partial y} &= \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y}, \\
\frac{M(x,D)}{2x^2D} &= \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD}, \\
M(x,y) &= x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right), \\
U &= xy(1+V)^2, \\
V &= y(1+U)^2.
\end{aligned}
$$

Degree Distribution

Asymptotic enumeration of planar graphs

$$
\begin{aligned}
b_n &= b \cdot \rho_1^{-n} n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right), \\
c_n &= c \cdot \rho_2^{-n} n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right), \\
g_n &= g \cdot \rho_2^{-n} n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\rho_1 &= 0.03819..., \\
\rho_2 &= 0.03672841..., \\
b &= 0.3704247487... \cdot 10^{-5}, \\
c &= 0.4104361100... \cdot 10^{-5}, \\
g &= 0.4260938569... \cdot 10^{-5}
\end{aligned}
$$

Degree Distribution

Generating functions for the degree distribution of planar graphs

$C^\bullet = \frac{\partial C}{\partial x}$ ... GF for graphs with a root vertex which is not counted

w ... additional variable that *counts* the **degree of the root vertex**

Generating functions:

$G^\bullet(x,y,w)$	**all rooted** planar graphs
$C^\bullet(x,y,w)$	**connected rooted** planar graphs
$B^\bullet(x,y,w)$	**2-connected rooted** planar graphs
$T^\bullet(x,y,w)$	**3-connected rooted** planar graphs

Note that $G^\bullet(x,y,1) = \dfrac{\partial G}{\partial x}(x,y)$ etc.

Degree Distribution

more precisely ....

$g^\bullet_{n,m,k}$ ... number of **vertex rooted labelled planar graphs** with $n+1$ vertices, m edges, where the (uncounted and unlabelled) **root vertex has degree** k.

$$G^\bullet(x,y,w) = \sum_{n,m,k} g^\bullet_{n,m,k} \frac{x^n}{n!} y^m w^k$$

$$\sum_k g^\bullet_{n-1,m,k} = n\, g_{n,m}, \qquad \boxed{p_{n,k} = \frac{g^\bullet_{n-1,m,k}}{n\, g_{n,m}}}$$

$$\sum_{k\ge 1} p_{n,k} w^k = \frac{1}{n\, g_{n,m}} \sum_{k\ge 1} g^\bullet_{n-1,m,k} w^k = \frac{[x^{n-1}]G^\bullet(x,1,w)}{[x^{n-1}]G^\bullet(x,1,1)}$$

$$\Longrightarrow \quad \boxed{p(w) = \sum_{k\ge 1} p_n w^k = \lim_{n\to\infty} \frac{[x^{n-1}]G^\bullet(x,1,w)}{[x^{n-1}]G^\bullet(x,1,1)}}$$

Degree Distribution

$$
\begin{aligned}
G^\bullet(x,y,w) &= \exp\left(C(x,y,1)\right) C^\bullet(x,y,w),\\
C^\bullet(x,y,w) &= \exp\left(B^\bullet\left(xC^\bullet(x,y,1),y,w\right)\right),\\
w\frac{\partial B^\bullet(x,y,w)}{\partial w} &= xyw\exp\left(S(x,y,w)+\frac{1}{x^2D(x,y,w)}T^\bullet\left(x,D(x,y,1),\frac{D(x,y,w)}{D(x,y,1)}\right)\right),\\
D(x,y,w) &= (1+yw)\exp\left(S(x,y,w)+\frac{1}{x^2D(x,y,w)}\times\right.\\
&\qquad\left.\times\, T^\bullet\left(x,D(x,y,1),\frac{D(x,y,w)}{D(x,y,1)}\right)\right)-1\\
S(x,y,w) &= xD(x,y,1)\left(D(x,y,w)-S(x,y,w)\right),\\
T^\bullet(x,y,w) &= \frac{x^2y^2w^2}{2}\left(\frac{1}{1+wy}+\frac{1}{1+xy}-1-\right.\\
&\qquad\left.-\frac{(u+1)^2\left(-w_1(u,v,w)+(u-w+1)\sqrt{w_2(u,v,w)}\right)}{2w(vw+u^2+2u+1)(1+u+v)^3}\right),\\
u(x,y) &= xy(1+v(x,y))^2,\quad v(x,y)=y(1+u(x,y))^2.
\end{aligned}
$$

Degree Distribution

with polynomials $w_1 = w_1(u, v, w)$ and $w_2 = w_2(u, v, w)$ given by

$$\begin{aligned} w_1 = &- uvw^2 + w(1 + 4v + 3uv^2 + 5v^2 + u^2 + 2u + 2v^3 + 3u^2v + 7uv) \\ &+ (u+1)^2(u + 2v + 1 + v^2), \end{aligned}$$

$$\begin{aligned} w_2 = &u^2v^2w^2 - 2wuv(2u^2v + 6uv + 2v^3 + 3uv^2 + 5v^2 + u^2 + 2u + 4v + 1) \\ &+ (u+1)^2(u + 2v + 1 + v^2)^2. \end{aligned}$$

Planar Maps vs. Planar Graphs

Whitney's Theorem

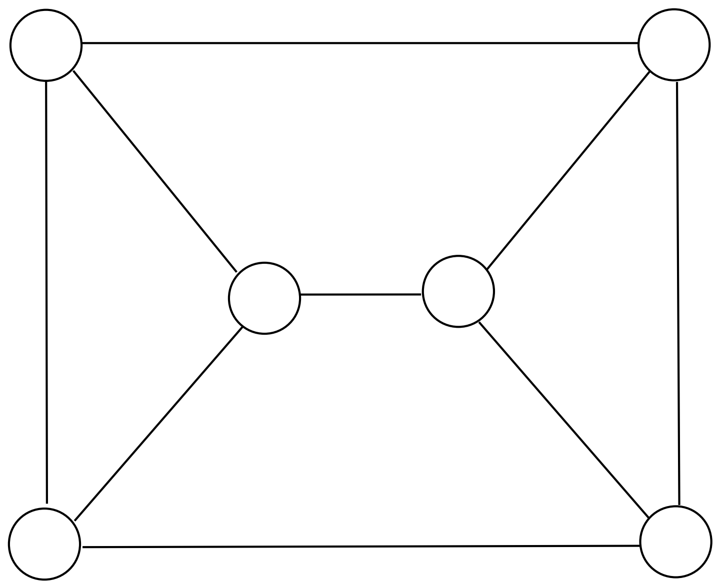
Every 3-connected planar graph has a unique embedding into the plane.

$\Longrightarrow$ The counting problem of **rooted 3-connected planar maps** is equivalent to the counting problem of **rooted (labelled) 3-connected planar graphs** (despite of a factor $(n-1)!$)

Furthermore, the counting problem of **rooted 3-connected planar maps** is equivalent to the counting problem of **rooted simple quadrangulations**.

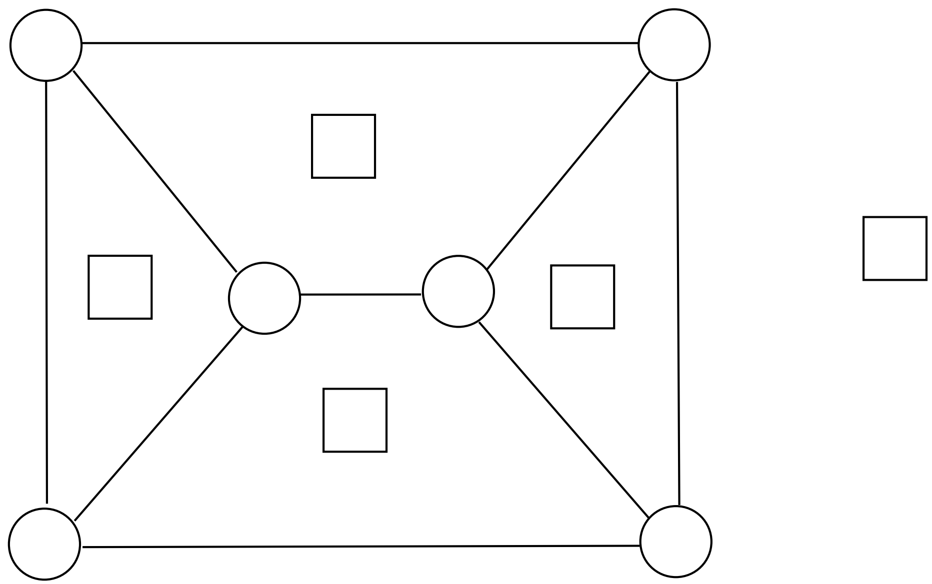
3-Connected Maps

3-connected map – simple quadrangulations



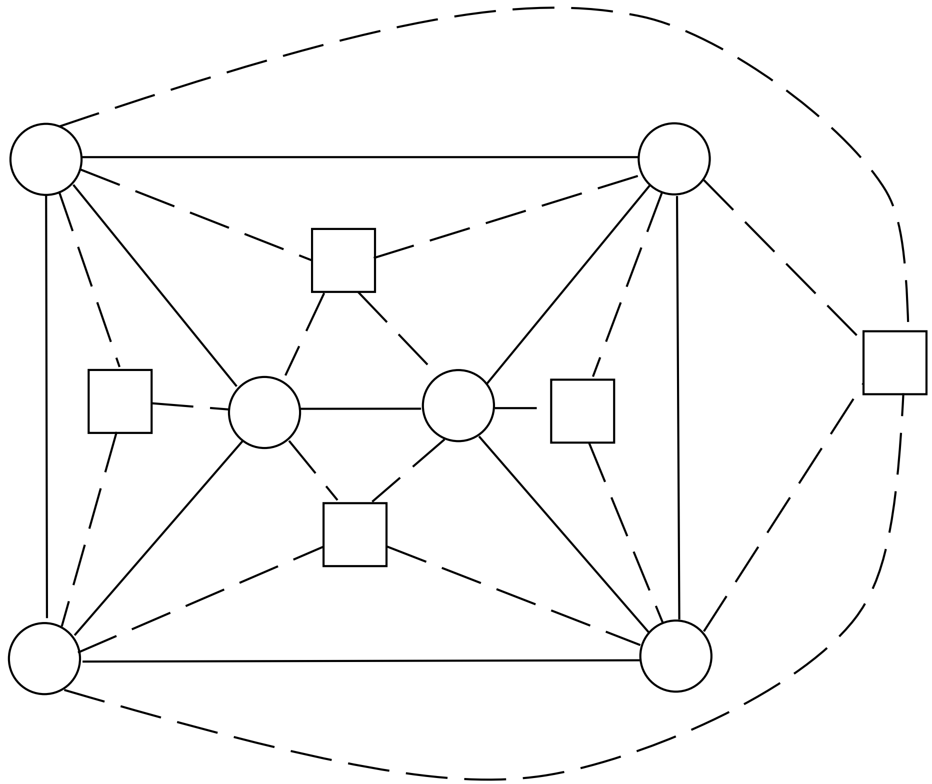
3-Connected Maps

3-connected map – simple quadrangulations



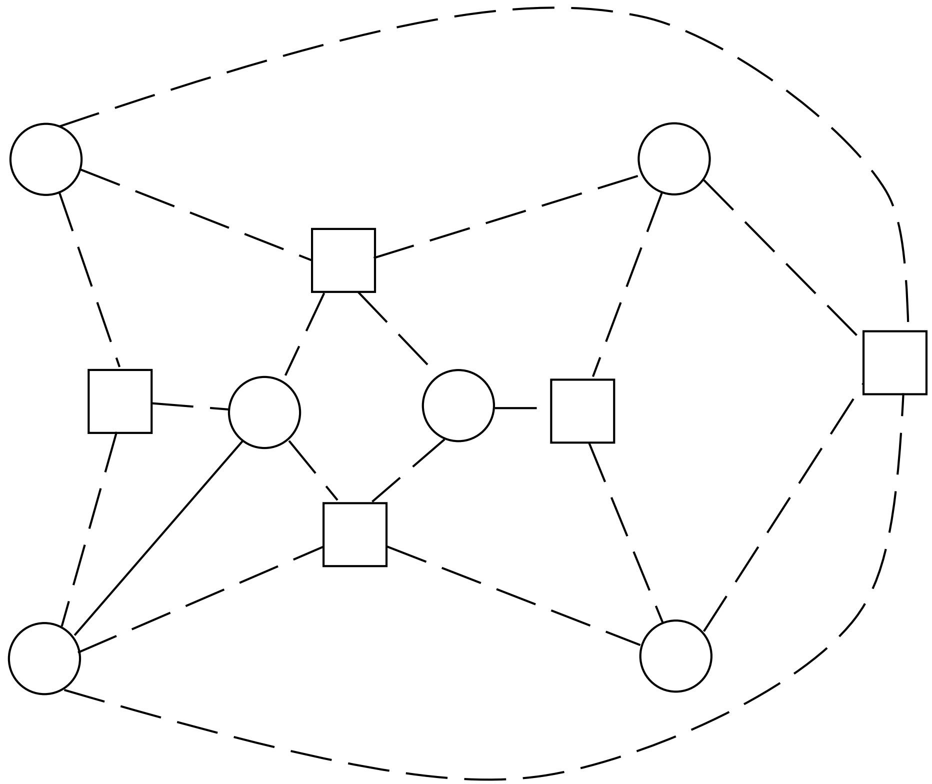
3-Connected Maps

3-connected map – simple quadrangulations



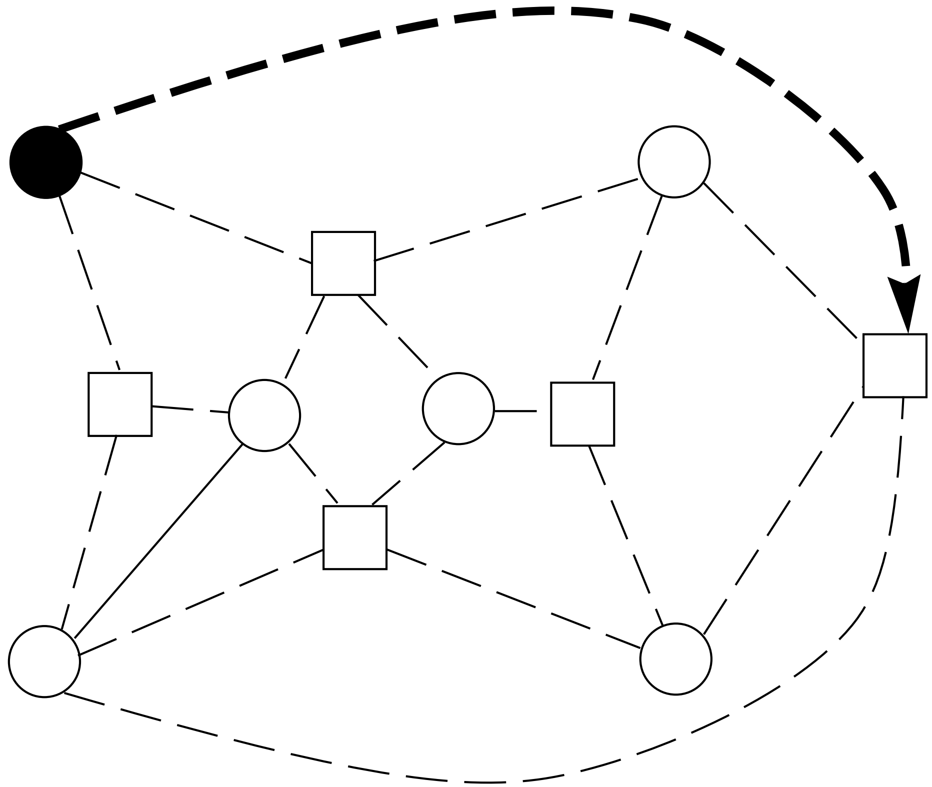
3-Connected Maps

3-connected map – simple quadrangulations



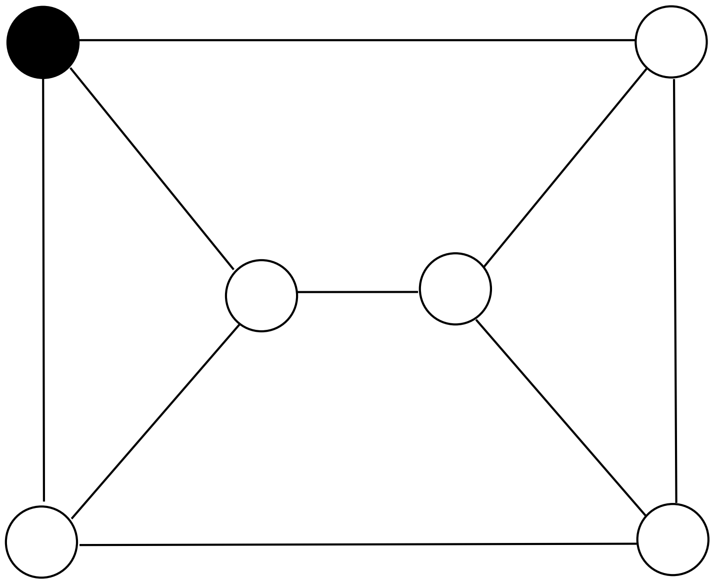
3-Connected Maps

3-connected map – simple quadrangulations



3-Connected Maps

3-connected map – simple quadrangulations



3-Connected Maps

q_{ijk} ... number of simple quadrangulations with
$i+1$ vertices of type 1 ($\circ$),
$j+1$ vertices of type 2 ($\square$), and with
root vertex of degree $k+1$

$$Q(x,y,w) = \sum_{i,j,k} q_{i,j,k} \cdot x^i \, y^j \, w^k$$

Theorem [Mullin+Schellenberg, D+Gimenez+Noy]

$$\boxed{Q(x,y,w) = xyw\left(\frac{1}{1+wy} + \frac{1}{1+x} - 1\right) - \frac{UV}{(1+U+V)^3} \cdot W(R,S,w)}$$

with ...

3-Connected Maps

with **algebraic function** $U = U(x,y)$, $V = V(x,y)$ given by

$$\boxed{U = x(V+1)^2}, \qquad \boxed{V = y(U+1)^2}$$

and

$$\boxed{W(U,V,w) = \frac{-w_1(U,V,w) + (U-w+1)\sqrt{w_2(U,V,w)}}{2(V+1)^2(Vw+U^2+2U+1)}}$$

with polynomials $w_1 = w_1(U,V,w)$ and $w_2 = w_2(U,V,w)$ given by

$$\begin{aligned}
w_1 =& -UVw^2 + w(1+4V+3UV^2+5V^2+U^2+2U+2V^3+3U^2V+7UV) \\
&+ (U+1)^2(U+2V+1+V^2), \\
w_2 =& U^2V^2w^2 - 2wUV(2U^2V+6UV+2V^3+3UV^2+5V^2+U^2+2U+4V+1) \\
&+ (U+1)^2(U+2V+1+V^2)^2.
\end{aligned}$$

3-Connected Planar Graphs

Corollary By Whitney's theorem:

$$T^\bullet(x,y,w) = \frac{xw}{2} Q(xy,y,w).$$

2-Connected Planar Graphs

Planar networks

A **network** N is a (multi-)graph with two distinguished vertices, called its poles (usually labelled 0 and ∞) such that the (multi-)graph $\hat{N}$ obtained from N by adding an edge between the poles of N is 2-connected.

Let M be a network and $X = (N_e, e \in E(M))$ a system of networks indexed by the edge-set $E(M)$ of M. Then $N = M(X)$ is called the **superposition** with core M and components N_e and is obtained by replacing all edges $e \in E(M)$ by the corresponding network N_e (and, of course, by identifying the poles of N_e with the end vertices of e accordingly).

A network N is called an h-**network** if it can be represented by $N = M(X)$, where the core M has the property that the graph $\hat{M}$ obtained by adding an edge joining the poles is 3-connected and has at least 4 vertices. Similarly $N = M(X)$ is called a p-**network** if M consists of 2 or more edges that connect the poles, and it is called an s-**network** if M consists of 2 or more edges that connect the poles in series.

2-Connected Planar Graphs

Planar networks

Trakhtenbrot's canonical network decomposition theorem: any network with at least 2 edges belongs to exactly one of the 3 classes of h-, p- or s-networks. Furthermore, any h-network has a unique decomposition of the form $N = M(X)$, and a p-network (or any s-network) can be uniquely decomposed into components which are not themselves p-networks (or s-networks).

2-Connected Planar Graphs

Planar networks

Lwt $D(x, y, w)$ and $S(x, y, w)$, respectively, the GFs of (planar) networks and series networks, with the same meaning for the variables x, y and w:

Then by a variant of [Walsh (1982)]

$$\begin{aligned}
D(x, y, w) &= (1 + yw) \exp\left(S(x, y, w) + \frac{1}{x^2 D(x, y, w)} T^{\bullet}\left(x, E(x, y), \frac{D(x, y, w)}{D(x, y, 1)}\right)\right) \\
S(x, y, w) &= xD(x, y, 1)\left(D(x, y, w) - S(x, y, w)\right),
\end{aligned}$$

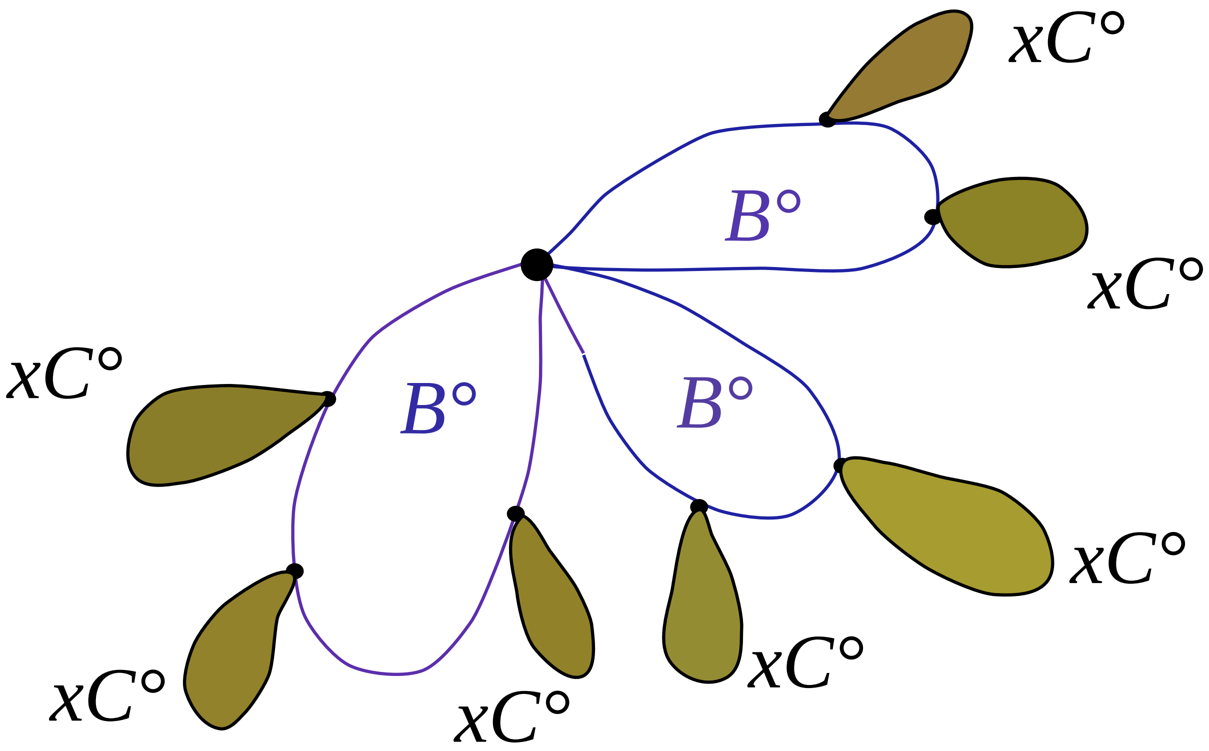
2-Connected Planar Graphs

A planar network with non-adjacent poles is obtained by distinguishing, orienting and then deleting any edge of an arbitrary 2-connected planar graph:

$$w\frac{\partial B^{\bullet}(x,y,w)}{\partial w} = xyw\,\mathsf{exp}\left(S(x,y,w) + \frac{1}{x^2D(x,y,w)}T^{\bullet}\left(x, D(x,y,1), \frac{D(x,y,w)}{D(x,y,1)}\right)\right)$$

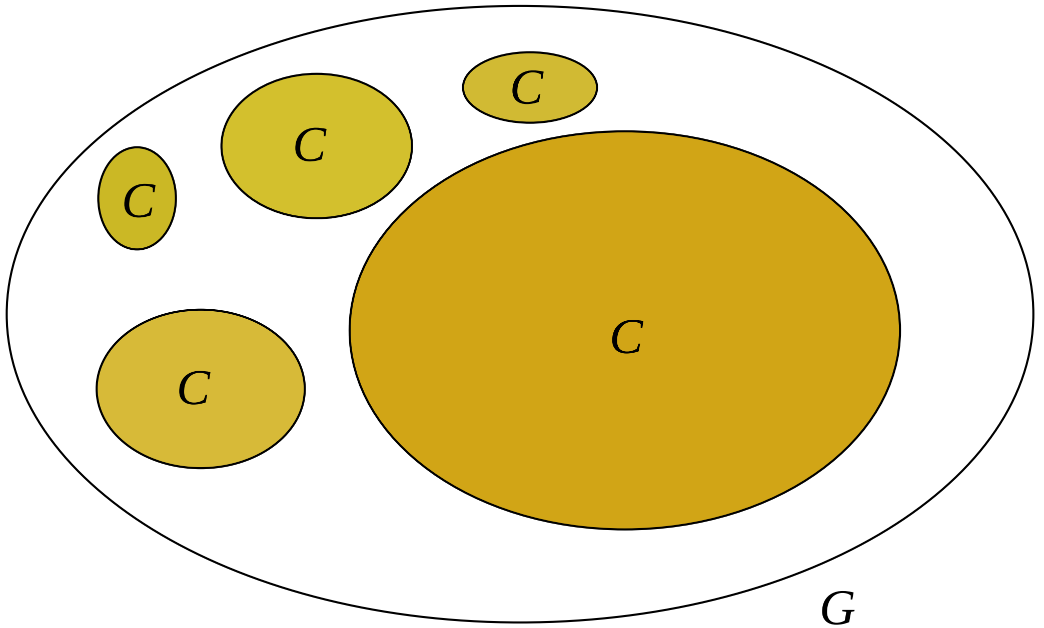
Connected Planar Graphs

$$C^{\bullet}(x,y,w) = e^{B^{\bullet}(xC^{\bullet}(x,y,1),x,w)}$$



All Planar Graphs

$$G^{\bullet}(x, y, w) = \exp\left(C(x, y, 1)\right)\, C^{\bullet}(x, y, w)$$



Asymptotics for Random Planar Graphs

Functional equations

Suppose that $\boxed{A(x,u) = \Phi(x,u,A(x,u))}$, where $\Phi(x,u,a)$ has a power series expansion at $(0,0,0)$ with non-negative coefficients and $\Phi_{aa}(x,u,a) \neq 0$.

Let $x_0 > 0$, $a_0 > 0$ (inside the region of convergence) satisfy the system of equations:

$$\boxed{a_0 = \Phi(x_0,1,a_0), \quad 1 = \Phi_a(x_0,1,a_0)}.$$

Then there exists analytic function $g(x,u), h(x,u)$, and $\rho(u)$ such that locally

$$\boxed{A(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}}.$$

Asymptotics for Random Planar Graphs

Asymptotics for coefficients

$$A(x) = g(x) - h(x)\sqrt{1 - \frac{x}{\rho}} \quad \text{(+ some technical conditions)}$$

$$\Longrightarrow \quad \boxed{[x^n]\, A(x) = \frac{h(\rho)}{2\sqrt{\pi}}\, \rho^{-n}\, n^{-\frac{3}{2}} \left(1 + O\left(\frac{1}{n}\right)\right)}.$$

Similarly:

$$A(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}} \quad \text{(+ some technical conditions)}$$

$$\Longrightarrow \quad [x^n]\, A(x,u) = \frac{h(\rho(u),u)}{2\sqrt{\pi}}\, \rho(u)^{-n}\, n^{-\frac{3}{2}} \left(1 + O\left(\frac{1}{n}\right)\right).$$

Asymptotics for Random Planar Graphs

Asymptotics for coefficients

and

$$A(x) = g(x) + h(x) \left(1 - \frac{x}{\rho}\right)^{\alpha} \quad \text{(+ some technical conditions)}$$

$$\Longrightarrow \quad [x^n]\, A(x) = \frac{h(\rho)}{\Gamma(-\alpha)}\, \rho^{-n}\, n^{-\alpha-1} \left(1 + O\left(\frac{1}{n}\right)\right).$$

Asymptotics for Random Planar Graphs

Singular expansion

$$
\begin{aligned}
A(x) &= \boxed{g(x) - h(x)\sqrt{1-\frac{x}{\rho}}} \\
&= \Big(g_0 + g_1(x-\rho) + g_2(x-\rho)^2 + \cdots\Big) \\
&\quad + \Big(h_0 + h_1(x-\rho) + h_2(x-\rho)^2 + \cdots\Big)\sqrt{1-\frac{x}{\rho}} \\
&= a_0 + a_1\left(1-\frac{x}{\rho}\right)^{\frac{1}{2}} + a_2\left(1-\frac{x}{\rho}\right)^{\frac{2}{2}} + a_3\left(1-\frac{x}{\rho}\right)^{\frac{3}{2}} + \cdots \\
&= \boxed{a_0 + a_1X + a_2X^2 + a_3X^3 + \cdots}
\end{aligned}
$$

with

$$X = \sqrt{1-\frac{x}{\rho}}.$$

Asymptotics for Random Planar Graphs

$$U(x,y) = xy(1+V(x,y))^2,$$
$$V(x,y) = y(1+U(x,y))^2$$

$$\Longrightarrow \quad U(x,y) = xy(1+y(1+U(x,y))^2)^2$$

$$\Longrightarrow \quad \boxed{U(x,y) = g(x,y) - h(x,y)\sqrt{1-\frac{y}{\tau(x)}}}$$

$$\Longrightarrow \quad V(x,y) = g_2(x,y) - h_2(x,y)\sqrt{1-\frac{y}{\tau(x)}}$$

$$M(x,y) = x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right)$$

$$!!! \quad \Longrightarrow \quad \boxed{M(x,y) = g_3(x,y) + h_3(x,y)\left(1-\frac{y}{\tau(x)}\right)^{\frac{3}{2}}}$$

due to cancellation of the $\sqrt{1-y/\tau(x)}$-term

Asymptotics for Random Planar Graphs

$$\frac{M(x,D)}{2x^2D} = \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD}$$

!!! $\Longrightarrow$ $$\boxed{D(x,y) = g_4(x,y) + h_4(x,y)\left(1 - \frac{x}{R(y)}\right)^{\frac{3}{2}}}$$

due to interaction of the singularities!!!

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\frac{1+D(x,y)}{1+y},$$

!!! $\Longrightarrow$ $$\boxed{B(x,y) = g_5(x,y) + h_5(x,y)\left(1 - \frac{x}{R(y)}\right)^{\frac{5}{2}}}$$

$\Longrightarrow$ $$\boxed{b_n \sim b \cdot R(1)^{-n} n^{-\frac{7}{2}} n!}$$

Asymptotics for Random Planar Graphs

$$B'(x,y) = g_6(x,y) + h_6(x,y)\left(1 - \frac{x}{R(y)}\right)^{\frac{3}{2}},$$

$$C'(x,y) = e^{B'(xC'(x,y),y)},$$

$$!!! \quad \Longrightarrow \quad \boxed{C'(x,y) = g_7(x,y) + h_7(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{3}{2}}}$$

due to interaction of the singularities!!!

$$\Longrightarrow \quad \boxed{C(x,y) = g_8(x,y) + h_8(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{5}{2}}}$$

$$\Longrightarrow \quad \boxed{c_n \sim c\, r(1)^{-n} n^{-\frac{7}{2}} n!}$$

Asymptotics for Random Planar Graphs

$$C(x,y) = g_8(x,y) + h_8(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{5}{2}}$$

$$\implies \quad G(x,y) = e^{C(x,y)} = g_9(x,y) + h_9(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{5}{2}}.$$

$$\implies \quad \boxed{g_n \sim g \cdot r(1)^{-n} n^{-\frac{7}{2}} n!}$$

Asymptotic Degree Distribution

3-connected planar graphs

$$T^{\bullet}(x,y,w) = \frac{x^2y^2w^2}{2}\left(\frac{1}{1+wy} + \frac{1}{1+xy} - 1 - \right.$$
$$\left. - \frac{(U+1)^2\left(-w_1(U,V,w) + (U-w+1)\sqrt{w_2(U,V,w)}\right)}{2w(Vw+U^2+2U+1)(1+U+V)^3}\right),$$

$$\tilde{u}_0(y) = -\frac{1}{3} + \sqrt{\frac{4}{9} + \frac{1}{3y}}, \quad r(y) = \frac{\tilde{u}_0(y)}{y(1+y(1+\tilde{u}_0(y))^2)^2},$$

$$\tilde{X} = \sqrt{1 - \frac{x}{r(y)}}$$

$$\Longrightarrow \quad \boxed{T^{\bullet}(x,y,w) = \tilde{T}_0(y,w) + \tilde{T}_2(y,w)\tilde{X}^2 + \tilde{T}_3(y,w)\tilde{X}^3 + O(\tilde{X}^4)}$$

due to cancellation of the $\sqrt{1 - x/r(z)}$-term.

Asymptotic Degree Distribution

Planar networks

$$D(x,y,w) = (1+yw)\exp\left(S(x,y,w) + \frac{1}{x^2 D(x,y,w)}\times\right.$$
$$\left.\times\, T^\bullet\left(x, D(x,y,1), \frac{D(x,y,w)}{D(x,y,1)}\right)\right) - 1$$
$$S(x,y,w) = xD(x,y,1)\,(D(x,y,w) - S(x,y,w))$$

$\tau(x)$ ... inverse function of $r(y)$

$$D(R(y),y,1) = \tau(R(y))$$

$$X = \sqrt{1 - \frac{x}{R(y)}}$$

$$\Longrightarrow \quad \boxed{D(x,y,w) = D_0(y,w) + D_2(y,w)X^2 + D_3(y,w)X^3 + O(X^4)},$$

Asymptotic Degree Distribution

2-connected planar graphs

$$w \frac{\partial B^\bullet(x,y,w)}{\partial w} = xyw \exp\left(S(x,y,w) + \frac{1}{x^2 D(x,y,w)} T^\bullet\left(x, D(x,y,1), \frac{D(x,y,w)}{D(x,y,1)}\right)\right)$$

$$\Longrightarrow \quad \boxed{B^\bullet(x,y,w) = B_0(y,w) + B_2(y,w)X^2 + B_3(y,w)X^3 + O(X^4)}$$

Remark. All these functions $B_j(y,w)$ can be *explicitly* computed.

Asymptotic Degree Distribution

connected planar graphs

$$C^\bullet(x,1,w) = \exp\left(B^\bullet\left(xC'(x),1,w\right)\right)$$

???

Asymptotic Degree Distribution

Lemma

$$f(x) = \sum_{n\ge 0} \boxed{a_n} \frac{x^n}{n!} = f_0 + f_2 X^2 + f_3 \boxed{X^3} + \mathcal{O}(X^4), \qquad X = \sqrt{1 - \frac{x}{\rho}},$$

$$H(x,z,w) = h_0(x,w) + h_2(x,w) Z^2 + h_3(x,w) \boxed{Z^3} + \mathcal{O}(Z^4),$$

$$Z = \sqrt{1 - \frac{z}{\boxed{f(\rho)}}},$$

$$f_H(x) = H(x, \boxed{f(x)}, w) = \sum_{n\ge 0} \boxed{b_n(w)} \frac{x^n}{n!}$$

$$\Longrightarrow \quad \boxed{\lim_{n\to\infty} \frac{b_n(w)}{a_n} = -\frac{h_2(\rho,w)}{f_0} + \frac{h_3(\rho,w)}{f_3} \left(-\frac{f_2}{f_0}\right)^{3/2}}.$$

Asymptotic Degree Distribution

connected planar graphs

$$C^\bullet(x,1,w) = \exp\left(B^\bullet\left(xC'(x),1,w\right)\right)$$

Application of the lemma with

$$f(x) = xC'(x)$$

and

$$H(x,z,w) = xe^{B^\bullet(z,1,w)}.$$

$$\Longrightarrow \quad p(w) = \lim_{n\to\infty} \frac{b_n(w)}{a_n}$$

$$= \boxed{-e^{B_0(1,w)-B_0(1,1)}B_2(1,w) + e^{B_0(1,w)-B_0(1,1)}\frac{1+B_2(1,1)}{B_3(1,1)}B_3(1,w)}$$

Random Planar Graphs

Classes of planar graphs

- **Outerplanar graphs**: all vertices are on the infinite face (equivalently no K_4 and no $K_{2,3}$ as a minor).

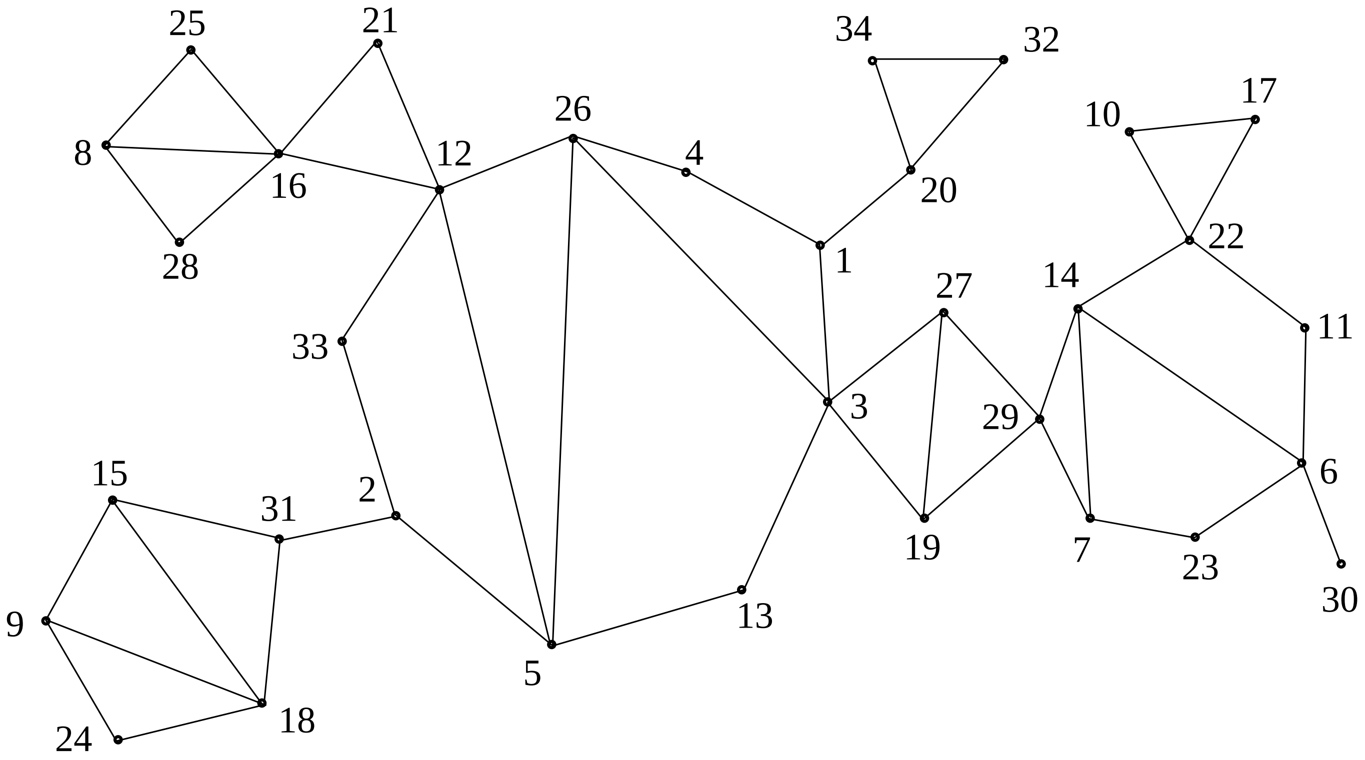
- **Series-parallel graphs**: series-parellel extension of a tree or forest (equivalently no K_4 as a minor).

- **Planar graphs**. (no K_5 and no $K_{3,3}$ as a minor)

Remark.

$$\text{outerplanar} \subseteq \text{series-parallel} \subseteq \text{planar}$$

Outerplanar Graphs



All vertices are on the infinite face.

Outerplanar Graphs

Theorem 1

Let $p_{n,k}$ be the probability that a random node in a random 2-connected, connected or unrestricted **outerplanar graph** with n vertices has degree k. Then the limit

$$\boxed{p_k := \lim_{n\to\infty} p_{n,k}}$$

exists. The probability generating function

$$\boxed{p(w) = \sum_{k\geq 1} p_k w^k}$$

can be explicitly computed.

Outerplanar Graphs

$$p(w) = \sum_{k \ge 1} p_k w^k$$

- **2-connected**

$$p(w) = \frac{2(3 - 2\sqrt{2})w^2}{(1 - (\sqrt{2} - 1)w)^2}$$

- **connected** or **unrestricted**:

$$p(w) = \frac{c_1 w^2}{(1 - c_2 w)^2} \exp\left(c_3 w + \frac{c_4 w^2}{(1 - c_2 w)}\right)$$

(with certain constants $c_1, c_2, c_3, c_4 > 0$).

Outerplanar Graphs

Theorem 2

$X_n^{(k)}$ … **number of vertices of degree** k in random 2-connected, connected or unrestricted **labelled outerplanar** graphs with n vertices.

$\Longrightarrow$ $X_n^{(k)}$ satisfies a **central limit theorem** with

$$\mathbb{E}\,X_n^{(k)} \sim \mu_k n \quad \text{and} \quad \mathbb{V}\,X_n^{(k)} \sim \sigma_k^2 n.$$

Remark. $\mu_k = p_k$.

Outerplanar Graphs

Theorem 3

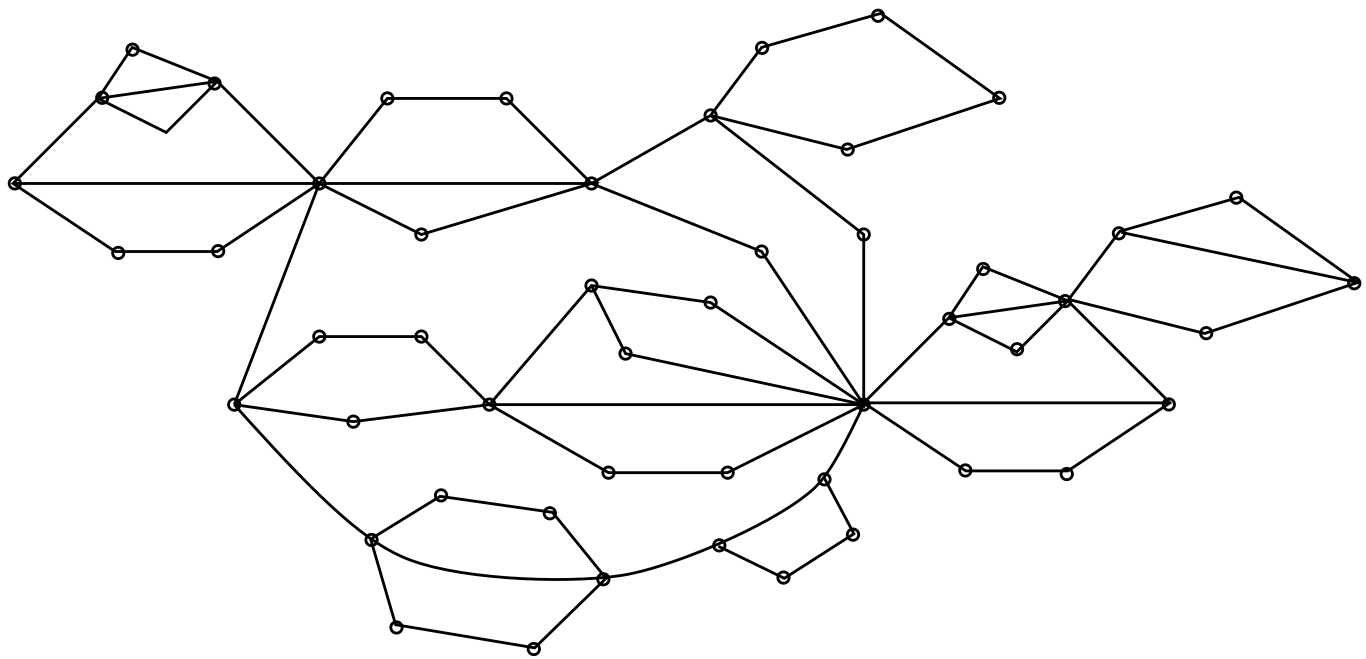
Δ_n ... **maximum degree** in random 2-connected, connected or unrestricted **labelled outerplanar** graphs with n vertices.

$$\Longrightarrow \quad \boxed{\frac{\Delta_n}{\log n} \to c \quad \text{in probability}}$$

$$\mathbb{E}\,\Delta_n \sim c \log n,$$

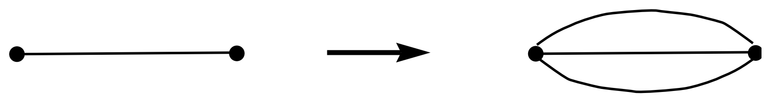
where $c = 1/\log(1/q)$ and $1/q$ in radius of convergence of $p(w)$.

Series-Parallel Graphs



Series-parallel extension of a tree or forest

Series-extension: 

Parallel-extension: 

Series-Parallel Graphs

Theorem 1

Let $p_{n,k}$ be the probability that a random node in a random 2-connected, connected or unrestricted series-parallel graph with n vertices has degree k. Then the limit

$$\boxed{p_k := \lim_{n\to\infty} p_{n,k}}$$

exists. The probability generating function

$$\boxed{p(w) = \sum_{k\geq 1} p_k w^k}$$

can be explicitly computed.

Series-Parallel Graphs

We just mention the case of

2-connected series-parallel graphs $p(w) = \sum_{k \geq 1} p_k w^k$:

$$\boxed{p(w) = \frac{B_1(1, w)}{B_1(1, 1)}},$$

where $B_1(y, w)$ is given by the following procedure ...

Series-Parallel Graphs

$$
\begin{aligned}
\frac{E_0(y)^3}{E_0(y)-1} &= \left(\log\frac{1+E_0(y)}{1+R(y)} - E_0(y)\right)^2, \\
R(y) &= \frac{\sqrt{1-1/E_0(y)}-1}{E_0(y)}, \\
E_1(y) &= -\left(\frac{2R(y)E_0(y)^2(1+R(y)E_0(y))^2}{(2R(y)E_0(y)+R(y)^2E_0(y)^2)^2+2R(y)(1+R(y)E_0(y))}\right)^{\frac{1}{2}}, \\
D_0(y,w) &= (1+yw)e^{\frac{R(y)E_0(y)}{1+R(y)E_0(y)}D_0(y,w)} - 1, \\
D_1(y,w) &= \frac{(1+D_0(y,w))^{\frac{R(y)E_1(y)D_0(y,w)}{1+R(y)E_0(y)}}}{1-(1+D_0(y,w))^{\frac{R(y)E_0(y)D_0(y,w)}{1+R(y)E_0(y)}}}, \\
B_0(y,w) &= \frac{R(y)D_0(y,w)}{1+R(y)E_0(y)} - \frac{R(y)^2E_0(y)D_0(y,w)^2}{2(1+R(y)E_0(y))}, \\
B_1(y,w) &= \frac{R(y)D_1(y,w)}{1+R(y)E_0(y)} - \frac{R(y)^2E_0(y)D_0(y,w)D_1(y,w)}{1+R(y)E_0(y)} \\
&\quad - \frac{R(y)^2E_1(y)D_0(y,w)(1+D_0(y,w)/2)}{(1+R(y)E_0(y))^2}.
\end{aligned}
$$

Series-Parallel Graphs

Theorem 2

$X_n^{(k)}$ ... **number of vertices of degree** k in random 2-connected, connected or unrestricted **labelled series-parallel** graphs with n vertices.

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Remark. $\mu_k = p_k$.

Series-Parallel Graphs

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References

Articles

O. Giménez and M. Noy,
Asymptotic enumeration and limit laws of planar graphs,
J. Amer. Math. Soc. (to appear).

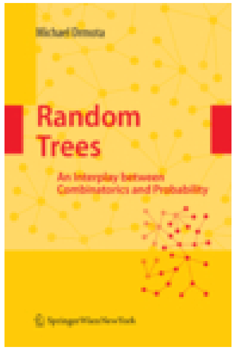
M. Drmota, O. Giménez, and M. Noy,
The number of vertices of given degree in series-parallel graphs
Random Structures and Algorithms (to appear).

M. Drmota, O. Giménez, and M. Noy,
Degree distribution in random planar graphs, *manuscript*.

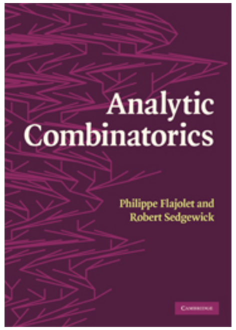
M. Drmota, O. Giménez, and M. Noy,
The maximum degree of planar graphs, *draft*.

General References

Books



Michael Drmota,
Random Trees, Springer, Wien-New York, 2009.



Philippe Flajolet and Robert Sedgewick,
Analytic Combinatorics, Cambridge University Press, 2009.
(http://algo.inria.fr/flajolet/Publications/books.html)

Thank You!